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Uncertainty in fatigue loading: consequences on statistical evaluation of reliability in service

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Abstract

A design-by-reliability approach is ever more required and accurate design inputs are needed to meet reliability targets while reducing costs. Different models have been proposed in the literature to describe fatigue life variation with respect to the applied stress by assuming the applied stress as a deterministic independent variable and the fatigue life as a dependent random variable.

In the paper also the applied stress is considered as a random variable with its own uncertainty and the procedure to evaluate the error on the estimation of parameters usually adopted in service reliability assessment of structural components is shown. Exact equations are proposed for some special cases and an illustrative example showing the reliability errors originated by applying the ASTM recommendations is given.

Keywords:

Random fatigue life; Random applied stress; Random fatigue limit; Fatigue life in service

Nomenclature

$f_{X_{a,serv}}, f_{X_a,X_m}, f_{X_l}, f_{X_{m,serv}}, f_Y, f_{Y_{true}}, f_{Y_{true},X_l}, f_{Y_{true}|X_l}, f_{Y_{true},X_l \leq x_a}, f_{Y_{true},X_l > x_a}$ = probability density functions

$F_Y, F_{Y_{aff}}, F_{Y_{aff}|(X_{a,serv},X_{m,serv})}, F_{Y_{true}}, F_{Y_{true}|(X_{a,serv},X_{m,serv})}$ = cumulative distribution functions

$g_{X_a}, g_{X_l}, g_{X_m}, g_{Y_{true}}, g_{Y_{true}|X_l}$ = standardized location-scale probability density functions

$R_{aff,serv_y}$ = in service affected reliability at y

R_{serv} = in service reliability

$R_{true,serv_y}$ = in service true reliability at y

$x_a, x_{a,serv}, x_l, x_m, x_{m,serv}, y$ = values assumed by random variables

$x_{a,des}, x_{m,des}, x_{des}$ = desired applied stresses

$x_{a,syst}, x_{m,syst}$ = systematic errors in the applied stresses

$\underline{x_a}, \underline{x_{a,serv}}, \underline{x_l}, \underline{x_m}, \underline{x_{m,serv}}$ = lower limits of integration

$\overline{x_a}, \overline{x_{a,serv}}, \overline{x_m}, \overline{x_{m,serv}}$ = upper limits of integration

X_a = random alternating stress

X_{axl} = random axial stress

$X_{a,serv}$ = in service random alternating stress

X_{bnd} = random bending stress

X_l = random fatigue limit

X_m = random mean stress

$X_{m,serv}$ = in service random mean stress

X_{MAX} = random maximum stress

$y_{aff_{R_{serv}}}, y_{aff_{1-R_{serv}}}$ = $(1 - R_{serv})$-th and R_{serv}-th quantiles of the affected random fatigue life

$y_{true_{R_{serv}}}, y_{true_{1-R_{serv}}}$ = $(1 - R_{serv})$-th and R_{serv}-th quantiles of the true random fatigue life

$y_{R_{serv},\%e}$ = percent error of the fatigue life

Y = random fatigue life

Y_{true} = random true fatigue life

Y_{aff} = random affected fatigue life

$z_{R_{serv}}$ = R_{serv}-th quantile of the standardized Normal distribution

β_0, β_1 = constant coefficients

$\phi(\cdot)$ = standardized Normal probability density function

$\Phi(\cdot)$ = standardized Normal cumulative distribution function

$\mu_{a,serv}, \mu_l, \mu_{m,serv}, \mu_{true}, \mu_{true|l}$ = location parameters

$\sigma_{aff}, \sigma_{a,aff}, \sigma_{a,serv}, \sigma_l, \sigma_{m,serv}, \sigma_{true}, \sigma_{a,true}, \sigma_{true|l}$ = scale parameters

$\boldsymbol{\vartheta}_{true}$ = vector of parameters of Y_{true}

$\cdot|\cdot$ = conditional event

1. Introduction

Stringent safety requirements are common in structural design. A design-by-reliability approach is required in ever more situations and accurate design inputs are needed to meet reliability targets while reducing costs.

Different models have been proposed in the literature to describe fatigue life variation with respect to the applied stress (for a summary, see [1]). Most of the proposed models assume the stress as a deterministic independent variable, while fatigue life is a dependent random variable. In some other cases [2-6], stress is considered as a random variable when conditioned on a given value of fatigue life.

Indeed, according to [7], in fatigue tests different sources of scatter should be taken into account simultaneously: scatter related to specimen manufacturing, to fatigue test conditions (e.g., test rig and clamping), and to the material itself. Kandil and Dyson [8,9], by analyzing inter-laboratory low cycle fatigue datasets [10], also showed that, even if a test is performed under the stringent conditions imposed by the ASTM standard E 606 – 80 [11] concerning the maximum allowable bending stress in axial fatigue tests (smaller than 5%), the ratio of the reproducibility limits of the fatigue life reaches the value of six suggesting a reappraisal of the British Standard BS 7270:1990 [12] and ASTM standards E 606 – 80 [11] and E 1012 – 89 [13]. The above cited results were confirmed in a further research supported by the Versailles Project on Advanced Materials and Standards [14,15][1].

The same uncertainty (5%) in the applied stress is accepted by the ASTM standard E 466 – 96 [16], concerning force controlled fatigue tests and, therefore, a similar reproducibility span should be expected in this case. Indeed, in [17,18], the results of an inter-laboratory fatigue test carried out along with the ASTM standard E 466 – 96 [16] showed that stress uncertainties due to load application misalignment are not negligible compared to the material scatter and should be considered in an uncertainty analysis [19].

In the paper an approach, which assumes that fatigue life is a dependent random variable and that applied stress is an independent random variable with its own uncertainty, is shown. The proposed approach allows to consider the random character of applied stresses in real cases and provides a model valid for any combination of mean and alternating stress.

2. Fatigue design curves

In fatigue testing, the applied cyclic stress can be uniquely defined by considering at least two stress quantities among, e.g., the applied alternating stress, the applied mean stress, the applied maximum stress, the applied minimum stress and the applied stress range. The applied alternating

[1] In [14], it was found that the main contribution to bending is the specimen orientation, rather than the machine misalignment: *"Further tests using a precision alignment system confirmed that specimen contribution is more significant than previously thought; in many instances it even exceeded the contribution due to the machine misalignment."* In this respect, even if specimens are carefully machined and the testing machine has been precisely aligned before the test series, fatigue loading is expected to vary randomly from specimen to specimen.

stress, here denoted as X_a, and the applied mean stress, here denoted as X_m, are considered in the following. This is a variable set commonly adopted in the literature [4,5,7]; in some other cases [1-3] the stress range (twice the alternating stress) is used in place of the alternating stress. In fatigue tests, the desired values of X_a and X_m are often affected by several noise factors: residual stresses and geometry uncertainties of tested specimens, geometry uncertainties of testing fixtures, uncertainty in the applied load, environmental conditions are only some examples. In most cases, these factors cannot be controlled during fatigue testing. For this reason their effects on the applied cyclic stress can only be taken into account by considering both X_a and X_m as random variables.

Experimental results show that specimens obtained from the same nominal material and tested under the same cyclic stress (i.e., suppose that X_a and X_m remain constant from specimen to specimen) may fail at different values of the fatigue life (logarithm of the number of cycles to failure), here denoted as Y. Material heterogeneity and random occurrence of both internal and external defects are the main causes of this variation. Since the scatter in specimen material properties cannot be taken under control during the testing campaign, the fatigue life corresponding to a specific applied cyclic stress is considered as a random variable, too. In particular, if X_a is supposed to be equal to value x_a and X_m is supposed to be equal to value x_m, then the random fatigue life is a conditional random variable, denoted as Y_{true} (i.e., $Y_{true} = Y|(X_a = x_a, X_m = x_m)$).

In classic fatigue life models (see [20] for a review), if x_a is smaller than a specific threshold value, the fatigue limit, then tested specimens are supposed to survive indefinitely. According to Murakami's model [21], the fatigue limit depends on material parameters (i.e., hardness, inclusion stiffness and geometry) that may vary from specimen to specimen and, according to well-known fatigue limit experimental studies (see [22] and the references therein), is influenced by the value of the applied mean stress, x_m. Therefore, as suggested by [5,23,24], the fatigue limit is considered as a random variable whose distribution varies with x_m and is here denoted as X_l.

Without loss of generality and in agreement with many fatigue models proposed in the literature [5,23,24], X_l, and Y_{true} are supposed to be location-scale random variables (e.g., Normal, Smallest Extreme Value, Logistic) with probability density functions (pdf's) denoted as f_{X_l} and $f_{Y_{true}}$, respectively. In particular, let μ_l and σ_l be the location and scale parameter for f_{X_l}, and μ_{true} and σ_{true} be the location and scale parameter for $f_{Y_{true}}$, then:

$$f_{X_l}(x_l; x_m, \mu_l, \sigma_l) = \frac{1}{\sigma_l} g_{X_l}\left(\frac{x_l - \mu_l}{\sigma_l}\right) \quad (1)$$

and

$$f_{Y_{true}}(y; x_a, x_m, \mu_{true}, \sigma_{true}) = \frac{1}{\sigma_{true}} g_{Y_{true}}\left(\frac{y - \mu_{true}}{\sigma_{true}}\right), \quad (2)$$

where $g_{X_l}(\cdot)$ and $g_{Y_{true}}(\cdot)$ are the standardized location-scale pdf's of X_l and Y_{true}, respectively.

In Equation (2), both μ_{true} and σ_{true} are general functions of x_a and of x_m since both the mean value and the scatter of the fatigue life strongly depend on the value assumed by the applied alternating stress and the applied mean stress. Moreover, if a random fatigue limit model is considered [23,24], then μ_{true} and σ_{true} may also depend on the value assumed by the fatigue limit, x_l. Therefore, the random fatigue life more generally depends on the values assumed by X_a, X_m, and X_l. In particular, if X_a is supposed equal to x_a, X_m is supposed equal to x_m, and X_l is supposed equal to x_l, then the random fatigue life is a conditional random variable, denoted as $Y_{true}|X_l$ (i.e., $Y_{true}|X_l = Y_{true}|(X_a = x_a, X_m = x_m, X_l = x_l)$). If, in agreement with the random fatigue limit model [23,24], $Y_{true}|X_l$ is supposed to be a location-scale random variable with pdf denoted as $f_{Y_{true}|X_l}$, then:

$$f_{Y_{true}|X_l}(y; x_a, x_m, x_l, \mu_{true|l}, \sigma_{true|l}) = \frac{1}{\sigma_{true|l}} g_{Y_{true}|X_l}\left(\frac{y-\mu_{true|l}}{\sigma_{true|l}}\right), \quad (3)$$

where $g_{Y_{true}|X_l}(\cdot)$ is the standardized location-scale pdf of $Y_{true}|X_l$, and $\mu_{true|l}$ and $\sigma_{true|l}$ denote the location and scale parameters of $f_{Y_{true}|X_l}$.

If the random fatigue limit model [23,24] is considered, the joint pdf of Y_{true} and X_l is given by multiplying Equations (1) and (3):

$$f_{Y_{true},X_l}(y, x_l; x_a, x_m, \mu_{true|l}, \sigma_{true|l}, \mu_l, \sigma_l) = \frac{1}{\sigma_l \sigma_{true|l}} g_{Y_{true}|X_l}\left(\frac{y-\mu_{true|l}}{\sigma_{true|l}}\right) g_{X_l}\left(\frac{x_l-\mu_l}{\sigma_l}\right). \quad (4)$$

By integrating the right-hand side of Equation (4) with respect to x_l, from the minimum value of X_l, $\underline{x_l}$, to the value assumed by X_a, x_a, it is possible to obtain the pdf of Y_{true} when the fatigue limit is smaller than the applied alternating stress, $f_{Y_{true},X_l \le x_a}$, as follows:

$$f_{Y_{true},X_l \le x_a}(y; x_a, x_m, \mu_{true|l}, \sigma_{true|l}, \mu_l, \sigma_l) = \int_{\underline{x_l}}^{x_a} \frac{1}{\sigma_l \sigma_{true|l}} g_{Y_{true}|X_l}\left(\frac{y-\mu_{true|l}}{\sigma_{true|l}}\right) g_{X_l}\left(\frac{x_l-\mu_l}{\sigma_l}\right) dx_l. \quad (5)$$

Since the probability of having a failure if the applied alternating stress is below the fatigue limit is equal to zero, then the pdf of Y_{true} when the fatigue limit is larger than the applied alternating stress, $f_{Y_{true},X_l > x_a}$, is equal to zero (i.e., $f_{Y_{true},X_l > x_a} = 0$). According to the Total Probability Theorem and taking into account Equation (5), the pdf of Y_{true} can be easily obtained:

$$f_{Y_{true}}(y; x_a, x_m, \mu_{true|l}, \sigma_{true|l}, \mu_l, \sigma_l) = f_{Y_{true},X_l \le x_a} + f_{Y_{true},X_l > x_a} = \\ = \int_{\underline{x_l}}^{x_a} \frac{1}{\sigma_l \sigma_{true|l}} g_{Y_{true}|X_l}\left(\frac{y-\mu_{true|l}}{\sigma_{true|l}}\right) g_{X_l}\left(\frac{x_l-\mu_l}{\sigma_l}\right) dx_l, \quad (6)$$

since the events $X_l \le x_a$ and $X_l > x_a$ form a partition of the event space.

For sake of clarity, the pdf of Y_{true} will be generically indicated with $f_{Y_{true}}$ in the following. If the random fatigue limit model holds, then $f_{Y_{true}}$ can be obtained through Equation (6); otherwise it can be obtained through Equation (2).

In experimental tests, X_a and X_m may be either dependent or independent random variables. If dependence exists between X_a and X_m, then a joint pdf of X_a and X_m, denoted as f_{X_a,X_m}, should be considered. However, if the joint pdf of X_a and X_m is Normal, then the conditional pdf of X_a given X_m (X_m given X_a) and the marginal pdf of X_m (X_a) are Normal and, as a consequence, f_{X_a,X_m} is the product of two Normal (location-scale) pdf's. The same result for f_{X_a,X_m} can be obtained if X_a and X_m are independent location-scale (not necessarily Normal) random variables.

For sake of simplicity and without loss of generality, in the following X_a and X_m are supposed to be either independent location-scale random variables (as an example, the independence assumption holds if the alternating stress and the mean stress are applied by means of independent testing equipments) or jointly Normal random variables (in agreement with the Central Limit Theorem, the Normality assumption can be adopted when uncertainty is due to many different simultaneous factors). In both cases, f_{X_a,X_m} is the product of two location-scale pdf's:

$$f_{X_a,X_m}(x_a, x_m; \mu_a, \sigma_a, \mu_m, \sigma_m) = \frac{1}{\sigma_a \sigma_m} g_{X_a}\left(\frac{x_a-\mu_a}{\sigma_a}\right) g_{X_m}\left(\frac{x_m-\mu_m}{\sigma_m}\right), \quad (7)$$

where $g_{X_a}(\cdot)$ and $g_{X_m}(\cdot)$ are the standardized location-scale pdf's of X_a (or X_a given X_m) and X_m, μ_a and σ_a are the location and scale parameters of X_a (or X_a given X_m), and μ_m and σ_m are the location and scale parameters of X_m.

By integrating the product $\left(f_{Y_{true}} \cdot f_{X_a,X_m}\right)$ with respect to x_a and x_m, over the ranges of variation of both X_a and X_m, the marginal pdf of Y, f_Y, can be obtained as:

$$f_Y(y; \mu_a, \sigma_a, \mu_m, \sigma_m, \boldsymbol{\vartheta}_{true}) = \\ = \int_{\underline{x_m}}^{\overline{x_m}} \int_{\underline{x_a}}^{\overline{x_a}} f_{X_a,X_m}(x_a, x_m; \mu_a, \sigma_a, \mu_m, \sigma_m) \cdot f_{Y_{true}}(y; x_a, x_m, \boldsymbol{\vartheta}_{true}) dx_a\, dx_m, \quad (8)$$

where $\underline{x_a}$ and $\overline{x_a}$ are the minimum and the maximum value of X_a, $\underline{x_m}$ and $\overline{x_m}$ are the minimum and the maximum value of X_m, and vector $\boldsymbol{\vartheta}_{true}$ denotes the set of parameters of Y_{true}. In particular, either $\boldsymbol{\vartheta}_{true} = (\mu_{true}, \sigma_{true})$ if model (2) holds, or $\boldsymbol{\vartheta}_{true} = \left(\mu_{true|l}, \sigma_{true|l}, \mu_l, \sigma_l\right)$ if model (6) holds.

By further integrating Equation (8) with respect to y, the expression of the cumulative distribution function (cdf) of Y, F_Y, can also be obtained:

$$F_Y(y; \mu_a, \sigma_a, \mu_m, \sigma_m, \boldsymbol{\vartheta}_{true}) = \\ = \int_{\underline{x_m}}^{\overline{x_m}} \int_{\underline{x_a}}^{\overline{x_a}} f_{X_a,X_m}(x_a, x_m; \mu_a, \sigma_a, \mu_m, \sigma_m) \cdot F_{Y_{true}}(y; x_a, x_m, \boldsymbol{\vartheta}_{true}) dx_a\, dx_m, \quad (9)$$

where $F_{Y_{true}}$ denotes the cdf of Y_{true} (i.e., $F_{Y_{true}} = \int_{-\infty}^{y} f_{Y_{true}} dy$).

In particular, according to model (7), Equation (9) yields:

$$F_Y(y; \mu_a, \sigma_a, \mu_m, \sigma_m, \boldsymbol{\vartheta}_{true}) =$$

$$= \int_{\underline{x_m}}^{\overline{x_m}} \frac{1}{\sigma_m} g_{X_m}\left(\frac{x_m-\mu_m}{\sigma_m}\right) \int_{\underline{x_a}}^{\overline{x_a}} \frac{1}{\sigma_a} g_{X_a}\left(\frac{x_a-\mu_a}{\sigma_a}\right) \cdot F_{Y_{true}}(y; x_a, x_m, \boldsymbol{\vartheta}_{true}) dx_a\, dx_m. \qquad (10)$$

Suppose that model (10) applies and the set of parameters $\boldsymbol{\vartheta}_{true}$ has been estimated by applying a statistically consistent estimation method (e.g., based on the Maximum Likelihood Principle) to the experimental data obtained from the specimens. In order to set aside the uncertainty in the parameter estimation and to focus the study only on the effects due to the uncertainty in the applied stress, the number of experimental data (i.e., the number of failures) can be considered large enough to give parameter estimates close to the true parameter values.

It is worth noting that, during fatigue testing, if the experimenter does not take into account the uncertainty in the applied stresses, he or she is induced to suppose that the random applied stresses, X_a and X_m, are instead deterministic variables equal to the desired applied stresses, $x_{a,des}$ and $x_{m,des}$. In particular, according to model (7), a true applied alternating stress which is unknown and is randomly drawn from a location-scale pdf, with location parameter μ_a and scale parameter σ_a, corresponds to a desired applied alternating stress equal to $x_{a,des}$, and a true applied mean stress which is still unknown and is randomly drawn from a location-scale pdf, with location parameter μ_m and scale parameter σ_m, corresponds to a desired applied mean stress equal to $x_{m,des}$. As an optimistic hypothesis, it can be assumed that the desired applied stresses are equal to the location parameters (corresponding to the modes of the Normal, the Smallest Extreme Value and the Logistic distributions) of the random applied stresses. However, calibration errors of the load cell and misalignment errors due uniquely to the testing machine [14] contribute in a systematic way to the stress level really experienced by specimens. Therefore, more generally, it can be assumed that the location parameters of the random applied stresses are equal to:

$$\mu_a = x_{a,des} + x_{a,syst}$$

and

$$\mu_m = x_{m,des} + x_{m,syst},$$

where $x_{a,syst}$ and $x_{m,syst}$ denote the systematic errors which affect the applied stress. It is worth noting that, if a correct and precise calibration of the testing machine is performed, then systematic errors are known (with their negligible uncertainties) and can be properly corrected (i.e., $x_{a,syst}$ and $x_{m,syst}$ tend to 0). Accidental variations, (e.g., due to the geometry of specimens, to residual stresses, to the orientation of the specimen) are instead taken into account with the scale parameters, σ_a and σ_m.

As a general result, if model (10) applies and the experimenter correctly considers the uncertainty in the applied stress, then it is possible to determine the true fatigue design curves (i.e., fatigue curves corresponding to different survival probabilities) corresponding to $F_{Y_{true}}$; otherwise, if the experimenter totally neglects the uncertainty and the systematic errors in the applied stresses, then it is only possible to determine the fatigue design curves corresponding to F_Y with $\mu_a = x_{a,des} + x_{a,syst}$ and $\mu_m = x_{m,des} + x_{m,syst}$. Therefore, let Y_{aff} denote the random fatigue

life affected by a neglected uncertainty in the applied stresses. Then, if model (10) applies, the cdf $F_{Y_{aff}}$ of Y_{aff} is given by:

$$F_{Y_{aff}}\left(y; x_{a,des}, x_{a,syst}, \sigma_a, x_{m,des}, x_{m,syst}, \sigma_m, \boldsymbol{\vartheta}_{true}\right) =$$
$$= \int_{\underline{x_m}}^{\overline{x_m}} \frac{1}{\sigma_m} g_{X_m}\left(\frac{x_m - (x_{m,des} + x_{m,syst})}{\sigma_m}\right) \int_{\underline{x_a}}^{\overline{x_a}} \frac{1}{\sigma_a} g_{X_a}\left(\frac{x_a - (x_{a,des} + x_{a,syst})}{\sigma_a}\right) \cdot F_{Y_{true}}(y; x_a, x_m \boldsymbol{\vartheta}_{true}) dx_a \, dx_m. \quad (11)$$

To the authors' best knowledge, in most cases the uncertainty in the applied stresses is totally neglected, thus model (11) should be adopted.

3. Reliability in service

In order to evaluate the reliability in service, it is necessary to make some assumptions on the applied service stress. In the following, the hypotheses assumed for model (11) are still considered valid. It is worth noting that, from the point of view of the designer, the fatigue design curves obtained by the experimenter represent the specimen strength distributions which have to be compared with the applied service stress in order to obtain the reliability in service.

3.1. Deterministic service load

As a first simplistic hypothesis the applied service stress can be considered as a deterministic variable. In particular, if the applied service mean stress is equal to $x_{m,serv}$ and the applied service alternating stress is equal to $x_{a,serv}$, then the true reliability in service corresponding to a specific service fatigue life y, $R_{true,serv_y}$, can be obtained [4] from $F_{Y_{true}}$, by setting $x_m = x_{m,serv}$ and $x_a = x_{a,serv}$. In particular, since $R_{true,serv_y}$ is the complementary cdf of the true fatigue life in service, then:

$$R_{true,serv_y}(y; x_{a,serv}, x_{m,serv}, \boldsymbol{\vartheta}_{true}) = 1 - F_{Y_{true}}(y; x_{a,serv}, x_{m,serv}, \boldsymbol{\vartheta}_{true}). \qquad (12)$$

It must be pointed out that $R_{true,serv_y}$ can be determined only if the true values of the applied stresses during fatigue testing are known. As already explained in Section 2, this is not really feasible, since the uncertainty in the applied stress prevents from knowing the true values. Therefore, the designer can only know fatigue design curves affected by the uncertainty in the applied stresses. In this respect, if the experimenter totally neglects the uncertainty in the applied stress, then the designer considers as true fatigue design curves those obtainable from model (11) and he or she is induced to consider the desired applied stresses $x_{a,des}$ and $x_{m,des}$ as the true applied stresses x_a and x_m and, thus, to set $x_{a,des} = x_{a,serv}$ and $x_{m,des} = x_{m,serv}$. In particular, the reliability in service evaluated at a specific service fatigue life y but affected by the uncertainty in the applied stress during testing, $R_{aff,serv_y}$, can be obtained from $F_{Y_{aff}}$ by setting $x_{m,des} = x_{m,serv}$ and $x_{a,des} = x_{a,serv}$, and is therefore equal to:

$$R_{aff,serv_y}(y; x_{a,serv}, x_{a,syst}, \sigma_a, x_{m,serv}, x_{m,syst}, \sigma_m, \boldsymbol{\vartheta}_{true}) = \\ = 1 - F_{Y_{aff}}(y; x_{a,serv}, x_{a,sist}, \sigma_a, x_{m,serv}, x_{m,syst}, \sigma_m, \boldsymbol{\vartheta}_{true}). \qquad (13)$$

Therefore, for the same applied service stresses two different values of the reliability in service can be computed. Usually, the uncertainty in the applied stress during fatigue testing is not taken into account and the scatter in the experimental data is totally assigned to the material heterogeneity. In this respect, the true reliability in service is usually underestimated, since it is calculated through Equation (13) rather than through Equation (12). In particular, the larger the uncertainty in the applied service stresses, the larger the underestimation of the reliability in service. Underestimation gives raise to conservative results and for this reason is commonly accepted; nevertheless, if it is too much large, it may lead to inacceptable oversized designs.

If a desired value of the reliability in service R_{serv} is given, two different values of the service fatigue life can be calculated by considering either the distribution of Y_{aff} or the distribution of Y_{true}. In particular, if $y_{aff_{R_{serv}}}$ denotes the $(1-R_{serv})$-th quantile of the distribution of Y_{aff} and $y_{true_{R_{serv}}}$ denotes the $(1-R_{serv})$-th quantile of the distribution of Y_{true}, then a service fatigue life percent error, $y_{R_{serv},\%e}$ can be computed as:

$$y_{R_{serv},\%e} = \frac{\left(y_{aff_{1-R_{serv}}} - y_{aff_{R_{serv}}}\right) - \left(y_{true_{1-R_{serv}}} - y_{true_{R_{serv}}}\right)}{y_{true_{1-R_{serv}}} - y_{true_{R_{serv}}}} 100, \quad (14)$$

where $y_{aff_{1-R_{serv}}}$ and $y_{true_{1-R_{serv}}}$ are the R_{serv}-th quantile of Y_{aff} and Y_{true}, respectively. The numerator of Equation (14) corresponds to the difference between the spans of the $(2R_{serv}-1)\%$ bilateral confidence intervals of the true and affected fatigue lives.

3.2. Random service load

In most real cases, the applied service load is not a deterministic value, but it may vary randomly. Therefore, the applied service mean stress, $X_{m,serv}$, and the applied service alternating stress, $X_{a,serv}$, can be more generally considered as random variables with pdf's equal to $f_{X_{m,serv}}$ and $f_{X_{a,serv}}$, respectively. In this respect, Y_{aff} and Y_{true} must be considered as the conditional service fatigue lives given that $X_{a,serv} = x_{a,serv}$ and $X_{m,serv} = x_{m,serv}$ and they must be more properly denoted as $Y_{aff}\left|\left(X_{a,serv}, X_{m,serv}\right)\right.$ and $Y_{true}\left|\left(X_{a,serv}, X_{m,serv}\right)\right.$, respectively; similarly, the corresponding cdf's, $F_{Y_{aff}}$ and $F_{Y_{true}}$, must be more properly denoted as $F_{Y_{aff}|(X_{a,serv},X_{m,serv})}$ and $F_{Y_{true}|(X_{a,serv},X_{m,serv})}$, respectively.

If, according to the assumptions made for X_m and X_a in Section 2, $X_{m,serv}$ and $X_{a,serv}$ are supposed to be independent location-scale random variables or jointly Normal random variables, then the cdf's of Y_{aff} and Y_{true} can be easily calculated as:

$$F_{Y_{aff}}\left(y; \mu_{a,serv}, \sigma_{a,serv}, x_{a,syst}, \sigma_a, \mu_{m,serv}, \sigma_{m,serv}, x_{m,syst}, \sigma_m, \boldsymbol{\vartheta}_{true}\right) = \int_{\underline{x_{m,serv}}}^{\overline{x_{m,serv}}} f_{X_{m,serv}} \int_{\underline{x_{a,serv}}}^{\overline{x_{a,serv}}} f_{X_{a,serv}} \cdot F_{Y_{aff}|(X_{a,serv},X_{m,serv})} dx_{a,serv}\, dx_{m,serv} \quad (15)$$

and

$$F_{Y_{true}}\left(y; \mu_{a,serv}, \sigma_{a,serv}, \mu_{m,serv}, \sigma_{m,serv}, \boldsymbol{\vartheta}_{true}\right) = \int_{\underline{x_{m,serv}}}^{\overline{x_{m,serv}}} f_{X_{m,serv}} \int_{\underline{x_{a,serv}}}^{\overline{x_{a,serv}}} f_{X_{a,serv}} \cdot F_{Y_{true}|(X_{a,serv},X_{m,serv})} dx_{a,serv}\, dx_{m,serv}, \quad (16)$$

where $\underline{x_{a,serv}}$ and $\overline{x_{a,serv}}$ are the minimum and the maximum values of $X_{a,serv}$, $\underline{x_{m,serv}}$ and $\overline{x_{m,serv}}$ are the minimum and the maximum values of $X_{m,serv}$, $\mu_{a,serv}$ and $\sigma_{a,serv}$ are the location and scale parameters of $X_{a,serv}$, $\mu_{m,serv}$ and $\sigma_{m,serv}$ are the location and scale parameters of $X_{m,serv}$.

It must be noticed that, if the applied service stresses are deterministic values, then both $F_{Y_{aff}}$ and $F_{Y_{true}}$ are functions of the true applied service stresses, $x_{m,serv}$ and $x_{a,serv}$; while, if the applied service stresses are random variables, then both $F_{Y_{aff}}$ and $F_{Y_{true}}$ result to be functions of the location and scale parameters of $X_{m,serv}$, and of the location and scale parameters of $X_{a,serv}$.

Finally, the definitions of $R_{true,serv_y}$, $R_{aff,serv_y}$ and $y_{R_{serv},\%e}$ given in Equations (12)-(14) remain valid if both the applied mean stress and the applied alternating stress are random variables.

4. Special Normal case

Suppose that the applied mean stress is set equal to x_m and is kept constant for all the tested specimens, that a simple linear model with constant scatter is considered (the well-known Wöhler model [1,25]) for Y_{true} and that X_a and Y_{true} are Normal random variables [4,5], then Equation (11) reduces to:

$$F_{Y_{aff}}\left(y; x_{a,des}, x_{a,syst}, \sigma_a, \beta_0, \beta_1, \sigma_{true}\right) = \frac{1}{\sigma_a}\int_0^{+\infty} \phi\left(\frac{x_a-\left(x_{a,des}+x_{a,syst}\right)}{\sigma_a}\right) \Phi\left(\frac{y-(\beta_0+\beta_1 x_a)}{\sigma_{true}}\right) dx_a, \quad (17)$$

where β_0 and β_1 are constant parameters, and ϕ and Φ denote the standardized Normal pdf and cdf, respectively. It can be easily demonstrated (see Appendix A) that, in this particular case, Y_{aff} is distributed as a Normal random variable with mean $\beta_0 + \beta_1\left(x_{a,des} + x_{a,syst}\right)$ and standard deviation $\sigma_{aff} = \sqrt{\sigma_{true}^2 + \beta_1^2\sigma_a^2}$. Therefore, $F_{Y_{aff}}$ is given by:

$$F_{Y_{aff}}\left(y; x_{a,des}, x_{a,syst}, \sigma_a, \beta_0, \beta_1, \sigma_{true}\right) = \Phi\left(\frac{y-\left(\beta_0+\beta_1\left(x_{a,des}+x_{a,syst}\right)\right)}{\sqrt{\sigma_{true}^2+\beta_1^2\sigma_a^2}}\right). \quad (18)$$

4.1. Deterministic service load

According to what stated in Section 3.1, if the applied service alternating stress is deterministic and equal to $x_{a,serv}$, then Equation (13) becomes:

$$R_{aff,serv_y}\left(y; x_{a,serv}, x_{a,syst}, \sigma_a, \beta_0, \beta_1, \sigma_{true}\right) = 1 - \Phi\left(\frac{y-\left(\beta_0+\beta_1\left(x_{a,serv}+x_{a,syst}\right)\right)}{\sqrt{\sigma_{true}^2+\beta_1^2\sigma_a^2}}\right),$$

and Equation (12) reduces to:

$$R_{true,serv_y}\left(y; x_{a,serv}, \beta_0, \beta_1, \sigma_{true}\right) = 1 - \Phi\left(\frac{y-\left(\beta_0+\beta_1 x_{a,serv}\right)}{\sigma_{true}}\right).$$

Since both Y_{aff} and Y_{true} are Normal distributed, then the $(1 - R_{serv})$-th quantile of the distributions of Y_{aff} and Y_{true} can be respectively computed as:

$$y_{aff_{R_{serv}}} = \beta_0 + \beta_1\left(x_{a,serv} + x_{a,syst}\right) - z_{R_{serv}}\sqrt{\sigma_{true}^2 + \beta_1^2\sigma_a^2}, \quad (19)$$

being $z_{R_{serv}}$ the R_{serv}-th quantile of the standardized Normal distribution, and

$$y_{true_{R_{serv}}} = \beta_0 + \beta_1 x_{a,serv} - z_{R_{serv}} \sigma_{true}. \quad (20)$$

By changing the sign of $z_{R_{serv}}$ in Equations (19) and (20), the R_{serv}-th quantile of the distributions of Y_{aff} and Y_{true} can be easily computed.

By considering Equations (19) and (20) and by taking into account Equation (14), the service fatigue life percent error, with easy passages, reduces to:

$$y_{R_{serv},\%e} = 100\left(\sqrt{1 + \left(\frac{\beta_1}{\sigma_{true}}\right)^2 \sigma_a^2} - 1\right).$$

4.2. Random service load

Consider the case of an applied service alternating stress, $X_{a,serv}$, which is a random variable. Moreover, suppose that $X_{a,serv}$ is a Normal random variable, then the cdf's of Y_{aff} and Y_{true} can be obtained from Equations (15) and (16) and are equal to:

$$F_{Y_{aff}}(y; \mu_{a,serv}, \sigma_{a,serv}, x_{a,syst}, \sigma_a, \beta_0, \beta_1, \sigma_{true}) =$$
$$= \frac{1}{\sigma_{a,serv}} \int_0^{+\infty} \phi\left(\frac{x_{a,serv} - \mu_{a,serv}}{\sigma_{a,serv}}\right) \Phi\left(\frac{y - \left(\beta_0 + \beta_1 (x_{a,serv} + x_{a,syst})\right)}{\sqrt{\sigma_{true}^2 + \beta_1^2 \sigma_a^2}}\right) dx_{a,serv} \quad (21)$$

and

$$F_{Y_{true}}(y; \mu_{a,serv}, \sigma_{a,serv}, \beta_0, \beta_1, \sigma_{true}) =$$
$$= \frac{1}{\sigma_{a,serv}} \int_0^{+\infty} \phi\left(\frac{x_{a,serv} - \mu_{a,serv}}{\sigma_{a,serv}}\right) \Phi\left(\frac{y - (\beta_0 + \beta_1 x_{a,serv})}{\sigma_{true}}\right) dx_{a,serv}, \quad (22)$$

respectively. It can be demonstrated (see Appendix A) that Y_{aff} is distributed as a Normal random variable with mean $\beta_0 + \beta_1(\mu_{a,serv} + x_{a,sist})$ and standard deviation $\sqrt{\sigma_{true}^2 + \beta_1^2(\sigma_a^2 + \sigma_{a,serv}^2)}$, and Y_{true} is distributed as a Normal random variable with mean $\beta_0 + \beta_1 \mu_{a,serv}$ and scale parameter $\sqrt{\sigma_{true}^2 + \beta_1^2 \sigma_{a,serv}^2}$. Therefore, $R_{aff,serv_y}$ and $R_{true,serv_y}$ become:

$$R_{aff,serv_y}(y; \mu_{a,serv}, \sigma_{a,serv}, x_{a,syst}, \sigma_a, \beta_0, \beta_1, \sigma_{true}) = 1 - \Phi\left(\frac{y - \left(\beta_0 + \beta_1 (\mu_{a,serv} + x_{a,syst})\right)}{\sqrt{\sigma_{true}^2 + \beta_1^2 (\sigma_a^2 + \sigma_{a,serv}^2)}}\right) \quad (23)$$

and

$$R_{true,serv_y}(y; \mu_{a,serv}, \sigma_{a,serv}, \beta_0, \beta_1, \sigma_{true}) = 1 - \Phi\left(\frac{y - (\beta_0 + \beta_1 \mu_{a,serv})}{\sqrt{\sigma_{true}^2 + \beta_1^2 \sigma_{a,serv}^2}}\right), \quad (24)$$

respectively.

Since both Y_{aff} and Y_{true} are Normal distributed, then the $(1 - R_{serv})$-th quantile of the distributions of Y_{aff} and Y_{true} can be easily computed as:

$$y_{aff_{R_{serv}}} = \beta_0 + \beta_1(\mu_{a,serv} + x_{a,syst}) - z_{R_{serv}}\sqrt{\sigma_{true}^2 + \beta_1^2(\sigma_a^2 + \sigma_{a,serv}^2)} \qquad (25)$$

and

$$y_{true_{R_{serv}}} = \beta_0 + \beta_1\mu_{a,serv} - z_{R_{serv}}\sqrt{\sigma_{true}^2 + \beta_1^2\sigma_{a,serv}^2}. \quad (26)$$

Again, by changing the sign of $z_{R_{serv}}$ in Equations (25) and (26), the R_{serv}-th quantile of the distributions of Y_{aff} and Y_{true} can be easily computed.

By considering Equations (25) and (26) and by taking into account Equation (14), the service fatigue life percent error can be obtained with easy passages and is given by:

$$y_{R_{serv},\%e} = 100\left(\sqrt{1 + \frac{\left(\frac{\sigma_a}{\sigma_{true}/\beta_1}\right)^2}{1+\left(\frac{\sigma_{a,serv}}{\sigma_{true}/\beta_1}\right)^2}} - 1\right). (27)$$

In Equation (27), the coefficient $\sigma_{true}/|\beta_1|$ corresponds to the standard deviation of the true fatigue strength [4,6,26] of the material, $\sigma_{a,true}$. Similarly, the standard deviation of the affected fatigue strength of the material, $\sigma_{a,aff}$, can be defined as $\sigma_{a,aff} = \sigma_{aff}/|\beta_1|$. Since $\sigma_{aff} = \sqrt{\sigma_{true}^2 + \beta_1^2\sigma_a^2}$, it can be shown that:

$$\sigma_{a,true}^2 = (\sigma_{true}/\beta_1)^2 = \sigma_{a,aff}^2 - \sigma_a^2. \qquad (28)$$

By taking into account Equation (28), Equation (27) can be finally rearranged as:

$$y_{R_{serv},\%e} = 100\left(\sqrt{1 + \frac{1}{(\sigma_{a,aff}/\sigma_a)^2 + (\sigma_{a,serv}/\sigma_a)^2 - 1}} - 1\right). \quad (29)$$

The uncertainty in the applied alternating service stress is quantified by $\sigma_{a,serv}$, in Equation (29). In practical applications, $\sigma_{a,serv}$ is generally larger than σ_a and, consequently, $\sigma_{a,serv}/\sigma_a$ is generally larger than 1. On the other end, if the applied alternating service stress is characterized by a too large standard deviation, reliability predictions should be made by adopting proper methods devoted to deal with variable amplitude loadings (i.e., damage accumulation rules must be considered in these cases). In fact, the approach proposed in the paper does not take into account any damage accumulation rule and is therefore correctly applicable only if the applied alternating service stress can be considered as constant. In this respect, it can be assumed that an applied alternating stress with standard deviation smaller than the true standard deviation of the material can be considered as constant and, as a consequence, for a correct application of Equation (29),

$\sigma_{a,serv}$ must be considered upper limited by $\sigma_{a,true}$. In particular, according to Equation (28), if $\sigma_{a,serv} \leq \sigma_{a,true}$ then $\sigma_{a,serv}^2 \leq \sigma_{a,aff}^2 - \sigma_a^2$ or, equivalently:

$$\sigma_{a,aff}^2 \geq \sigma_{a,serv}^2 + \sigma_a^2. \qquad (30)$$

By taking into account Equation (29), Equation (30) finally yields:

$$y_{R_{serv},\%e} \leq 100\left(\sqrt{1 + \frac{1}{2\left(\sigma_{a,serv}/\sigma_a\right)^2}} - 1\right), \quad (31)$$

where the maximum value of $y_{R_{serv},\%e}$ is equal to 22.5% and is achieved for $\sigma_{a,serv} = \sigma_a$.

Figure 1 depicts, for different values of the ratio $\sigma_{a,aff}/\sigma_a$ (ranging from 1.5 to 10), the variation of $y_{R_{serv},\%e}$ with respect to the ratio $\sigma_{a,serv}/\sigma_a$. As shown in Figure 1, for values of $\sigma_{a,aff}/\sigma_a$ larger than 3.1, $y_{R_{serv},\%e}$ is always smaller than 5%, while for values of $\sigma_{a,aff}/\sigma_a$ smaller than 1.8, $y_{R_{serv},\%e}$ is always larger than 10%.

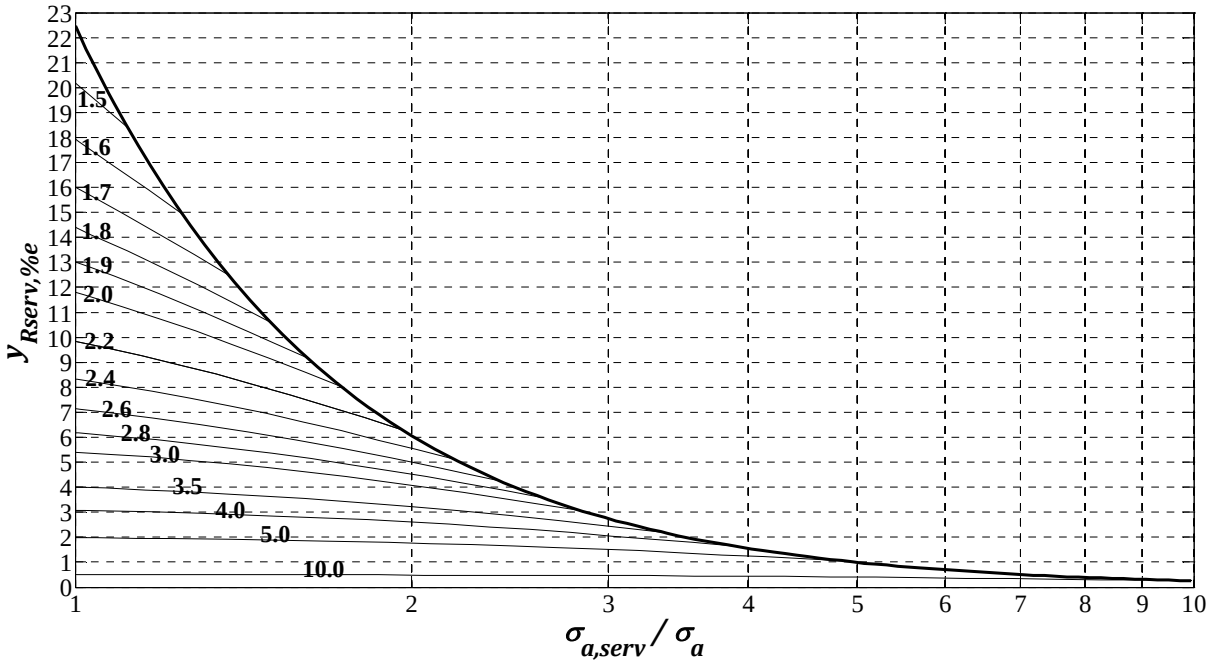

Figure 1: Plot of the service fatigue life percent error with respect to the ratio $\sigma_{a,serv}/\sigma_a$ for different values of the ratio $\sigma_{a,aff}/\sigma_a$. The thick line limits the region of error evaluation.

Indeed, if the ratio $\sigma_{a,aff}/\sigma_a$ is small enough then the uncertainty in the applied alternating stress significantly contributes to the scatter of the test results. In particular, if $\sigma_{a,aff}/\sigma_a$ approaches 1 then the scatter in test results is totally due to the uncertainty in the applied alternating stress and does not depends on the material variability. As depicted in Figure 1, in the domain of application of Equation (29) (i.e., in the area below the thick line representing Equation (31)), the percent error $y_{R_{serv},\%e}$ is monotone decreasing with $\sigma_{a,serv}/\sigma_a$ and the variation increases for decreasing values of the ratio $\sigma_{a,aff}/\sigma_a$.

It is worth noting that, following Equation (29), $y_{R_{serv},\%e}$ does not depend on the systematic errors in the applied load during testing. Indeed, $y_{R_{serv},\%e}$ is directly related to accidental effects (e.g., specimen geometry and orientation) that cannot be corrected and it thus corresponds to the minimum error occurring after elimination or correction of every systematic contributions.

5. Example

In [17,18], axial fatigue tests were performed by six laboratories according to [16], with stress ratio equal to $.1$, at three different nominal maximum stress levels, namely 460 MPa, 430 MPa and 400 MPa, with four specimens at each stress level, at a test frequency between 10 and 30 Hz, with a run-out limit at $5 \cdot 10^6$ cycles and in a normal laboratory climate. Specimens were cut from SS 1650 steel plates with thickness equal to 6 mm and width equal to 26 mm.

As explained in Section 2, the applied stress levels may be different from the nominal ones, since they are generally affected by a number of experimental uncertainties.

For sake of simplicity, the minimum applied stress level is supposed to be a deterministic value equal to the desired nominal minimum stress level: e.g., if a test is run at a nominal maximum stress level equal to 460 MPa, then the minimum applied stress level is exactly equal to 46 MPa. The maximum applied stress level, X_{MAX}, is instead supposed to be affected by the experimental uncertainties cited in [16]. In particular, according to [16], "*The varying stress amplitude […] should be maintained at all times within 2% of the desired test value.*" (i.e., $X_{axl} = (1 \pm 1\%)x_{des}$, where X_{axl} is the applied axial stress and x_{des} is the desired axial stress level), and "*The bending stresses […] should be limited to less than 5% of the greater of the range, maximum or minimum stresses […]*" (i.e., by assuming that the bending stresses, X_{bnd}, can be equally subdivided in a systematic effect due to the machine misalignment and in an accidental effect due to the specimen orientation, $X_{bnd} = (2.5\% \pm 2.5\%)x_{des}$).

By taking into account the recommendations given in [16] and by applying the procedure specified in [19], it is possible to compute the uncertainty of the maximum applied stress level, as shown in Table 1.

Table 1: Computation of X_{MAX} uncertainty according to [19].

Source	Symbol X_i	Type	Value x_i	Range Δ_i	Probability distribution	Divisor d_i	Sensitivity coefficient $c_i = \partial X_{MAX}/\partial x_i$	$u_i(X_{MAX}) = c_i\,\Delta_i/d_i$
Axial stress	X_{axl}	B	$x_{des} \pm 1\%\ x_{des} =$ $= (460 \pm 4.6)$ MPa	0	-	-	1	0
Bending stress	X_{bnd}	B	$2.5\%\ x_{des} =$ $= 11.5$ MPa	$\pm 2.5\%\ x_{des} =$ $= \pm 11.5$ MPa	Normal	3	1	$\frac{.025}{3} x_{des} =$ $= 3.8$ MPa
	$X_{MAX} = X_{axl} + X_{bnd}$		$x_{MAX} = 466.9$ MPa		Normal	Combined Standard Uncertainty $u_{X_{MAX}} = \sqrt{u_{X_{axl}}^2 + u_{X_{bnd}}^2}$		3.8 MPa

Note: Computations are made by assuming $x_{des} = 460$ MPa and the mean value of X_{axl} equal to $x_{des} - 1\%\ x_{des}$.

Computation is carried out by optimistically assuming that the uncertainty due to X_{axl} is entirely attributable to the testing machine (i.e., X_{axl} contributes systematically to the uncertainty of X_{MAX}) and that the systematic effect due to X_{axl} compensates the systematic effect due to X_{bnd}.

Therefore, according to Table 1 and by considering as an example $x_{des} = 460$ MPa, X_{MAX} is finally supposed to be Normal distributed with mean equal to 467 MPa (i.e., $\mu_a = 467$ MPa) and standard deviation equal to 3.8 MPa (i.e., $\sigma_a = 3.8$ MPa).

Failure data obtained during the fatigue test inter-laboratory comparison are depicted in Figure 2.

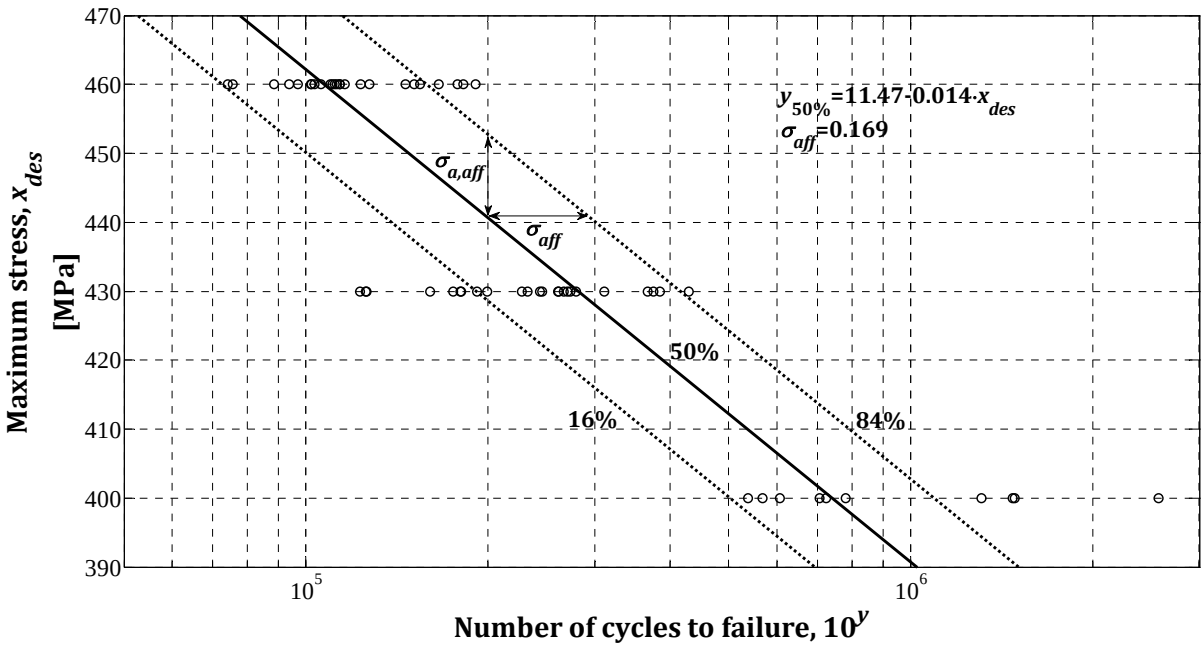

Figure 2: Semi-logarithmic plot of experimental data taken from [17] and estimated linear fits.

Linear regression allows to obtain the estimates of β_0, β_1, and of the standard deviation of the experimental data. Since data are supposed to be affected by a neglected uncertainty in the maximum applied stresses, the standard deviation obtained with linear regression is, indeed, an estimate of σ_{aff}, rather than of σ_{true}.

In particular, by setting the parameter values equal to the corresponding estimates (i.e., $\beta_1 = -0.014$ and $\sigma_{aff} = .169$), it is possible to obtain an estimate of $\sigma_{a,aff}$ as follows:

$$\sigma_{a,aff} = \sigma_{aff}/|\beta_1| = 12.1 \text{ MPa}.$$

In case of a Normal distributed applied service stress with mean $\mu_{a,serv} = 460$ MPa and standard deviation $\sigma_{a,serv}$ equal to σ_a, the maximum value of $y_{R_{serv},\%e}$ can be computed (Equation (29)):

$$y_{R_{serv},\%e} = 100\left(\sqrt{1 + \frac{1}{(12.1/3.8)^2}} - 1\right) = 4.9\%.$$

It can be approximately estimated through Figure 1 by conservatively considering the curve corresponding to $\sigma_{a,aff}/\sigma_a = 3.0$ and $\sigma_{a,serv}/\sigma_a = 1$. In case $\sigma_{a,serv} = \sigma_{a,true} = \sqrt{\sigma_{a,aff}^2 - \sigma_a^2}$, the minimum value of $y_{R_{serv},\%e}$ can be computed (Equation (31)):

$$y_{R_{serv},\%e} = 100\left(\sqrt{1 + \frac{1}{2((12.1/3.8)^2-1)}} - 1\right) = 2.8\%.$$

It can be approximately estimated also through Figure 1 by conservatively considering the intersection between the curve corresponding to $\sigma_{a,aff}/\sigma_a = 3.0$ and the thick line.

Through Equations (23) and (24) and according to the computation made in Table 1 (i.e., by setting $x_{a,syst} = 11.5 - 4.6 = 6.9$ MPa), the values of $R_{aff,serv_y}$ and $R_{true,serv_y}$ can be calculated. The reliabilities in service are obtained for a fatigue life y equal to $\log_{10} 60000$ and for an applied service stress with $\mu_{a,serv} = 460$ MPa and with different values of $\sigma_{a,serv}$ ranging from σ_a to $\sigma_{a,true}$.

Figure 3 shows the variation of $R_{aff,serv_y}$ (black line) and $R_{true,serv_y}$ (grey line) with respect to the value of the standard deviation of the applied service stress.

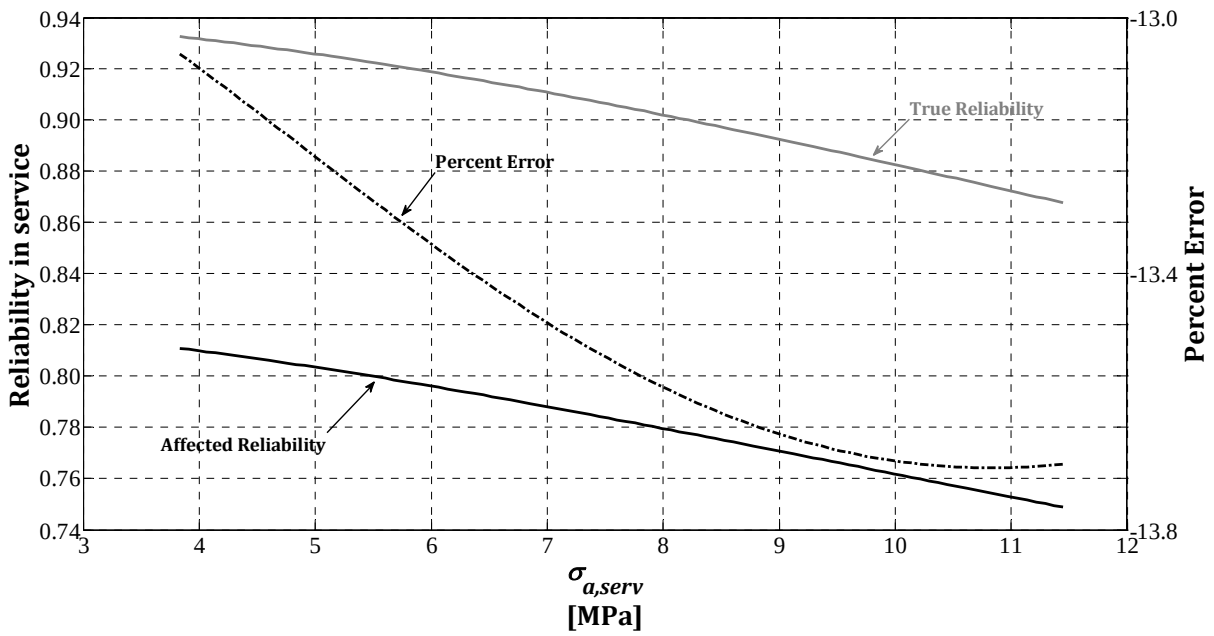

Figure 3: Reliability in service corresponding to 60000 cycles to failure for different values of $\sigma_{a,serv}$

In particular, Figure 3 shows that the affected reliability significantly suffers from the uncertainty in the applied load during testing: the percent error in the evaluation of the reliability in service (i.e., $100\left(R_{aff,serv_y} - R_{true,serv_y}\right)/R_{true,serv_y}$) is negative (i.e., predictions are conservative) with absolute values larger than 13%.

As a consequence, if recommendations of [16] are strictly followed and if it is optimistically supposed that systematic errors due to the machine misalignment compensate calibration errors of the testing equipment, computation of reliability in service suffers from large inaccuracy. These errors cannot be neglected and slightly increase with $\sigma_{a,serv}$.

Differently from what pointed out by Figure 3, the percent error $y_{R_{serv},\%e}$ is smaller than 5%. Therefore, according to $y_{R_{serv},\%e}$, it could be concluded that recommendations of [16] are sufficient for accurate reliability predictions. However, it is worth to recall that, following Equation (29), $y_{R_{serv},\%e}$ does not depend on the systematic errors in the applied load during testing. Indeed, $y_{R_{serv},\%e}$ is directly related to accidental effects (e.g., specimen geometry and orientation) that cannot be corrected and it thus corresponds to the minimum error occurring after elimination or correction of every systematic contributions.

6. Conclusions

In fatigue data analysis, different sources of scatter should be taken into account: scatter related to specimen manufacturing, to fatigue test conditions (e.g., test rig and clamping), to the material itself, and to the applied stress. The latter scatter source has not been commonly considered in the analysis, even if it can play a major role.

The paper showed the procedure that should be applied in order to evaluate the error on the parameters usually adopted to determine in service reliability of structural components, due to the uncertainty in the applied stress. An exact and an approximated simplified equation are proposed.

If recommendations imposed by the ASTM standard E 466 [16] are taken into account for determining the uncertainty on the applied stress during fatigue testing, computation of reliability in service undergoes large errors, that can be significantly reduced only if systematic contributions (e.g., calibration of load cell and testing machine misalignment) are properly eliminated or corrected. If systematic contributions are taken into account, then computation of reliability in service suffers uniquely from errors due to accidental contributions in the applied load during testing (e.g., specimen geometry and orientation). In any case, errors in computation of reliability in service reduce if scatter in the applied service increases. However, if the applied service stresses are characterized by a too large coefficient of variation, reliability should be predicted by adopting proper methods developed to deal with variable amplitude random fatigue.

Appendix A.
Evaluation of the cdf's given in the right-hand side of Equations (18), (23) and (24)

Let X be a Normal random variable with mean μ_X and standard deviation σ_X, then the pdf of X, f_X, is given by:

$$f_X = \frac{1}{\sigma_X}\phi\left(\frac{x-\mu_X}{\sigma_X}\right). \qquad \text{(A.1)}$$

Let $Y|X$ be a Normal random variable with mean $\mu_{Y|X}$ and standard deviation $\sigma_{Y|X}$. Suppose that $\mu_{Y|X}$ linearly depends on the value, x, assumed by the random variable X:

$$\mu_{Y|X} = \beta_0 + \beta_1 x, \qquad \text{(A.2)}$$

where β_0 and β_1 are two constant coefficients. As a consequence the cdf of $Y|X$, $F_{Y|X}$, is given by:

$$F_{Y|X} = \Phi\left(\frac{y-(\beta_0+\beta_1 x)}{\sigma_{Y|X}}\right). \qquad \text{(A.3)}$$

Since the Normal distribution is a location-scale distribution, both X and $Y|X$ are linear transformations of standard Normal random variables as follows:

$$X = \mu_X + \sigma_X Z_X, \qquad \text{(A.4)}$$

and

$$Y|X = (\beta_0 + \beta_1 x) + \sigma_{Y|X} Z_{Y|X}, \qquad \text{(A.5)}$$

where Z_X and $Z_{Y|X}$ are two independent, standard Normal random variables and $\mu_{Y|X}$ has been substituted by $\beta_0 + \beta_1 x$, according to Equation (A.2).

If, in Equation (A.5), the value x is replaced by the random variable X, then the random variable $Y|X$ is no more conditioned by the specific value assumed by X. In this respect, $Y|X$ turns out to be the marginal random variable Y, which can be expressed as:

$$Y = (\beta_0 + \beta_1 X) + \sigma_{Y|X} Z_{Y|X}, \text{(A.6)}$$

and the cdf of Y, F_Y, corresponds to the marginal cdf of $Y|X$:

$$F_Y = \frac{1}{\sigma_X}\int_{-\infty}^{+\infty} \phi\left(\frac{x-\mu_X}{\sigma_X}\right)\Phi\left(\frac{y-(\beta_0+\beta_1 x)}{\sigma_{Y|X}}\right)dx, \quad \text{(A.7)}$$

where f_X and $F_{Y|X}$ are replaced by the expressions given in Equations (A.1) and (A.3).

By replacing Equation (A.4) in Equation (A.6), Y finally becomes:

$$Y = (\beta_0 + \beta_1 \mu_X) + \beta_1 \sigma_X Z_X + \sigma_{Y|X} Z_{Y|X}. \qquad \text{(A.8)}$$

Since Z_X and $Z_{Y|X}$ are two independent, standard Normal random variables, the linear combination $\beta_1\sigma_X Z_X + \sigma_{Y|X} Z_{Y|X}$ in Equation (A.8) is Normal distributed with mean equal to 0 and standard deviation equal to $\sqrt{(\beta_1\sigma_X)^2 + \sigma_{Y|X}^2}$. Therefore, Y can be expressed as:

$$Y = (\beta_0 + \beta_1\mu_X) + Z\sqrt{\beta_1^2\sigma_X^2 + \sigma_{Y|X}^2}, \qquad \text{(A.9)}$$

where Z is a standard Normal random variable. Since, according to Equation (A.9), Y is a linear transformation of Z, it follows that Y is a Normal random variable with mean equal to $\beta_0 + \beta_1\mu_X$ and standard deviation equal to $\sqrt{\beta_1^2\sigma_X^2 + \sigma_{Y|X}^2}$. Consequently, the cdf of Y, F_Y, is finally given by:

$$F_Y = \Phi\left(\frac{y-(\beta_0+\beta_1\mu_X)}{\sqrt{\beta_1^2\sigma_X^2+\sigma_{Y|X}^2}}\right). \qquad \text{(A.10)}$$

If, in Equations (A.7) and (A.10), μ_X is equal to $\left(x_{a,des} + x_{a,sist}\right)$, σ_X is equal to σ_a and $\sigma_{Y|X}$ is equal to σ_{true}, then the cdf in Equation (18) follows. If, in Equations (A.7) and (A.10), μ_X is equal to $\left(\mu_{a,serv} + x_{a,sist}\right)$, σ_X is equal to $\sigma_{a,serv}$ and $\sigma_{Y|X}$ is equal to $\sqrt{\sigma_{true}^2 + \beta_1^2\sigma_a^2}$, then the cdf in the right-hand side of Equation (23) follows. Finally, if, in Equations (A.7) and (A.10), μ_X is equal to $\mu_{a,serv}$, σ_X is equal to $\sigma_{a,serv}$ and $\sigma_{Y|X}$ is equal to σ_{true}, then the cdf in the right-hand side of Equation (24) follows. It is worth noting that, since the applied alternating stress cannot be negative, in Equations (17), (21) and (22) the lower limit of integration is equal to 0 rather than to $-\infty$ as in Equation (A.7). However, if the pdf of X is assumed to tend to 0 for negative values of X (i.e., the probability of having negative values approaches 0), then setting the lower limit of integration in Equation (A.7) equal to 0 gives raise to negligible approximations which do not modify the obtained results.

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