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A NOTE ON INTEGER POLYNOMIALS WITH SMALL INTEGRALS

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ABSTRACT. The smart method of Gelfond–Shnirelman–Nair allows to obtain in elementary way a lower bound for the prime counting function $\pi(x)$ in terms of integrals of suitable integer polynomials. In this paper we studied the properties of the class of integer polynomials relevant for the method.

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1. INTRODUCTION

Let $\pi(x)$ be the number of primes not exceeding x . The Prime Number Theorem (PNT), independently proved in 1896 by Hadamard and the de la Vallée Poussin, states that

$$\pi(N) \sim \frac{N}{\log N} \quad N \rightarrow +\infty.$$

In 1851, Chebyshev [6] made the first step towards the PNT by proving that, given $\varepsilon > 0$,

$$(c_1 - \varepsilon) \frac{N}{\log N} \leq \pi(N) \leq (c_2 + \varepsilon) \frac{N}{\log N}$$

where $c_1 = \log(2^{1/2}3^{1/3}5^{1/5}/30^{1/30})$, $c_2 = 6c_1/5$ and N is sufficiently large. This result was proved using an elementary approaches, i.e. without use of complex analysis and the Riemann zeta function. A survey of elementary methods in the study of the distribution of prime numbers may be found in Diamond [7].

In 1936 Gelfond and Shnirelman, see Gelfond's editorial remarks in the 1944 edition of Chebyshev's Collected Works [6, pag. 287–288], proposed a new elementary and clever method for deriving a lower bound for the prime-counting functions $\pi(x)$ and $\psi(x)$. In 1982 the Gelfond–Shnirelman method was rediscovered and developed by Nair, see [9] and [10]. The method of Gelfond–Shnirelman–Nair allows to obtain in elementary way a lower bound for $\pi(x)$ in terms of integrals of suitable integer polynomials and runs as follows.

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Let d_N denote the least common multiple of the integers $1, 2, \dots, N$ and observe that

$$d_N \leq \prod_{p \leq N} p^{\log N / \log p},$$

where p belongs to the set of prime numbers. Taking the logarithm of both sides gives

$$\log d_N \leq \log \left(\prod_{p \leq N} p^{\log N / \log p} \right) = \sum_{p \leq N} \log \left(p^{\log N / \log p} \right) = \pi(N) \log N$$

and then

$$(1) \quad \pi(N) \geq \frac{\log d_N}{\log N}.$$

From this we can obtain a lower bound for the prime counting function $\pi(N)$ from a lower bound for the least common multiple d_N . An elementary and smart way to proceed is to consider a polynomial with integral coefficients

$$P(x) = \sum_{n=0}^{N-1} a_n x^n$$

and let

$$I(P) = \int_0^1 P(x) dx = \sum_{n=0}^{N-1} \frac{a_n}{n+1}.$$

Since $I(P)$ is a rational number whose denominator divides d_N , we see that $I(P)d_N$ is an integer, and hence if $I(P) \neq 0$ we have

$$d_N |I(P)| \geq 1$$

and then

$$d_N \geq \frac{1}{|I(P)|}.$$

Form the above and (1) we get

$$(2) \quad \pi(N) \geq \frac{\log(1/|I(P)|)}{\log N}.$$

The easiest way to proceed is to bound the absolute value of the integral $I(P)$

$$(3) \quad |I(P)| = \left| \int_0^1 P(x) dx \right| \leq \int_0^1 |P(x)| dx$$

and

$$(4) \quad \int_0^1 |P(x)| dx \leq \max_{0 \leq x \leq 1} |P(x)| = \|P\|_{[0,1]},$$

obtaining

$$\pi(N) \geq \frac{\log(1/\|P\|_{[0,1]})}{\log N}.$$

If we could find a sequence of integer polynomials p_n , of degree n , with sufficiently small supremum norms such that

$$\lim_{n \rightarrow +\infty} \log \left(\|p_n\|_{[0,1]}^{-1/n} \right) = \lim_{n \rightarrow +\infty} -\frac{1}{n} \log \|p_n\|_{[0,1]} = 1,$$

we can obtain the best possible lower bound consistent with the Prime Number Theorem.

This motivates the study of the integer polynomials $P_N(x)$ and the quantities C_N such that

$$\|P_N\|_{[0,1]} = \min_{\substack{P(x) \in \mathbb{Z}[x] \\ \deg(P)=N, \|P\|_{[0,1]} > 0}} \|P\|_{[0,1]}$$

and

$$C_N = -\frac{1}{N} \log \|P_N\|_{[0,1]},$$

the so-called integer Chebyshev problem. Much is known about $P_N(x)$ and C_N . It was proved by Snirelman, see [11], that the sequence C_N converges to a limit C . Borwein and Erdélyi [5] showed that $C \in (0.85866, 0.86577)$ and the lower bound was improved by Flammang [8] to 0.85912. The best known result to date, due to Pritsker [12], is that $C \in (0.85991, 0.86441)$. See also [1], [2], [3], [4], [5] and [14].

Therefore, following this line, we can get a lower bound in the form

$$\pi(N) \geq C \frac{N}{\log N},$$

only for constant C less than 0.87, which is quite far from what is expected by the PNT.

In order to avoid the trouble above, in this paper we deal with the problem in a different way. From the definition of $I(P)$ we have that

$$|I(P)| = \left| \int_0^1 P(x) dx \right| = \left| \sum_{n=0}^{N-1} \frac{a_n}{n+1} \right| = \frac{1}{d_N} \left| \sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n \right|.$$

Since $d_N/(n+1)$ and a_n are integers for every $n = 0, 1, \dots, N-1$, we have that

$$\left| \sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n \right|$$

is an integer and then the small positive value of $|I(P)|$ is $1/d_N$ and it is reached if

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} a_n = \pm 1.$$

Without loss of generality we can deal with the linear diophantine equation

$$\sum_{n=0}^{N-1} \frac{d_N}{n+1} x_n = 1$$

with integer coefficients $d_N/(n+1)$. Observing that the integer coefficients $d_N, d_N/2, \dots, d_N/N$ are relatively prime, we obtain that for every N there exists at least one polynomial of degree $N-1$ such that $I(P) = 1/d_N$. Note that the set of the integer polynomials of fixed

degree with integral on $[0, 1]$ equal to zero is a vector space and then the set of the integer polynomials of fixed degree with integrals on $[0, 1]$ equal to a constant is an affine space. This leads to define the following affine space of the polynomials with positive and minimal integral on $[0, 1]$.

Definition. Let $S_N = \{P(x) \in \mathbb{Z}[x], \deg(P) = N - 1, I(P) = 1/d_N\}$

In this paper we studied the properties of such a class of integer polynomials.

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2. SOME PROPERTIES OF THE SET S_N

In the set S_N there are integer polynomials with many of the first coefficients equal to zero, and then with $x = 0$ as a root of great degree.

Theorem 1. For every N , there exists an integer polynomial

$$P(x) = \sum_{n=K(N)}^{N-1} a_n x^n \in S_N$$

with

$$K(N) \sim \frac{N}{2}.$$

Proof. As usual, (a_1, a_2, \dots, a_j) denotes the greatest common divisor of the integers a_1, a_2, \dots, a_j . We start to observe that if we have

$$\left(\frac{d_N}{k}, \frac{d_N}{k+1}, \dots, \frac{d_N}{N} \right) = 1,$$

for a fixed natural k , it follows that

$$\left(\frac{d_N}{i}, \frac{d_N}{i+1}, \dots, \frac{d_N}{N} \right) = 1,$$

for every $1 \leq i \leq k$ and for the same reason if we have

$$\left(\frac{d_N}{k}, \frac{d_N}{k+1}, \dots, \frac{d_N}{N} \right) > 1,$$

for a fixed natural k , it follows that

$$\left(\frac{d_N}{i}, \frac{d_N}{i+1}, \dots, \frac{d_N}{N} \right) > 1,$$

for every $k \leq i \leq N$. This allows to define $K(N)$ as the natural number such that

$$(5) \quad \left(\frac{d_N}{K(N)+1}, \frac{d_N}{K(N)+2}, \dots, \frac{d_N}{N} \right) = 1$$

and

$$(6) \quad \left(\frac{d_N}{K(N)+2}, \frac{d_N}{K(N)+3}, \dots, \frac{d_N}{N} \right) > 1.$$

From (5) it follows that the linear diophantine equation

$$\sum_{n=K(N)}^{N-1} \frac{d_N}{n+1} x_n = 1$$

has solutions and this implies that there exists an integer polynomial

$$P(x) = \sum_{n=K(N)}^{N-1} a_n x^n \in S_N.$$

Now we prove that

$$(7) \quad K(N) = \min \{p^m : p \text{ prime}, m \geq 1, p^m > N/2\} - 1$$

Let $q = p^m$ such that $N/2 < q = p^m < N$. $q \leq N$ implies q/d_N and then

$$\left(\frac{d_N}{q+1}, \frac{d_N}{q+2}, \dots, \frac{d_N}{N} \right) \geq p,$$

since every natural number between $q+1$ and N has strictly less than m factors p in his prime decomposition. This prove

$$(8) \quad K(N) \leq \min \{p^m : p \text{ prime}, m \geq 1, p^m > N/2\} - 1.$$

On the other hand, by the definition of $K(N)$, we have

$$\left(\frac{d_N}{K(N)+2}, \frac{d_N}{K(N)+3}, \dots, \frac{d_N}{N} \right) > 1$$

which implies that there exists a prime number p such that

$$p \mid \frac{d_N}{K(N)+2}, p \mid \frac{d_N}{K(N)+3}, \dots, p \mid \frac{d_N}{N}.$$

Let $m = \max\{i : p^i \mid d_N\}$ and therefore $p^m \leq N$. From this follows

$$p^m \nmid (K(N)+2), p^m \nmid (K(N)+3), \dots, p^m \nmid N$$

and then

$$(9) \quad K(N) \geq \min \{p^m : p \text{ prime}, m \geq 1, p^m > N/2\} - 1.$$

From (8) and (9) it follows (7). Now the difference between $K(N)$ and $N/2$ can be bound by the maximum of the difference between consecutive elements of the set $\{p^m \leq N : p \text{ prime}, m \geq 1\}$, which is less than the maximum of the difference between consecutive primes less than N . This allow to write

$$K(N) = \frac{N}{2} + O(N^{7/12+\varepsilon}),$$

for every $\varepsilon > 0$, which concludes the proof of the theorem. \square

Corollary 2. *For every N , there exists an integer polynomial $P(x) \in S_N$ with $x = 1$ as a root of degree $K(N)$ and*

$$K(N) \sim \frac{N}{2}.$$

Proof. The corollary follows immediately from the Theorem 1, observing that the change of variable $x \rightarrow (1 - x)$ don't change the absolute value of the integral $I(P)$. \square

The second result is about the number of roots and the number of changes of sign of the integer polynomials in S_N .

Theorem 3. *For all even N , there exists an integer polynomial $P(x) \in S_N$ with $N - 1$ roots on $(0, 1)$ and $N - 1$ changes of sign.*

Proof. Let N even number and $R(x) = (Nx - 1)(Nx - 2) \cdots (Nx - (N - 1))$. $R(x)$ is a polynomial with integer coefficients of degree $N - 1$, has $N - 1$ roots on $(0, 1)$, $(N - 2)/2$ local maxima, $(N - 2)/2$ local minima and

$$I(R) = \int_0^1 R(x) dx = 0,$$

since the symmetry of the function. Let $P(x)$ a fixed polynomial in S_N , $k \in \mathbb{Z}$ and $Q_k(x) = P(x) + kR(x)$. For every $k \in \mathbb{Z}$ we have $I(Q_k) = I(P) = 1/d_N$ and then $Q_k(x) \in S_N$. For every N there exists a constant k such that $Q_k(x)$ has $N - 1$ roots on $(0, 1)$ and $N - 1$ changes of sign. \square

Corollary 4. *For every N , there exists an integer polynomial $P(x) \in S_N$ with at least $N - 2$ roots on $(0, 1)$ and $N - 2$ changes of sign.*

On the other side we can prove that in the set S_N there are also integer polynomials with at most one root and one change of sign.

Theorem 5. *For every N , there exists an integer polynomial $P(x) \in S_N$ with at most one root on $(0, 1)$ and at most one change of sign on $(0, 1)$.*

Proof. Let $P(x)$ a fixed polynomial in S_N , $k \in \mathbb{Z}$ and $Q_k(x) = P(x) + k(2x - 1)$. For every $k \in \mathbb{Z}$ we have $I(Q_k) = I(P) = 1/d_N$ and then $Q_k(x) \in S_N$. Now we observe that $Q_k(0) = P(0) - k$, $Q_k(1) = P(1) + k$ and $Q'_k(x) = P'(x) + 2k$ for every $x \in [0, 1]$.

For every N there exists a constant k such that $Q_k(0) < 0$, $Q_k(1) > 0$ and $Q'_k(x) > 0$ for every $x \in [0, 1]$ and this implies that the polynomial $Q_k(x)$ has exactly one root and one change of sign on $(0, 1)$. \square

3. OPEN PROBLEM

In the standard method of Gelfond–Shnirelman–Nair we bound the absolute value of the integral

$$(10) \quad |I(P)| = \left| \int_0^1 P(x) dx \right| \leq \int_0^1 |P(x)| dx$$

and then

$$(11) \quad \int_0^1 |P(x)| dx \leq \max_{0 \leq x \leq 1} |P(x)| = \|P\|_{[0,1]},$$

to obtain

$$\pi(N) \geq \frac{\log(1/\|P\|_{[0,1]})}{\log N}.$$

As observed in the introduction, following this line we can get a lower bound in the form

$$\pi(N) \geq C \frac{N}{\log N},$$

only for constant C much less than 1. It is not clear if this is only a consequence of the use of supremum norm on the interval $[0, 1]$ in (11) or if the inequality (10) is also involved.

If the set S_N contains polynomials of constant sign in $(0, 1)$ for all N , or at least for infinite values of N , the limit of the method would be only due to the inequality (11).

It is simple to verify that for very small values of N these positive polynomials exist. For S_3 , $\deg(P) = 2$ and $d_3 = 6$, we have the positive polynomial $P(x) = x(1 - x)$ and for S_4 , $\deg(P) = 3$ and $d_3 = 12$, we have the positive polynomial $P(x) = x^2(1 - x)$. For S_N with greater values of N is not simple to determine what happens, and this leads to the following question.

Problem: for every N , or at least for infinite values of N , there exists an integer polynomial $P(x) \in S_N$ such that $P(x) \geq 0$?

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