On Small-type formulae for curves in PSL(2,C)

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Differential Geometry — *On Small-type formulae for curves in* $\mathrm{PSL}(2,\mathbb{C})$, by

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Abstract. — Taking the point of view of Legendrian curves in complex projective 3-space and using the classical Lie–Klein duality of algebraic geometry, we revisit various meromorphic representation formulae for Legendrian and null curves in $\mathrm{PSL}(2,\mathbb{C})$. These include Small's formula for null curves, the formulae for Legendrian curves given by Kokubu, Umehara and Yamada, by Gálvez, Martínez and Milán, and more recently by Ejiri and Takahashi. We discuss their relationships and provide a dictionary for their meromorphic data. Some examples related to the W-curves of complex projective 3-space are discussed.

Key words: Constant mean curvature one surfaces, flat fronts, null curves, Legendrian curves, Lie–Klein correspondence.

Mathematics Subject Classifications: 53C42, 53A10.

1. Introduction

It is well known that the holomorphicity of a suitable curve in $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$ characterizes both surfaces of constant mean curvature one (CMC 1) and flat fronts in hyperbolic 3-space H^3. Actually, Bryant [3] proved that any holomorphic null immersion into $\mathrm{PSL}(2,\mathbb{C})$ projects to a CMC 1 surface in H^3, and that every such surface locally lifts to a holomorphic null immersion into $\mathrm{PSL}(2,\mathbb{C})$. Analogously, Gálvez, Martínez and Milán [8] proved that any holomorphic Legendrian map into $\mathrm{PSL}(2,\mathbb{C})$ projects to a flat surface in H^3, and that any flat surface locally arises in this way. This result has later been extended by Kokubu, Umehara and Yamada [10], [11] to the case of flat fronts, namely flat surfaces admitting a special type of singularities.

In [14], Small obtained a representation formula that generates null meromorphic curves in $\mathrm{PSL}(2,\mathbb{C})$ from pairs of meromorphic functions on a Riemann surface, called the hyperbolic and the secondary Gauss maps. Small's formula only involves the derivation of the two Gauss maps. A simpler proof of the Small formula was given by Kokubu, Umehara and Yamada in [10]. Alternative proofs were also given by de Lima and Roitman [6], using a method that goes back to Bianchi, and by Sa Earp and Toubiana [13].

In [10], Kokubu, Umehara and Yamada, besides proving Small's formula, provided a Small-type representation formula for Legendrian meromorphic

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curves into $\mathrm{PSL}(2,\mathbb{C})$ in terms of two meromorphic functions on a Riemann surface, which are called the hyperbolic Gauss maps. A special instance of this formula provides a representation formula which appeared implicitly in the work of Gálvez, Martínez and Milán [8]. A third representation formula for meromorphic Legendrian maps into $\mathrm{PSL}(2,\mathbb{C})$ has recently been obtained by Ejiri and Takahashi [7] as an application of the Bryant formula for Legendrian curves in $\mathbb{C}P^3$ (cf. [2]). Unlike the other known representation formulae, this formula, which we refer to as the Ejiri–Takahashi formula, only involves the derivation of the meromorphic data.

One purpose of this paper is to clarify the relationships between the meromorphic data of the Kokubu–Umehara–Yamada formulae and those of the Ejiri–Takahashi formula for Legendrian curves into $\mathrm{PSL}(2,\mathbb{C})$ and provide a dictionary to translate the ones into the others. This is done in Section 2. In Section 3, using an explicit construction relating to the classical Lie–Klein duality of algebraic geometry, we associate with a Legendrian curve in $\mathrm{PSL}(2,\mathbb{C})$ a null curve in $\mathrm{PSL}(2,\mathbb{C})$ and express its Small meromorphic data in terms of the meromorphic data of Kokubu, Umehara ad Yamada and of Ejiri–Takahashi of the original Legendrian curve (cf. Theorem 6). Finally, some examples related to the W-curves of $\mathbb{C}P^3$ are discussed.

2. Representation formulae for Legendrian curves in $\mathrm{PSL}(2,\mathbb{C})$

2.1. Legendrian curves in $\mathrm{PSL}(2,\mathbb{C})$

Let $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I_2\}$ be the projective special linear group. The elements of $\mathrm{PSL}(2,\mathbb{C})$ are written as equivalent classes $[A]$ of 2×2 unimodular complex matrices A and the Lie algebra of $\mathrm{PSL}(2,\mathbb{C})$ is identified with $\mathfrak{sl}(2,\mathbb{C})$.

Let M be a Riemann surface. A meromorphic map $\Phi : M \to \mathrm{PSL}(2,\mathbb{C})$ is a map which is represented as

$$\Phi = \left[\begin{pmatrix} \Phi_1^1 & \Phi_2^1 \\ \Phi_1^2 & \Phi_2^2 \end{pmatrix}\right] = \left[\frac{1}{\sqrt{\hat\Phi_1^1\hat\Phi_2^2 - \hat\Phi_1^2\hat\Phi_1^1}}\begin{pmatrix} \hat\Phi_1^1 & \hat\Phi_2^1 \\ \hat\Phi_1^2 & \hat\Phi_2^2 \end{pmatrix}\right],$$

where $\Phi_1^1\Phi_2^2 - \Phi_1^2\Phi_1^1 = 1$ and the $\hat\Phi_j^i$ are meromorphic functions on M (cf. [10]).

Let $\Omega = (\Omega_j^i)$ be the Maurer–Cartan form of $\mathrm{PSL}(2,\mathbb{C})$. From the structure equations $d\Omega + \Omega\wedge\Omega = 0$, it follows that the 1-form Ω_1^1 defines a holomorphic contact form on $\mathrm{PSL}(2,\mathbb{C})$. Accordingly, a meromorphic map $\Phi : M \to \mathrm{PSL}(2,\mathbb{C})$ is called a *Legendrian curve* (or a *contact curve*) if the pull-back of Ω_1^1 by Φ vanishes identically, i.e.

$$\Phi_2^2\, d\Phi_1^1 - \Phi_2^1\, d\Phi_1^2 = 0.$$

For a Legendrian curve Φ, two meromorphic functions

$$G_0 := \frac{\Phi_1^1}{\Phi_1^2}, \quad G_1 := \frac{\Phi_2^1}{\Phi_2^2} \tag{1}$$

are defined, which in [8] are called the *hyperbolic Gauss maps*. A meromorphic 1-form on M is also defined by

$$\omega := \begin{cases} d\Phi_1^1/\Phi_2^1 & (\text{if } d\Phi_1^1 \neq 0 \text{ or } \Phi_2^1 \neq 0), \\ d\Phi_1^2/\Phi_2^2 & (\text{if } d\Phi_1^2 \neq 0 \text{ or } \Phi_2^2 \neq 0), \end{cases}$$

which is called the *canonical form*, see [10]. Here $d\Phi_1^1 \neq 0$ (respectively, $\Phi_2^1 \neq 0$) means that the 1-form $d\Phi_1^1$ (respectively, the function Φ_2^1) is not identically zero on M.

Remark 1. Hyperbolic 3-space may be viewed as

$$H^3 = \mathrm{PSL}(2,\mathbb{C})/\mathrm{PSU}(2) = \{AA^* : A \in \mathrm{PSL}(2,\mathbb{C})\} \quad (A^* = \bar{A}^T). \tag{2}$$

It has been shown in [8] (cf. also [11]) that if $\Phi : M \to \mathrm{PSL}(2,\mathbb{C})$ is a holomorphic Legendrian immersion of a Riemann surface M, then $\varphi = \Phi\Phi^* : M \to H^3$ is a flat front, that is a flat surface with special type of singularities (cf. [11] for details). Conversely, any flat front is locally the hyperbolic projection of a holomorphic Legendrian immersion into $\mathrm{PSL}(2,\mathbb{C})$.

The following representation formula for meromorphic Legendrian curves was obtained by Kokubu, Umehara and Yamada.

Theorem 1 [10]. *Let G_0 and G_1 be two nonconstant meromorphic functions on a Riemann surface M such that $G_0 \neq G_1$. Assume that*

1. *all poles of the* 1*-form $dG_0/(G_0 - G_1)$ are of order* 1*;*
2. $\int_\alpha dG_0/(G_0 - G_1) \in \pi i\mathbb{Z}$ *for each loop α in M.*

Then

$$\Phi = \left[\begin{pmatrix} G_0/\xi & \xi G_1/(G_0 - G_1) \\ 1/\xi & \xi/(G_0 - G_1) \end{pmatrix}\right], \tag{3}$$

where

$$\xi := c \exp \int_{z_0}^{z} \frac{dG_0}{G_0 - G_1}, \quad c \in \mathbb{C}\backslash\{0\} \tag{4}$$

and $z_0 \in M$ is a base point, is a nonconstant Legendrian curve in $\mathrm{PSL}(2,\mathbb{C})$ *whose hyperbolic Gauss maps are G_0 and G_1. Moreover, the canonical form is given by*

$$\omega = -dG_0/\xi^2.$$

Conversely, any meromorphic Legendrian curve $\Phi : M \to \mathrm{PSL}(2, \mathbb{C})$ *from a Riemann surface* M *into* $\mathrm{PSL}(2, \mathbb{C})$ *with nonconstant hyperbolic Gauss maps* G_0 *and* G_1 *arises in this way.*

Definition 1. The pair of meromorphic functions (G_0, G_1) of Theorem 1 will be referred to as the *Kokubu–Umehara–Yamada data* of the Legendrian curve Φ (KUY-data for short).

As a corollary of Theorem 1, Kokubu-Umehara-Yamada [10] obtain a second representation formula for Legendrian curves, due essentially to Gálvez, Martínez and Milán [8].

Theorem 2 ([8], [10]). *For an arbitrary pair* (G_0, ω) *of a nonconstant meromorphic function* G_0 *and a non-zero meromorphic* 1*-form* ω *on Riemann surface* M*, the meromorphic map given by*

$$\Phi = \left[\begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix}\right], \quad C = i\sqrt{\frac{\omega}{dG_0}}, \quad A = G_0 C, \tag{5}$$

defines a Legendrian curve in $\mathrm{PSL}(2, \mathbb{C})$*, whose hyperbolic Gauss map and canonical form are* G_0 *and* ω*, respectively. Conversely, any meromorphic Legendrian curve in* $\mathrm{PSL}(2, \mathbb{C})$ *defined on a Riemann surface* M *with nonconstant hyperbolic Gauss map* G *and non-zero canonical for* ω *is written as in* (5).

Definition 2. The pair (G_0, ω) of Theorem 2 will be referred to as the *Gálvez–Martínez–Milán data* of the Legendrian curve Φ (GMM-data for short).

We now recall a third representation formula for Legendrian curves into the projective group $\mathrm{PSL}(2, \mathbb{C})$ which is due to Ejiri and Takahashi [7]. It does not involve any integration and relies on the representation formula for Legendrian curves in $\mathbb{C}P^3$ given by Bryant [2].

Theorem 3 [7]. *If* f, g *are meromorphic functions on a connected Riemann surface* M *with* g *nonconstant and* $fg' - f'g \neq 0$, *then*

$$\Phi(f, g) = \left[\sqrt{\frac{g'}{fg' - f'g}}\begin{pmatrix} 1 & g \\ \frac{1}{2}\frac{f'}{g'} & f - \frac{1}{2}g\frac{f'}{g'} \end{pmatrix}\right] \tag{6}$$

defines a Legendrian curve $\Phi(f, g) : M \to \mathrm{PSL}(2, \mathbb{C})$.[1]

Conversely, up to multiplication by an element of $\mathrm{PSL}(2, \mathbb{C})$*, any nonconstant Legendriam curve* Φ *from* M *into* $\mathrm{PSL}(2, \mathbb{C})$ *is either of the form* $\Phi(f, g)$ *for*

[1] Throughout, $'$ denotes the derivative with respect to a local complex coordinate on M.

some unique meromorphic functions f and g on M, or takes the form

$$\Phi(h,a,c) = \left[\sqrt{\frac{1}{a-2ch}}\begin{pmatrix} 1 & c \\ h & a-ch \end{pmatrix}\right], \tag{7}$$

for some constants a, c, and a meromorphic function h.

PROOF OF THEOREM 3. If we choose homogeneous coordinate (x_1, x_2, x_3, x_4) on $\mathbb{C}P^3$, the complex 1-form

$$\theta = x_1\,dx_4 - x_4\,dx_1 + x_2\,dx_3 - x_3\,dx_2 \tag{8}$$

induces a complex contact structure on $\mathbb{C}P^3$ through the standard projection $[\cdot] : \mathbb{C}^4\backslash\{0\} \to \mathbb{C}P^3$. More precisely, on the coordinate domain $U_1 = \{x_1 \neq 0\} \subset \mathbb{C}P^3$, θ induces the local contact form

$$\theta_1 = dx_4 + x_2\,dx_3 - x_3\,dx_2. \tag{9}$$

A map $\gamma = [v] : M \to \mathbb{C}P^3$ is a Legendrian (or contact) curve if the pull-back of the 1-form θ by v vanishes identically. We identify $\mathbb{C}^4$ with the vector space $\mathfrak{gl}(2,\mathbb{C})$ of 2×2 complex matrices by[2]

$$\mathbb{C}^4 \ni (x_1, x_2, x_3, x_4)^T \leftrightarrow \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathfrak{gl}(2,\mathbb{C})$$

and think of $\mathbb{C}P^3$ as the projectivization of $\mathfrak{gl}(2,\mathbb{C})$. Then, the projective group $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm I\}$ may be identified with $\mathbb{C}P^3\backslash Q_2$, the complement in $\mathbb{C}P^3$ of the nonsingular quadric surface $Q_2 = \{[x] \in \mathbb{CP}^3 : \det x = 0\}$, by the mapping

$$\imath : \mathrm{PSL}(2,\mathbb{C}) \ni [x] \mapsto [x] \in \mathbb{C}P^3\backslash Q_2,$$

whose inverse is given by

$$\imath^{-1} : \mathbb{C}P^3\backslash Q_2 \ni [x] \mapsto \left[\frac{x}{\sqrt{\det x}}\right] \in \mathrm{PSL}(2,\mathbb{C}).$$

The maps $\imath$ and $\imath^{-1}$ are compatible with the contact structures determined by θ on $\mathbb{C}P^3\backslash Q_2$ and by Ω_1^1 on $\mathrm{PSL}(2,\mathbb{C})$. In fact, on the one hand, writing $\theta = d(x_1x_4 - x_2x_3) - 2(x_4\,dx_1 - x_2\,dx_3)$, it follows immediately that

$$\imath^*\theta = -2\Omega_1^1. \tag{10}$$

[2] We will often abuse notation and denote by the same symbol either elements. The context will make it clear whether a 4-vector or a 2×2 matrix is intended.

On the other hand, a simple computation shows that

$$(11)\qquad (\imath^{-1})^*\Omega_1^1 = \frac{x_4}{\sqrt{\det x}}\, d\Big(\frac{x_1}{\sqrt{\det x}}\Big) - \frac{x_2}{\sqrt{\det x}}\, d\Big(\frac{x_3}{\sqrt{\det x}}\Big) = -\frac{1}{2\det x}\theta.$$

Now, according to Theorem F in [2], if f and g are meromorphic functions with g nonconstant, the curve defined by

$$(12)\qquad \gamma = \left[\begin{pmatrix} 1 & g \\ \frac{1}{2}\frac{f'}{g'} & f - \frac{1}{2} g \frac{f'}{g'} \end{pmatrix}\right]$$

is Legendrian in $\mathbb{C}P^3$. The fact that (6) defines a meromorphic Legendrian curve in $\mathrm{PSL}(2,\mathbb{C})$ follows from the above discussion.

Conversely, let $\Phi = [x] = \left[\frac{v}{\sqrt{\det v}}\right] : M \to \mathrm{PSL}(2,\mathbb{C})$ be a meromorphic Legendrian curve into $\mathrm{PSL}(2,\mathbb{C})$. Then, by (11), the map

$$\gamma = \left[\begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}\right] : M \to \mathbb{C}P^3 \backslash Q_2$$

is Legendrian. By possibly replacing x by Ax, for $A \in \mathrm{SL}(2,\mathbb{C})$, we may assume that $x_1 \neq 0$, hence $v_1 \neq 0$, and then reduce the discussion to the case $v_1 \equiv 1$. Again, we argue as in the proof of Theorem F in [2]. If v_2 is not constant, set $g = v_2$ and $f = v_4 + v_2 v_3$. Now, the contact condition for γ can be written as $d(v_4 + v_2 v_3) - 2 v_3\, dv_2 = 0$. This yields $v_3 = \frac{1}{2} f'/g'$, and then $v_4 = f - \frac{1}{2} g(f'/g')$, from which (6) follows. Next, if $v_2 = c$ is a constant, the contact condition for γ reduces to $d(v_4 + v_2 v_3) = 0$ and thus $v_4 = a - c v_3$, for some constant a. By setting $h = v_3$, $\Phi = \left[\frac{v}{\sqrt{\det v}}\right]$ takes the form (7), as required. □

DEFINITION 3. The pair of meromorphic functions (f, g) of Theorem 3 will be referred to as the *Ejiri–Takahashi data* of the Legendrian curve Φ (ET-data for short).

2.2. Comparison of meromorphic data for Legendrian curves

In this section we express the KUY-data in terms of the ET-data, and conversely.

PROPOSITION 4. *Let M be a Riemann surface and let $\Phi : M \to \mathrm{PSL}(2,\mathbb{C})$ be a meromorphic Legendrian curve. If (G_0, G_1) are the* KUY-data *of Φ, then the meromorphic functions*

$$(13)\qquad g = \frac{G_1 \xi^2}{G_0(G_0 - G_1)}, \quad f = \frac{G_0 G_1 \xi^2}{G_0^2 (G_0 - G_1)}$$

give the ET-data *of Φ, where ξ is as in* (4).

On the other hand, if (f, g) *are the* ET-data *of* Φ, *then*

$$G_0 = 2\frac{g'}{f'}, \quad G_1 = \frac{G_0 g}{G_0 f - g} \quad \left(\xi = \sqrt{G_0(G_0 f - 2g)}\right) \tag{14}$$

are the KUY-data *of* Φ. *Moreover,*

$$G_0 = 2\frac{g'}{f'}, \quad \omega = -\frac{dG_0}{G_0(G_0 f - 2g)} \tag{15}$$

are the GMM-data *of* Φ, *where*

$$C(z) = \frac{1}{\sqrt{G_0(G_0 f - 2g)}}, \quad A(z) = G_0 C,$$

Proof. If G_0 and G_1 are the KUY-data of Φ and f and g are as in (13), then a direct computation yields

$$[v] := \left[\begin{pmatrix} 1 & g \\ \frac{1}{2}\frac{df}{dg} & f - \frac{1}{2}g\frac{df}{dg} \end{pmatrix}\right] = \left[\begin{pmatrix} 1 & \frac{\xi^2 G_1}{G_0(G_0 - G_1)} \\ \frac{1}{G_0} & \frac{\xi^2}{G_0(G_0 - G_1)} \end{pmatrix}\right],$$

and hence

$$\left[\frac{v}{\sqrt{\det v}}\right] = \left[\begin{pmatrix} G_0/\xi & \xi G_1/(G_0 - G_1) \\ 1/\xi & \xi/(G_0 - G_1) \end{pmatrix}\right],$$

which is (3).

On the other hand, if f and g are the ET-data of Φ and G_0, G_1 are as in (14), then

$$\frac{\xi}{G_0}\begin{pmatrix} G_0/\xi & \xi G_1/(G_0 - G_1) \\ 1/\xi & \xi/(G_0 - G_1) \end{pmatrix} = \begin{pmatrix} 1 & g \\ \frac{1}{2}\frac{df}{dg} & f - \frac{1}{2}g\frac{df}{dg} \end{pmatrix},$$

which implies

$$\left[\begin{pmatrix} G_0/\xi & \xi G_1/(G_0 - G_1) \\ 1/\xi & \xi/(G_0 - G_1) \end{pmatrix}\right] = \left[\sqrt{\frac{g'}{fg' - f'g}}\begin{pmatrix} 1 & g \\ \dfrac{1}{2}\dfrac{f'}{g'} & f - \dfrac{1}{2}g\dfrac{f'}{g'} \end{pmatrix}\right],$$

and G_0, G_1 are the KUY-data of Φ as required. Finally, if G_0 and ω are as in (15), it easily seen that

$$\left[\begin{pmatrix} A & dA/\omega \\ C & dC/\omega \end{pmatrix}\right] = \left[\sqrt{\frac{g'}{fg' - f'g}}\begin{pmatrix} 1 & g \\ \dfrac{1}{2}\dfrac{f'}{g'} & f - \dfrac{1}{2}g\dfrac{f'}{g'} \end{pmatrix}\right],$$

which implies that G_0, ω are the GMM-data of Φ. $\square$

3. Representation formulae for null curves in $\mathrm{PSL}(2,\mathbb{C})$

Let M be a Riemann surface. A meromorphic map $\Psi : M \to \mathrm{PSL}(2,\mathbb{C})$ is said a *null curve* if the pull-back by Ψ of the Killing form of $\mathfrak{sl}(2,\mathbb{C})$ vanishes, that is, $\det(\Psi^{-1}\, d\Psi) = 0$, or equivalently, if $\det \Psi' = 0$, at every point of M.

For a nonconstant null curve $\Psi = (\Psi^i_j) : M \to \mathrm{PSL}(2,\mathbb{C})$ two meromorphic functions

$$\mathscr{G}_0 := \begin{cases} \dfrac{\Psi_1^{1\prime}}{\Psi_1^{2\prime}} & (\text{if } (d\Phi_1^1, d\Phi_1^2) \neq (0,0)), \\[2ex] \dfrac{\Psi_2^{1\prime}}{\Psi_2^{2\prime}} & (\text{if } (d\Phi_2^1, d\Phi_2^2) \neq (0,0)), \end{cases} \tag{16}$$

$$\mathscr{G}_1 := \begin{cases} -\dfrac{\Psi_2^{1\prime}}{\Psi_1^{1\prime}} & (\text{if } (d\Phi_1^1, d\Phi_2^1) \neq (0,0)), \\[2ex] -\dfrac{\Psi_2^{2\prime}}{\Psi_1^{2\prime}} & (\text{if } (d\Phi_1^2, d\Phi_2^2) \neq (0,0)) \end{cases} \tag{17}$$

are defined, which are called the *hyperbolic Gauss map* and the *secondary Gauss map*, respectively (cf. [10, 17]).

Note that since Φ is null, if $(d\Phi_1^1, d\Phi_1^2) \neq (0,0)$ and $(d\Phi_2^1, d\Phi_2^2) \neq (0,0)$, then $\Psi_1^{1\prime}/\Psi_1^{2\prime} = \Psi_2^{1\prime}/\Psi_2^{2\prime}$, and similarly, if $(d\Phi_1^1, d\Phi_2^1) \neq (0,0)$ and $(d\Phi_1^2, d\Phi_2^2) \neq (0,0)$, then $-\Psi_2^{1\prime}/\Psi_1^{1\prime} = -\Psi_2^{2\prime}/\Psi_1^{2\prime}$. Moreover, according to Lemma 2.2 of [10], if either $d\Phi_1^1 = d\Phi_1^2$ or $d\Phi_2^1 = d\Phi_2^2$ are identically zero, then the hyperbolic Gauss map $\mathscr{G}_0$ is constant. Similarly, if either $d\Phi_1^1 = d\Phi_2^1$ or $d\Phi_1^2 = d\Phi_2^2$ are identically zero, then the secondary Gauss map $\mathscr{G}_1$ is constant.

Remark 2. It is well known that if $\Psi : M \to \mathrm{PSL}(2,\mathbb{C})$ is a holomorphic null immersion of a Riemann surface M, then $f = \Psi\Psi^* : M \to H^3$ is a CMC 1 surface, and that, conversely, any CMC 1 surface arises locally as the hyperbolic projection of a holomorphic null immersion into $\mathrm{PSL}(2,\mathbb{C})$ (cf. [3] and also [15], [16]). For a related study of null curves in $\mathrm{SL}(2,\mathbb{R})$, see [12].

In [14] (cf. also [10]) Small obtained a representation formula for null curves in terms of the Gauss maps.

Theorem 5 (Small [14, 10]). *If $\mathscr{G}_0$ and $\mathscr{G}_1$ are arbitrary nonconstant meromorphic function on a Riemann surface M such that $\mathscr{G}_1 \neq \frac{b_1^1\mathscr{G}_0 + b_2^1}{b_1^2\mathscr{G}_0 + b_2^2}$ for any $(b^i_j) \in$* $\mathrm{PSL}(2,\mathbb{C})$, *then the meromorphic map Ψ given by*

$$\Psi = \left[\begin{pmatrix} \mathscr{G}_0 \frac{da}{d\mathscr{G}_0} - a & \mathscr{G}_0 \frac{db}{d\mathscr{G}_0} - b \\ \frac{da}{d\mathscr{G}_0} & \frac{db}{d\mathscr{G}_0} \end{pmatrix}\right], \quad a := \sqrt{\frac{d\mathscr{G}_0}{d\mathscr{G}_1}},\ b := -\mathscr{G}_1 a, \tag{18}$$

is a nonconstant null curve in $\mathrm{PSL}(2,\mathbb{C})$ *whose hyperbolic and secondary Gauss maps are, respectively, $\mathscr{G}_0$ and $\mathscr{G}_1$.*

Conversely, any meromorphic null curve in $\mathrm{PSL}(2,\mathbb{C})$ *whose hyperbolic and secondary Gauss maps are non constant arises in this way.*

Definition 4. The pair of meromorphic functions $(\mathscr{G}_0, \mathscr{G}_1)$ of Theorem 5 will be referred to as the *Small data* of the null curve Ψ.

The aim of this section is to prove the following.

Theorem 6. *Let* (f,g) *be the ET-data of a Legendrian curve* $\Phi(f,g) : M \to \mathrm{PSL}(2,\mathbb{C})$. *Then, the pair* $(\mathscr{G}_0, \mathscr{G}_1)$ *of meromorphic functions given by*

$$\mathscr{G}_0 = -\frac{f'}{2gg'}, \quad \mathscr{G}_1 = \frac{2g'}{gf' - 2fg'} \tag{19}$$

are the Small data of a null curve Ψ *into* $\mathrm{PSL}(2,\mathbb{C})$ *which*[3] *takes the form*

$$\left[\begin{pmatrix} \mathscr{G}_0 \frac{da}{d\mathscr{G}_0} - a & \mathscr{G}_0 \frac{db}{d\mathscr{G}_0} - b \\ \frac{da}{d\mathscr{G}_0} & \frac{db}{d\mathscr{G}_0} \end{pmatrix}\right] = \left[\begin{pmatrix} \frac{g'[(f')^2 - 2ff''] + 2ff'g''}{2(gg'f'' - f'[(g')^2 + gg''])} & -\frac{g'f'' - g''f'}{gg'f'' - f'[(g')^2 + gg'']} \\ g + \frac{(g')^2(2fg' - gf')}{gg'f'' - f'[(g')^2 + gg'']} & \frac{2(g')^3}{gg'f'' - f'[(g')^2 + gg'']} \end{pmatrix}\right].$$

Moreover, the KUY-data (G_0, G_1) *of the Legendrian curve* Φ *are related to the Small data* $(\mathscr{G}_0, \mathscr{G}_1)$ *of the null curve* Ψ *by the relation*

$$G_0 G_1 = \frac{\mathscr{G}_1}{\mathscr{G}_0}. \tag{20}$$

For the proof of Theorem 6 we need to recall some preliminary material. Let W be the complex 5-dimensional vector space of all 4×4 skew-symmetric complex matrices $A = (a_{ij})$ given by

$$W = \{A = (a_{ij}) \in \mathfrak{so}(4,\mathbb{C}) : a_{14} + a_{23} = 0\}.$$

Endow W with the quadratic form q given by the Pfaffian,

$$q(A) := \mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} - a_{14}^2,$$

and let Q^3 be the 3-quadric in $\mathbb{P}(W) = \mathbb{C}P^4$ given by

$$Q^3 = \{[A] \in \mathbb{P}(W) : q(A) = 0\}.$$

Since the Pfaffian of A is the square root of the determinant of A, $\det(A) = \mathrm{Pf}(A)^2$, then $q(A) = 0$ if and only if A is singular. Now skew-symmetric matrices

[3] away from the zeroes of a suitable function (cf. (24) below)

have even rank, so if $q(A) = 0$, but $A \neq 0$, then A must have rank 2. It is easily seen that any skew-symmetric 4×4 matrix of rank 2 is decomposable, that is, has the form

$$x \times y := yx^T - xy^T,$$

for some linearly independent (column) vectors $x, y \in \mathbb{C}^4$. Note that if $A = x \times y \neq 0$, then the column space $\operatorname{Im} A$ of A is spanned by x and y. The condition $a_{14} + a_{23} = 0$ applied to a decomposable skew-symmetric matrix $A = x \times y$ is equivalent to

$$x^T \mathbb{J} y = 0,$$

where $\mathbb{J}$ is the non-singular skew-symmetric matrix given by

$$\mathbb{J} = \begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{21}$$

Thus, the 3-quadric Q^3 identifies with the Grassmannian of Lagrangian 2-planes in $\mathbb{C}^4$ with respect to the symplectic structure induced by $\mathbb{J}$. We will write

$$Q^3 = \left\{ [a_1, a_2, a_3, a_4, a_5] := \left[\begin{pmatrix} 0 & -a_1 & -a_2 & a_3 \\ a_1 & 0 & -a_3 & -a_4 \\ a_2 & a_3 & 0 & -a_5 \\ -a_3 & a_4 & a_5 & 0 \end{pmatrix} \right] : a_1 a_5 - a_2 a_4 - a_3^2 = 0 \right\}.$$

A 2-dimensional subspace $L \subseteq W$ on which q vanishes identically is called a *q-null 2-plane*. Each q-null 2-plane is a plane of decomposable skew-symmetric matrices in W.

LEMMA 7. *A 2-plane $L \subseteq W$ spanned by A and B is q-null if and only if* $\dim(\operatorname{Im} A \cap \operatorname{Im} B) = 1$.

PROOF OF LEMMA 7. If L is q-null, then $q(A) = q(B) = q(A + B) = 0$. In particular, A and B are decomposable and we can write $A = x \times y$ and $B = \tilde{x} \times \tilde{y}$. A direct calculation shows that

$$\det(x, y, \tilde{x}, \tilde{y}) = 2g(A, B) = q(A + B), \tag{22}$$

where g is the bilinear form on W obtained from q by polarization. Therefore $\det(x, y, \tilde{x}, \tilde{y}) = 0$, that is, the four vectors x, y, $\tilde{x}$, $\tilde{y}$ are linearly dependent. Since $\operatorname{Im} A = \operatorname{span}\{x, y\}$ and $\operatorname{Im} B = \operatorname{span}\{\tilde{x}, \tilde{y}\}$, we have to exclude that $\dim(\operatorname{Im} A \cap \operatorname{Im} B) = 0$. That $\dim(\operatorname{Im} A \cap \operatorname{Im} B) = 2$ must also be excluded, because A and B form a basis. Therefore, the only possibility is that $\dim(\operatorname{Im} A \cap \operatorname{Im} B) = 1$.

Conversely, if $\operatorname{Im} A \cap \operatorname{Im} B = \operatorname{span}\{v\}$, for some non-zero vector $v \in \mathbb{C}^4$, we can choose y and $\tilde{y}$ so that $A = v \times y$, $B = v \times \tilde{y}$ and then $A + B = v \times (y + \tilde{y})$. Thus, $q(A + B) = 0$, which implies that $L = \operatorname{span}\{A, B\}$ is q-null. $\square$

Lemma 7 establishes a correspondence between the points of $\mathbb{CP}^3$ and the set of q-null 2-planes in W or, equivalently, the set of lines in Q^3.

Remark 3. Let $L \subseteq W$ be a q-null 2-plane spanned by

$$A = \begin{pmatrix} 0 & -a_1 & -a_2 & a_3 \\ a_1 & 0 & -a_3 & -a_4 \\ a_2 & a_3 & 0 & -a_5 \\ -a_3 & a_4 & a_5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -b_1 & -b_2 & b_3 \\ b_1 & 0 & -b_3 & -b_4 \\ b_2 & b_3 & 0 & -b_5 \\ -b_3 & b_4 & b_5 & 0 \end{pmatrix}.$$

We know that A and B have rank 2. Assume that $\operatorname{Im} A$ and $\operatorname{Im} B$ are spanned by the first two columns of A and B, respectively. Using that $\dim(\operatorname{Im} A \cap \operatorname{Im} B) = 1$, we find that the vector

$$v = (b_1(a_1b_3 - a_3b_1), b_1(a_2b_1 - a_1b_2), a_1(a_1b_3 - a_3b_1), a_1(a_2b_1 - a_1b_2))^T \tag{23}$$

is a generator of the intersection.

Definition 5. Let $\psi : M \to Q^3$ be a holomorphic map of a Riemann surface M into Q^3. In terms of a local complex coordinate z, we write $\psi(z) = [A(z)]$, with A a holomorphic function taking values in $W\backslash\{0\}$. A holomorphic map $\psi = [A] : M \to Q^3$ from M to Q^3 is said a *q-null curve* if $\operatorname{span}\{A(z), A'(z)\}$ is a q-null 2-plane in W for all z.

We are now in a position to prove Theorem 6.

Proof of Theorem 6. Let $\Phi = \left[\frac{1}{\sqrt{\det v}} v\right] : M \to \mathrm{PSL}(2,\mathbb{C})$ be a Legendrian curve, where

$$v = \begin{pmatrix} 1 & g \\ \frac{1}{2}\frac{f'}{g'} & f - \frac{1}{2}g\frac{f'}{g'} \end{pmatrix}$$

and consider the Legendrian curve γ into $\mathbb{C}P^3$ given by $\gamma = [v]$. The Legendrian condition of γ amounts to the condition

$$v^T \mathbb{J} v' = 0,$$

where $\mathbb{J}$ is as in (21). This means that, at every point, the 2-plane spanned by $v(z)$ and $v'(z)$ is a Lagrangian 2-plane in $\mathbb{C}^4$ with respect to the symplectic structure induced by $\mathbb{J}$. Let $A = v \times v'$. Since $\operatorname{span}\{v(z), v'(z)\}$ is Lagrangian, $A = v \times v' : M \to W$ and then the map $\psi_\gamma := [A] : M \to Q^3$. Since $A' = v \times v''$ and the vectors v, v', v'' are linearly independent (γ is nonlinear), we have that $\dim(\operatorname{Im} A \cap \operatorname{Im} A') = 1$. Thus, by Lemma 7, the plane spanned by $A(z)$ and $A'(z)$ is a q-null 2-plane in W, which implies that the curve ψ_γ is q-null.

Explicitly, the matrix $A = v \times v'$ representing ψ_γ is expressed in terms of the ET-data by

$$\begin{pmatrix} 0 & -g' & -\frac{g'f''-g''f'}{2(g')^2} & \frac{gg'f''-f'[(g')^2+gg'']}{2(g')^2} \\ g' & 0 & -\frac{gg'f''-f'[(g')^2+gg'']}{2(g')^2} & g'f-gf'+\frac{g^2(g'f''-g''f')}{2(g')^2} \\ \frac{g'f''-g''f'}{2(g')^2} & \frac{gg'f''-f'[(g')^2+gg'']}{2(g')^2} & 0 & -\frac{g'[(f')^2-2ff'']+2ff'g''}{4(g')^2} \\ -\frac{gg'f''-f'[(g')^2+gg'']}{2(g')^2} & gf'-g'f-\frac{g^2(g'f''-g''f')}{2(g')^2} & \frac{g'[(f')^2-2ff'']+2ff'g''}{4(g')^2} & 0 \end{pmatrix}.$$

Next, consider the projection of the affine part $Q^3 \cap \{a_3 \neq 0\}$ of the quadric Q^3 onto $\mathrm{PSL}(2,\mathbb{C})$ given by

$$\pi : Q^3 \ni \left[\begin{pmatrix} 0 & -a_1 & -a_2 & a_3 \\ a_1 & 0 & -a_3 & -a_4 \\ a_2 & a_3 & 0 & -a_5 \\ -a_3 & a_4 & a_5 & 0 \end{pmatrix}\right] \mapsto \left[\begin{pmatrix} a_5/a_3 & -a_2/a_3 \\ -a_4/a_3 & a_1/a_3 \end{pmatrix}\right] \in \mathrm{PSL}(2,\mathbb{C}).$$

Now, the meromorphic $\mathrm{PSL}(2,\mathbb{C})$-valued map

$$\Psi = \pi \circ \psi_\gamma = \left[\begin{pmatrix} a_5/a_3 & -a_2/a_3 \\ -a_4/a_3 & a_1/a_3 \end{pmatrix}\right] \tag{24}$$

that corresponds, away from the zeroes of a_3, to the null curve $\psi_\gamma : M \to Q^3$, is a null curve of $\mathrm{PSL}(2,\mathbb{C})$ in the sense specified above.

In terms of the ET-data of Φ, the null curve Ψ is given by

$$\Psi(f,g) = \left[\begin{pmatrix} \frac{g'[(f')^2-2ff'']+2ff'g''}{2(gg'f''-f'[(g')^2+gg''])} & -\frac{g'f''-g''f'}{gg'f''-f'[(g')^2+gg'']} \\ g+\frac{(g')^2(2fg'-gf')}{gg'f''-f'[(g')^2+gg'']} & \frac{2(g')^3}{gg'f''-f'[(g')^2+gg'']} \end{pmatrix}\right]. \tag{25}$$

According to (16) and (17), the hyperbolic and secondary Gauss maps of Ψ are computed to be, respectively,

$$\mathscr{G}_0 = -\frac{f'}{2gg'}, \quad \mathscr{G}_1 = \frac{2g'}{gf'-2fg'}, \tag{26}$$

which expresses the Small data of Ψ in terms of the ET-data of Φ. In view of (26), we have that

$$a = \sqrt{\frac{d\mathscr{G}_0}{d\mathscr{G}_1}} = \frac{1}{2}\sqrt{\frac{(gf'-2fg')}{gg'}}, \quad b = -\mathscr{G}_1 a,$$

from which follows that

$$\left[\begin{pmatrix} \mathscr{G}_0\frac{da}{d\mathscr{G}_0}-a & \mathscr{G}_0\frac{db}{d\mathscr{G}_0}-b \\ \frac{da}{d\mathscr{G}_0} & \frac{db}{d\mathscr{G}_0} \end{pmatrix}\right] = \left[\begin{pmatrix} \frac{g'[(f')^2-2ff'']+2ff'g''}{2(gg'f''-f'[(g')^2+gg''])} & -\frac{g'f''-g''f'}{gg'f''-f'[(g')^2+gg'']} \\ g+\frac{(g')^2(2fg'-gf')}{gg'f''-f'[(g')^2+gg'']} & \frac{2(g')^3}{gg'f''-f'[(g')^2+gg'']} \end{pmatrix}\right],$$

as required.

Combining the equations (14) with the equations (26) above, we have

$$\mathscr{G}_0 G_0 = -\frac{1}{g}, \quad G_1 = -g\mathscr{G}_1,$$

and then the relation (20) between the KUY-data of Φ and the Small data of Ψ, as claimed. □

REMARK 4 (Symplectic equivariance). Let $\mathbb{J}$ be the skew-symmetric matrix given by (21). The corresponding symplectic group

$$\mathrm{Sp}(4,\mathbb{C}) = \left\{ X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(4,\mathbb{C}) \mid X^T\mathbb{J}X = \mathbb{J} \right\},$$

acts on $\mathbb{C}P^3$ by

$$X\cdot[v] = [Xv], \quad [v]\in\mathbb{CP}^3,$$

and on Q^3 by

$$X\cdot[A] = [XAX^T], \quad [A]\in Q^3.$$

These two actions induce actions on the sets of nonlinear Legendrian curves in $\mathbb{C}P^3$ and of null curves in Q^3. In fact, if $\gamma=[v]: M\to\mathbb{C}P^3$ is a contact curve, also $X\cdot\gamma$ is contact. Similarly, if $\psi=[A]: M\to Q^3$ is a q-null curve, also $X\cdot\psi$ is q-null. In particular, the correspondence established in the proof of Theorem 6 is equivariant with respect to the action of $\mathrm{Sp}(4,\mathbb{C})$. In fact, if $\gamma=[v]$, then $X\cdot\gamma=[Xv]$ and $\psi_{X\cdot\gamma}=[Xv\times Xv']=[X(v\times v')X^T]=X\cdot\psi_\gamma$.

EXAMPLE 1 (W-curves and the curve of Veronese). Let $M=\mathbb{C}\backslash\{0\}$ and take

$$g(z)=\sqrt{-3}z, \quad f(z)=-2z^3.$$

Then (12) gives the Legendrian curve

$$\gamma(z)=[v(z)]=\left[\begin{pmatrix} 1 & \sqrt{-3}z \\ \sqrt{-3}z^2 & z^3 \end{pmatrix}\right]. \tag{27}$$

This is essentially the Veronese curve (the rational normal cubic) in $\mathbb{CP}^3$ in the inhomogeneous coordinate z. The corresponding Legendrian curve in $\mathrm{PSL}(2,\mathbb{C})$

is given by

$$\Phi(z) = \left[\frac{1}{2}\begin{pmatrix} z^{-3/2} & \sqrt{-3}z^{-1/2} \\ \sqrt{-3}z^{1/2} & z^{3/2} \end{pmatrix}\right].$$

More generally, taking

$$g(z) = \frac{\sqrt{m+p}}{\sqrt{m-p}}z^m, \quad f(z) = \frac{2m}{m-p}z^{m+p},$$

for integers $1 \le m < p$, then (12) produces a Legendrian curve $\gamma_{m,p} : M \to \mathbb{CP}^3$,

$$\gamma_{m,p}(z) = [v_{m,p}(z)] := \left[\begin{pmatrix} 1 & \frac{\sqrt{m+p}}{\sqrt{m-p}}z^m \\ \frac{\sqrt{m+p}}{\sqrt{m-p}}z^p & z^{m+p} \end{pmatrix}\right]. \tag{28}$$

Such a curve generalizes the Veronese curve $\gamma_{1,2}$ and is known in the literature as a W-curve in $\mathbb{CP}^3$ (cf. Chern [5]). The corresponding Legendrian curve in $\mathrm{PSL}(2,\mathbb{C})$ is

$$\Phi_{m,p}(z) = [v_{m,p}/\sqrt{\det v_{m,p}}] = \left[\frac{\sqrt{p-m}}{\sqrt{2p}}\begin{pmatrix} z^{-(m+p)/2} & \frac{\sqrt{m+p}}{\sqrt{m-p}}z^{(m-p)/2} \\ \frac{\sqrt{m+p}}{\sqrt{m-p}}z^{(p-m)/2} & z^{(m+p)/2} \end{pmatrix}\right].$$

The q-null curve corresponding to $\gamma_{m,p}$ constructed in the proof of Theorem 6 is the curve

$$\psi_{m,p} = [A] : M \to Q^3,$$

where A is given by

$$\begin{pmatrix} 0 & -m\sqrt{\frac{m+p}{m-p}}z^{m-1} & -p\sqrt{\frac{m+p}{m-p}}z^{p-1} & -(m+p)z^{m+p-1} \\ m\sqrt{\frac{m+p}{m-p}}z^{m-1} & 0 & (m+p)z^{m+p-1} & -p\sqrt{\frac{m+p}{m-p}}z^{2m+p-1} \\ p\sqrt{\frac{m+p}{m-p}}z^{p-1} & -(m+p)z^{m+p-1} & 0 & -m\sqrt{\frac{m+p}{m-p}}z^{m+2p-1} \\ (m+p)z^{m+p-1} & p\sqrt{\frac{m+p}{m-p}}z^{2m+p-1} & m\sqrt{\frac{m+p}{m-p}}z^{m+2p-1} & 0 \end{pmatrix}.$$

Note that the Veronese curve $\gamma_{1,2}$ yields the holomorphic null curve $\psi_{1,2} : M \to Q^3 \subset \mathbb{CP}^4$,

$$\psi_{1,2}(z) = [\sqrt{-3}, 2\sqrt{-3}z, -3z^2, 2\sqrt{-3}z^3, \sqrt{-3}z^4],$$

which is the Veronese map into $\mathbb{CP}^4$.

Finally, according to Theorem 6, the null curve $\Psi_{m,p}$ of PSL$(2,\mathbb{C})$ corresponding to $\Phi_{m,p}$ is computed to be

$$\Psi_{m,p} = \left[\begin{pmatrix} \frac{-m}{\sqrt{m^2-p^2}} z^p & \frac{p}{\sqrt{m^2-p^2}} z^{-m} \\ \frac{p}{\sqrt{m^2-p^2}} z^m & \frac{-m}{\sqrt{m^2-p^2}} z^{-p} \end{pmatrix}\right].$$

Remark 5. In the classical literature, a W-curve of $\mathbb{CP}^3$ is a curve parametrized by $z \mapsto [1, z^m, z^p, z^{m+p}]$, for integers $1 \leq m < p$. It easily seen that such a curve is a Legendrian curve with respect to the symplectic structure of $\mathbb{C}^4$ given by the 2-form $dx_1 \wedge dx_4 + \left(\frac{m+p}{m-p}\right) dx_2 \wedge dx_3$.

Example 2 (Flat fronts of revolution (cf. [8], [10])). Let $M = \mathbb{C}\backslash\{0\}$ and set

$$k = \sqrt{\frac{h-1}{h+1}}, \quad G_0 = kz, \quad G_1 = \frac{1}{k} z \quad (h \in \mathbb{R}_+\backslash\{1\})$$

with

$$\xi = -i \frac{\sqrt{2}}{\sqrt{h+1}} \exp \int \frac{dG_0}{G_0 - G_1}.$$

The corresponding Legendrian curve into PSL$(2,\mathbb{C})$, see (3), is represented by the multi-valued map

$$\Phi(z) = \left[\frac{-i}{\sqrt{2}} \begin{pmatrix} \sqrt{h-1} z^{(h+1)/2} & \sqrt{h+1} z^{-(h-1)/2} \\ \sqrt{h+1} z^{(h-1)/2} & \sqrt{h-1} z^{-(h+1)/2} \end{pmatrix}\right].$$

The corresponding flat front $\varphi = \Phi\Phi^* : M \to H^3$ (cf. Remark 1) is well-defined on M and is a surface of revolution (cf. [10] and Example 3 below). Using the expressions (13) of the ET-data f and g in terms of the KUY-data (G_0, G_1), we find

$$f(z) = \frac{2h}{h-1} z^{-(h+1)}, \quad g(z) = \frac{\sqrt{h+1}}{\sqrt{h-1}} z^{-h}, \tag{29}$$

from which

$$\gamma(f,g) = [v(f,g)] = \left[\begin{pmatrix} 1 & \frac{\sqrt{h+1}}{\sqrt{h-1}} z^{-h} \\ \frac{\sqrt{h+1}}{\sqrt{h-1}} z^{-1} & z^{-(h+1)} \end{pmatrix}\right]$$

and hence $[v/\sqrt{\det v}] = \Phi$. For $0 < h < 1$, the flat front $\varphi : M \to H^3$ is called an hourglass, while, for $h > 1$, it is called a snowman.

Example 3 (W-curves and flat fronts of revolution). With reference to Example 1, we observe that the flat front $\varphi_{m,p} = \Phi_{m,p}\Phi^*_{m,p}$ induced by the Legendrian curve $\Phi_{m,p}$ is a surface of revolution. In fact, a direct computation shows that

$$\varphi_{m,p}(z) = \frac{p-m}{2p}\begin{pmatrix} (z\bar{z})^{-(m+p)/2} + \frac{m+p}{m-p}(z\bar{z})^{(m-p)/2} & \frac{\sqrt{m+p}}{\sqrt{m-p}}\left(\frac{(z\bar{z})^{(m+p)/2}}{z^p} + \frac{(z\bar{z})^{(p-m)/2}}{z^p}\right) \\ \frac{\sqrt{m+p}}{\sqrt{m-p}}\left(\frac{(z\bar{z})^{(m+p)/2}}{\bar{z}^p} + \frac{(z\bar{z})^{(p-m)/2}}{\bar{z}^p}\right) & (z\bar{z})^{(m+p)/2} + \frac{m+p}{m-p}(z\bar{z})^{(p-m)/2} \end{pmatrix}.$$

Now, in terms of polar coordinates $z = re^{i\theta}$, $r > 0$, we have that

$$\varphi_{m,p}(z) = R(\theta)P(r)R(\theta)^*,$$

where

$$R(\theta) = \begin{pmatrix} e^{-i(p/2)\theta} & 0 \\ 0 & e^{i(p/2)\theta} \end{pmatrix}$$

is a 1-parameter family of rotations, and $P : \mathbb{R}^+ \to H^3$,

$$P(r) = \frac{p-m}{2p}\begin{pmatrix} r^{-(m+p)} + \frac{m+p}{m-p}r^{m-p} & \frac{\sqrt{m+p}}{\sqrt{m-p}}(r^m + r^{-m}) \\ \frac{\sqrt{m+p}}{\sqrt{m-p}}(r^m + r^{-m}) & r^{m+p} + \frac{m+p}{m-p}r^{p-m} \end{pmatrix}$$

is a planar profile curve. This means that φ is a surface of revolution, as claimed.

REMARK 6. Actually, all flat fronts of revolution are of the form $\varphi_{m,p}$, for real $m,p > 0$, $m \neq p$. In fact, if we perform the change of coordinate $z(w) = w^{-1/p}$ and set $h = m/p$, then a straightforward computation shows that

$$f(z(w)) = \frac{2m}{m-p}z^{m+p} = \frac{2h}{h-1}w^{-(h+1)},$$
$$g(z(w)) = \frac{\sqrt{m+p}}{\sqrt{m-p}}z^m = \frac{\sqrt{h+1}}{\sqrt{h-1}}w^{-h},$$

which are the ET-data (29) inducing the flat fronts of revolution.

EXAMPLE 4 (W-curves, flat fronts of revolution and catenoid cousins). With reference to Examples 1 and 3, let $\gamma_{m,m+1} : M = \mathbb{C}\backslash\{0\} \to \mathbb{CP}^3$ be the Legendrian curve given by

$$\gamma_{m,m+1}(z) = [v_{m,m+1}(z)] = \left[\begin{pmatrix} 1 & \sqrt{-2m-1}\,z^m \\ \sqrt{-2m-1}\,z^{m+1} & z^{2m+1} \end{pmatrix}\right].$$

The corresponding q-null curve $\psi_m = [A] : M \to Q^3$ constructed in Theorem 6 is given by

(30) $\quad [a_1, a_2, a_3, a_4, a_5] = [m, (m+1)z, \sqrt{-2m-1}\,z^{m+1}, (m+1)z^{2m+1}, mz^{2m+2}].$

The projection of ψ_m onto PSL$(2,\mathbb{C})$ yields the holomorphic null curve

$$\Psi = \left[\begin{pmatrix} a_5/a_3 & -a_2/a_3 \\ -a_4/a_3 & a_1/a_3 \end{pmatrix}\right] \tag{31}$$
$$= \left[\frac{1}{\sqrt{-2m-1}}\begin{pmatrix} mz^{(m+1)} & -(m+1)z^{-m} \\ -(m+1)z^{m} & mz^{-(m+1)} \end{pmatrix}\right],$$

that, as a map on the universal cover of M, defines a holomorphic null immersion. The hyperbolic projection $f_m = \Psi\Psi^* : M \to H^3$ is a well-defined immersion and gives rise to a rotational CMC 1 immersion which is a Bryant's catenoid cousin (cf. [2] and [16]). Thus, under the correspondence established in the proof of Theorem 6, the flat fronts induced by $\gamma_{m,m+1}$ correspond to CMC 1 catenoid cousins. In Example 3, we have already observed that the Legendrian curves $\gamma_{m,m+1}$ produce flat fronts of revolution.

Note that a catenoid cousin has two ends at $z = 0, \infty$. Moreover, among the 1-parameter family of catenoid cousins (cf. [2]), the countable family corresponding to integral parameters $m \in \mathbb{N}\backslash\{0\}$ share the property of having smooth ends. This means, in the Poincaré model for hyperbolic space, that the CMC 1 surface $f_m(M) \subset H^3$ compactifies to the image of a smooth immersion of the whole 2-sphere S^2 by adding two points on the ideal boundary of hyperbolic space. Actually, the ends of f_m are smooth if and only if $m \in \mathbb{N}\backslash\{0\}$ (cf. Bohle–Peters [1]).

Remark 7. The Legendrian curve

$$\gamma_{m,m+1} : S^2 \to \mathbb{CP}^3$$

from the Riemann sphere $S^2 \cong \mathbb{CP}^1$ into $\mathbb{CP}^3$ is a nonlinear curve of degree $2m+1$ and ramification degree $2m-2$ for all integer $m \geq 1$. In particular, it is not an immersion for $m > 1$. The map $\psi_m : S^2 \to Q^3$ is instead a holomorphic null immersion of degree $2m-1$. This is a consequence of the Plücker relations (cf. [4], [9]).

References

[1] C. Bohle - G. P. Peters, *Bryant surfaces with smooth ends*, Comm. Anal. Geom. 17 (2009), no. 4, 587–619; arXiv:math/0411480v2.

[2] R. L. Bryant, *Conformal and minimal immersions of compact surfaces into the 4-sphere*, J. Differential Geom. 17 (1982), no. 3, 455–473

[3] R. L. Bryant, *Surfaces of mean curvature one in hyperbolic space*, Théorie des variétés minimales et applications (Palaiseau, 1983–1984), *Astérisque* 154–155 (1987), 321–347.

[4] R. L. Bryant, *Surfaces in conformal geometry*, Proc. Sympos. Pure Math. 48 (1988), 227–240, Amer. Math. Soc., Providence, RI.

[5] S.-S. Chern, *On minimal spheres in the four-sphere*, Studies and Essays (presented to Yu-Why Chen on his 60th Birthday, April 1, 1970) Math. Res. Center, Nat. Taiwan Univ, Taipei, 1970, pp. 137–150.

[6] Levi L. de Lima - P. Roitman, *Constant mean curvature one surfaces in hyperbolic 3-space using the Bianchi-Calò method*, An. Acad. Brasil. Cienc. 74 (2002), 19–24.

[7] N. Ejiri - M. Takahashi, *The Lie transform between null curves in* $\mathrm{SL}(2, \mathbf{C})$ *and contact curves in* $\mathrm{PSL}(2, \mathbf{C})$, Riemann surfaces, harmonic maps and visualization, 265–277, OCAMI Stud., 3, Osaka Munic. Univ. Press, Osaka, 2010.

[8] J. A. Gálvez - A. Martínez - F. Milán, *Flat surfaces in the hyperbolic 3-space*, Math. Ann. 316 (2000), 419–435.

[9] P. Griffiths - J. Harris, *Principles of algebraic geometry*, Pure and Applied Mathematics, Wiley-Interscience, New York, 1978.

[10] M. Kokubu - M. Umehara - K. Yamada, *An elementary proof of Small's formula for null curves in* $\mathrm{PSL}(2, \mathbb{C})$ *and an analogue for Legendrian curves in* $\mathrm{PSL}(2, \mathbb{C})$, Osaka J. Math. 40 (2003), no. 3, 697–715.

[11] M. Kokubu - M. Umehara - K. Yamada, *Flat fronts in hyperbolic 3-space*, Pacific J. Math. 216 (2004), 149–175.

[12] E. Musso - L. Nicolodi, *Closed trajectories of a particle model on null curves in anti-de Sitter 3-space*, Classical Quantum Gravity 24 (2007), no. 22, 5401–5411; arXiv:0709.2017 [math.DG].

[13] R. Sa Earp - E. Toubiana, *Meromorphic data for mean curvature one surfaces in hyperbolic three-space*, Tohoku Math. J. (2) 56 (2004), 27–64.

[14] A. J. Small, *Surfaces of constant mean curvature 1 in* H^3 *and algebraic curves on a quadric*, Proc. Amer. Math. Soc. 122 (1994), 1211–1220.

[15] M. Umehara - K. Yamada, *A parametrization of the Weierstrass formulae and perturbation of complete minimal surfaces in* $\mathbb{R}^3$ *into the hyperbolic 3-space*, J. Reine Angew. Math. 432 (1992), 93–116.

[16] M. Umehara - K. Yamada, *Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space*, Ann. of Math. (2) 137 (1993), no. 3, 611–638.

[17] M. Umehara - K. Yamada, *A duality on CMC-1 surfaces in hyperbolic space, and a hyperbolic analogue of the Osserman inequality*, Tsukuba J. Math. 21 (1997), no. 1, 229–237.

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