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Mathematical Models in Landscape Ecology: Stability Analysis and Numerical Tests

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Abstract In the present paper a review of some mathematical models for the ecological evaluation of environmental systems is considered. Moreover a new model, capable to furnish more detailed information at the level of landscape units, is proposed. Numerical tests are then performed for a case study in the province of Viterbo (central Italy).

Keywords Landscape ecology · Mathematical models · Stability analysis

Mathematics Subject Classification (2000) 34D05 · 92F05

1 Introduction

The European Landscape Convention [1] encourages all European countries to define their landscape quality objectives on the ground of management and planning of territory. Thus, the Convention is a reference point for territorial government, conservation and protection of landscapes in order to assure an increase of life quality of a population. In this context

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Landscape Ecology [2–4] is a rather new discipline that provides tools for a quantitative evaluation of the ecology of an environmental system.

Landscape Ecology is an interdisciplinary field of research needing integration between theoretical development, empirical testing and mathematical modelling. In this framework landscape is considered as a spatially extended heterogeneous complex system determined by nonlinear connections among its components (the so-called *Landscape Units* (LU) [5]) through flows of materials and bio-energy [4]. In general, a landscape stands in a meta-stable equilibrium, that is it responds stably only to a limited range of perturbations, otherwise it may evolve towards significant environmental modifications [6, 7]. In this context simulation models may be useful and reliable tools to give information about the trend of environments towards future scenarios, presenting even bifurcation phenomena, arising by some criticality of territorial parameters [8, 9].

The state of an environment is well represented by the so-called *Ecological Graph* (EG) [10], a kind of landscape graph [11], determinable by the *Geographic Information System* (GIS), which gives quantitative information about the territory under investigation. Mainly an EG furnishes data relative to production of biological energy, due to the biomass present in each LU, and to transmission of such an energy to the other LUs. However an EG gives a static representation of the environment ecological state, so that evolution models, as already said, may be viewed as powerful tools to analyze the nonlinear dynamics of the system itself.

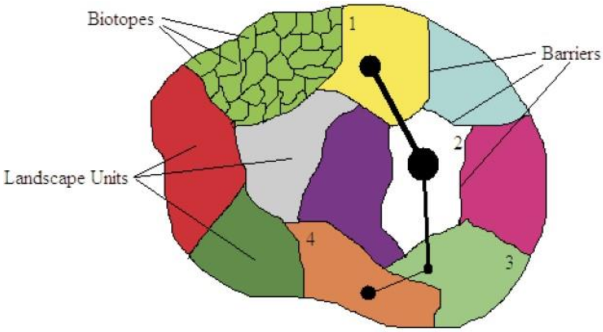
A first attempt to construct an evolution model was proposed in 2007 [12], and further analyzed in paper [13] where it was shown the appearance of bifurcations between quite different equilibrium states in correspondence of environmental threshold values. Moreover, numerical simulations were performed for a case study of the province of Cremona (north Italy). A modified version of this model has been successively proposed in paper [14] which shows the interesting property to admit solutions corresponding to strongly fragmented landscapes, the most recurrent territorial settlement nowadays. This modified version of the first model has been applied to the ecological evaluation of two environments, the first in the region of Cuneo (north Italy) [14] and the second in that of Viterbo (central Italy) [15]. In this last paper a uniform procedure for the implementation of the model in a general landscape was also derived.

The state variables of these models are given by two quantities M and V . The former is proportional to the *Biological Territorial Capacity* (BTC) [4, 5], and measures at the same time production and diffusivity of bio-energy in the whole environmental system. The latter is the percentage of land characterized by green areas with high value of bio-energy production. The model is represented by two nonlinear ODEs which include several parameters that are deduced by the EG, which, at the same time, provides the initial data for the equations themselves. The future scenarios of the environment under investigation are represented by the equilibrium solutions of the ODEs for which, obviously, a stability analysis is necessary.

Starting from the results of these models, in the present paper we propose a new model which represents not only the evolution of the whole environmental system but also that of each LU, characterized by time-dependent parameters. Let us underline that this new development is important since the evaluation is now carried on in each portion of the environmental system, so that it is possible to recognize where are specifically the critical areas of the whole system itself.

In details the paper is organized as follows: in Sect. 2 we discuss the construction of the EG; in Sect. 3 we summarize the principal mathematical features of the two models of papers [13] and [14]; in Sect. 4 we present the new model, giving then in the last section some simulations for a system of several LUs in the province of Viterbo.

Fig. 1 The environmental system and its ecological graph



Let us finally underline that a correct interpretation of such simulations consists in comparing the effect of some different actions or strategies on landscape equilibrium conditions and to identify the best choice, so that the proposed procedure may be really considered a reliable tool for environmental planning and estimate of possible scenarios evolution [15].

2 The Ecological Graph

In landscape ecology the landscape itself is defined as a heterogenous land composed of interacting ecosystems that exchange energy and matter. In this paper an environmental system will be considered as a territory subdivided in a given number, n, of ecological patches (the so-called LUs) separated from each other by natural or anthrop *barriers*. Examples of barriers are railroads, viaducts, highways, national and municipal roads, compact edified and industrial grounds, urban sprawl, rivers, lakes, ridges .... According to [5, 16] each barrier has been classified by an index of permeability $p \in [0,1]$, $p = 0$ and $p = 1$ indicating complete impermeability or permeability to bio-energy fluxes, respectively. Moreover each LU is once more divided into land patches (see Fig. 1), called *biotopes*, classified according to the actual use of its land cover; in other words each biotope is characterized by bio-energy production, defined by the BTC index B^b measured in Mcal/m^2/year and running from 0 to a value $B^b_{\max}$, generally considered, in the European climatical zone, equal to 6.5. In particular, the values of the index B^b are generally divided in five classes, A,...,E [4, 10, 16, 17], i.e.

$$A = [0,0.4), \qquad B = [0.4,1.2], \qquad C = (1.2,2.4],$$

$$D = (2.4,4.0], \qquad E = (4.0,6.5].$$

In Appendix A, at the end of the paper, a table with the values of the index B^b for different types of land cover is provided.

Therefore the total value of BTC of each i-th LU, i = 1,...,n, can be given by the following formula

$$B_{i0} = \sum_{j=1}^{m_i} B^b_{ji} s_{ji} \tag{1}$$

where s_{ji} is the area of j-th biotope, j = 1,...,m_i, belonging to the i-th LU, having BTC index B^b_{ji}. Moreover the mean value of BTC of the whole environmental system will be given by the average

$$B_0 = \frac{1}{n} \sum_{i=1}^{n} B_{i0} \tag{2}$$

According to [5], in constructing the EG, a generalized bio-energy (hereinafter indicated with the acronym GBE) is considered in order to include in each LU, beside the actual energy production, also its capacity to be diffused into the other neighbor LUs. Therefore, we shall denote by M_{i0} the GBE, which takes into account several morphological, physical and biological characters of the LU itself, i.e.

$$M_{i0} = (1 + \kappa_i)B_{i0}, \tag{3}$$

where κ_i is a dimensionless environmental parameter with values in the range [0, 1], which may augment the actual value B_{i0} of BTC if the corresponding LU has high capacity of bio-energy diffusion. In paper [14] the parameter κ_i has been assumed to depend upon the borders shape, the barriers permeability and the landscape diversity of each LU. In paper [15] also dependence upon sun exposition and relative humidity of the land cover has been taken into account. For the actual computation of the parameters κ_i the reader is addressed to Appendix B at the end of the paper.

Equivalently to the mean value of BTC, also the mean value of the GBE for the whole environmental system can be defined by

$$\mathcal{M}_0 = \frac{1}{n} \sum_{i=1}^{n} \mathcal{M}_{i0} \tag{4}$$

Hereinafter in order to handle with normalized variables the quantity M will be substituted by

$$M = \frac{\mathcal{M}}{\mathcal{M}_{\max}}, \quad \mathcal{M}_{\max} = 2 \max_{i=1,\dots,n} \{B_{i0}\}, \tag{5}$$

so that M ∈ [0,1] and where, obviously, the factor 2 is the maximum value assumed by the BTC correction term $(1 + \kappa_i)$.

Once the GBE has been computed for each LU, it is possible to derive as well the energy fluxes F_{ik} between two neighbor LUs i and k; in formula

$$F_{ik} = \frac{(\mathcal{M}_{i0} + \mathcal{M}_{k0}) L_{ik} p_{ik}}{2(P_i + P_k)}, \tag{6}$$

where L_{ik} is the length of the border, P_i and P_k are the perimeters of the two LUs and p_{ik} represents the mean permeability of the barrier whose value, as already said above, depends on the type of the barrier itself (see the table reported in Appendix C where for several types of barriers the permeability index p is furnished). The actual number of fluxes F_{ik} depends of course on the number of LUs and on their disposition inside the environmental system.
Let us denote such a number by Λ and by $F^\ell,\ \ell = 1, \ldots, \Lambda$, the re-ordered values of the fluxes F_{ik}.

From the knowledge of fluxes an important landscape parameter, called *connectivity index*, can be defined, i.e.

$$c = \frac{1}{\Lambda} \sum_{\ell=1}^{\Lambda} \frac{F^\ell}{\max_\ell\{F^\ell\}}, \qquad c \in [0, 1] \tag{7}$$

Let us remark that, in case of an environment presenting between its LUs several barriers with low permeability, c may be close to zero. Therefore the value of such a parameter can be considered in some sense a measure of the environment fragmentation.

Once all the above quantities have been computed by the GIS, the EG can be drawn. As shown in Fig. 1 for the LUs 1,...,4, the GBEs and their corresponding fluxes can be represented by a graph where the nodes are circles whose diameters are proportional to the quantities M_{i0} and the edges have thickness proportional to the fluxes F_{ik}.

The EG, of course, gives a static representation of the state of the environment. Starting from such a state in the next sections dynamical models will be introduced in order to show the possible evolution of the system scenarios.

3 On the Dynamical Modelling of an Environment

In paper [13], for the purposes already discussed in the Introduction, a time-evolution model has been proposed assuming as state variables the quantities M(t) and V(t), $t \in R_+$, V being the portion of the whole environment characterized by a green area with high value of BTC, say $B^b \in [3.5, 6.5]$. The model is represented by the following set of ODEs

$$\begin{cases} \mathcal{M}'(t) = c\mathcal{M}(t)\left[1 - \dfrac{\mathcal{M}(t)}{\mathcal{M}_{\max}}\right] - h_B\left[1 - \dfrac{\mathcal{V}(t)}{A}\right]\mathcal{M}_{\max} \\ \mathcal{V}'(t) = \dfrac{\mathcal{M}(t)}{\mathcal{M}_{\max}}\mathcal{V}(t)\left[1 - \dfrac{\mathcal{V}(t)}{A}\right] - h_R U_0 \mathcal{V}(t), \end{cases} \tag{8}$$

where A is the total area of the environmental system.

The first equation of the model is given by a balance between a logistic term, driven by the connectivity index and accounting for energy growth, and a correspondent energy decrease due to the presence in the environment of impermeable barriers and low BTC areas. The parameter h_B is the ratio between the length of such impermeable barriers and that of the perimeter of the whole system.

The second equation for V is obtained as well by a balance between a logistic increasing term proportional to the actual value of energy production and a negative quantity proportional to the variable V itself. The parameter h_R is the ratio between the perimeter of the edified areas (those with a BTC of class A) and the total perimeter of the environment; in such a way high values of h_R indicate high dispersion of buildings all around the territory (for other details on the model parameters the reader is addressed to the bibliography [12– 16]). Finally, $U_0 \in [0,1]$ is the ratio between the surface of the edified areas and that of the whole system. Let us underline that in the edified areas in the present paper are included also infrastructures as roads, railroads, bridges, viaducts ....

By introducing the new dimensionless and normalized variables

$$M(t) = M(t)/M_{max}, \qquad V(t) = V(t)/A,$$

$\{ M(t),V(t)\} \in [0,1] \times [0,1]$ $\forall t$, system (8) assumes the simpler form

$$\begin{cases} M'(t) = cM(t)[1 - M(t)] - h_B[1 - V(t)] \\ V'(t) = M(t)V(t)[1 - V(t)] - h_R U_0 V(t), \end{cases} \qquad (9)$$

to be joined to the initial data directly obtainable by the EG, i.e.

$$M(t = 0) = M_0/M_{max}, \qquad V(t = 0) = V(t = 0)/A.$$

In what follows the main mathematical properties of the model are summarized. The stability analysis [18] is rather straightforward, but with some tedious calculations, and can be found together with the proof of lemmas in the afore-mentioned paper [13].

Let us start analyzing the equilibrium solutions (M^e,V^e) of the model. System (9) provides two families of equilibria, the former with $V^e = 0$, the latter with $V^e = 0$.

Lemma 1 *The first family admits the following two equilibria* $(M_1^e \le M_2^e)$:

$$\begin{cases} M_1^e = \dfrac{1}{2} - \sqrt{\dfrac{1}{4} - \dfrac{h_B}{c}}, & V_1^e = 0 \\ M_2^e = \dfrac{1}{2} + \sqrt{\dfrac{1}{4} - \dfrac{h_B}{c}}, & V_2^e = 0 \end{cases} \qquad (10)$$

provided that $c \ge 4h_B.$

Remark The equilibria given by $(M_{1,2}^e, 0)$ correspond to a territorial settlement with a lack of areas at high value of biological activity ($V_{1,2}^e =$ 0). Nevertheless the condition of low impermeability, i.e. $h_B \le c/4 \le 1/4$, allows to have some energy production so that $M_{1,2}^e$ is different from zero.

Lemma 2 *The second family is obtained by finding the solutions of the following third order equation* [19]:

$$M^3 - M^2 + H = 0, \quad H = h_B h_R U_0/c, \tag{11}$$

with the corresponding values of V^e *given by*

$$V^e = 1 - \frac{h_R U_0}{M^e}, \tag{12}$$

which are meaningful only if $M^e > h_R U_0$.

If $H > 4/27$, (11) *admits* [19] *two complex conjugate solutions and a unique negative real one*, *say* M_3^e.

If $H = 4/27$, (11) *admits three real solutions*, *one negative*, *equal to* $-1/3$, *and two positive*, *both equal to* $2/3$.

If $H < 4/27$, (11) *provides three real solutions*, *one of which is negative*, *say* M_4^e, *and the other two*, *say* $M_5^e \le M_6^e$, *positive*.

Let us now deal with the stability conditions of the afore-mentioned equilibria, recalling the following two lemmas.

Lemma 3 *For what concerns the behavior of the solutions corresponding to* $V^e = 0$ *and* $c \geq 4h$, *for given* h_R *and* U_0, *we have*

1. *stable nodes if* $\frac{1}{2} < M^e < h_R U_0$,
2. *unstable nodes if* $h_R U_0 < M^e < \frac{1}{2}$,

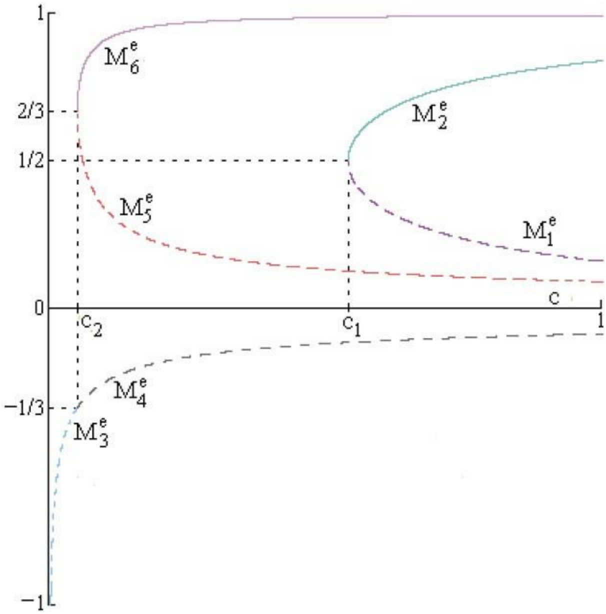


Fig. 2 Bifurcations diagram

3. *unstable saddle points if*

(i) $M^e > h_R U_0$ *and* $M^e > \frac{1}{2}$ *or* (ii) $M^e < h_R U_0$ *and* $M^e < \frac{1}{2}$.

Lemma 4 *The equilibria corresponding to* $M^e > h_R U_0$, $V^e = 1 - h_R U_0/M^e$ *and* $c \geq 27 h_B h_R U_0/4$ (*i.e.* $H < 4/27$) *are*:

(a) *unstable saddle points if* $0 < M^e < \frac{2}{3}$, (b) *stable nodes if* $M^e > \frac{2}{3}$.

Finally, concerning system bifurcations, the following lemma holds.

Lemma 5 *Assuming* c *as the control variable of the dynamical system* (9), *the stationary bifurcation points are*:

$$\left(M_1^{bif}, V_1^{bif}, c_1\right) = \left(\frac{1}{2}, 0, 4h_B\right),$$

$$\left(M_2^{bif}, V_2^{bif}, c_2\right) = \left(\frac{2}{3}, 1 - \frac{3}{2} h_R U_0, \frac{27}{4} h_B h_R U_0\right).$$

More in details, the point $\left(M_1^{bif}, V_1^{bif}, c_1\right)$ *is a simple bifurcation point, while the other one* $\left(M_2^{bif}, V_2^{bif}, c_2\right)$ *is a turning or hysteresis point*.

The results of Lemma 5 are contained in the bifurcation diagram of Fig. 2 where the stable (solid lines) and the unstable or negative (dashed lines) equilibria of M are plotted versus c in a case where $c_1 > c_2$.

Remark As shown in [13], for some initial data and/or values of the parameter c the negative equilibria $M_{3,4}^e$ may play a role in forcing M to become negative: such a possible trend corresponds to an ecological collapse.

As discussed in the Introduction, from the previous analysis emerges how it is interesting to study the equilibria of environmental systems, because they can give, depending on suitable values of the parameters, indications on the level of meta-stability of the environment itself. As shown in Lemmas 1–4 by the rather rich variety of stable and unstable equilibrium solutions, changes in bio-energy, bio-diversity and connectivity may produce territorial modifications toward which, for instance, individual landscapes may provide critical thresholds that result in radical changes in the ecological state of the system and therefore in its future scenarios. In a simple way one can say that meta-stability means that an ecological system can keep itself over a limited range of changes in environmental conditions but may eventually undergo significant alterations if environmental constraints continue to change [20], as shown in Lemma 5 by the existence of bifurcation points. The more or less meta-stability, i.e. the more or less resistance to disturbances, is related to the more or less presence of bio-energy, bio-diversity and connectivity.

We now turn to the modified model [14] which present more different evolution scenarios.

Such a model has almost the same mathematical structure of that represented by (8), but with some simplifications, and is given by

$$\begin{cases} \mathcal{M}'(t) = c\mathcal{M}(t)\left[1 - \dfrac{\mathcal{M}(t)}{\mathcal{M}_{\max}}\right] - h_B\left[1 - \dfrac{\mathcal{V}(t)}{A}\right]\mathcal{M} \quad (t) \\ \mathcal{V}'(t) = b_T\mathcal{V}(t)\left[1 - \dfrac{\mathcal{V}(t)}{A}\right] - h_R U_0 \mathcal{V}(t). \end{cases} \tag{13}$$

In particular the modifications to (8) are the following: in the first equation the term accounting for decrease of GBE depends now, for ∀t, on the time-dependent variable M(t) itself and not on the constant quantity M_{max}; the second equation is set, for simplicity, uncoupled from the first one since the logistic term is not multiplied anymore by M(t) but by the constant quantity b_T, defined by

$$b_T = \frac{B_0}{B_{\max}}, \qquad B_{\max} = \max_{i=1,\ldots,n}\{B_{i0}\}.$$

For other details concerning the reasons of such modifications the reader is addressed to paper [14].

By substituting again in (13) the dimensionless variables M(t) and V(t), the model now reads

$$\begin{cases} M'(t) = cM(t)\big[1 - M(t)\big] - h_B\big[1 - V(t)\big]M(t) \\ V'(t) = b_T V(t)\big[1 - V(t)\big] - h_R U_0 V(t). \end{cases} \tag{14}$$

According to the standard methods of ODEs [18], the equilibrium solutions of system (14) and their stability are determined by the following two lemmas.

Lemma 6 *System* (14) *admits four equilibria, given by*

$$\begin{aligned} &\text{(I)} && M_1^e = 0, && V_1^e = 0 \\ &\text{(II)} && M_2^e = 0, && V_2^e = \frac{b_T - h_R U_0}{b_T} \\ &\text{(III)} && M_3^e = \frac{c - h_B}{c}, && V_3^e = 0 \\ &\text{(IV)} && M_4^e = \frac{b_T c - h_R h_B U_0}{b_T c}, && V_4^e = \frac{b_T - h_R U_0}{b_T} \end{aligned} \tag{15}$$

Lemma 7 *Concerning stability of system* (14), *the following results hold*:

equilibrium (I) *is a stable node if* $c < h_B$ *and* $b_T < h_R U_0$; *equilibrium* (II) *is a stable node if* $b_T c < h_R h_B U_0$ *and* $b_T > h_R U_0$; *equilibrium* (III) *is a stable node if* $c > h_B$ *and* $b_T < h_R U_0$; *equilibrium* (IV) *is a stable node if* $b_T c > h_R h_B U_0$ *and* $b_T > h_R U_0$.

According to these lemmas, this model provides four different possible scenarios, each stable under the conditions stated by the last one. Indeed, beside the worst and best scenarios, respectively represented by equilibria (I) and (IV), the model admits two other scenarios frequently present in landscapes: equilibrium (III), already provided by model (9), and equilibrium (II) which corresponds to an environment where there is a lack of GBE diffusion ($M(t) \to 0$) between the LUs, but, at the same time, there are still, in some LUs, areas (*islands*) of high ecological quality ($V^e = 0$); in other words equilibrium (II) represents a situation of a landscape strongly fragmented.

Remark Contrary to the solution of model (9) the stability analysis carried out in paper [14] has shown that for positive initial data the model (14) exhibits at any time a solution (M,V) inside the square $[0,1] \times [0,1]$.

Moreover, let us note that if the connectivity index c is assumed as the parameter measuring the environment critical conditions, then, for $b_T < h_R U_0$, $c = h_B$ is the transition point which separates the branch of equilibria (I) from the other (III). In a similar way, when $b_T > h_R U_0$ the transition point between the branches of (II) and (IV) is given by $c = h_R h_B U_0 / b_T$.

Thanks to the presence of equilibrium (II), this model has been applied to some environments [14, 15, 21] with the purpose of evaluating the territory fragmentation.

Beside their interesting results, both models present some drawbacks: parameters b_T and c are time-independent (this assumption is not realistic since bio-energy production and connectivity must change during environment transformation); moreover they furnish possible future scenarios only for the whole system whereas, as already discussed in the Introduction, the investigation should analyze its evolution at the level of each LU, since different LUs may have different trends to equilibrium. For these reasons in the next section a new model, overcoming these simplifications, is proposed.

4 A New Model

The new model will be determined on the basis of the following hypotheses:

(1) the equations are written at the level of each LU and not anymore at that of the whole environment under investigation;
(2) the connectivity must be re-defined and be time-dependent so that the links between the LUs are updated at any time;
(3) the dimensionless variables are now defined with respect to absolute quantities.

With regard to this last hypothesis we define a new quantity M^{max}_i (*maximum producible* GBE of each LU)

$$\mathcal{M}_i^{\max} = 2B^b_{\max} A_i,$$

where A_i is the area of the i-th LU and $B^b_{\max}$ the absolute maximum value of the BTC index which, as already mentioned, is assumed to be equal to 6.5.

By taking into account hypotheses (1) and (2) system (13) can be re-written in 2n dimensions in the following way

$$\begin{cases} \mathcal{M}_i'(t) = c_i(t)\mathcal{M}_i(t)\left(1 - \dfrac{\mathcal{M}_i(t)}{\mathcal{M}_i^{\max}}\right) - \nu_i\left(1 - \dfrac{\mathcal{V}_i(t)}{A_i}\right)\mathcal{M}_i \\ \mathcal{V}_i'(t) = \dfrac{\mathcal{M}_i(t)}{\mathcal{M}_i^{\max}}\mathcal{V}_i(t)\left(1 - \dfrac{\mathcal{V}_i(t)}{A_i}\right) - \mu_i U_i \mathcal{V}_i(t), \end{cases} \quad (t)$$

(16)

where the constants v_i, μ_i and U_i play almost the same role of h_B, h_R and U_0, but this time are referred to each LU, i = 1,...,n, so that

(a) v_i are the ratios between the sum of all the perimeters of the impermeable barriers inside the i-th LU and the perimeter P_i of the LU itself;
(b) μ_i are the ratios between the sum of the perimeters of all the compact edified areas (those with BTC belonging to class A) inside the i-th LU and P_i;
(c) U_i are the ratios between the sum of the surfaces of all the edified areas inside the i-th LU and A_i.

Beside the fact that the equations on V_i are now coupled with those on M_i through the term M_i/M^{max}_i, as in the model (8), the main modification regards the connectivity indexes $c_i(t)$ for which it is necessary to state a new definition.

First of all let us define the flux between two neighbor LUs, say i and k. Such a flux will be given by

$$F_{ik} = \frac{\mathcal{M}_i + \mathcal{M}_k}{2(P_i + P_k)} \sum_{r=1}^{s} L_{ik}^r p^r , \tag{17}$$

where L_{ik}^r are the lengths of the LUs border characterized by the permeability index $p^r \in [0,1]$ and divided into s tracts which may present a different permeability. As usual P_i and P_k are the perimeters of the two LUs. Moreover, let us introduce the absolute maximum flux F_{ik}^{max} between two LUs, which may occur if all the borders have permeability index equal to one and each LU produces the maximum possible value of GBE. Thus, one gets

$$F_{ik}^{\max} = \frac{\mathcal{M}_i^{\max} + \mathcal{M}_k^{\max}}{2(P_i + P_k)} L_{ik} , \tag{18}$$

where L_{ik} is the length of the entire border.

After these definitions the connectivity indexes c_{ik} between two LUs i and k, as well as the total connectivity index c_i between the i-LU and all its neighbors can be defined by the following formulas

$$c_{ik} = \frac{F_{ik}}{F_{ik}^{\max}} = \frac{\mathcal{M}_i + \mathcal{M}_k}{(\mathcal{M}_i^{\max} + \mathcal{M}_k^{\max}) L_{ik}} \sum_{r=1}^{s} L_{ik}^r p^r , \tag{19}$$

$$c_i = \sum_{k \in I_i} c_{ik}, \tag{20}$$

where I_i is the set of the neighbors of the i-th LU. Let us note that the indexes c_i, contrary to the connectivity index c, defined in Sect. 2, can be greater than one.

The last expression can be written in a more explicit form by introducing the quantity

$$H_{ik} = \frac{1}{L_{ik}} \sum_{r=1}^{s} L_{ik}^r p^r \quad , \tag{21}$$

that can be computed once for all by the EG; thus the total connectivity index can be finally written as

$$c_i = \sum_{k \in I_i} \frac{\mathcal{M}_i + \mathcal{M}_k}{\mathcal{M}_i^{\max} + \mathcal{M}_k^{\max}} H_{ik} \quad , \tag{22}$$

where the quantities H_{ik} e $\mathcal{M}^{max}_k$ are computed for all the neighbors of the i-th LU. Moreover, since $\mathcal{M}_i = \mathcal{M}_i(t)$ and $\mathcal{M}_k = \mathcal{M}_k(t)$, the connectivity index (22) results to be timedependent, i.e. $c_i = c_i(t)$, and, through it, all the equations on $\mathcal{M}_i$ are coupled with those on $\mathcal{M}_k$, $k \in I_i$.

In order to get normalized solutions, (16) will be now re-written in terms of the new variables

$$M_i = \frac{\mathcal{M}_i}{\mathcal{M}_i^{\max}} \le 1, \qquad V_i = \frac{\mathcal{V}_i}{A_i} \le 1 \quad . \tag{23}$$

Therefore, dividing the first equation of (16) by $\mathcal{M}^{max}_i$ and the second by A_i, and taking into account the expression of $c_i(t)$ given by (22), the following final version of the model is obtained by

$$\begin{cases} M_i' = \left[\sum_{k \in I_i} \frac{M_i \mathcal{M}_i^{\max} + M_k \mathcal{M}_k^{\max}}{\mathcal{M}_i^{\max} + \mathcal{M}_k^{\max}} H_{ik} \right] M_i (1 - M_i) - \nu_i (1 - V_i) M_i \\ \\ V_i' = M_i \, V_i (1 - V_i) - \mu_i U_i V_i . \end{cases} \tag{24}$$

In order to perform quantitative results system (24) will be joined to the following initial data

$$M_i(t = 0) = M_{i0} = \mathcal{M}_{i0}/\mathcal{M}^{max}_i \quad , \qquad V_i(t = 0) = V_{i0} = \mathcal{V}_i/A_i, \qquad i = 1,...,n$$

which can be recovered directly by the EG (in particular $\mathcal{M}_{i0}$ from (3)).

Once the variables $M_i(t)$ and $V_i(t)$ are known from (24), one can recover, at each time t, the corresponding variables at the level of the whole environmental system; in particular the non-dimensionless variables $\mathcal{M}$ and $\mathcal{V}$ can be computed by

$$\mathcal{M}(t) = \frac{1}{n} \sum_{i=1}^{n} M_i(t) \mathcal{M}_i^{\max}, \qquad \mathcal{V}(t) = \sum_{i=1}^{n} V_i(t) A_i \tag{25}$$

whereas the dimensionless ones M and V are given by

$$M(t) = \mathcal{M}(t)/\mathcal{M}^{max}, \qquad V(t) = \mathcal{V}(t)/A, \tag{26}$$

where A is the total area of the environment and $\mathcal{M}^{max}$ is the *maximum producible* GBE of the whole system, defined by

$$\mathcal{M}^{\max} = 2 B_{\max}^b A,$$

which, conversely to definition (5) and according to the hypothesis (3) made at the beginning of this section, is now referred to the absolute quantity $B^b_{\max}$.

Finally, let us comment that conversely to the solutions of (14) it is not evident that the solutions of model (24) are always non-negative, provided that the initial data are positive. In fact the great number of equations of this model and their rather complicated coupling do not permit to find easily invariant regions in the phase space. Nevertheless all the numerical tests performed have shown that each pair (M_i,V_i), $i = 1,...,n$ remains $\forall t$ in the square $[0,1] \times [0,1]$.

5 An Application in the Province of Viterbo

The model derived in the last section has been applied to a study case corresponding to a subset of the LUs identified in the Traponzo river catchment, in the province of Viterbo in central Italy. Traponzo river originates in the Cimini mountains and flows into the Marta river so that Traponzo watershed, a sub-basin of the Marta River, has a total area of 475 Km^2, a mean elevation of 526 m a.s.l. and a maximum of 979 m a.s.l. The climate of this area is quite Mediterranean, with a mean annual temperature of about 15 $^{\circ}C$ and a mean annual precipitation rate of approximately 970 mm. Urban fabric covers 3.4 km^2 of the whole basin, which is characterized also be the presence of the Civitavecchia-Orte freeway that divides the watershed into two almost equal parts.

As already recalled the environmental evaluation of the whole watershed has been performed in [15] using the model (14) whereas the new model, presented here, has been implemented only on a partial number of LUs composing the watershed. The subset of LUs taken into account consists of 8 LUs (Fig. 3) confined in the south-west part of Traponzo watershed. For this area a GIS database was constructed to set up the model and it was updated through a manual digitalization process using aerial orthophotos from 1999, with particular reference to isolated buildings outside the already mapped urban areas (afterwards defined as urban sprawl), road and railway networks and hedges. The total area covered by urban sprawl is about 21 % of the total urban area. Road and railway networks are highly developed (183.7 Km), whereas hedges cover an area of 4.5 km^2.

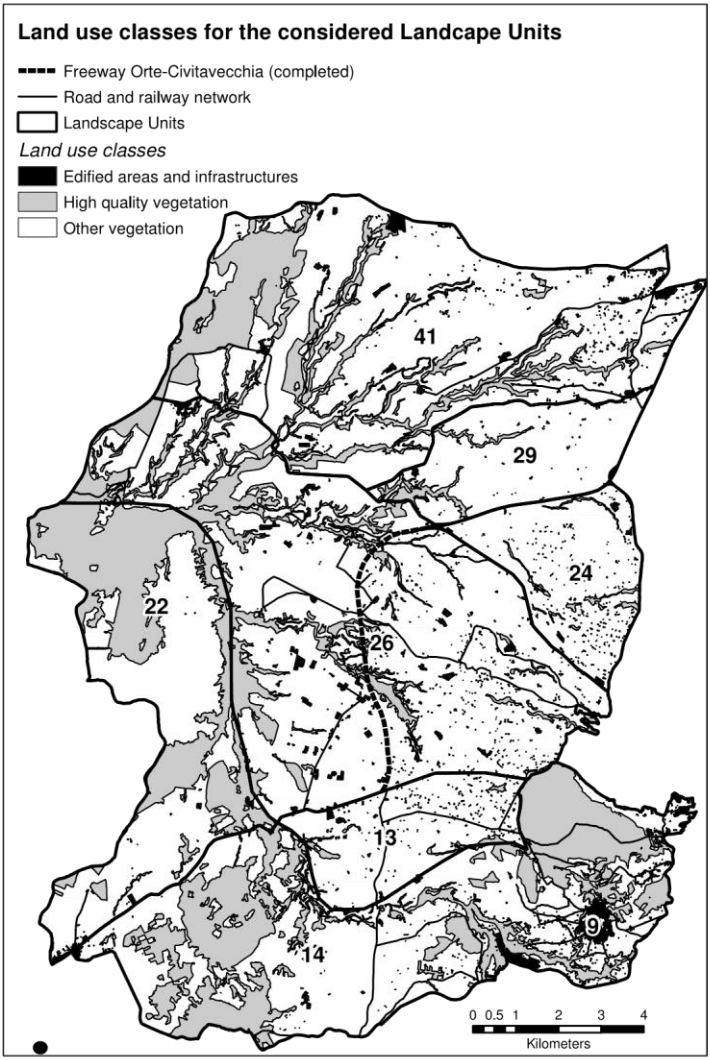


Fig. 3 The study case: map of the landscape units

A new stretch of the Orte-Civitavecchia freeway (dashed line in Fig. 3) was completed during the year 2011 and it crosses the LU No. 26. Thus, in order to take into account the effects of this infrastructure on the ecological connections of the study area, two simulations have been carried out: the first one (scenario A) with the aim of modeling the LUs before the completion of the works (i.e. without the last stretch of the freeway Orte-Civitavecchia), while the second simulation (scenario B) has the purpose to represent the actual conditions of landscape with the completed freeway. The simulations have been performed solving system (24) with the well assessed ODE45 solver of MATLAB.

The table below shows the values of initial data and of the model parameters for each simulated LU in scenario A.

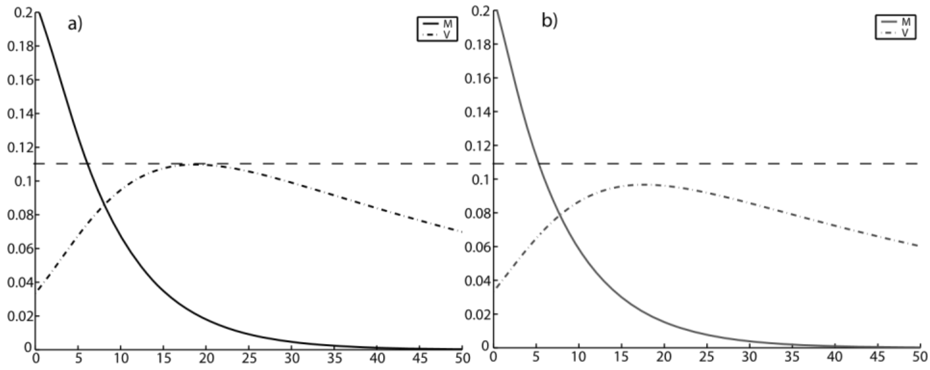


Fig. 4 Time-evolution of V and M for LU No. 24: (**a**) scenario A; (**b**) scenario B

LUNo.	V_{i0}	M_{i0}	U_i	μ_i	ν_i
9	0.000	0.044	0.689	1.317	1.859
13	0.032	0.191	0.014	0.084	0.286
14	0.389	0.423	0.013	0.222	0.835
22	0.433	0.457	0.006	0.005	0.562
24	0.033	0.206	0.021	0.909	0.566
26	0.161	0.257	0.016	0.347	1.517
29	0.062	0.185	0.014	0.912	0.341
41	0.223	0.311	0.016	0.177	1.029

In scenario B the values of initial data and parameters of the LU No. 26 are, respectively, substituted by 0.160, 0.256, 0.017, 0.746, 1.956. The parameters of the above table characterize the landscape accounting for the presence of compact (U_i) and spread (μ_i) edified areas, and for the presence of impervious barriers inside the i-th LU (ν_i). Referring to Scenario A high values of U_i, μ_i and ν_i for LU No. 9 reflect the significant impact of compact edified areas present there, whereas high values of μ_i for LUs No. 24 and No. 29 are related with a wide urban sprawl phenomenon involving those landscapes. LUs No. 14, No. 26 and No. 41 present also high values of ν_i due to the density of their road networks.

Let us recall that V_{i0} and M_{i0} represent the initial percentage of high BTC land and of GBE, respectively. LU No. 9 has the lowest values of V_{i0} (close to zero) and M_{i0} since it is the most urbanized unit and its border is characterized by almost impervious barriers. LUs No. 14 and No. 22 show the highest value of V_{i0} corresponding to extended forested areas. LU No. 26 presents a rather large value of M_{i0} even with a not so high value of V_{i0}: the quite permeable barriers characterizing the border of this LU allow the passage of bio-energy to the neighbor units.

In the simulation of scenario A, looking at the evolution of each single LU, the LUs No. 13, No. 14 and No. 22 exhibit an increase of V_i and M_i, whereas all the other LUs show a marked decay of the two variables. The evolution trend of the variables $V_i(t)$ and $M_i(t)$ for some explicative LUs is reported in Fig. 4a (LU No. 24), Fig. 5a (LU No. 26) and Fig. 6b (LU No. 22).

Both LU No. 24 and LU No. 26 show a quick decay of the production of GBE and a smooth decrease of the variable V_i. The graphs confirm the characteristics of

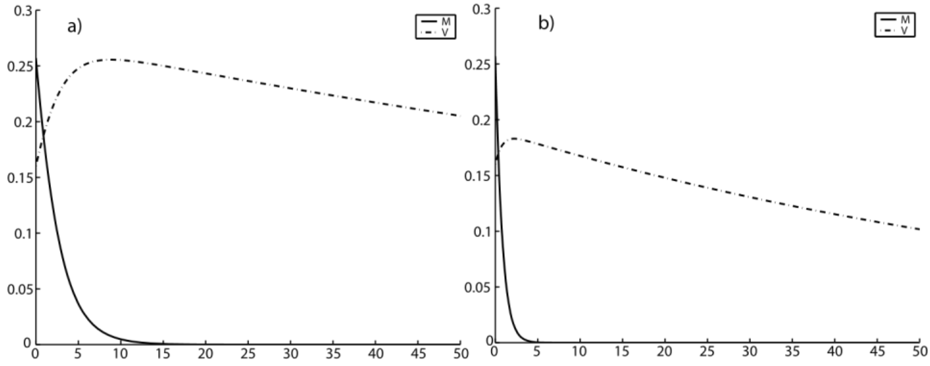


Fig. 5 Time-evolution of V and M for LU No. 26: (**a**) scenario A; (**b**) scenario B

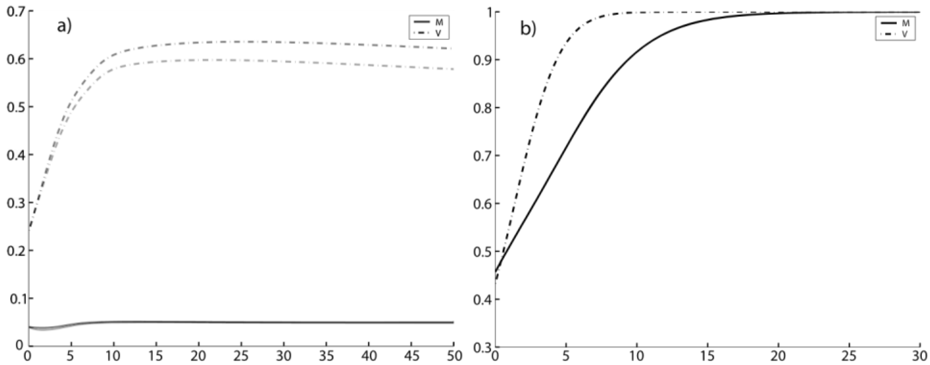


Fig. 6 Time-evolution of V and M: (**a**) for the whole system in scenario A (*black line*) and scenario B (*grey line*); (**b**) for LU No. 22 in scenario A

landscape fragmentation undergoing these two LUs, due to the dense road network crossing them. LU No. 22 shows, on the contrary, a growing trend of both variables M_i and V_i due to the presence of vegetation at high BTC values and of the limited urban and road network development.

According to (26), the total environmental quality of the territory determines the evolution trend reported in Fig. 6a where the overall variables V(t) and M(t) increase slowly towards the equilibrium state. In general, the obtained results for scenario A (i.e. before the completion of the new stretch of freeway Orte-Civitavecchia) underline a high fragmentation of the considered territory in which only some islands present moderate production of bio-energy. These restricted areas at high BTC values may be even reduced by the presence of new anthrop barriers that may heighten the quite enough critical fragmentation of the environmental system. The second simulation was carried out to model scenario B which takes into account the completion of works for the freeway Orte-Civitavecchia. The effects of this stretch of freeway on the landscape can be deduced from the graphs representing the evolution trends of the variables V(t) and M(t) for each LU (e.g. Fig. 4b and 5b) and for the whole environmental system (Fig. 6a). The comparison between the evolution trends of the variables V(t) and M(t) for scenarios A and B, respectively, points out a global reduction of bio-energy production that represents the effect of the new infrastructure on landscape. However, looking at the evolution of each single LU, only some LUs, neighbor to the LU No. 26, show significant variation of M(t) and V(t) trend. In particular, LU No. 26 exhibits an abrupt decrease in the production of bio-energy and in the percentage of areas at high BTC as a consequence of the presence of the freeway on the equilibrium state of the LU (Fig. 5b). On the contrary, the impact of the enhanced road network on the neighbor LUs is relatively evident: in scenario B, the LU No. 24 is characterized by a more decreasing evolution trend of the simulated variables (see Fig. 4b). This LU shows an initial fragmented state in scenario A so that a new infrastructure, in its neighbor LU No. 26, has however some impact on it. On the contrary, LU No. 22, located west of LU No. 26, does not suffer the influence of the new construction since it is characterized by rather great initial percentage of high BTC areas and by a good production of bio-energy, so that in scenario B the variables $V_i(t)$ and $M_i(t)$ exhibit almost the same trend shown in scenario A, as reported by Fig. 6b.

6 Conclusions

The new proposed model studies equilibrium conditions for landscapes by analyzing spatial data at the level of each LU. It works with two state variables, allowing to point out possible local fragmentation or local critical condition in terms of ecological functionality. This new formulation of the model could be of help in land planning since it can provide a reliable tool to estimate the effects of actions and strategies on the landscape equilibrium conditions not only at the whole landscape scale but also at that of each LU. Due to the natural heterogeneity and complexity of landscape, the response of the whole environmental system to external constraints (e.g. anthrop actions) derives from the interactions between its internal components. Simulating the whole landscape behavior in terms of a unique variable trend for all the system, could hide local critical environmental quality that could be balanced by the response of another portion of the studied territory. So if it is true that to better understand the complex mechanism of cause and effect underlying landscape evolution dynamics, a holistic approach should be pursued (as claimed in [7, 15, 22, 23]), it is also true that the local critical values of the variables chosen to describe the health of the landscape can be pointed out only recurring to the simulation of the evolution of the same variables at local level, namely at the level of each LU. Furthermore, possible future scenarios of the environment, as consequences of different planning strategies, can be predicted through the mathematical model proposed here. Namely, this model considers the effect of the environment spatial scale and structure through the state variables and parameters. Indeed, the parameters and indices of the model can represent suitably the ecological health of the landscape and can be used alone or in combination to assess and compare landscape scenarios. Finally, all the parameters required by the mathematical model can be obtained from GIS data, which are usually available to land managers. Further effort is needed to accurately test this new dynamical model to real-life applications in order to develop a more helpful tool for “what if” scenarios analysis and planning strategy conception.

Appendix A

In the following table (see [4]) the BTC classes and indexes, considered in this paper, are reported for each land cover.

Land cover	BTC class	BTC index
continuous and dense urban fabric	A	0.0
sprawl urban fabric	A	0.0
industrial, commercial, transport units	A	0.0
mineral extraction sites	A	0.0
dump sites and mine deposits	A	0.0
highways and freeways	A	0.0
rivers and streams	A	0.1
cemeteries	A	0.3
leisure and sport facilities	B	0.4
non-irrigated arable land	B	1.0
nurseries in non-irrigated areas	B	0.8
areas of glass or plastic greenhouses	B	0.8
irrigated arable land	B	1.2
nurseries in irrigated areas	B	1.0
pastures	B	1.0
annual crops and permanent crops	B	1.0
natural grassland	B	0.8
vineyards	C	1.8
fruit trees and berries plantations	C	1.8
olive groves	C	1.8
complex cultivation patterns	C	1.8
agricultural and natural areas	C	1.8
moors and heath-land	C	1.8
recolonization areas	D	3.2
broad-leaved forests	E	6.5
coniferous forests	E	6.5

Appendix B

In this Appendix the computation of the parameters κ_i defined in (3) and necessary to determine the initial data M_{i_0} for (24) is given.

As already mentioned the parameter κ_i takes into account several features of the LU border and of the biotopes belonging to the LU itself. Here we define six parameters [10, 15] that are included in κ_i and have been used throughout several papers. For a complete and specific list of indicators characterizing a landscape the reader may be addressed to paper [24].

The first one κ_i^{sh} takes into account the shape of the LU through the formula

$$\mathcal{K}_i^{sh} = 1 - P_i^c/P_i = 1 - 2\sqrt{\pi A_i}/P_i,$$

where P_i^c is the perimeter of a circle having the same area A_i of the LU. In such a way if the ratio P_i^c/P_i is very small the parameter κ_i^{sh} tends to one. Thus, the larger is the LU perimeter the larger is the bio-energy transmitted to the neighbor LUs.

The second parameter κ_i^{pe} is referred to the permeability of the LUs border, i.e.

$$\mathcal{K}_i^{pe} = \frac{1}{P_i}\sum_{k\in I_i}\sum_{r=1}^{s} L_{ik}^r p^r, \quad \sum_{k\in I_i}\sum_{r=1}^{s} L_{ik}^r = P_i,$$

so that if the border is completely permeable ($p^r = 1, \forall r$) then $\kappa_i^{pe} = 1$.

The third parameter κ_i^{ld} is relevant to landscape diversity which takes into account that the biotopes are defined to belong to the afore mentioned five classes of BTC, A,...,E. Then κ_i^{ld} is computed by a Shannon-type entropy formula given by

$$\mathcal{K}_i^{ld} = \left(\sum_{\kappa=\mathrm{A}}^{\mathrm{E}} \frac{m_i^\kappa}{m_i}\log\frac{m_i^\kappa}{m_i}\right)/\log(1/5),$$

where m^κ_i are the number of biotopes of class κ in the i-th LU. The last expression must be computed by setting the log equal to zero if $m^\kappa_i = 0$, so that $\kappa_i^{ld} = 0$ when all the biotopes in the LU are of the same class and $\mathcal{K}_i^{ld} = 1$ if the biotopes are therein equally distributed.

The fourth parameter $\mathcal{K}_i^{ec}$ takes into account the length of the ecotone, that is the land cover along the biotope borders. The length of the ecotones has a relevant influence on bio-diversity and we will take it into account by means of the following formula

$$\mathcal{K}_i^{ec} = 1 - P_i/\sum_{j=1}^{m_i} P_{ji},$$

where P_{ji} is the perimeter of the j-th biotope belonging to the i-th LU. From the above computation, however, the biotope perimeter tracts composed by anthrop barriers must be excluded. Obviously κ_i^{ec} must be put equal to zero if the whole LU includes only land cover types of BTC class A.

The last two parameters $\mathcal{K}_i^{hu}$ and $\mathcal{K}_i^{se}$ refer, respectively, to climate condition (De Martonne aridity index) and sun exposition. They are defined by

$$\mathcal{K}_i^{hu} = \left(w_1 A_i^h + w_2 A_i^s\right)/A_i$$

$$\mathcal{K}_i^{se} = \left(w_3 A_i^{SES} + w_4 A_i^W + w_5 A_i^{NE}\right)/A_i,$$

where A^h_i, A^s_i, A^{SES}_i, A^W_i and A^{NE}_i are, respectively, the fractions of land characterized by humid and sub-humid climate classification, south-east/south, west and north/north-east exposition; the coefficients w are suitable weights such that $w_1 + w_2 = 1$ and $w_3 + w_4 + w_5 = 1$.

Once the above six parameters have been determined, then the global one K_i can be computed as their average.

In papers [12–14, 21] the average has been computed taking into account the parameters K_i^{sh}, K_i^{pe}, K_i^{ld}, whereas in article [15] also the parameters K_i^{hu} and K_i^{se} have been included in the average.

In this paper for the case study of Sect. 5 only the parameters $\mathcal{K}_i^{ld}$, $\mathcal{K}_i^{ec}$, $\mathcal{K}_i^{hu}$, $\mathcal{K}_i^{se}$ have been considered, since, in authors' opinion, it is more correct to include in the parameter K_i only quantities related to biotopes. In fact shape and permeability of the LUs border are already taken into account in the formula of the total connectivity indexes c_i.

Appendix C

In the following table (see [10]) the permeability indexes of the different types of anthrop and natural barriers considered in this paper are reported.

Layers	Barrier type	Permeability
edified areas & infrastructures	compact urban texture	0.05
	linear urban texture	0.4
	diffuse urban texture	0.5
	freeway	0.05
	state road	0.05
	provincial road	0.4
	secondary road	0.5
	high-speed railway	0.05
	railway	0.5
	viaduct	0.5
	small roads and channels	0.7
	dirt roads	0.9
pedology	volcanic/alluvial soil change	0.9
altimetry	hill/mountain zones change	0.95
	structurally defined ridges	0.7
rivers	main rivers	0.85
	rivers with cemented banks	0.4

References

1. Council of Europe: European Landscape Convention. Treaty Series no. 176, Florence, 2000 October20th (2000)
2. Naveh, Z., Lieberman, A.: Landscape Ecology: Theory and Applications. Springer, New York (1984)
3. Turner, M.G., Gardnerl, R.H.: Quantitative Methods in Landscape Ecology. Springer, New York (1990) 4. Ingegnoli, V.: Landscape Ecology: A Widening Foundation. Springer, New York (2002)
5. Fabbri, P.: Principi Ecologici per la Progettazione del Paesaggio. Franco Angeli Editore, Milano (2007)
6. Turner, M.G., Romme, V.H., Gardnerl, R.H., O'Neil, R.V., Kratz, T.K.: A revised concept of landscapeequilibrium: disturbance and stability on scaled landscapes. Landsc. Ecol. **8**(3), 213–227 (1993)
7. Pelorosso, R., Della Chiesa, S., Tappeiner, U., Leone, A., Rocchini, D.: Stability analysis for definingmanagement strategies in abandoned mountain landscapes of the Mediterranean basin. Landsc. Urban Plan. **103**, 335–346 (2011)
8. Petit, C., et al. (eds.): Landscape Analysis and Visualisation-Spatial Models for Natural and Resourcesan Planning. Springer, New York (2008)
9. Vermaat, J.E., Eppink, F., Van den Bergh, J.C., Barendregt, A., Van Belle, J.: Matching of scales in spatialeconomic and ecological analysis. Ecol. Econ. **52**, 229–237 (2005)
10. Fabbri, P.: Paesaggio, Pianificazione, Sostenibilità. Alinea, Firenze (2003)
11. Urban, D.L., Minor, E.S., Treml, E.A., Schick, R.S.: Graph models of habitat mosaics. Ecol. Lett. **12**, 260–273 (2009)
12. Lauro, G., Monaco, R., Servente, G.: A model for the evolution of bioenergy in an environmental system.In: Sammartino, M., Ruggeri, T. (eds.) Asymptotic Methods in Non-linear Wave Phenomena, pp. 96– 106. World Scientific, Singapore (2007)
13. Lauro, G., Lisi, M., Monaco, R.: A modeling framework for analysis of landscape stability and bifurcation phenomena. Rend. Sem. Mat. Univ. Politec. Torino **68**(4), 399–413 (2010)
14. Finotto, F., Monaco, R., Servente, G.: Un modello per la valutazione della produzione e della diffusivitàdi energia biologica in un sistema ambientale. Sci. Reg. **9**(3), 61–84 (2010)
15. Gobattoni, F., Lauro, G., Leone, A., Monaco, R., Pelorosso, R.: A procedure for mathematical analysisof landscape evolution and equilibrium scenarios assessment. Landsc. Urban Plan. **103**, 289–302 (2011)
16. Monaco, R., Servente, G.: Introduzione ai Modelli Matematici nelle Scienze Territoriali, 2nd edn. Celid,Torino (2011)
17. Ingegnoli, V., Giglio, E.: Ecologia del Paesaggio. Manuale per conservare, gestire e pianificarel'ambiente. Esselibri, Napoli (2005)
18. Jordan, D.W., Smith, P.: Nonlinear Ordinary Differential Equations. Clarendon Press, Oxford (1977)
19. Abramowitz, M., Stegun, I.A. (eds.): Handbook of Mathematical Functions. Dover, New York (1980) 20. O'Neill, R.V., Johnson, A.R., King, A.W.: A hierarchical framework for the analysis of scale. Landsc. Ecol. **3**, 193–205 (1989)
21. Navino, D.: Valutazione della BTC mediante un modello dinamico relativo al grafo ecologico. GraduateThesis in Urban Planning, Politecnico di Torino (2010)
22. Li, B.: Why is the holistic approach becoming so important in landscape ecology? Landsc. Urban Plan.**50**, 27–41 (2000)
23. Naveh, Z.: Ten major premises for a holistic conception of multifunctional landscapes. Landsc. UrbanPlan. **57**, 269–284 (2001)
24. Finotto, F.: Landscape assessment: the ecological profile. In: Cassatella, C., Peano, A. (eds.) Landscape Indicators. Assessing and Monitoring Landscape Quality, pp. 47–75. Springer, Dordrecht (2011)