Gevrey local solvability in locally integrable structures

GEVREY LOCAL SOLVABILITY IN LOCALLY INTEGRABLE STRUCTURES

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ABSTRACT. We consider a locally integrable real-analytic structure, and we investigate the local solvability in the category of Gevrey functions and ultradistributions of the complex d' naturally induced by the de Rham complex. We prove that the so-called condition $Y(q)$ on the signature of the Levi form, for local solvability of $\mathrm{d}'u = f$, is still necessary even if we take f in the classes of Gevrey functions and look for solutions u in the corresponding spaces of ultradistributions.

1. Introduction and statement of the results

Consider a real-analytic manifold M of dimension $m+n$. A real-analyitc locally integrable structure on M, of rank n, is defined by a real analytic subbundle $\mathcal{V} \subset \mathbb{C}TM$ of rank n, satisfying the Frobenius condition and such that the subbundle $T' \subset \mathbb{C}T^*M$ orthogonal to $\mathcal{V}$ is locally spanned by exact differentials. As usual we will denote by $T^0 = T' \cap T^*M$ the so-called *characteristic set*. For any open subset $\Omega \subset M$ and $s > 1$ the space $G^s(\Omega, \Lambda^{p,q})$ of (p,q)-forms with Gevrey coefficients of order s is then defined (see Section 3 below and Treves [28]) and the de Rham differential induces a map

$$\mathrm{d}' : G^s(\Omega, \Lambda^{p,q}) \to G^s(\Omega, \Lambda^{p,q+1}).$$

Similarly, the de Rham differential induces a complex on the space of "ultra-currents" $\mathcal{D}'_s(\Omega, \Lambda^{p,q})$, i.e. forms with ultradistribution coefficients:

$$\mathrm{d}' : \mathcal{D}'_s(\Omega, \Lambda^{p,q}) \to \mathcal{D}'_s(\Omega, \Lambda^{p,q+1}).$$

When $\mathcal{V} \cap \overline{\mathcal{V}} = 0$ the structure is called CR and d' is the so-called tangential Cauchy-Riemann operator.

We are interested in necessary conditions for the Gevrey local solvability problem for the complex d' to hold near a given point x_0.

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Definition 1.1. *We say that the complex* d' *is locally solvable near* x_0 *and in degree* q*,* $1 \leq q \leq n$*, in the sense of ultradistribution of order* s*, if for every sufficiently small open neighborhood* Ω *of* x_0 *and every cocycle* $f \in G^s(\Omega, \Lambda^{0,q})$ *there exists an open neighbourhood* $V \subset U$ *of* x_0 *and a ultradistribution section* $u \in \mathcal{D}'_s(V, \Lambda^{0,q-1})$ *solving* $\mathrm{d}'u = f$ *in* V*.*

The analogous problem in the setting of smooth functions and Schwartz distributions has been extensively considered, see e.g. [1, 2, 3, 8, 9, 10, 11, 12, 20, 22, 23, 24, 26, 28], inspired by the results in [16, 17] for scalar operators of principal type; see also [18, 19] as general references for the problem of local solvability of scalar linear partial differential operators.

Several geometric invariants were there introduced, e.g. the signature of the Levi form recalled below, which represent obstructions to the solvability in the sense of distributions, that is, for some smooth $f \in C^\infty(U, \Lambda^{0,q})$ there is no distribution solution $u \in \mathcal{D}'(V, \Lambda^{0,q-1})$ to $\mathrm{d}'u = f$ in V, for every neighbourhood $V \subset U$ of x_0.

It is therefore natural to wonder whether, under the same condition as in the smooth category, d' is still non-solvable even if we choose f in the smaller class of Gevrey functions $G^s(U, \Lambda^{0,q}) \subset C^\infty(U, \Lambda^{0,q})$ and we look for solutions in the larger class of ultradistributions $\mathcal{D}'_s(V, \Lambda^{0,q}) \supset \mathcal{D}'(V, \Lambda^{0,q})$, as in Definition 1.1. In this note we present a result in this direction.

Let us note that general sufficient conditions for local solvability in the Gevrey category have been recently obtained in [5]; see also [4, 21].

We recall that at any point $(x_0, \omega_0) \in T^0$ it is well defined a sesquilinear form $\mathcal{B}_{(x_0,\omega_0)} : \mathcal{V}_{x_0} \times \mathcal{V}_{x_0} \to \mathbb{C}$, ($\mathcal{V}_{x_0}$ is the fibre above x_0) by

$$\mathcal{B}_{(x_0,\omega_0)}(\mathbf{v}_1, \mathbf{v}_2) = \left\langle \omega_0, (2\imath)^{-1}[V_1, \overline{V_2}]|_{x_0} \right\rangle,$$

with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}_{x_0}$, where V_1 and V_2 are smooth sections of $\mathcal{V}$ such that $V_1|_{x_0} = \mathbf{v}_1$, $V_2|_{x_0} = \mathbf{v}_2$. The associated quadratic form $\mathcal{V}_{x_0} \ni \mathbf{v} \mapsto \mathcal{B}_{(x_0,\omega_0)}(\mathbf{v}, \mathbf{v})$, or $\mathcal{B}_{(x_0,\omega_0)}$ itself, is known as *Levi form.*

Here is our result.

Theorem 1.2. *Let* $(x_0, \omega_0) \in T^0$*,* $\omega_0 \neq 0$*. Suppose that* $\mathcal{B}_{(x_0,\omega_0)}$ *has exactly* q *positive eigenvalues,* $1 \leq q \leq n$*, and* $n - q$ *negative eigenvalues, and that its restriction to* $\mathcal{V}_{x_0} \cap \overline{\mathcal{V}}_{x_0}$ *is non-degenerate.*

Then, for every $s > 1$*,* d' *is not locally solvable in the sense of ultradistributions of order* s*, near* x_0 *and in degree* q*.*

This result therefore strengthens the analogous one in the category of smooth functions and Schwartz distributions, which was proved in [1] for CR manifolds and in [28, Theorem XVIII.3.1] for general locally integrable structure; see also [15, 25] for partial results when the Levi form is degenerate. As general reference for related results about scalar operators on Gevrey spaces see [27].

2. Preliminaries

2.1. **Gevrey functions and ultradistributions.** Let us briefly recall the definition of the classes of Gevrey functions and corresponding ultradistributions; see e.g. [27, Chapter 1] for details.

Let $s > 1$ be a real number and Ω be an open subset of $\mathbb{R}^n$; let C be a positive constant. We denote by $G^s(\Omega, C)$ the space of smooth functions f in $\mathbb{R}^n$ such that for every compact $K \subset \Omega$,

$$||f||_{K,C} := \sup_{\alpha} C^{-|\alpha|}(\alpha!)^{-s} \sup_{x\in K} |\partial^\alpha f(x)| < \infty.$$

This is a Fréchet space endowed with the above seminorms. We set $G^s(\Omega)$ for the usual Gevrey space of order s, i.e. $f \in G^s(\Omega)$ if f is smooth in Ω and for every compact $K \subset \Omega$ there exists $C > 0$ such tha $||f||_{K,C} < \infty$. We will also consider the space $G^s_0(K, C)$ of functions in $G^s(\Omega, C)$ supported in the compact K; it is a Banach space with the norm $||u||_{K,C}$. Finally we set

$$G^s_0(\Omega) = \bigcup_{K\subset\Omega,\ C>0} G^s_0(K, C).$$

The space of $\mathcal{D}'_s(\Omega)$ of ultradistributions of order s in Ω is by definition the dual of $G^s_0(\Omega)$, i.e. an element $u \in \mathcal{D}'_s(\Omega)$ is a linear functional on $G^s_0(\Omega)$ such that for every compact $K \subset \Omega$ and every constant $C > 0$ there exists a constant $C' > 0$ such that

$$|\langle u, f\rangle| \le C'\|f\|_{K,C},$$

namely $u \in (G^s_0(K, C))'$ for every K, C. Clearly, $\mathcal{D}'_s(\Omega)$ contains the usual space $\mathcal{D}'(\Omega)$ of Schwartz distributions.

We will need the following estimate for Gevrey seminorms of exponential functions.

Proposition 2.1. *Let ψ be a real-analytic function in a neighborhood Ω of 0 in $\mathbb{R}^n$; then for every compact subset K of Ω and every $C > 0$, $s > s' > 1$, there exists a constant $C' > 0$ such that*

$$||\exp(\iota\rho\psi)||_{K,C} \le C'\exp(a\rho + \rho^{1/s'}) \tag{2.1}$$

for every $\rho > 0$, where $a = \sup\{-\operatorname{Im}\psi(x) : x \in K\}$.

Proof. By the Faà di Bruno formula (see e.g. [14, page 16]) we have, for $|\alpha| \geq 1$,

$$\partial^\alpha e^{\iota\rho\psi(x)} = \sum_{j=1}^{|\alpha|} \frac{\exp(\iota\rho\psi)}{j!} \sum_{\substack{\gamma_1+\ldots+\gamma_j=\alpha \\ |\gamma_k|\geq 1}} \frac{\alpha!}{\gamma_1!\ldots\gamma_j!} |\partial^{\gamma_1}(\iota\rho\psi(x))|\ldots|\partial^{\gamma_j}(\iota\rho\psi(x))|.$$

By assumption there exists a constant $C_1 > 0$ such that $|\partial^\gamma\psi(x)| \leq C_1^{|\gamma|}\gamma!$ for $x \in K$, $|\gamma| \geq 1$. Hence for every α,

$$\sup_{x\in K} |\partial^\alpha e^{\iota\rho\psi(x)}| \leq e^{a\rho}\alpha! C_2^{|\alpha|} \sum_{j=0}^{|\alpha|} \frac{\rho^j}{j!}$$

with $C_2 = 2^{n+1}C_1$, where we used

$$\sum_{\substack{\gamma_1+\ldots+\gamma_j=\alpha \\ |\gamma_k|\geq 1}} 1 \leq \prod_{k=1}^{n} \binom{\alpha_k + j - 1}{j-1} \leq 2^{|\alpha|+n(j-1)} \leq 2^{(n+1)|\alpha|}.$$

Hence we have

$$\begin{aligned} ||\exp(\iota\rho\psi)||_{K,C} &\leq e^{a\rho}(\alpha!)^{1-s}(C_2/C)^{|\alpha|} \sum_{j=0}^{|\alpha|} \frac{\rho^j}{j!} \\ &\leq e^{a\rho}(|\alpha|!)^{1-s}(C_2C_3/C)^{|\alpha|} \sum_{j=0}^{|\alpha|} \frac{\rho^j}{j!}, \end{aligned}$$

because $|\alpha|! \leq n^{|\alpha|}\alpha!$. Now, we have $(|\alpha|!)^{1-s'}(j!)^{s'-1} \leq 1$ and by Stirling formula $(|\alpha|!)^{s'-s}(C_2C_3/C)^{|\alpha|} \leq C'$, so that

$$||\exp(\iota\rho\psi)||_{K,C} \leq C'e^{a\rho} \sum_{j=0}^{|\alpha|} \frac{\rho^j}{(j!)^{s'}} \leq C'e^{a\rho} \sum_{j=0}^{|\alpha|} \left(\frac{\rho^{j/s'}}{j!}\right)^{s'} \leq C'e^{a\rho+s'\rho^{1/s'}}.$$

Since this holds for every $1 < s' < s$, we can replace the constant s' in front of $\rho^{1/s'}$ by 1, possibly for a new constant C' and for a slightly lower value of s'. Hence (2.1) is proved. □

2.2. **Locally integrable structures.** Consider a real-analytic manifold M of dimension $N = m + n$. A real-analyitc locally integrable structure on M, of rank n, is defined by a real analytic subbundle $\mathcal{V} \subset \mathbb{C}TM$ of rank n, satisfying the Frobenius condition and such that subbundle $T' \subset \mathbb{C}T^*M$ orthogonal to $\mathcal{V}$ is locally spanned by exact differentials. As usual we will denote by $T^0 = T' \cap T^*M$ the so-called *characteristic set*. Let k be a positive integer, we denote by $\Lambda^k \mathbb{C}T^*M$ the k-th exterior power of $\mathbb{C}T^*M$. Let us consider complex exterior algebra

$$\Lambda \mathbb{C}T^*M = \oplus_{k=0}^{N} \Lambda^k \mathbb{C}T^*M,$$

for any pair of positive integers p, q we denote by

$$T'^{p,q}$$

the homogeneous of degree $p+q$ in the ideal generated by the p-th exterior power of T', $\Lambda^p T'$. We have the inclusion

$$T'^{p+1,q-1} \subset T'^{p,q}$$

which allows us to define

$$\Lambda^{p,q} = T'^{p,q}/T'^{p+1,q-1}.$$

If ϕ is a smooth section of T' over an open subset $\Omega \subset M$, its exterior derivative $d\phi$ is section of $T'^{1,1}$. In other words

$$dT' \subset T'^{1,1}.$$

It follows at once from this that, if σ is a smooth section of $T'^{p,q}$ over Ω, then $d\sigma$ is a section of $T'^{p,q+1}$ i.e.

$$dT'^{p,q} \subset T'^{p,q+1}.$$

Let $s > 1$, the space $G^s(\Omega, \Lambda^{p,q})$ of (p,q)-forms with Gevrey coefficients of order s is defined, as well as $G^s(\Omega, C; \Lambda^{p,q})$, $G_0^s(K, C; \Lambda^{p,q})$, etc, with notation analogous to the scalar case.

The de Rham differential induces then a map

$$\mathrm{d}' : G^s(\Omega, \Lambda^{p,q}) \to G^s(\Omega, \Lambda^{p,q+1}).$$

(see Treves [28, Section I.6] for more details). Similarly, the de Rham differential induces a complex on the space of "ultra-currents" $\mathcal{D}'_s(\Omega, \Lambda^{p,q})$, i.e. forms with ultradistribution coefficients:

$$\mathrm{d}' : \mathcal{D}'_s(\Omega, \Lambda^{p,q}) \to \mathcal{D}'_s(\Omega, \Lambda^{p,q+1}).$$

Namely, consider for simplicity the case when Ω is orientable (in fact, in the sequel we will work in a local chart). Stokes' theorem implies that

$$\int_\Omega \mathrm{d}'u \wedge v = (-1)^{p+q-1} \int_\Omega u \wedge \mathrm{d}'v$$

if $u \in G^s(\Omega, \Lambda^{p,q})$, $v \in G_0^s(\Omega, \Lambda^{m-p,n-q-1})$, and accordingly we can define

$$\langle \mathrm{d}'u, v \rangle = (-1)^{p+q-1} \langle u, \mathrm{d}'v \rangle$$

if $u \in D'_s(\Omega, \Lambda^{p,q})$, $v \in G_0^s(\Omega, \Lambda^{m-p,n-q-1})$.

3. Local solvability estimates

We now show that local solvability implies an a priori-estimate. This is analogous to the estimates of Hörmander [16], Andreotti, Hill and Nacinovich [1], Treves [28, Lemma VIII.1.1], in the framework of Schwartz distributions.

Proposition 3.1. *Suppose that, for some $s > 1$, the complex* d' *is locally solvable near x_0 and in degree q, in the sense of ultradistributions of order s (see Definition 1.1). Then for every sufficiently small open neighborhood Ω of x_0, every $C_1 > 0$, $0 < \epsilon < C_2$ there exist a compact $K \subset \Omega$, an open neighbourhood $\Omega' \subset\subset \Omega$ of x_0 and a constant $C' > 0$, such that*

$$|\int_\Omega f \wedge v| \le C' \|f\|_{K,C_1} \|\mathrm{d}'v\|_{\overline{\Omega'},C_2}, \tag{3.1}$$

for every cocycle $f \in G^s(\Omega, C_1; \Lambda^{0,q})$ and every $v \in G^s_0(\overline{\Omega'}, C_2 - \epsilon; \Lambda^{m,n-q})$.

It will follow from the proof that $\|\mathrm{d}'v\|_{\overline{\Omega'},C_2} < \infty$ if $v \in G^s_0(\overline{\Omega'}, C_2 - \epsilon; \Lambda^{m,n-q})$.

Proof. Let $V_{j+1} \subset V_j \subset\subset \Omega$, $j = 1, 2, \ldots$, be a fundamental system of neighborhoods of x_0. Fix $C_1 > 0$, $0 < \epsilon < C_2$ and consider the space

$$F_j = \{(f,u) \in G^s(\Omega, C_1; \Lambda^{0,q}) \times G^s_0(\overline{V_j}, C_2; \Lambda^{0,q-1})' : \\ \mathrm{d}'f = 0 \text{ in } \Omega,\ \mathrm{d}'u = f \text{ in } G^s_0(\overline{V_j}, C_2 - \epsilon; \Lambda^{0,q})'\}.$$

The last condition means $\langle \mathrm{d}'u, v\rangle = \langle f, v\rangle$ for every $v \in G^s_0(\overline{V_j}, C_2 - \epsilon; \Lambda^{m,n-q})$, which makes sense by transposition, because differentiation maps $G^s_0(\overline{V_j}, C_2 - \epsilon) \to G^s_0(\overline{V_j}, C_2)$ (see e.g. [27, Proposition 2.4.8]) and multiplication by analytic functions preserves the latter space.

Now, by direct inspection one sees that F_j is a closed subspace of $G^s(\Omega, C_1; \Lambda^{0,q-1}) \times G^s_0(\overline{V_j}, C_2; \Lambda^{0,q})'$, therefore Fréchet.

Let

$$\pi_j : F_j \to \{f \in G^s(\Omega, C_1; \Lambda^{0,q-1}) : \mathrm{d}'f = 0\}$$

be the canonical projection $(f,u) \mapsto f$. The assumption of local solvability implies that

$$\{f \in G^s(\Omega, C_1; \Lambda^{0,q-1}) : \mathrm{d}'f = 0\} = \cup_j \pi_j(F_j).$$

By the Baire theorem, there exists j_0 such that $\pi_{j_0}(F_{j_0})$ is of second category. By the open mapping theorem, we see that π_{j_0} is onto and open: there exists a compact $K \subset \Omega$ and a constant $C' > 0$ such that for every cocycle

$f \in G^s(\Omega, C_1; \Lambda^{0,q})$, there exists $u \in G_0^s(\overline{V_{j_0}}, C_2; \Lambda^{0,q-1})'$ satisfying $\mathrm{d}'u = f$ in $G_0^s(\overline{V_{j_0}}, C_2 - \epsilon; \Lambda^{0,q})'$ and

$$|u|_{\overline{V_{j_0}},C_2} := \sup_{||v||_{\overline{V_{j_0}},C_2}=1} |\langle u, v\rangle| \leq C'||f||_{K,C_1}.$$

Consider now the bilinear functional $(f, v) \mapsto \int_\Omega f \wedge v = \langle f, v\rangle$, for $f \in G^s(\Omega, C_1; \Lambda^{0,q})$ cocycle, and $v \in G_0^s(\overline{V_{j_0}}, C_2 - \epsilon; \Lambda^{m,n-q})$. Given such a f, we take u as before, and we get

$$|\langle f, v\rangle| = |\langle \mathrm{d}'u, v\rangle| = |\langle u, \mathrm{d}'v\rangle| \leq |u|_{\overline{V_{j_0}},C_2} \|\mathrm{d}'v\|_{\overline{V_{j_0}},C_2} \leq C'\|f\|_{K,C_1}\|\mathrm{d}'v\|_{\overline{V_{j_0}},C_2}.$$

□

4. Proof of Theorem 1.2

We work in a sufficiently small neighborhood Ω of the point x_0 (to be chosen later), where local solvability holds. We also take x_0 as the origin of the coordinates, i.e. $x_0 = 0$. Moreover we make use of the special coordinates, whose existence is proved in see section I.9 of [28]. Namely, let $n = \dim_{\mathbb{C}} \mathcal{V}_0$, $d = \dim_{\mathbb{R}} T_0^0$, $\nu = n - \dim_{\mathbb{C}}(\mathcal{V}_0 \cap \overline{\mathcal{V}}_0)$. We have the following result.

Proposition 4.1. *Let $(0, \omega_0) \in T^0$, $\omega_0 \neq 0$, and suppose that the restriction of the Levi form $\mathcal{B}_{(0,\omega_0)}$ to $\mathcal{V}_0 \cap \overline{\mathcal{V}}_0$ is non-degenerate. There exist real-analytic coordinates x_j, y_j, s_k and t_l, $j = 1, \ldots, \nu$, $k = 1, \ldots, d$, $l = 1, \ldots, n - \nu$, and smooth real valued and real-analitic functions $\phi_k(x, y, s, t)$, $k = 1, \ldots d$, in a neighborhood $\mathcal{O}$ of 0, satisfying*

$$\phi_k|_0 = 0 \quad \text{and} \quad d\phi_k|_0 = 0, \tag{4.1}$$

such that

$$\begin{cases} z_j := x_j + \iota y_j, \; j = 1, \ldots, \nu, \\ w_k := s_k + \iota\phi_k(x, y, s, t), \; k = 1, \ldots, d, \end{cases}$$

define a system of first integrals for $\mathcal{V}$, i.e. their differential span $T'|_{\mathcal{O}}$.

Moreover, with respect to the basis

$$\left\{ \left.\frac{\partial}{\partial \overline{z}_j}\right|_0, \left.\frac{\partial}{\partial t_l}\right|_0; j = 1, \ldots, \nu, \; l = 1, \ldots, n - \nu \right\}$$

of $\mathcal{V}_0$ the Levi form $\mathcal{B}_{(0,\omega_0)}$ reads

$$\sum_{j=1}^{p''} |\zeta_j|^2 - \sum_{j=p''+1}^{\nu} |\zeta_j|^2 + \sum_{l=1}^{p'} |\tau_l|^2 - \sum_{l=p'+1}^{n-\nu} |\tau_l|^2. \tag{4.2}$$

Remark 4.2. In particular

$$\mathrm{d}' z_j = 0,\ \mathrm{d}' w_k = 0, \quad j = 1, \dots, \nu;\ k = 1, \dots, d. \tag{4.3}$$

In these coordinates we have $T_0^0 = \operatorname{span}_{\mathbb{R}}\{ds_k|_0;\ k = 1, \dots, d\}$, so that $\omega_0 = \sum_{k=1}^d \sigma_k ds_k|_0$, with $\sigma_k \in \mathbb{R}$. By (I.9.2) of [28] we have $\mathcal{B}_{(0,\omega_0)}(\mathbf{v}_1, \mathbf{v}_2) = \sum_{k=1}^d \sigma_k (V_1 \overline{V}_2 \phi_k)|_0$, with V_1 and V_2 smooth sections of $\mathcal{V}$ extending $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Upon setting $\Phi = \sum_{k=1}^d \sigma_k \phi_k$ we can suppose, in addition, that

$$\Phi = \sum_{j=1}^{p''} |z_j|^2 - \sum_{j=p''+1}^{\nu} |z_j|^2 + \frac{1}{2} \sum_{l=1}^{p'} t_l^2 - \frac{1}{2} \sum_{l=p'+1}^{n-\nu} t_l^2 + O(|s|(|z|+|s|+|t|)+|z|^3+|t|^3); \tag{4.4}$$

see [28, Section I.9] and [28, (XVIII.3.2)] for details.

We can now prove Theorem 1.2. We may assume, without loss of generality, that $\sigma = (1, 0, \dots, 0)$. Consequently, from (4.4) (after the change of variables $t \mapsto t/\sqrt{2}$) we have

$$\phi_1(x, y, s, t) = |z'|^2 - |z''|^2 + |t'|^2 - |t''|^2 + O(|s|(|z|+|s|+|t|)+|z|^3+|t|^3), \tag{4.5}$$

where we set

$$\begin{cases} z' = (z_1, \dots, z_{p''}), \\ z'' = (z_{p''+1}, \dots, z_\nu) \\ t' = (t_1, \dots, t_{p'}), \\ t'' = (t_{p'+1}, \dots, t_{n-\nu}). \end{cases}$$

Moreover, we choose a function $\chi(x, y, s, t)$ in $G_0^s(\mathbb{R}^{2\nu+d+(n-\nu)})$, $\chi = 0$ away from a neighborhood $V \subset\subset \Omega$ of 0 and $\chi = 1$ in a neighborhood $U \subset\subset V$ of 0, where V and U will be chosen later on. We set, for $\rho > 0, \lambda > 0$,

$$f_{\rho,\lambda} = e^{\rho h_{1,\lambda}} d\overline{z}' \wedge dt'$$
$$v_{\rho,\lambda} = \rho^{(m+n)/2} \chi e^{\rho h_{2,\lambda}} d\overline{z}'' \wedge dt'' \wedge dz \wedge dw,$$

where, with $\lambda > 1$,

$$h_{1,\lambda} := -\iota s_1 + \phi_1 - 2|z'|^2 - 2|t'|^2 - \lambda \sum_{k=1}^d (s_k + \iota \phi_k)^2, \tag{4.6}$$

and

$$h_{2,\lambda} := \iota s_1 - \phi_1 - 2|z''|^2 - 2|t''|^2 - \lambda \sum_{k=1}^d (s_k + \iota \phi_k)^2. \tag{4.7}$$

Now, we have $\chi \in G_0^s(\overline{V}, C/2)$, for some $C > 0$. We then apply Proposition 3.1 with $f_{\rho,\lambda}$ and $v_{\rho,\lambda}$ in place of f and v respectively, and $C_1 = C_2 = C$, $\epsilon = C/2$, taking V small enough to be contained in the neighborhood Ω' which arises in the conclusion of Proposition 3.1. Observe that, in fact, $f_{\rho,\lambda} \in G^s(\Omega, C; \Lambda^{0,q})$ since this form is in fact real-analytic and $p'' + p' = q$ by hypothesis, whereas $v_{\rho,\lambda} \in G_0^s(\overline{V}, C/2; \Lambda^{m,n-q}) \subset G_0^s(\overline{\Omega'}, C/2; \Lambda^{m,n-q})$ (recall, $m = \dim M - n$). We prove now that f_ρ is a cocycle (i.e. $\mathrm{d}' f_{\rho,\lambda} = 0$), so that Proposition 3.1 can in fact be applied. However, we will show that (3.1) fails for every choice of C' when $\rho \to +\infty$, if λ is large enough, obtaining a contradiction.
Now, by (4.3)

$$\mathrm{d}' f_{\rho,\lambda} = 0,$$

and

$$\mathrm{d}' v_{\rho,\lambda} = \rho^{(m+n)/2} e^{\rho h_{2,\lambda}} \mathrm{d}'\chi \wedge d\overline{z}'' \wedge dt'' \wedge dz \wedge dw. \tag{4.8}$$

In order to estimate the right hand side of (3.1) we observe that, by (4.5) and (4.6)

$$\mathrm{Re}\, h_{1,\lambda} = -|z'|^2 - |z''|^2 - |t|^2 - \lambda |s|^2 + \mathcal{R}(z,s,t) + O(|z|^3 + |t|^3) + \lambda O(|z|^4 + |t|^4),$$

where

$$|\mathcal{R}(z,s,t)| = O(|s|(|z| + |s| + |t|)) \leq \tilde{C} \left(\frac{\epsilon}{2}(|z| + |s| + |t|)^2 + \frac{1}{2\epsilon}|s|^2 \right), \tag{4.9}$$

for every $\epsilon > 0$. Hence, if ϵ and then $1/\lambda$ are small enough we see that, possibly after replacing Ω with a smaller neighborhood,

$$\sup_{\Omega} \mathrm{Re}\, h_{1,\lambda} \leq 0. \tag{4.10}$$

Similarly,

$$\begin{aligned} \mathrm{Re}\, h_{2,\lambda} = -|z'|^2 - |z''|^2 - |t|^2 - \lambda |s|^2 & \\ + \mathcal{R}'(z,s,t) + O(|z|^3 + |t|^3) + \lambda O(|z|^4 + |t|^4), \end{aligned}$$

with $\mathcal{R}'$ satisfying the same estimate (4.9). Therefore if λ is sufficiently large, in Ω we have

$$\mathrm{Re}\, h_{2,\lambda} \leq -\frac{1}{2}(|z|^2 + |t|^2 + \lambda |s|^2) + \tilde{C}_1(|z|^3 + |t|^3) + \tilde{C}_2 \lambda (|z|^4 + |t|^4).$$

Hence, possibly for a smaller V, since $U \subset\subset V$ is a neighborhood of 0, there exists a constant $c > 0$ such that

$$\sup_{\overline{V} \setminus U} h_{2,\lambda}(z,s,t) \leq -c. \tag{4.11}$$

As a consequence of Proposition 2.1 and (4.10), (4.11), for every compact subset $K \subset \Omega$ it turns out

$$\|f_{\rho,\lambda}\|_{K,C} \leq C' e^{\rho^{1/s'}}, \tag{4.12}$$

$$\|\mathrm{d}' v_{\rho,\lambda}\|_{K,C} \leq C'' \rho^{(m+n)/2} e^{-c\rho + \rho^{1/s'}}, \tag{4.13}$$

for any $1 < s' < s$, where the constants C', C'' are independent of ρ. It follows that

$$\|f_{\rho,\lambda}\|_{K,C} \|\mathrm{d}' v_{\rho,\lambda}\|_{K,C} \leq C' C'' \rho^{(m+n)/2} e^{-c\rho + 2\rho^{1/s'}} \longrightarrow 0 \text{ as } \rho \to +\infty, \tag{4.14}$$

because $s' > 1$. On the other hand, for the right-hand side of (3.1) it is easily seen that

$$\int f_{\rho,\lambda} \wedge v_{\rho,\lambda} \longrightarrow c' \neq 0,$$

as $\rho \to +\infty$ (see the end of the proof of [28, Theorem XVIII.3.1]), which together with (4.14) contradicts (3.1).

This completes the proof of Theorem 1.2.

Remark 4.3. The above machinery can be applied to prove other necessary conditions for Gevrey local solvability in the spirit of analogous results valid in the framework of smooth functions and Schwartz distributions.

As an example, consider the special case of local solvability when $q = n$, namely in top degree. In this case, the Cordaro-Hounie condition $(\mathcal{P})_{n-1}$ (see [7] and [9]) is known to be necessary for the local solvability in the smooth category, and it is conjectured to be sufficient as well. Consider the following analytic variant. Let L_j, $j = 1, ..., n$, be real-analytic independent vector fields which generates $\mathcal{V}$ at any point of Ω.

We say that the real-analytic condition $(\mathcal{P})_{n-1}$ is satisfied at x_0 if there exists an open neighbourhood $U \subset \Omega$ of x_0 such that, given any open set $V \subset U$ and given any real-analytic $h \in C^\infty(V)$ satisfying $L_j h = 0$, $j = 1, ..., n$, then $\mathrm{Re}\, h$ does not assume a local minimum[1] *over any nonempty compact subset of V.*

One could then prove that *if the real-analytic condition $(\mathcal{P})_{n-1}$ is not satisfied at x_0, then for every $s > 1$, d' is not locally solvable in the sense of ultradistributions of order s, near x_0 and in degree n.*

For the sake of brevity we omit the proof, which goes on along the same lines as that in [9, Theorem 1.2], using the local solvability estimates in Proposition 3.1, combined with Proposition 2.1.

[1] A real-valued function f defined on a topological space X is said to assume a local minimum over a compact set $K \subset X$ if there exist $a \in \mathbb{R}$ and $K \subset V \subset X$ open such that $f = a$ on K and $f > a$ on $V \setminus K$.

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