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Review

Theoretical Foundations and Mathematical Formalism of the Power-Law Tailed Statistical Distributions

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Abstract: We present the main features of the mathematical theory generated by the κ-deformed exponential function $\exp_\kappa(x) = (\sqrt{1+\kappa^2x^2} + \kappa x)^{1/\kappa}$, with $0 \leq \kappa < 1$, developed in the last twelve years, which turns out to be a continuous one parameter deformation of the ordinary mathematics generated by the Euler exponential function. The κ-mathematics has its roots in special relativity and furnishes the theoretical foundations of the κ-statistical mechanics predicting power law tailed statistical distributions, which have been observed experimentally in many physical, natural and artificial systems. After introducing the κ-algebra, we present the associated κ-differential and κ-integral calculus. Then, we obtain the corresponding κ-exponential and κ-logarithm functions and give the κ-version of the main functions of the ordinary mathematics.

Keywords: κ-statistical mechanics; κ-mathematics; κ-exponential; κ-logarithm; power-law tailed statistical distributions

PACS: 05.90.+m; 05.20.-y; 51.10.+y; 03.30.+p

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1. Introduction

Undoubtedly the most interesting feature of the statistical distribution function:

$$f_i = \exp_\kappa(-\beta E_i + \beta\mu) \tag{1}$$

where the κ-exponential is defined as:

$$\exp_\kappa(x) = (\sqrt{1+\kappa^2x^2} + \kappa x)^{1/\kappa} \quad , \quad 0 \leq \kappa < 1 \tag{2}$$

Proof. From the definition of $\stackrel{\kappa}{\oplus}$, the following properties follow.

(1) associativity: $(x \stackrel{\kappa}{\oplus} y) \stackrel{\kappa}{\oplus} z = x \stackrel{\kappa}{\oplus} (y \stackrel{\kappa}{\oplus} z)$,

(2) neutral element: $x \stackrel{\kappa}{\oplus} 0 = 0 \stackrel{\kappa}{\oplus} x = x$,

(3) opposite element: $x \stackrel{\kappa}{\oplus} (-x) = (-x) \stackrel{\kappa}{\oplus} x = 0$,

(4) commutativity: $x \stackrel{\kappa}{\oplus} y = y \stackrel{\kappa}{\oplus} x$.

□

Remark 1. *The κ-sum is a one parameter continuous deformation of the ordinary sum, which recovers in the classical limit $\kappa \to 0$,* i.e., $x \stackrel{0}{\oplus} y = x + y$. *The κ-sum in Equation (4) is the additivity law of the dimensionless relativistic momenta of special relativity, while the real parameter $-1 < \kappa < 1$ is the reciprocal of the dimensionless light speed* [3,9]. The κ-difference $\stackrel{\kappa}{\ominus}$ is defined as $x \stackrel{\kappa}{\ominus} y = x \stackrel{\kappa}{\oplus} (-y)$.

Theorem 2. *Let $x, y \in \mathbf{R}$ and $-1 < \kappa < 1$. The composition law, $\stackrel{\kappa}{\otimes}$, defined through:*

$$x \stackrel{\kappa}{\otimes} y = \frac{1}{\kappa} \sinh \left(\frac{1}{\kappa} \operatorname{arcsinh}(\kappa x) \operatorname{arcsinh}(\kappa y) \right) \tag{5}$$

is a generalized product, called the κ-product, and the algebraic structure $(\mathbf{R}, \stackrel{\kappa}{\otimes})$ forms an abelian group.

Proof. From the definition of $\stackrel{\kappa}{\otimes}$, the following properties follow.

(1) associativity: $(x \stackrel{\kappa}{\otimes} y) \stackrel{\kappa}{\otimes} z = x \stackrel{\kappa}{\otimes} (y \stackrel{\kappa}{\otimes} z)$,

(2) neutral element: is defined through $x \stackrel{\kappa}{\otimes} I = I \stackrel{\kappa}{\otimes} x = x$ and is given by $I = \kappa^{-1} \sinh \kappa$,

(3) inverse element: is defined through $x \stackrel{\kappa}{\otimes} \overline{x} = \overline{x} \stackrel{\kappa}{\otimes} x = I$ and is given by $\overline{x} = \kappa^{-1} \sinh(\kappa^2 / \operatorname{arcsinh} \kappa x)$,

(4) commutativity: $x \stackrel{\kappa}{\otimes} y = y \stackrel{\kappa}{\otimes} x$.

□

Remark 2. *The κ-product reduces to the ordinary product as $\kappa \to 0$,* i.e., $x \stackrel{0}{\otimes} y = xy$. *The κ-division $\stackrel{\kappa}{\oslash}$ is defined through $x \stackrel{\kappa}{\oslash} y = x \stackrel{\kappa}{\otimes} \overline{y}$.*

Theorem 3. *Let $x, y \in \mathbf{R}$ and $-1 < \kappa < 1$. The κ-sum $\stackrel{\kappa}{\oplus}$ defined in Equation (4) and the κ-product $\stackrel{\kappa}{\otimes}$ defined in Equation (5) obey the distributive law:*

$$z \stackrel{\kappa}{\otimes} (x \stackrel{\kappa}{\oplus} y) = (z \stackrel{\kappa}{\otimes} x) \stackrel{\kappa}{\oplus} (z \stackrel{\kappa}{\otimes} y) \tag{6}$$

and then, the algebraic structure $(\mathbf{R}, \stackrel{\kappa}{\oplus}, \stackrel{\kappa}{\otimes})$ forms an abelian field.

Proof. The relationship in Equation (6) follows directly from the definitions of the κ-product in Equation (5) and of the κ-sum in Equation (4), which can be written also in the form:

$$x \overset{\kappa}{\oplus} y = \frac{1}{\kappa} \sinh \Big(\mathrm{arcsinh}\,(\kappa x) + \mathrm{arcsinh}\,(\kappa y) \Big) \tag{7}$$

□

Theorem 4. *The abelian fields,* $(\mathbf{R}, \overset{\kappa}{\oplus}, \overset{\kappa}{\otimes})$ *and* $(\mathbf{R}, +, .)$*, are isomorphic.*

Proof. After introducing the function, $\{x\} \in C^\infty(\mathbf{R})$, through:

$$\{x\} = \frac{1}{\kappa} \mathrm{arcsinh}\,(\kappa x) \tag{8}$$

whose inverse function, $[x] \in C^\infty(\mathbf{R})$, *i.e.*, $[\{x\}\,] = \{\,[x]\} = x$, is given by:

$$[x] = \frac{1}{\kappa} \sinh\,(\kappa x) \tag{9}$$

we can write Equations (7) and (5) in the form:

$$\{x \overset{\kappa}{\oplus} y\} = \{x\} + \{y\} \tag{10}$$
$$\{x \overset{\kappa}{\otimes} y\} = \{x\} \cdot \{y\} \tag{11}$$

or, equivalently, as:

$$[x] \overset{\kappa}{\oplus} [y] = [x + y] \tag{12}$$
$$[x] \overset{\kappa}{\otimes} [y] = [x \cdot y] \tag{13}$$

□

Theorem 5. *Let* $x \in \mathbf{R}$ *and* n *be an arbitrary nonnegative integer. It holds:*

$$\underbrace{x \overset{\kappa}{\oplus} x \overset{\kappa}{\oplus} ... \overset{\kappa}{\oplus} x}_{n\ times} = [n] \overset{\kappa}{\otimes} x \tag{14}$$

Proof. The function, $[x]$, and its inverse, $\{x\}$, obey the condition $[\{x\}] = \{[x]\} = x$. Furthermore, we take into account Equations (10) and (13). Then, we have:

$$\begin{aligned} x \overset{\kappa}{\oplus} x \overset{\kappa}{\oplus} ... \overset{\kappa}{\oplus} x &= [\{x \overset{\kappa}{\oplus} x \overset{\kappa}{\oplus} ... \overset{\kappa}{\oplus} x\}] \\ &= [\{x\} + \{x\} + ... + \{x\}] \\ &= [n \cdot \{x\}] \\ &= [\{[n]\} \cdot \{x\}] \\ &= [n] \overset{\kappa}{\otimes} [\{x\}] \\ &= [n] \overset{\kappa}{\otimes} x \end{aligned} \tag{15}$$

□

3. κ-Differential Calculus

3.1. κ-Differential

The κ-differential of x, indicated by $d_\kappa x$, is defined through:

$$(x + dx) \overset{\kappa}{\ominus} x = d_\kappa x + 0((dx)^2) \tag{16}$$

and results to be:

$$d_\kappa x = \frac{d\,x}{\sqrt{1 + \kappa^2\, x^2}} \tag{17}$$

In order to better understand the origin of the expression of $d_\kappa x$, we recall that the variable, x, is a dimensionless momentum. Then, the quantity $\gamma(x) = \sqrt{1 + \kappa^2\, x^2}$ is the Lorentz factor of relativistic physics, in the momentum representation. Therefore, we can write:

$$d_\kappa x = \frac{dx}{\gamma(x)} \tag{18}$$

Moreover, it holds:

$$d_\kappa x = d\{x\} = \frac{d\{x\}}{dx}\; dx \tag{19}$$

3.2. κ-Derivative

We define the κ-derivative of the function, $f(x)$, through:

$$\frac{d\,f(x)}{d_\kappa\, x} = \lim_{z \to x} \frac{f(z) - f(x)}{z \overset{\kappa}{\ominus} x} \approx \lim_{dx \to 0} \frac{f(x + dx) - f(x)}{(x + dx) \overset{\kappa}{\ominus} x} \tag{20}$$

We observe that $df(x)/d_\kappa x$, which reduces to $df(x)/dx$ as the deformation parameter, $\kappa \to 0$, can be written in the form:

$$\frac{d\,f(x)}{d_\kappa\, x} = \sqrt{1 + \kappa^2\, x^2}\;\; \frac{d\,f(x)}{d\,x} \tag{21}$$

From the latter relationship, it follows that the κ-derivative obeys the Leibniz's rules of the ordinary derivative. After introducing the $\gamma(x)$ Lorentz factor, the κ-derivative can be written also in the form:

$$\frac{d}{d_\kappa\, x} = \gamma(x)\, \frac{d}{d\,x} \tag{22}$$

3.3. κ-Integral

We define the κ-integral as the inverse operator of the κ-derivative through:

$$\int d_\kappa x\;\, f(x) = \int \frac{d\,x}{\sqrt{1 + \kappa^2\, x^2}}\;\, f(x) \tag{23}$$

and note that it is governed by the same rules of the ordinary integral, which recovers when $\kappa \to 0$.

3.4. Connections with Physics

We indicate with $p = |\mathbf{p}|$ and $x = p/mv_*$ the moduli of the particle momentum in dimensional and dimensionless form, respectively, and define $\kappa = v_*/c$. The classical relationship linking x with the dimensionless kinetic energy $\mathcal{W} = x^2/2$ follows from the kinetic energy theorem, which in differential form reads:

$$\frac{d}{dx}\mathcal{W} = x \tag{24}$$

The latter equation after replacing the ordinary derivative by the derivative, $d/d_\kappa x$, *i.e.*,:

$$\frac{d}{d_\kappa x}\mathcal{W} = x \tag{25}$$

transforms into the corresponding relativistic equation. This differential equation with the condition $\mathcal{W}(x = 0) = 0$ admits as a unique solution $\mathcal{W} = \left(\sqrt{1+\kappa^2 x^2} - 1\right)/\kappa^2$, defining the relativistic kinetic energy.

Let us consider the four-dimensional Lorentz invariant integral:

$$I = \int d^4p\; \theta(p_0)\; \delta(p^\mu p_\mu - m^2c^2)\; F(p) \tag{26}$$

$p^\mu = (p^0, \mathbf{p}) = \left(\sqrt{m^2c^2 + p^2}, \mathbf{p}\right)$, $\theta(.)$ being the Heaviside step function and $\delta(.)$, the Dirac delta function. It is trivial to verify that the latter integral transforms into the one-dimension integral:

$$I \propto \int d_\kappa x\, f(x) \tag{27}$$

being $f(x) = 4\pi x^2 F(x)$. Then, the κ-integral is essentially the Lorentz invariant integral of special relativity.

4. The Function $\exp_\kappa(x)$

4.1. Definition

We recall that the ordinary exponential $f(x) = \exp(x)$ emerges as a solution both of the functional equation $f(x+y) = f(x)f(y)$ and of the differential equation $(d/dx)f(x) = f(x)$. The question to determine the solution of the generalized equations:

$$f(x \overset{\kappa}{\oplus} y) = f(x)f(y) \tag{28}$$

$$\frac{d\, f(x)}{d_\kappa x} = f(x) \tag{29}$$

reducing in the $\kappa \to 0$ limit to the ordinary exponential naturally arises. This solution is unique and represents a one-parameter generalization of the ordinary exponential.

Solution of Equation (28): We write this equation explicitly:

$$f\left(x\sqrt{1+\kappa^2y^2} + y\sqrt{1+\kappa^2x^2}\right) = f(x)f(y) \tag{30}$$

which, after performing the change of variables, $f(x) = \exp(g(\kappa x))$, $z_1 = \kappa x$, $z_2 = \kappa y$, transforms as:

$$g\left(z_1\sqrt{1+z_2^2} + z_2\sqrt{1+z_1^2}\right) = g(z_1) + g(z_2) \tag{31}$$

and admits as a solution $g(x) = A\,\mathrm{arcsinh}x$. Then, it results that $f(x) = \exp(A\,\mathrm{arcsinh}\,\kappa x)$. The arbitrary constant, A, can be fixed through the condition $\lim_{\kappa\to 0} f(x) = \exp(x)$, obtaining $A = 1/\kappa$. Therefore, $f(x)$ assumes the form $f(x) = \exp_\kappa(x)$, being:

$$\exp_\kappa(x) = \exp\left(\frac{1}{\kappa}\,\mathrm{arcsinh}\,\kappa x\right) \tag{32}$$

Solution of Equation (29): According to Equation (29), the function $f(x) = \exp_\kappa(x)$ is defined as the eigenfunction of $d/d_\kappa x$, *i.e.*,:

$$\frac{d\,\exp_\kappa(x)}{d_\kappa x} = \exp_\kappa(x) \tag{33}$$

After recalling that $d_\kappa x = d\{x\}$ with $\{x\} = \kappa^{-1}\,\mathrm{arcsinh}\,\kappa x$, Equation (33) can be written in the form:

$$\frac{d\,\exp_\kappa(x)}{d\{x\}} = \exp_\kappa(x) \tag{34}$$

The solution of the latter equation with the condition $\exp_\kappa(0) = 1$ follows immediately:

$$\exp_\kappa(x) = \exp\left(\{x\}\right) \tag{35}$$

After taking into account that $\mathrm{arcsinh}\,x = \ln(\sqrt{1+x^2} + x)$, we can write $\exp_\kappa(x)$ in the form:

$$\exp_\kappa(x) = \left(\sqrt{1+\kappa^2 x^2} + \kappa x\right)^{1/\kappa} \tag{36}$$

which will be used in the following. We remark that $\exp_\kappa(x)$ given by Equation (36) is the solution of both Equations (28) and (29) and, therefore, represents a generalization of the ordinary exponential.

4.2. Basic Properties

From the definition in Equation (36) of $\exp_\kappa(x)$ follows that:

$$\exp_0(x) \equiv \lim_{\kappa\to 0} \exp_\kappa(x) = \exp(x) \tag{37}$$

$$\exp_{-\kappa}(x) = \exp_\kappa(x) \tag{38}$$

Like the ordinary exponential, $\exp_\kappa(x)$ has the properties:

$$\exp_\kappa(x) \in C^\infty(\mathbf{R}) \tag{39}$$

$$\frac{d}{d\,x}\,\exp_\kappa(x) > 0 \tag{40}$$

$$\exp_\kappa(-\infty) = 0^+ \tag{41}$$

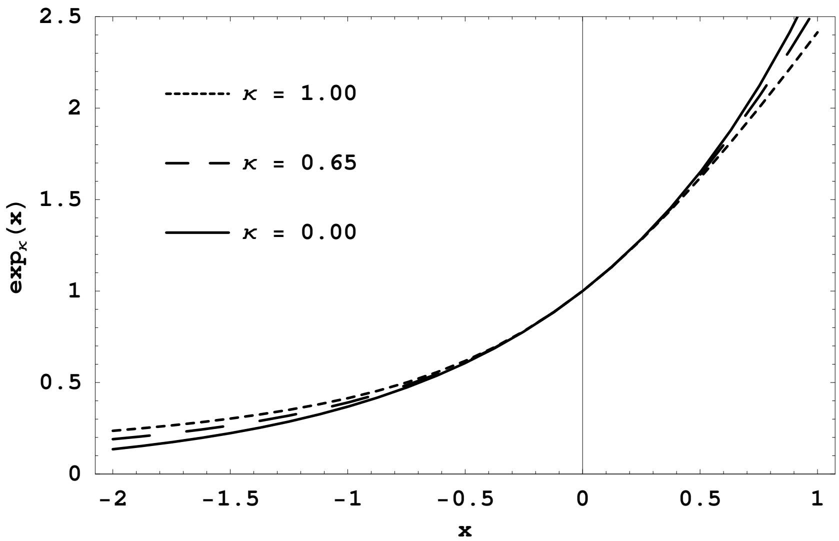
$$\exp_\kappa(0) = 1 \tag{42}$$

$$\exp_\kappa(+\infty) = +\infty \tag{43}$$

$$\exp_\kappa(x)\,\exp_\kappa(-x) = 1 \tag{44}$$

In Figure 1 is plotted the function, $\exp_\kappa(x)$, defined in Equation (36) for three different values of the parameter of κ. The continuous curve corresponding to $\kappa = 0$ is the ordinary exponential function, $\exp(x)$.

Figure 1. Plot of the function, $\exp_\kappa(x)$, defined in Equation (36) for three different values of the parameter of κ. The continuous curve corresponding to $\kappa = 0$ is the ordinary exponential function, $\exp(x)$.



Property in Equation (44) emerges as a particular case of the more general one:

$$\exp_\kappa(x)\exp_\kappa(y) = \exp_\kappa(x \overset{\kappa}{\oplus} y) \tag{45}$$

Furthermore, $\exp_\kappa(x)$ has the property:

$$(\exp_\kappa(x))^r = \exp_{\kappa/r}(rx) \tag{46}$$

with $r \in \mathbf{R}$, which in the limit, $\kappa \to 0$, reproduces one well known property of the ordinary exponential.

We remark the following convexity property:

$$\frac{d^2}{d\,x^2}\exp_\kappa(x) > 0 \;\; ; \;\; x \in \mathbf{R} \tag{47}$$

holding when $\kappa^2 < 1$.

Undoubtedly, one of the more interesting properties of $\exp_\kappa(x)$ is its power law asymptotic behavior:

$$\exp_\kappa(x) \underset{x\to\pm\infty}{\sim} \big|\, 2\kappa x \big|^{\pm 1/|\kappa|} \tag{48}$$

4.3. Mellin Transform

Let us consider the incomplete Mellin transform of the $\exp_\kappa(-t)$:

$$\mathcal{M}_\kappa(r,x) \quad = \int_0^x t^{r-1} \, \exp_\kappa(-t)\, dt \tag{49}$$

After performing the change of integration variable $y = \left(\sqrt{1+\kappa^2 t^2} - |\kappa| t\right)^2$ and after taking into account that $t = \frac{1}{2|\kappa|}\left(y^{-1/2} - y^{1/2}\right)$ and $\exp_\kappa(-t) = y^{1/2|\kappa|}$, the function, $\mathcal{M}_\kappa(r,x)$, can be written in the form:

$$\mathcal{M}_\kappa(r,x) \quad = \frac{1}{2}\,|2\kappa|^{-r} \int_X^1 y^{\frac{1}{2|\kappa|}-\frac{r}{2}-1} \,(1-y)^{r-1}\,(1+y)\; dy \tag{50}$$

with:

$$X = \left(\sqrt{1+\kappa^2 x^2} - |\kappa| x\right)^2 \tag{51}$$

When r is an integer greater than zero, $\mathcal{M}_\kappa(r,x)$ can be calculated analytically. For instance, it results:

$$\mathcal{M}_\kappa(1,x) = \frac{1}{1-\kappa^2} - \frac{\kappa^2 x + \sqrt{1+\kappa^2x^2}}{1-\kappa^2}\, \exp_\kappa(-x) \tag{52}$$

$$\mathcal{M}_\kappa(2,x) = \frac{1}{1-4\kappa^2} - \frac{1+2\kappa^2x^2 + x\sqrt{1+\kappa^2x^2}}{1-4\kappa^2}\, \exp_\kappa(-x) \tag{53}$$

In general, the function, $\mathcal{M}_\kappa(r,x)$, can be written as:

$$\mathcal{M}_\kappa(r,x) = \frac{1}{2}\,|2\kappa|^{-r} \left[I_1(r) - I_X(r)\right] \tag{54}$$

with:

$$I_X(r) = \int_0^X y^{\frac{1}{2|\kappa|}-\frac{r}{2}-1}\,(1-y)^{r-1}\; dy + \int_0^X y^{\frac{1}{2|\kappa|}-\frac{r}{2}}\,(1-y)^{r-1}\; dy \tag{55}$$

After recalling the definition of the Beta incomplete function $B_X(s,r) = \int_0^X y^{s-1}(1-y)^{r-1}\,dy$, the integral, $I_X(r)$, becomes:

$$I_X(r) = B_X\left(\frac{1}{2|\kappa|} - \frac{r}{2}\,,\, r\right) + B_X\left(\frac{1}{2|\kappa|} - \frac{r}{2} + 1\,,\, r\right) \tag{56}$$

The function, $I_1(r)$, can be expressed in terms of the Beta functions $B(s,r) = \int_0^1 y^{s-1}(1-y)^{r-1}\,dy$ and, then, in terms of Gamma functions, obtaining:

$$I_1(r) = \frac{2\,\Gamma(r)}{1+|\kappa| r}\, \frac{\Gamma\left(\frac{1}{2|\kappa|} - \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2|\kappa|} + \frac{r}{2}\right)} \tag{57}$$

Finally, the incomplete Mellin transform, $\mathcal{M}_\kappa(r,x)$, of $\exp_\kappa(-t)$ assumes the form:

$$\begin{aligned}\mathcal{M}_\kappa(r,x) \quad &= \frac{|2\kappa|^{-r}}{1+|\kappa| r}\, \frac{\Gamma\left(\frac{1}{2|\kappa|} - \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2|\kappa|} + \frac{r}{2}\right)}\,\Gamma(r) \\ &-\frac{1}{2}\,|2\kappa|^{-r}\, B_X\left(\frac{1}{2|\kappa|} - \frac{r}{2}\,,\, r\right) \\ &-\frac{1}{2}\,|2\kappa|^{-r}\, B_X\left(\frac{1}{2|\kappa|} - \frac{r}{2} + 1\,,\, r\right)\end{aligned} \tag{58}$$

The Mellin transform of $\exp_\kappa(-t)$, namely:

$$\mathcal{M}_\kappa(r) = \int_0^\infty t^{r-1} \exp_\kappa(-t)\, dt \tag{59}$$

can be calculated from Equation (58) by posing $x = \infty$. The explicit expression of $\mathcal{M}_\kappa(r)$ holding for $0 < r < 1/|\kappa|$ is given by:

$$\mathcal{M}_\kappa(r) = \frac{|2\kappa|^{-r}}{1+|\kappa|r} \frac{\Gamma\left(\frac{1}{2|\kappa|} - \frac{r}{2}\right)}{\Gamma\left(\frac{1}{2|\kappa|} + \frac{r}{2}\right)} \Gamma(r) \tag{60}$$

From the latter relationship, one can verify easily the property:

$$\mathcal{M}_\kappa(r+2) = \frac{r(r+1)}{1-\kappa^2(r+2)^2} \mathcal{M}_\kappa(r) \tag{61}$$

4.4. Taylor Expansion

The Taylor expansion of $\exp_\kappa(x)$ given in [3] can be written also in the following form:

$$\exp_\kappa(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!_\kappa} \quad ; \quad \kappa^2 x^2 < 1 \tag{62}$$

where the symbol, $n!_\kappa$, representing the κ-generalization of the ordinary factorial, $n!$, recovered for $\kappa = 0$, is given by:

$$n!_\kappa = \frac{n!}{\xi_n(\kappa)} \tag{63}$$

and the polynomials, $\xi_n(\kappa)$, are defined as:

$$\xi_0(\kappa) = \xi_1(\kappa) = 1 \tag{64}$$

$$\xi_n(\kappa) = \prod_{j=1}^{n-1} \Big[1 - (2j-n)\,\kappa\Big] \;\; ; \;\; n > 1 \tag{65}$$

The polynomials, $\xi_n(\kappa)$, for $n > 1$, when n is odd, are of the degree $n-1$, with respect to the variable, κ, while when n is even, the degree of $\xi_n(\kappa)$ is $n-2$. The degree of $\xi_n(\kappa)$ is always an even number, and it results:

$$\xi_{2m}(\kappa) = \prod_{j=0}^{m-1} \Big[1 - (2j)^2\kappa^2\Big] \;\; ; \;\; m > 0 \tag{66}$$

$$\xi_{2m+1}(\kappa) = \prod_{j=0}^{m-1} \Big[1 - (2j+1)^2\kappa^2\Big] \;\; ; \;\; m > 0 \tag{67}$$

The polynomials, $\xi_n(\kappa)$, can be generated by the following simple recursive formula:

$$\xi_0(\kappa) = \xi_1(\kappa) = 1 \tag{68}$$

$$\xi_{n+2}(\kappa) = (1 - n^2\kappa^2)\, \xi_n(\kappa) \;\; ; \;\; n \geq 0 \tag{69}$$

The first nine polynomials read as:

$$\xi_0(\kappa) = \xi_1(\kappa) = \xi_2(\kappa) = 1 \tag{70}$$
$$\xi_3(\kappa) = 1 - \kappa^2 \tag{71}$$
$$\xi_4(\kappa) = 1 - 4\kappa^2 \tag{72}$$
$$\xi_5(\kappa) = (1 - \kappa^2)(1 - 9\kappa^2) \tag{73}$$
$$\xi_6(\kappa) = (1 - 4\kappa^2)(1 - 16\kappa^2) \tag{74}$$
$$\xi_7(\kappa) = (1 - \kappa^2)(1 - 9\kappa^2)(1 - 25\kappa^2) \tag{75}$$
$$\xi_8(\kappa) = (1 - 4\kappa^2)(1 - 16\kappa^2)(1 - 36\kappa^2) \tag{76}$$

After noting that for a given value of κ, the maximum natural number, N, satisfying the condition $N < 2 + 1/|\kappa|$ is defined univocally, we can verify easily that for $n = 0, 1, 2, ..., N$, it results $\xi_n(\kappa) > 0$ and, then, $n!_\kappa > 0$. For $n > N$, the sign of $\xi_n(\kappa)$ and, then, of $n!_\kappa$ alternates with periodicity $- - + + - - + + ...$

From Equations (63) and (69) follows the recursive formula:

$$(n+2)!_\kappa = \frac{(n+1)(n+2)}{1 - n^2\kappa^2}\, n!_\kappa \tag{77}$$

By direct comparison of Equations (61) and (77), we obtain the relationship:

$$n!_\kappa = \left(1 - \kappa^2 n^2\right) n \int_0^\infty t^{n-1} \exp_\kappa(-t)\, dt \tag{78}$$

It is remarkable that the first three terms in the Taylor expansion of $\exp_\kappa(x)$ are the same as the ordinary exponential:

$$\exp_\kappa(x) = 1 + x + \frac{x^2}{2} + (1 - \kappa^2)\, \frac{x^3}{3!} + ... \tag{79}$$

4.5. The Function $\Gamma_\kappa(x)$

The $\Gamma_\kappa(n)$ with an n integer is defined through:

$$\Gamma_\kappa(n) = (n-1)!_\kappa \tag{80}$$

and represents a generalization of the Euler $\Gamma(n)$ function. In particular, we have $\Gamma_\kappa(1) = \Gamma_\kappa(2) = 1$ and $\Gamma_\kappa(3) = 2$. This definition and the relationship in Equation (78) suggests the following one parameter generalization of the Euler $\Gamma(x)$ function, *i.e.*, $\Gamma_\kappa(x)$, given by:

$$\Gamma_\kappa(x) = \left[1 - \kappa^2(x-1)^2\right](x-1) \int_0^\infty t^{x-2} \exp_\kappa(-t)\, dt \tag{81}$$

The explicit expression of $\Gamma_\kappa(x)$ in terms of the ordinary $\Gamma(x)$ is given by:

$$\Gamma_\kappa(x) = \frac{1 - |\kappa|(x-1)}{|2\kappa|^{x-1}}\; \frac{\Gamma\left(\frac{1}{|2\kappa|} - \frac{x-1}{2}\right)}{\Gamma\left(\frac{1}{|2\kappa|} + \frac{x-1}{2}\right)}\, \Gamma(x) \tag{82}$$

and can be used as the definition of $\Gamma_\kappa(x)$ when x is a complex variable. Clearly, in the $\kappa \to 0$ limit, it results $\Gamma_0(x) = \Gamma(x)$. An expression of $\Gamma_\kappa(x)$ in terms of the Beta function is the following:

$$\Gamma_\kappa(x) = \frac{1-|\kappa|(x-1)}{|2\kappa|^{x-1}}\,(x-1)\,B\left(\frac{1}{|2\kappa|} - \frac{x-1}{2}\,,\,x-1\right) \tag{83}$$

From Equations (61) and (81) follows the property:

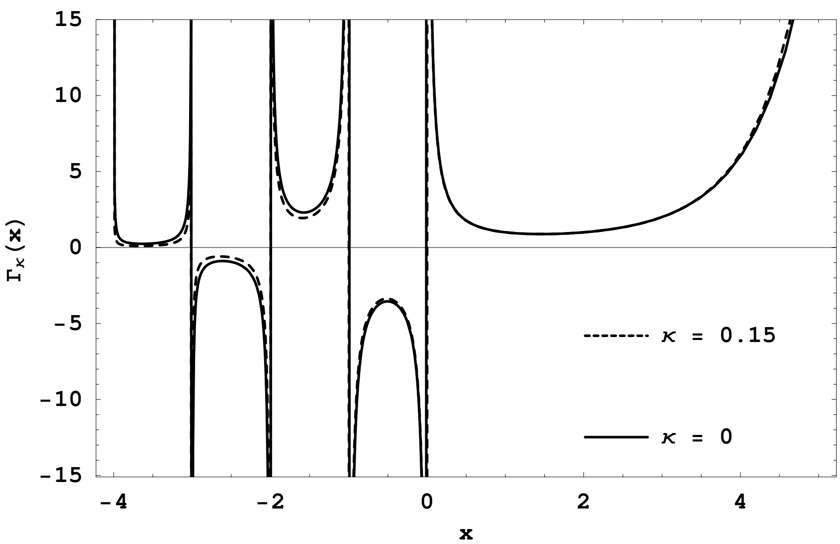
$$\Gamma_\kappa(x+2) = \frac{x\,(x+1)}{1-\kappa^2(x-1)^2}\,\Gamma_\kappa(x) \tag{84}$$

The Taylor expansion of $\exp_\kappa(x)$ can be written also in the form:

$$\exp_\kappa(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma_\kappa(n+1)} \quad ; \quad \kappa^2 x^2 < 1 \tag{85}$$

In Figures 2 and 3 is plotted the function, $\Gamma_\kappa(x)$, defined in Equation (82) in the ranges $-4 < x < 4$ and $9 < x < 12$, respectively, for $\kappa = 0$ and $\kappa = 0.15$. The continuous curve corresponding to $\kappa = 0$ is the ordinary Gamma function, $\Gamma(x)$.

Figure 2. Plot of the function, $\Gamma_\kappa(x)$, defined in Equation (82) in the range $-4 < x < 4$ for $\kappa = 0$ and $\kappa = 0.15$. The continuous curve corresponding to $\kappa = 0$ is the ordinary Gamma function, $\Gamma(x)$.



The incomplete $\gamma_\kappa(r,x)$ and $\Gamma_\kappa(r,x)$ are defined as:

$$\gamma_\kappa(r,x) = \left[1-\kappa^2(r-1)^2\right](r-1)\int_0^x t^{r-2}\,\exp_\kappa(-t)\,dt \tag{86}$$

$$\Gamma_\kappa(r,x) = \left[1-\kappa^2(r-1)^2\right](r-1)\int_x^\infty t^{r-2}\,\exp_\kappa(-t)\,dt \tag{87}$$

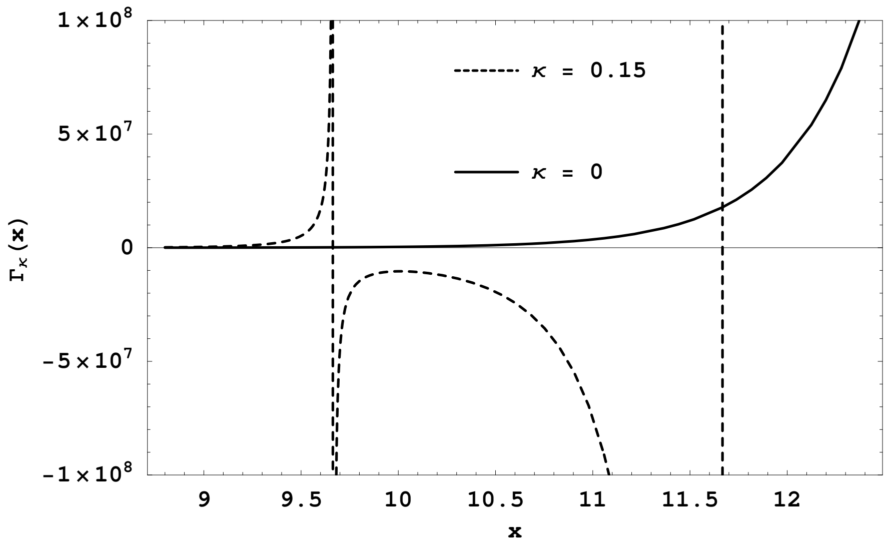
and hold the following relationships:

$$\gamma_\kappa(r,x) + \Gamma_\kappa(r,x) = \Gamma_\kappa(r) \tag{88}$$

$$\gamma_\kappa(r,\infty) = \Gamma_\kappa(r) \tag{89}$$

$$\gamma_\kappa(r,x) = (r-1)\left[1-\kappa^2(r-1)^2\right]\mathcal{M}_\kappa(r-1,x) \tag{90}$$

Figure 3. Plot of the function, $\Gamma_\kappa(x)$, defined in Equation (82) in the range $9 < x < 12$ for $\kappa = 0$ and $\kappa = 0.15$. The continuous curve corresponding to $\kappa = 0$ is the ordinary Gamma function, $\Gamma(x)$.



4.6. Expansion in Ordinary Exponentials

Starting from Expression (32) in Equation $\exp_\kappa(x)$ and the Taylor expansion of the function, $\operatorname{arcsinh}(x)$, we obtain:

$$\exp_\kappa(x) = \exp\left(\sum_{n=0}^{\infty} c_n\,\kappa^{2n}x^{2n+1}\right) \quad , \quad \kappa^2x^2 \le 1 \tag{91}$$

with:

$$c_n = \frac{(-1)^n\,(2n)!}{(2n+1)\,2^{2n}\,(n!)^2} \tag{92}$$

Exploiting this relationship, we can write $\exp_\kappa(x)$ as an infinite product of ordinary exponentials:

$$\exp_\kappa(x) = \prod_{n=0}^{\infty} \exp\left(c_n\,\kappa^{2n}x^{2n+1}\right) \tag{93}$$

On the other hand, $\exp_\kappa(x)$ can be viewed as a continuous linear combination of an infinity of standard exponentials. Namely, for $\operatorname{Re} s \ge 0$, the following Laplace transform holds:

$$\exp_\kappa(-s) = \int_0^\infty \frac{1}{\kappa x}\, J_{1/\kappa}\!\left(\frac{x}{\kappa}\right) \exp\left(-sx\right) dx \tag{94}$$

$J_\nu(x)$ being the Bessel function.

4.7. The κ-Laplace Transform

The following κ-Laplace transform emerges naturally:

$$F_\kappa(s) = \mathcal{L}_\kappa\{f(t)\}(s) = \int_0^\infty f(t)\, [\exp_\kappa(-t)]^s \, dt \tag{95}$$

as a generalization of the ordinary Laplace transform. The inverse κ-Laplace transform is given by:

$$f(t) = \mathcal{L}_\kappa^{-1}\{F_\kappa(s)\}(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_\kappa(s)\, \frac{[\exp_\kappa(t)]^s}{\sqrt{1+\kappa^2 t^2}}\, ds \tag{96}$$

In [34], the mathematical properties of the κ-Laplace transform have been investigated systematically. In Table 1 are reported the main properties of the κ-Laplace transform, which in the $\kappa \to 0$ limit reduce to the corresponding ordinary Laplace transform properties.

Table 1. Properties of the κ-Laplace transform.

$f(t)$	$F_\kappa(s)$
$a\, f(t) + b\, g(t)$	$a\, F_\kappa(s) + b\, G_\kappa(s)$
$f(at)$	$\frac{1}{a}\, F_{\kappa/a}(\frac{s}{a})$
$f(t)\, [\exp_\kappa(-t)]^a$	$F_\kappa(s-a)$
$\frac{d\, f(t)}{dt}$	$s\, \mathcal{L}_\kappa \left\{ \frac{f(t)}{\sqrt{1+\kappa^2 t^2}} \right\}(s) - f(0)$
$\frac{d}{dt}\, \sqrt{1+\kappa^2 t^2}\, f(t)$	$s\, F_\kappa(s) - f(0)$
$\frac{1}{\sqrt{1+\kappa^2 t^2}}\, \int_0^t f(w) dw$	$\frac{1}{s}\, F_\kappa(s)$
$f(t)\, [\ln(\exp_\kappa(t))]^n$	$(-1)^n \frac{d^n F_\kappa(s)}{ds^n}$
$f(t)\, [\ln(\exp_\kappa(t))]^{-n}$	$\int_s^{+\infty} dw_n \int_{w_n}^{+\infty} dw_{n-1} ... \int_{w_3}^{+\infty} dw_2 \int_{w_2}^{+\infty} dw_1\, F_\kappa(w_1)$

Furthermore, the initial value theorem holds:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF_\kappa(s) \tag{97}$$

and the final value theorem:

$$\lim_{t\to\infty} |\kappa| t f(t) = \lim_{s\to 0} sF_\kappa(s) \tag{98}$$

The κ-convolution of two functions, $f \overset{\kappa}{*} g = (f \overset{\kappa}{*} g)(t)$, is defined as:

$$f \overset{\kappa}{*} g = \int_0^t f(t \overset{\kappa}{\ominus} \tau)\, g(\tau)\, \frac{1-\kappa^2 \tau(t-\tau)}{\sqrt{1+\kappa^2 \tau^2}}\, d\tau \tag{99}$$

and has the following properties:

$$f \overset{\kappa}{*} (ag + bh) = a(f \overset{\kappa}{*} g) + b(f \overset{\kappa}{*} h) \tag{100}$$

$$f \overset{\kappa}{*} g = g \overset{\kappa}{*} f \tag{101}$$

$$f \overset{\kappa}{*} (g \overset{\kappa}{*} h) = (f \overset{\kappa}{*} g) \overset{\kappa}{*} h \tag{102}$$

the following κ-convolution theorem holds:

$$\mathcal{L}_\kappa\{f \overset{\kappa}{*} g\} = \mathcal{L}_\kappa\{f\}\, \mathcal{L}_\kappa\{g\} \tag{103}$$

In Table 2 are reported the κ-Laplace transforms for the delta function, for the unit function and for the power function. We note that the κ-Laplace transform of the power function $f(t) = t^{\nu-1}$ involves the κ-generalized Gamma function. All the κ-Laplace transforms in the $\kappa \to 0$ limit reduce to the corresponding ordinary Laplace transforms.

Table 2. The κ-Laplace transform of the Dirac delta-function, of the Heaviside unit function and of the power function.

$f(t)$	$F_\kappa(s)$
$\delta(t-\tau)$	$[\exp_\kappa(-\tau)]^s$
$u(t-\tau)$	$\frac{s\sqrt{1+\kappa^2\tau^2}+\kappa^2\tau}{s^2-\kappa^2}\,[\exp_\kappa(-\tau)]^s$
$t^{\nu-1}$	$\frac{s^2}{s^2-\kappa^2\nu^2}\,\frac{\Gamma_{\frac{\kappa}{s}}(\nu+1)}{\nu\, s^\nu} = \frac{s}{s+\lvert\kappa\rvert\nu}\,\frac{\Gamma(\nu)}{\lvert 2\kappa\rvert^\nu}\,\frac{\Gamma\left(\frac{s}{\lvert 2\kappa\rvert}-\frac{\nu}{2}\right)}{\Gamma\left(\frac{s}{\lvert 2\kappa\rvert}+\frac{\nu}{2}\right)}$
$t^{2m-1},\;\; m \in Z^+$	$\frac{(2m-1)!}{\prod_{j=1}^{m}[s^2-(2j)^2\kappa^2]}$
$t^{2m},\;\; m \in Z^+$	$\frac{(2m)!\,s}{\prod_{j=1}^{m+1}[s^2-(2j-1)^2\kappa^2]}$

5. The Function $\ln_\kappa(x)$

5.1. Definition and Basic Properties

The function, $\ln_\kappa(x)$, is defined as the inverse function of $\exp_\kappa(x)$, namely:

$$\ln_\kappa(\exp_\kappa x) = \exp_\kappa(\ln_\kappa x) = x \tag{104}$$

and is given by:

$$\ln_\kappa(x) = [\ln x] \tag{105}$$

and then:

$$\ln_\kappa(x) = \frac{1}{\kappa}\, \sinh\left(\kappa \ln x\right) \tag{106}$$

or more properly:

$$\ln_\kappa(x) = \frac{x^\kappa - x^{-\kappa}}{2\kappa} \tag{107}$$

It results that:

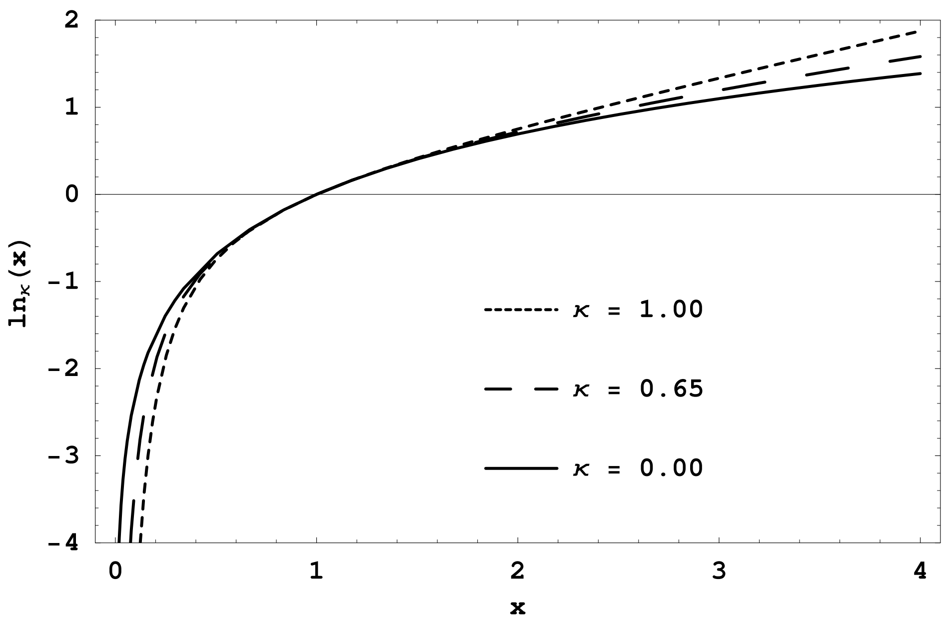
$$\ln_0(x) \equiv \lim_{\kappa\to 0} \ln_\kappa(x) = \ln(x) \tag{108}$$
$$\ln_{-\kappa}(x) = \ln_\kappa(x) \tag{109}$$

The function, $\ln_\kappa(x)$, just as the ordinary logarithm, has the properties:

$$\ln_\kappa(x) \in C^\infty(\mathbf{R}^+) \tag{110}$$
$$\frac{d}{dx}\ln_\kappa(x) > 0 \tag{111}$$
$$\ln_\kappa(0^+) = -\infty \tag{112}$$
$$\ln_\kappa(1) = 0 \tag{113}$$
$$\ln_\kappa(+\infty) = +\infty \tag{114}$$
$$\ln_\kappa(1/x) = -\ln_\kappa(x) \tag{115}$$

In Figure 4 is plotted the function, $\ln_\kappa(x)$, defined in Equation (107) for three different values of the parameter of κ. The continuous curve corresponding to $\kappa = 0$ is the ordinary logarithm function, $\ln(x)$.

Figure 4. Plot of the function, $\ln_\kappa(x)$, defined by Equation (107) for three different values of the parameter of κ. The continuous curve corresponding to $\kappa = 0$ is the ordinary logarithm function, $\ln(x)$.



Furthermore, $\ln_\kappa(x)$ has the two properties:

$$\ln_\kappa(x^r) = r\ln_{r\kappa}(x) \tag{116}$$
$$\ln_\kappa(x\,y) = \ln_\kappa(x) \overset{\kappa}{\oplus} \ln_\kappa(y) \tag{117}$$

with $r \in \mathbf{R}$. Note that property in Equation (115) follows as a particular case of property in Equation (117).

We remark the following concavity properties:

$$\frac{d^2}{d\,x^2}\,\ln_\kappa(x) < 0 \tag{118}$$

$$\frac{d^2}{d\,x^2}\,x\,\ln_\kappa(x) < 0 \tag{119}$$

A very interesting property of this function is its power law asymptotic behavior:

$$\ln_\kappa(x) \underset{x\to 0^+}{\sim} -\frac{1}{2\,|\kappa|}\,x^{-|\kappa|} \tag{120}$$

$$\ln_\kappa(x) \underset{x\to +\infty}{\sim} \frac{1}{2\,|\kappa|}\,x^{|\kappa|} \tag{121}$$

After recalling the integral representation of the ordinary logarithm:

$$\ln(x) = \frac{1}{2}\int_{1/x}^{x} \frac{1}{t}\,dt \tag{122}$$

one can verify that the latter relationship can be generalized easily in order to obtain $\ln_\kappa(x)$, by replacing the integrand function $y_0(t) = t^{-1}$ by the new function $y_\kappa(t) = t^{-1-\kappa}$, namely:

$$\ln_\kappa(x) = \frac{1}{2}\int_{1/x}^{x} \frac{1}{t^{1+\kappa}}\,dt \tag{123}$$

5.2. Taylor Expansion

The Taylor expansion of $\ln_\kappa(1+x)$ converges if $-1 < x \leq 1$ and assumes the form:

$$\ln_\kappa(1+x) = \sum_{n=1}^{\infty} b_n(\kappa)\,(-1)^{n-1}\,\frac{x^n}{n} \tag{124}$$

with $b_1(\kappa) = 1$, while for $n > 1$, $b_n(\kappa)$ is given by:

$$\begin{aligned} b_n(\kappa) \quad &= \frac{1}{2}\Big(1-\kappa\Big)\Big(1-\frac{\kappa}{2}\Big)...\Big(1-\frac{\kappa}{n-1}\Big) \\ &+\frac{1}{2}\Big(1+\kappa\Big)\Big(1+\frac{\kappa}{2}\Big)...\Big(1+\frac{\kappa}{n-1}\Big) \end{aligned} \tag{125}$$

It results $b_n(0) = 1$ and $b_n(-\kappa) = b_n(\kappa)$. The first terms of the expansion are:

$$\ln_\kappa(1+x) = x - \frac{x^2}{2} + \left(1+\frac{\kappa^2}{2}\right)\frac{x^3}{3} - ... \tag{126}$$

5.3. The Function $\Gamma_\kappa(x)$

The following integral is useful:

$$\int_0^1 \left(\ln_\kappa \frac{1}{x} \right)^{r-1} dx = \frac{|2\kappa|^{1-r}}{1+(r-1)|\kappa|} \, \frac{\Gamma\left(\frac{1}{|2\kappa|} - \frac{r-1}{2}\right)}{\Gamma\left(\frac{1}{|2\kappa|} + \frac{r-1}{2}\right)} \, \Gamma(r) \tag{127}$$

Starting from the definition of the generalized Euler gamma function, *i.e.*, $\Gamma_\kappa(x)$ given in the previous section, we can write it also in the following alternative, but equivalent form:

$$\Gamma_\kappa(x) = \left[1 - \kappa^2 (x-1)^2\right] \int_0^1 \left(\ln_\kappa \frac{1}{t} \right)^{x-1} dt \tag{128}$$

An expression of $\Gamma_\kappa(x)$, where the parameter κ enters exclusively through the function, $\ln_\kappa(.)$, follows easily:

$$\Gamma_\kappa(x) = (x-1) \, \frac{\int_0^1 \left(\ln_\kappa \frac{1}{t}\right)^{x-1} dt}{\int_0^1 \ln_\kappa \left(\frac{1}{t}\right)^{x-1} dt} \tag{129}$$

From the latter relationships, it follows that $n!_\kappa$ is given by:

$$n!_\kappa = \left(1 - \kappa^2 n^2\right) \int_0^1 \left(\ln_\kappa \frac{1}{t} \right)^{n} dt = n \, \frac{\int_0^1 \left(\ln_\kappa \frac{1}{t}\right)^{n} dt}{\int_0^1 \ln_\kappa \left(\frac{1}{t}\right)^{n} dt} \tag{130}$$

5.4. $\ln_\kappa(x)$ *as the Solution of a Functional Equation*

The logarithm $y(x) = \ln(x)$ is the only existing function, unless a multiplicative constant, which results in being the solution of the function equation $y(x_1 x_2) = y(x_1) + y(x_2)$. Let us consider now the generalization of this equation, obtained by substituting the ordinary sum by the generalized sum:

$$y(x_1 x_2) = y(x_1) \overset{\kappa}{\oplus} y(x_2) \tag{131}$$

We proceed by solving this equation, which assumes the explicit form:

$$\begin{aligned} y(x_1 x_2) &= y(x_1) \sqrt{1 + \kappa^2 \, y(x_2)^{\,2}} \\ &+ y(x_2) \sqrt{1 + \kappa^2 \, y(x_1)^{\,2}} \end{aligned} \tag{132}$$

After performing the substitution $y(x) = \kappa^{-1} \sinh \kappa g(x)$, we obtain that the auxiliary function, $g(x)$, obeys the equation $g(x_1 x_2) = g(x_1) + g(x_2)$ and, then, is given by $g(x) = A \ln x$. The unknown function, $y(x)$, becomes $y(x) = \kappa^{-1} \sinh(\kappa \ln x)$ where we have set $A = 1$ in order to recover, in the limit $\kappa \to 0$, the classical solution $y(x) = \ln(x)$. Then, we can conclude that the solution of Equation (131) is given by:

$$y(x) = \ln_\kappa(x) \tag{133}$$

5.5. $\ln_\kappa(x)$ *as the Solution of a Differential-Functional Equation*

The following first order differential-functional equation emerges in statistical mechanics within the context of the maximum entropy principle:

$$\frac{d}{dx}\left[\,x\;f\,(x)\,\right]=\frac{1}{\gamma}\,f\,(\epsilon x) \tag{134}$$
$$f(1)=0 \tag{135}$$
$$f'(1)=1 \tag{136}$$
$$f\,(1/x)=-f\,(x) \tag{137}$$

The latter problem admits two solutions [3,9]. The first is given by $f(x)=\ln(x)$ and $\gamma=1$, $\epsilon=e$. The second solution is given by:

$$f(x)=\ln_\kappa(x) \tag{138}$$

and:

$$\gamma=\frac{1}{\sqrt{1-\kappa^2}} \tag{139}$$
$$\epsilon=\left(\frac{1+\kappa}{1-\kappa}\right)^{1/2\kappa} \tag{140}$$

The constant, γ, is the Lorentz factor corresponding to the reference velocity v_*, while the constant, $\epsilon=\exp_\kappa(\gamma)$, represents the κ-generalization of the Napier number, e.

5.6. The Entropy

A physically meaningful link between the functions, $\ln_\kappa(x)$ and $\exp_\kappa(x)$, is given by a variational principle. The following theorem holds:

Theorem 6. *Let* $g(x)$ *be an arbitrary real function and* $y(x)$*, a probability distribution function of the variable,* $x\in A$*. The solution of the variational equation:*

$$\frac{\delta}{\delta y(x)}\left[-\int_A dx\;y(x)\ln_\kappa y(x)+\int_A dx\;y(x)\,g(x)\right]=0 \tag{141}$$

is unique and is given by:

$$y(x)=\frac{1}{\epsilon}\,\exp_\kappa\!\big(\gamma\,g(x)\big) \tag{142}$$

the constants γ *and* ϵ *being defined by Equations (139) and (140), respectively.*

The proof of the theorem is trivial and employs Equations (134). This theorem permits us to interpret the functional:

$$S_\kappa=-\int_A dx\;y(x)\ln_\kappa y(x) \tag{143}$$

which can be written also in the form:

$$S_\kappa = \int_A dx \; \frac{y(x)^{1-\kappa} - y(x)^{1+\kappa}}{2\kappa} \tag{144}$$

as the entropy associated with the function, $\exp_\kappa(x)$. It is remarkable that in the $\kappa \to 0$ limit, as $\ln_\kappa(y)$ and $\exp_\kappa(x)$ approach $\ln(y)$ and $\exp(x)$, respectively, the new entropy reduces to the old Boltzmann-Shannon entropy.

It is shown that the entropy S_κ has the standard properties of Boltzmann-Shannon entropy: it is thermodynamically and Lesche stable and obeys the Khinchin axioms of continuity, maximality, expandability and generalized additivity.

6. κ-Trigonometry

6.1. κ-Hyperbolic Trigonometry

The κ-hyperbolic trigonometry can be introduced by defining the κ-hyperbolic sine and κ-hyperbolic cosine:

$$\sinh_\kappa(x) = \frac{\exp_\kappa(x) - \exp_\kappa(-x)}{2} \tag{145}$$

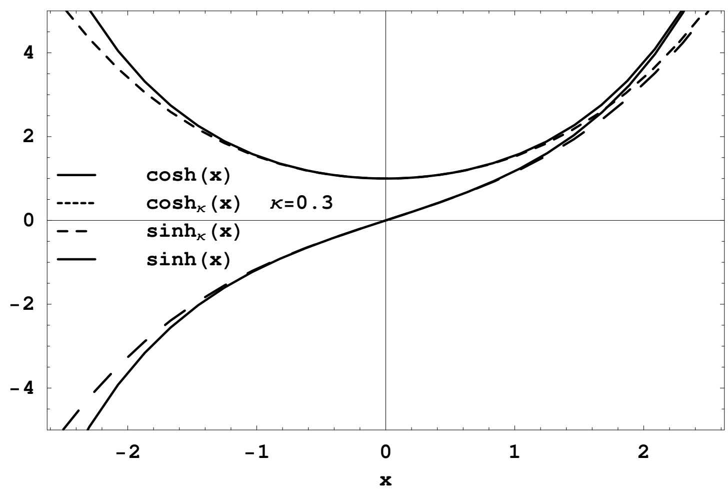
$$\cosh_\kappa(x) = \frac{\exp_\kappa(x) + \exp_\kappa(-x)}{2} \tag{146}$$

starting from the κ-Euler formula:

$$\exp_\kappa(\pm x) = \cosh_\kappa(x) \pm \sinh_\kappa(x) \tag{147}$$

In Figure 5 are plotted the functions, $\sinh_\kappa(x)$ and $\cosh_\kappa(x)$, for $\kappa = 0.3$ (dashed lines). For comparison, in the same figure are reported the corresponding ordinary functions, $\sinh(x)$ and $\cosh(x)$ (continuous lines).

Figure 5. Plot of the functions, $\sinh_\kappa(x)$ and $\cosh_\kappa(x)$, for $\kappa = 0.3$ (dashed lines) defined through Equations (145) and (146), respectively. For comparison, in the same plot are reported the corresponding ordinary functions, $\sinh(x)$ and $\cosh(x)$ (continuous lines).



The κ-hyperbolic tangent and cotangent functions are defined through:

$$\tanh_\kappa(x) = \frac{\sinh_\kappa(x)}{\cosh_\kappa(x)} \tag{148}$$

$$\coth_\kappa(x) = \frac{\cosh_\kappa(x)}{\sinh_\kappa(x)} \tag{149}$$

Holding the relationships:

$$\sinh_\kappa(x) = \sinh\left(\{x\}\right) \tag{150}$$

$$\cosh_\kappa(x) = \cosh\left(\{x\}\right) \tag{151}$$

$$\tanh_\kappa(x) = \tanh\left(\{x\}\right) \tag{152}$$

$$\coth_\kappa(x) = \coth\left(\{x\}\right) \tag{153}$$

it is straightforward to verify that κ-hyperbolic trigonometry preserves the same structure of the ordinary hyperbolic trigonometry, which recovers as a special case in the limit $\kappa \to 0$. For instance, from the κ-Euler formula and from $\exp_\kappa(-x)\exp_\kappa(x) = 1$, the fundamental formula of the κ-hyperbolic trigonometry follows:

$$\cosh^2_\kappa(x) - \sinh^2_\kappa(x) = 1 \tag{154}$$

All the formulas of the ordinary hyperbolic trigonometry still hold, after proper generalization. Taking into account that $\{x \overset{\kappa}{\oplus} y\} = \{x\} + \{y\}$, it is easy to verify that the generalization of a given formula can be obtained starting from the corresponding ordinary formula and, then, by making in the arguments of the hyperbolic trigonometric functions the substitutions $x + y \to x \overset{\kappa}{\oplus} y$ and $x - y \to x \overset{\kappa}{\ominus} y$. For instance, it results:

$$\sinh_\kappa(x \overset{\kappa}{\oplus} y) + \sinh_\kappa(x \overset{\kappa}{\ominus} y) = 2\sinh_\kappa(x)\cosh_\kappa(y) \tag{155}$$

$$\cosh_\kappa(x \overset{\kappa}{\oplus} y) = \cosh_\kappa(x)\cosh_\kappa(x) + \sinh_\kappa(x)\sinh_\kappa(y) \tag{156}$$

$$\tanh_\kappa(x) + \tanh_\kappa(y) = \frac{\sinh_\kappa(x \overset{\kappa}{\oplus} y)}{\cosh_\kappa(x)\cosh_\kappa(y)} \tag{157}$$

and so on.

Obviously, the substitution $nx \to x \overset{\kappa}{\oplus} x \overset{\kappa}{\oplus} \ldots \overset{\kappa}{\oplus} x = [n] \overset{\kappa}{\otimes} x$ is required, so that, for instance, it holds the formula:

$$\sinh^4_\kappa(x) = \frac{1}{8}\left[\cosh_\kappa\left([4] \overset{\kappa}{\otimes} x\right) - 4\cosh_\kappa\left([2] \overset{\kappa}{\otimes} x\right) + 3\right] \tag{158}$$

and so on.

The κ-De Moivre formula involving hyperbolic trigonometric functions having arguments of the type rx, with $r \in \mathbf{R}$, assumes the form:

$$\left[\cosh_\kappa(x) \pm \sinh_\kappa(x)\right]^r = \cosh_{\kappa/r}(rx) \pm \sinh_{\{\kappa/r\}}(rx) \tag{159}$$

Furthermore, the formulas involving the derivatives of the hyperbolic trigonometric function still hold, after being properly generalized. For instance, we have:

$$\frac{d\ \sinh_\kappa(x)}{d_\kappa x} = \cosh_\kappa(x) \tag{160}$$

$$\frac{d\ \tanh_\kappa(x)}{d_\kappa x} = \cosh_\kappa^{-2}(x) \tag{161}$$

and so on.

The κ-inverse hyperbolic functions can be introduced starting from the corresponding ordinary functions. It is trivial to verify that κ-inverse hyperbolic functions are related to the κ-logarithm by the usual formulas of the ordinary mathematics. For instance, we have:

$$\mathrm{arcsinh}_\kappa(x) = \ln_\kappa\left(\sqrt{1+x^2}+x\right) \tag{162}$$

$$\mathrm{arccosh}_\kappa(x) = \ln_\kappa\left(\sqrt{x^2-1}+x\right) \tag{163}$$

$$\mathrm{arctanh}_\kappa(x) = \ln_\kappa\sqrt{\frac{1+x}{1-x}} \tag{164}$$

$$\mathrm{arccoth}_\kappa(x) = \ln_\kappa\sqrt{\frac{1-x}{1+x}} \tag{165}$$

and, consequently, hold:

$$\mathrm{arcsinh}_\kappa(x) = \mathrm{arccosh}_\kappa\sqrt{1+x^2} \tag{166}$$

$$\mathrm{arcsinh}_\kappa(x) = \mathrm{arctanh}_\kappa\frac{x}{\sqrt{1+x^2}} \tag{167}$$

$$\mathrm{arcsinh}_\kappa(x) = \mathrm{arccoth}_\kappa\frac{\sqrt{1+x^2}}{x} \tag{168}$$

From Equation (162) follows the relationship:

$$\exp_\kappa\left(\mathrm{arcsinh}_\kappa\, x\right) = \exp\left(\mathrm{arcsinh}\, x\right) \tag{169}$$

Furthermore, the relationship:

$$\mathrm{arcsinh}_\kappa\,(x) = \frac{1}{\kappa}\,\sinh_{1/\kappa}(\kappa x) \tag{170}$$

involving the function, $\mathrm{arcsinh}_\kappa(x)$, follows from Equation (162). Analogous formulas involving $\mathrm{arccosh}_\kappa(x)$, $\mathrm{arctanh}_\kappa(x)$ or $\mathrm{arccoth}_\kappa(x)$ do not hold, instead.

6.2. κ-Cyclic Trigonometry

By employing the generalized κ-Euler formula:

$$\exp_\kappa(\pm ix) = \cos_\kappa(x) \pm i\sin_\kappa(x) \tag{171}$$

we introduce the κ-cyclic sine and κ-cosine as:

$$\sin_\kappa(x) = \frac{\exp_\kappa(ix)-\exp_\kappa(-ix)}{2i} \tag{172}$$

$$\cos_\kappa(x) = \frac{\exp_\kappa(ix)+\exp_\kappa(-ix)}{2} \tag{173}$$

while the κ-cyclic tangent and κ-cotangent functions are defined through:

$$\tan_\kappa(x) = \frac{\sin_\kappa(x)}{\cos_\kappa(x)} \tag{174}$$

$$\cot_\kappa(x) = \frac{\cos_\kappa(x)}{\sin_\kappa(x)} \tag{175}$$

After noting that:

$$\exp_\kappa(ix) = \exp\left(i\{x\}\right) \tag{176}$$

with:

$$\{x\} = \frac{1}{\kappa} \arcsin \kappa x \tag{177}$$

it follows that the cyclic functions are defined in the interval, $-1/\kappa \leq x \leq 1/\kappa$. The function:

$$[x] = \frac{1}{\kappa} \sin \kappa x \tag{178}$$

is defined as the inverse of $\{x\}$, i.e, $[\{x\}] = \{[x]\} = x$. The κ-sum $\overset{\kappa}{\oplus}$ and κ-product $\overset{\kappa}{\otimes}$ given by:

$$x \overset{\kappa}{\oplus} y = x\sqrt{1-\kappa^2 y^2} + y\sqrt{1-\kappa^2 x^2} \tag{179}$$

$$x \overset{\kappa}{\otimes} y = \frac{1}{\kappa} \sin\left(\frac{1}{\kappa} \arcsin\left(\kappa x\right) \arcsin\left(\kappa y\right)\right) \tag{180}$$

are isomorphic operations to the ordinary sum and product respectively, *i.e.*:

$$\{x \overset{\kappa}{\oplus} y\} = \{x\} + \{y\} \tag{181}$$

$$\{x \overset{\kappa}{\otimes} y\} = \{x\} \cdot \{y\} \tag{182}$$

Holding the relationships:

$$\sin_\kappa(x) = \sin\left(\{x\}\right) \tag{183}$$

$$\cos_\kappa(x) = \cos\left(\{x\}\right) \tag{184}$$

$$\tan_\kappa(x) = \tan\left(\{x\}\right) \tag{185}$$

$$\cot_\kappa(x) = \cot\left(\{x\}\right) \tag{186}$$

It is straightforward to verify that the generalized cyclic trigonometry preserves the same structure of the ordinary cyclic trigonometry, which recovers as a special case in the limit $\kappa \to 0$. For instance, the following generalized formulas hold:

$$\cos_\kappa^2(x) + \sin_\kappa^2(x) = 1 \tag{187}$$

$$\sin_\kappa(x \overset{\kappa}{\oplus} y) = \sin_\kappa(x)\cos_\kappa(y) + \cos_\kappa(x)\sin_\kappa(y) \tag{188}$$

$$\cos_\kappa^5(x) = \frac{1}{16}\Big[\cos_\kappa\Big([5] \overset{\kappa}{\otimes} x\Big) + 5\cos_\kappa\Big([3] \overset{\kappa}{\otimes} x\Big) + 10\cos_\kappa(x)\Big] \tag{189}$$

and so on.

After introducing the following κ-deformed derivative operator:

$$\frac{d}{d_\kappa x} = \sqrt{1-\kappa^2 x^2}\ \frac{d}{d\,x} \tag{190}$$

we can obtain, easily, further formulas on the cyclic κ-trigonometry emerging as generalizations of the corresponding formulas of the ordinary trigonometry. For instance, we have:

$$\frac{d\ \cos_\kappa(x)}{d_\kappa x} = -\sin_\kappa(x) \tag{191}$$

$$\frac{d\ \cot_\kappa(x)}{d_\kappa x} = -\ \sin_\kappa^{-2}(x) \tag{192}$$

and so on.

The κ-inverse cyclic functions can be calculated by inversion of the corresponding direct functions and are given by:

$$\mathrm{arcsin}_\kappa(x) = -i\ \ln_\kappa\left(\sqrt{1-x^2}+ix\right) \tag{193}$$

$$\mathrm{arccos}_\kappa(x) = -i\ \ln_\kappa\left(\sqrt{x^2-1}+x\right) \tag{194}$$

$$\mathrm{arctan}_\kappa(x) = i\ \ln_\kappa\sqrt{\frac{1-ix}{1+ix}} \tag{195}$$

$$\mathrm{arccot}_\kappa(x) = i\ \ln_\kappa\sqrt{\frac{ix+1}{ix-1}} \tag{196}$$

Finally, we note that the κ-cyclic and κ-hyperbolic trigonometric functions are linked through the relationships:

$$\sin_\kappa(x) = -i\sinh_\kappa(ix) \tag{197}$$

$$\cos_\kappa(x) = \cosh_\kappa(ix) \tag{198}$$

$$\tan_\kappa(x) = -i\tanh_\kappa\,(ix) \tag{199}$$

$$\cot_\kappa(x) = i\coth_\kappa\,(ix) \tag{200}$$

$$\mathrm{arcsin}_\kappa(x) = -i\,\mathrm{arcsinh}_\kappa(ix) \tag{201}$$

$$\mathrm{arccos}_\kappa(x) = -i\,\mathrm{arccosh}_\kappa(x) \tag{202}$$

$$\mathrm{arctan}_\kappa(x) = -i\,\mathrm{arctanh}_\kappa(ix) \tag{203}$$

$$\mathrm{arccot}_\kappa(x) = i\,\mathrm{arccoth}_\kappa(ix) \tag{204}$$

which, in the $\kappa \to 0$ limit, reduce to the standard formulas involving the ordinary cyclic and hyperbolic functions.

Conflicts of Interest

The author declares no conflict of interest.

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