BLACK HOLES IN SUPERGRAVITY AND HAMILTON-JACOBI FORMALISM

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BLACK HOLES IN SUPERGRAVITY AND HAMILTON-JACOBI FORMALISM

Supervisor
Riccardo D'Auria

Candidate:
Paolo Giaccone

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Chapter 1

Introduction

The physics of black holes[1, 2], with its phenomenological and theoretical implications, has a great impact on many branches of natural science, such as: cosmology, astrophysics, particle physics, mathematical physics and quantum information theory[3, 4]. This is not so strange in view of the fact that, owing to "The singularities of gravitational collapse and cosmology" of S. W. Hawking and R. Penrose[5], the existence of black holes seems to be a consequence of Einstein's theory of general relativity[3] and its generalizations such as supergravity[6, 7, 8], superstrings and M-Theory[9, 10].

A fundamental aspect of black hole physics is in their thermodynamic properties that seem to encode fundamental insights of a not yet "discovered final theory" of quantum gravity. In this context a "fascinating role" is played by "Bekenstein-Hawking", or $B - H$ (o BH) entropy formula [11, 12]:

$$S_{BH} = \frac{k_B}{\ell_p^2}\frac{A}{4}; \tag{1.1}$$

where $\ell_p^2 = G\hbar/c^3$ is the squared Planck length and k_B is Boltzmann's constant. The subscript BH stands either for "Bekenstein-Hawking" or "black hole". The black hole entropy is proportional to the area of its event horizon A.

This relation between a geometric quantity (A) and a thermodynamic quantity (S_{BH}) is a fundamental aspect that motivated much theoretical work in the last decades. In fact a microscopic statistical explanation of BH entropy formula, related to microstate counting, has been regarded as possible only within a satisfactory and consistent formulation of quantum gravity [3]. Superstring theory is a serious candidate for a theory of quantum gravity and, as such, should eventually provide such a microscopic explanation of the entropy formula [13, 14, 15, 16]. Since black holes are a nonperturbative phenomonon, perturbative string theory could say very little about their entropy. Progress in this direction came by the early 1990s [17], through the recognition of the role of string dualities [3]. These dualities allow one to relate the strong coupling regime of one string model to the weak coupling regime of another. String duality is a class of symmetries in physics that link different superstring theories. Interestingly enough, there is evidence that the non-perturbative and perturbative string dualities are all encoded in the global symmetry group, called the U-duality group, of the low energy supergravity effective action [18].

Let us introduce the topic of spherically symmetric, asymptotically flat extremal black hole solutions in supergravity. This theory has a history of almost twenty years. In the mid nineties a broad interest was raised by two discoveries:

- The attractor mecchanism [19, 20] in BPS black holes, where the scalar fields of the supergravity flow to fixed values Φ^a_{fix} at the event-horizon, independent of the boundary values at infinity Φ^a_∞ and solely determined by the eletromagnetic dyonic charges $Q = \{P^\Lambda, Q_\Lambda\}$, $(\Lambda = 1, \ldots, N)$, of the N gauge fields of the theory [21, 20, 22, 23]. The area of the horizon, $A_H = A_H(Q)$ is a function of the quantized charges only [24]:

$$A_H(Q) = 4\pi\sqrt{|I_4(Q)|}; \tag{1.2}$$

  where $I_4(Q)$ is a certain quadratic invariant of the duality group U and of the dyonic charge vector Q and depends on the particular theory under consideration;

- The statistical interpretation of black hole entropy. The horizon area of BPS black holes can be interpreted as:

$$\frac{A_H(Q)}{4\ell_p^2} = log(N_s); \tag{1.3}$$

  where N_s denotes the number of microstates that correspond to the same classical solution of the effective supergravity Lagrangian [21, 13]. In the context of superstring theory, the microstates are given by the possible superstring configurations corresponding to the given effective supergravity description of the black hole.

These two points have a strong conceptual link pivoted around the interpretation of the entropy as the square root of the quartic invariant. In view of these perspectives the search and analysis of BPS black hole solutions was extensively pursued in the nineties in all versions of supergravity [25, 26, 27, 28, 21]. A basic tool in these theories was the use of the first order Killing spinor equations obtained by imposing that a fraction of the original supersymmetry should be preserved by the classical solution [29, 30, 31].

The bridge between the two aspects of the theory, namely the microscopic and the macroscopic one, was constantly provided by the algebraic and geometric structure of supergravity theories dictating the properties of the supersymmetry field dependent central charges Z^A and of the U-duality [21]. In this context the most investigated case of study was that of $N = 2$ supergravity where the geometric structure of scalar sector, i.e. special Kähler geometry [32, 33, 34, 35, 36], provides a mathematical framework to formulate and investigate all the fundamental questions about black hole construction and properties [21].

Renewed interest in the topics of supergravity black holes and a new wave of extended research activities developed in the last decade as soon as it was realized that the attractor mechanism is not limited to the BPS black holes but occurs also for the non BPS ones [37, 38, 39, 40, 41, 3]. In this situation there emerged the concept of fake superpotential, that is a function, $\mathcal{W}$, also named prepotential [24, 42, 43, 44, 45]. A first order description of the scalar fields coupled to asymptotically flat black holes in supergravities, called “gradient-flow” was considered, using the identification of the function $\mathcal{W}$ with Hamilton’s characteristic function of a corresponding Hamiltonian formulation of the effective one-dimensional theory. The first order differential equations obtained by gradient-flow equations for static, spherically symmetric black holes take the form [42, 43]:

$$\frac{d\phi^a}{dr} = G^{ab}(\phi)\left(\frac{\partial\mathcal{W}}{\partial\phi^b}\right), \tag{1.4}$$

where the function $\mathcal{W}$ and the symmetric tensor G_{ab} are suitable real functions of the scalar fields. Let us observe that the evolution variable here is the radial variable r. [43, 42, 24, 46, 47, 48, 49].

One of the most significant significant points in these new developments is that equation (1.4) is reminiscent of the Hamilton-Jacobi (H-J) formulation of classical mechanics. This fact was first observed and exploited in the paper “First order description of black holes in moduli space” by L. Andrianopoli, R. D’Auria, E. Orazi and M. Trigiante [42] in the context of supergravity black holes to derive general properties of function $\mathcal{W}$ like its duality invariance [43, 42, 24, 46, 47, 48, 49]. Considering the radial variable as a “time”, the prepotential plays the role of Hamilton principal function of an associated Hamiltonian system, while the set of fields $\phi^a = \{\varphi^a; \frac{\partial\mathcal{L}}{\partial\partial_r\varphi^a}\}$ is assimilated to the coordinates of phase-space [42]. This opens an entirely new perspective on the nature of the black hole construction problem[21]. Indeed the existence of the function $\mathcal{W}$, alias Hamilton–Jacobi equation, is guaranteed for a system of $2n$ variables ϕ^a equipped with an underlying Poisson structure, namely with a Poisson bracket [21]:

$$\{\phi^a, \phi^b\} = -\{\phi^b, \phi^a\}; \tag{1.5}$$

if this latter is Liouville integrable, namely if there exist n Hamilton functions $\mathfrak{H}^\alpha(\phi)$ in involution:

$$\{\mathfrak{H}^\alpha, \mathfrak{H}^\beta\} = 0 \;\; \forall\, \alpha\,,\, \beta\,; \tag{1.6}$$

whose set includes the "Hamiltonian" $\mathfrak{H}_0$ defining the field equations of the dynamical sistem[21]:

$$\frac{d\phi^a}{dr} = \{\mathfrak{H}_0, \phi^a\} \,. \tag{1.7}$$

Naturally, in order for the remarks to make sense, the crucial issue is the existence of a Poissonian equations and of a Hamiltonian allowing to recast the supergravity field equations into the form of a "classical dynamical system". From a physical point of view, the first order description of static, spherically symmetric black holes appeared as a very appropriate tool to study extremal solutions, supersymmetric or not, which exhibit an attractive behavior, but also as a powerful tool to better understand solutions out of the extremality.

Our main aim in the present thesis will be to extend the first order description of black holes to more general solutions, in particular to axisymmetric black holes far from extremality and also to analyze their extremal limit.

A peculiarity of static, spherically symmetric solutions is that one can exploit the symmetries to reduce the Lagrangian to a one-dimensional effective Lagrangian, where the evolution variable is the radial one [23, 50]. However, when considering four dimensional solutions with less symmetries, in particular stationary solutions where only the time-like Killing vector ∂_t is present, an effective three-dimensional Lagrangian can be obtained upon compactification along the time coordinate [51, 52, 53, 54, 21, 55, 56, 57]. The fields in the effective Lagrangian now depend on the three space variables x^i, $(i = 1, 2, 3)$. In particular, for stationary axisymmetric solutions, the presence of an azimuthal angular Killing vector ∂_φ allows a further dimensional reduction to two dimensions.

An important issue in my research work was to extend the Hamilton-Jacobi formalism from mechanical models, whose degrees of freedom depend on just one variable, to field theories where the degrees of freedom (the fields) depend on two or more variables. This problem was addressed and developed in generality in field theory from several points of view (a useful review is given by [58, 59]), but not much was known in the context of gravitational theories. Our main aim in the present thesis is to apply such extended formalism to the study of black holes. We will adhere to the so-called De Donder-Weyl-Hamilton-Jacobi theory, hereafter referred to as DWHJ, which is the simplest extension of the classical Hamilton-Jacobi approach in mechanics [58, 59]. One important difference with respect to the case of classical mechanics consists in the replacement of the Hamilton principal function S, directly related to the fake-superpotential of static black holes, with a *Hamilton principal* 1*-form*, that is with a covariant vector S_i. In this case the issue of integrability is more involved than in mechanics since to find solutions to the Euler–Lagrange equations strong constraints, which are trivial in the one dimensional case, have to be imposed on the vector S_i.

A first achievement in my thesis is to formulate the physics of rotating black holes (Kerr, Kerr-Newman or their extensions in the presence of scalar matter) in terms of an effective two dimensional Euclidean Lagrangian, whose independent variables are the radial variable r and the angular variable θ. It is particularly useful to formulate the theory in such a way that all the propagating degrees of freedom have been reduced to scalars by use of 3D Hodge-dualization [51]. In this way, the effective 3-dimensional Lagrangian has the form of a non linear sigma model, whose scalars include the degrees of freedom of the space-time metric and of the electric and magnetic components of the gauge vectors. Note that the effective three-dimensional description of axisymmetric black holes:

- allows a simplified effective Lagrangian description of the physics and consequently also a Hamiltonian one [59, 18, 56];
- the scalars ϕ^r parametrize a tangent space with metric $G_{IJ}(\Phi)$ of indefinite signature [60], since the kinetic terms of the degrees of freedom corresponding to four-dimensional vector fields contribute to the σ-model with negative-definite terms;
- the third important consequence of the lower dimensional description is that the isometry group $G_{(3)}$ of the σ-model metric $G_{ab}(z)$ contains as non trivial subgroups the 4-dimensional U-duality group $G_{(4)}$ times the

group $SL(2,\mathbb{R})$ (the Ehlers group) under which the degrees of freedom of the 4d metric transform. The simplest 3D model is the one originating form a pure 4D Einstein–Maxwell gravitational theory with a single time-like Killing vector. In this case $G_{(4)} = \mathrm{U}(1)$ and the 3D σ-model has the homogeneous-symmetric target space:

$$\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)}\,. \tag{1.8}$$

Its field content consists of four scalars belonging to a non-compact version of the universal hypermultiplet, dubbed the universal *pseudo-hypermultiplet* [59].

The thesis is organized as follows:
In Chapter 2 we will discuss rotating and non-rotating black hole solutions of gravity theory, including an introduction to the thermodynamics of black holes[61, 62, 1, 2, 7, 63]. It is shown how familiar concepts such as temperature and entropy apply to systems containing black holes. The thermodynamic connection is based on Hawking's celebrated application of quantum theory to black holes[11, 12].
In Chapter 3 we report on the main features of the physics of extremal, static and spherically symmetric black holes embedded in supersymmetric theories of gravitation. In particular, we present a detailed derivation of the effective one-dimensional Lagrangian, which encodes the dynamics of this class of solution.
In Chapter 4 we present the application of the Hamilton-Jacobi equation to the first order description of four dimensional static and spherically symmetric black holes. In particular we show that the prepotential characterizing the flow coincides with the Hamiltonian principal function associated with the one-dimensional Lagrangian[24].
In Chapter 5, which contains the main results reached during this thesis work, we present the extension of the Hamilton-Jacobi theory to field theory, following the DWHJ approach, and give a general formula to find the Hamilton principal 1-form; our main focus is on stationary axisymmetric black

holes, whose description, following [51], is two dimensional. We review the construction of the two-dimensional effective Lagrangian and the expression of the characteristic physical quantities associated with the four-dimensional solution in terms of Nöether currents of the 3D sigma-model. We also write the angular momentum in terms of the 3D sigma-model Nöether currents and introduce, besides Q, the matrix Q_ψ, which allows to describe in a $G_{(3)}$-invariant fashion the rotational properties of the solution. We also discuss the *under-rotating* extremal limit of a non-extremal solution in the $G_{(3)}$-orbit of the Kerr-black hole. Then we find a manifestly (three-dimensional) duality invariant expression for the principal functions S_m $(m = 1, 2)$. Finally we restrict our attention to the KN-Taub-NUT solution, making use of the so called Ernst potentials written in terms of the inhomogeneous fields (u, v) parametrizing the $\mathrm{SU}(1,2)/\mathrm{U}(1,1)$ coset and give the explicit form of the principal functions S_m in terms of the fields and the two-dimensional spherical coordinates.

We end in Chapter 6 with some concluding remarks.

In Appendix A we introduce the idea of duality in supergravity theory [64, 65, 8, 66, 67]. In Appendix B we will discuss the Taub-NUT solution, while Appendix C contains the explicit form of the algebra $\mathrm{SU}(1,2)$. In Appendix D we present the surface gravity, and finally in Appendix E we discuss the geometry of the Special Kähler manifold of the $D = 4$, $N = 2$ model.

Chapter 2

Black holes and black hole thermodynamics

This chapter includes an introduction to the thermodynamics of black holes[61, 62, 1, 2, 7, 63]. It is shown how familiar concepts such as temperature and entropy apply to systems containing black holes. The thermodynamic connection is based on Hawking's celebrated application of quantum theory to black holes[11, 12].

In the first part of the chapter, we will discuss rotating and non-rotating black hole solutions. A rotating black hole is a black hole that possesses spin angular momentum[62, 1, 2, 7, 63]. There are four known black hole solution to Einstein's equations, which describe gravity coupled to electromagnetic in general relativity. Two of these, the Kerr and Kerr-Newman black holes, rotate. We will see that stable black holes can be completely described by these quantities:

- mass-energy;
- linear momentum;
- angular momentum;
- electric charge[1].

[1]Electric charge, or else more generally electric and magnetic charge.

In the second part we present the concept of black hole thermodynamics. In physics, thermodynamics of black holes is an area of study that seeks to reconcile the laws of thermodynamics with the existence of black holes event horizons[61, 1, 2, 63]. Much as the study of statistical mechanics of black body radiation led to the advent of the theory of quantum mechanics, the effort to understand the statistical mechanics of black holes has had a deep impact upon the understanding of quantum gravity, leading to the formulation of the holographic principle[68, 69, 70, 71].

2.1 Types of black holes

Black holes are classical solutions of Einstein-Maxwell equations defined by a spacetime metric asymptotically flat with a singularity hidden by an event horizon. There are four known black hole solutions to Einstein-Maxwell field equations, which describe gravity in general relativity. Two of these, the Kerr and Kerr-Newman black holes, rotate; and, by no-hair theorem, any stable black holes can be completely described these quantities:

- mass-energy;
- linear momentum;
- angular momentum;
- electric charge.

These quantities represent the conserved attributes of a physical body and can be determined by examining its gravitational and electromagnetic field by are asymptotically distant observer. All the other physical quantities of the black hole will either escape to infinity or be swallowed up by the black hole. This is because anything happening inside the back hole horizon can not affects events outside it[72, 73].

In terms of these physical properties, the four types of stable black holes are:

	Nonrotating ($J = 0$)	Rotating ($J > 0$)
Uncharged ($Q = 0$)	Schwarzschild	Kerr
Charged ($Q \neq 0$)	Reissner-Nordström	Kerr-Newman

The following subsections will analyze these four types of black holes.

2.1.1 Schwarzschild black hole

According to Birkhoff's theorem, the Schwarzschild vacuum is the most general spherically symmetric, vacuum solution of the Einstein field equations[7]. A Schwarzschild black hole is a black hole that has no angular momentum or charge. A Schwarzschild black hole has a Schwarzschild metric, and can not be distinguished from any other Schwarzschild black hole except by its mass[62, 1, 2, 7].

The Schwarzschild black hole is characterized by a surrounding surface, which is spherical, called the event horizon which is situated at the Schwarzschild radius. The Schwarzschild radius is often called the radius of the black hole[62, 1, 2, 7].

In the polar spherical coordinates, the Schwarzschild metric has the form[62, 1, 2, 7]:

$$ds^2 = c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\varphi^2) \quad (2.1)$$

where:

- τ is the proper time[2];

[2] In relativity, proper time is time measured by an ideal clock that is carried along with a specified particle, and is based on the invariant timelike spacetime intervals between points along the particle's trajectory[7].

- c is the speed of light;
- t is the time coordinate;
- r is the radial coordinate;
- θ is the colatitude;
- φ is the longitude;
- r_s is the Schwarzschild radius, of the "massive body" which is related to its mass M by:

$$r_s = \frac{2GM}{c^2} \tag{2.2}$$

- G is the gravitational constant[3].

The Schwarzschild solution appears to have two singularities at $r = 0$ and $r = r_s$. Since the Schwarzschild metric is only expected to be valid for radii larger than the radius R of the gravitational body, there is no problem as long as $R > r_s$. One could naturally wonder what happens when the radius R becomes equal or less then to the Schwarzschild radius r_s; one can prove that the Schwarzschild solution still makes sense in this case, although it has some rather strange properties[62, 1]. The apparent singularity at $g_{tt} = 0$ $(r = r_s)$ is actually an instance of what is called a coordinate singularity[7, 74]. As the name implies, the singularity arises from a non-optimal choice of coordinates. By choosing another set of suitable coordinates, for example Kruskal-Szekeres coordinates, one can show that the metric is well-defined at the Schwarzschild radius[7, 74].

For the Schwarzschild black hole the case $r = 0$ is different from the $r = r_s$. If one asks that the solution be valid for all r one runs into a gravitational singularity, or true physical singularity, at the origin. To see that this is a true singularity one must look at quantities that are independent of the choice

[3] We will use Planck units: $c = G = \hbar = 1$.

of coordinates. One such important quantity is the Kretsschmann invariant, which is given by[75]:

$$K = R_{klij}R^{klij} = \frac{12{r_s}^2}{r^6} = \frac{48M^2}{r^6}. \tag{2.3}$$

At $r = 0$ the Kretsschmann scalar blows up, becomes infinite, indicating the presence of a singularity. At this point spacetime, and the metric itself, is no longer well-defined. For a long time it was thought that such a solution was non-physical. However, a greater understanding of general relativity led to the realization that such singularities were a generic feature of the theory and not just a special case. Such solutions are now believed to exist and are called black holes.

The Schwarzschild solution, taken to be valid for all $r > 0$, is called a Schwarzschild black hole. It is a perfectly valid solution of the Einstein field equations, although it has some rather strange properties. For $r < r_s$ the Schwarzschild radial coordinate r becomes timelike and the time coordinates t becomes spacelike. The $r = r_s$ demarcates what is called the event horizon of the black hole. It represents the point past which light can no longer escape the gravitational field. Any physical body whose radius R becomes less than or equal to the Schwarzschild radius will undergo gravitational collapse and become a black hole.

2.1.2 Reissner-Nordström black hole

In this section we are going to initiate the study of the black hole solution of the Einstein equations in the presence of electricmagnetic field so that the stress-energy tensor is now nonzero.

The Reissner-Nordström metric[76, 77] is a static solution to the Einstein-Maxwell field equations, which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric object of mass M[62, 1, 2, 7].

We start by implementing the standard field equations of general relativity. The Einstein field equations (EFE) may be written in the form[7]:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4}T_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.4}$$

where:

- $R_{\mu\nu}$ is the Ricci curvature tensor;
- R is the scalar curvature;
- $g_{\mu\nu}$ is the metric tensor;
- $T_{\mu\nu}$ is the stress-energy tensor of the matter field;

The EFE is a tensor equation relating a set of a symmetric 4 x 4 tensor; so that this tensor equation has ten independent components.

Despite the simple appearance of the equations they are, in fact, quite complicated. Given a specified distribution of energy and matter in the form of a stress-energy tensor, the EFE are understood to be equations for the metric tensor $g_{\mu\nu}$, as both the scalar curvature and Ricci tensor depend on the metric in a complicated nonlinear manner.

One can write the Einstein field equations in a more compact form by defining Einstein tensor:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = 8\pi T_{\mu\nu} \tag{2.5}$$

or:

$$G_{\mu\nu} = 8\pi T_{\mu\nu} \tag{2.6}$$

which is a symmetric second-rank tensor that is function of the metric, where the cosmological term Λ is taken to be zero in conventional relativity theory[7]. The expression on the right represents the energy-matter content of spacetime and the expression on the left represents the curvature of spacetime as determined by the metric. The EFE can then be interpreted as a set of equations dictating how the curvature of spacetime is related to the energy-matter content of the universe.

The electromagnetic field admits a coordinate-independent geometric description, and Maxwell's equations expressed in terms of these geometrical quantity are the same in any spacetime curved or not.

The electromagnetic field is a covariant vector A_α ($\alpha = 0, 1, 2, 3$); as a covariant vector, its rule for transforming from one coordinate system to another is:

$$A'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} A_\beta. \tag{2.7}$$

The electromagnetic field is a covariant antisymmetric rank two tensor which can be defined in terms of the electromagnetic potential by:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.8}$$

This equation is covariant:

$$F'_{\alpha\beta} = \frac{\partial A'_\beta}{\partial x'_\alpha} - \frac{\partial A'_\alpha}{\partial x'_\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} F_{\mu\nu}. \tag{2.9}$$

In the vacuum, the action for the electromagnetic field in curved spacetime is given the Einstein-Hilbert term S_G plus a term S_{EM} describing the electromagnetic field:

$$S = S_G + S_{EM}; \tag{2.10}$$

with, in Planck units:

$$S_G = \frac{1}{16\pi} \int d^4x \sqrt{-g} R; \tag{2.11}$$

$$S_{EM} = \frac{-1}{8\pi} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu}; \tag{2.12}$$

and the stress-energy tensor is:

$$T_{\mu\nu} = +\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{4\pi} \left[F_{\mu\lambda} F_\nu{}^\lambda - \frac{g_{\mu\nu}}{4} F^2 \right]; \tag{2.13}$$

where:

$$F^2 = F_{\mu\nu} F^{\mu\nu}. \tag{2.14}$$

The Einstein field equations assume the following form:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu} = 2 \left[F_{\mu\lambda} F_\nu{}^\lambda - \frac{g_{\mu\nu}}{4} F^2 \right]. \tag{2.15}$$

The Reissner-Nordström is concerned with a single single electrostatic field:

$$F_{tr} = \frac{Q}{r^2}, \tag{2.16}$$

so that the stress-energy tensor is:

$$8\pi T_{\mu\nu} = \begin{pmatrix} \frac{Q^2\Sigma}{r^4} & 0 & 0 & 0 \\ 0 & -\frac{Q^2}{r^4\Sigma} & 0 & 0 \\ 0 & 0 & \frac{Q^2}{r^2} & 0 \\ 0 & 0 & 0 & \frac{Q^2}{r^2} sin^2\theta \end{pmatrix} \tag{2.17}$$

with

$$\Sigma = \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) \tag{2.18}$$

The Reissner-Nordström (R-N) metric[76, 77] is a static solution to the Einstein-Maxwell field equations (2.15), which corresponds to the gravitational field of a charged, non-rotating, spherically symmetric object of mass M[62, 1, 2, 7], where metric is given by:

$$ds^2 = \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + \\ -r^2(d\theta^2 + sin^2\theta d\varphi^2)\,; \tag{2.19}$$

where:

- r_s is the Schwarzschild radius: of the "massive body" which is related to its mass M by $r_s = 2MG/c^2 = 2M$;
- r_Q is a length-scale corresponding to the electric Q of the mass:

$$r_Q^2 = \frac{Q^2 G}{c^4} = Q^2\,. \tag{2.20}$$

Note that later this metric will be rewritten as follows:

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + \\ -r^2(d\theta^2 + sin^2\theta d\varphi^2). \tag{2.21}$$

In the limit that the charge the length-scale r_Q or equivalently the charge Q goes zero, one recovers the Schwarzschild black hole.

Although charged black holes with $r_Q \ll r_s$ are similar to the Schwarzschild black hole, they have two horizons: the event horizon and a Cauchy horizon[74]. As usual the event horizons for the spacetime are located where g_{tt} diverges, or $(g_{rr})^{-1} = 0$:

$$(g_{rr})^{-1} = 0 \Longrightarrow \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) = 0; \tag{2.22}$$

from which it follows that:

$$r_\pm = \frac{1}{2}\left(r_s \pm \sqrt{(r_s^2 - 4r_Q^2)}\right) \Longrightarrow r_\pm = M \pm \sqrt{(M^2 - Q^2)}. \tag{2.23}$$

There are therefore three cases to consider:

- $M < |Q|$
  If $M < |Q|$ the two horizons disappear and we have a naked singularity. In classical general relativity people have postulated the so-called cosmic censorship conjecture[78, 25, 27]: spacetime singularities should always be hidden inside a horizon. The conjecture implies, in the Reissner-Nordström case, the bound;

$$M \geq |Q|;$$

- $M > |Q|$
  $(g_{rr})^{-1} = 0$ vanishes at $r = r_+$ and $r = r_-$, so metric is singular there, but these, as in the Schwarzschild case, are coordinate singularities. The important quantity is the Kretsschmann invariant, which is given

by[75, 79]:

$$\begin{aligned} K &= R_{klij}R^{klij} = \\ &= \frac{8}{r^{12}}(6M^2r^6 - 12MQ^2r^5 + 7Q^4r^4); \end{aligned} \tag{2.24}$$

and for the Reissner-Nordström black hole at $r = 0$ the Kretsschmann scalar blows up indicating the presence of a singularity, while at $r = r_+$ and $r = r_-$ the Kretsschmann invariant does not become infinite.

- $M = |Q|$
  These concentric event horizons becomes degenerate for:

$$r_s = 2r_Q \Longrightarrow M = Q; \tag{2.25}$$

which corresponds to an extremal black hole[27]. In particular the extremal configurations, that is the configurations that saturate the bound $M = |Q|$, have some special properties[27, 3]. One is that, in that case the two horizons r_+ and r_- coincide and:

$$r_+ = r_- = \frac{1}{2}r_s \Longrightarrow r_+ = r_- = M, \tag{2.26}$$

and the region where the metric components charge sign is reduced to the event horizon

$$r_H \equiv r_+ = r_- = M = |Q|. \tag{2.27}$$

Moreover, the extremal Reissner-Nordström configuration, whose metric is conveniently rewritten in terms of the distance from the horizon $\tilde{r} = r - r_H$:

$$ds^2 = \left(1 + \frac{Q}{\tilde{r}}\right)^{-2} dt^2 - \left(1 + \frac{Q}{\tilde{r}}\right)^2 (d\tilde{r}^2 + \tilde{r}^2 d\Omega^2); \tag{2.28}$$

with $d\Omega^2 = (d\theta^2 + sin^2\theta d\varphi^2)$. Introducing the harmonic function:

$$H(\tilde{r}) = \left(1 + \frac{Q}{\tilde{r}}\right), \tag{2.29}$$

we have

$$ds^2 = H^{-2}(\tilde{r})dt^2 - H^2(\tilde{r})(d\overrightarrow{x} \cdot d\overrightarrow{x}); \tag{2.30}$$

where $\tilde{r}^2 = \overrightarrow{x} \cdot \overrightarrow{x}$.

As equation (2.28) shows, the extremal Reissner-Nordström configuration may be considered as a soliton of classical general relativity, interpolating between two vacua of the theory: the flat Minkowski spacetime, asymptotically reached at spatial infinity $\tilde{r} \to \infty$, and the Bertotti-Robinson metric (B-R metric)[80, 81], describing the conformally flat geometry AdS_2 x S^2 near the horizon $\tilde{r} \to 0$[25, 3]:

$$ds^2_{B-R} = \frac{\tilde{r}^2}{M^2_{B-R}}dt^2 - \frac{M^2_{B-R}}{\tilde{r}^2}(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2); \tag{2.31}$$

where:

$$M^2_{B-R}d\Omega^2 \tag{2.32}$$

is the two-sphere (S^2) of radius M_{B-R} and

$$ds^2_{AdS_2} = \frac{\tilde{r}^2}{M^2_{B-R}}dt^2 - \frac{M^2_{B-R}}{\tilde{r}^2}d\tilde{r}^2 \tag{2.33}$$

is the metric of the two-dimensional anti-de Sitter (AdS_2) spacetime.

Finally, let us note that the condition $|Q| = M$ can be regarded as a no-force condition between the gravitational attraction $F_g = \frac{M}{r^2}$ and the electric repulsion $F_Q = -\frac{Q}{r^2}$ on a unit mass carrying a unit charge[3].

The electromagnetic potential is

$$A^\alpha = \left(\frac{Q}{r}, 0, 0, 0\right). \tag{2.34}$$

If magnetic monopoles are included into the theory, then a generalization to include magnetic charge P is obtained by replacing Q^2 by $Q^2 + P^2$ in the metric and including the term $Pcos\theta d\varphi$ in the electromagnetic potential.

2.1.3 Kerr black hole

In the two previous subsections we have studied the asymptotically flat, static and spherically symmetric Schwarzschild and Reissner-Nordström solutions. To find more solutions, we have to relax these conditions or couple to gravity more general types of energy-matter, as we will do later on. In Einstein-Maxwell theory, one possibility is to look for static and axially symmetric solutions and another possibility is to relax the condition of of staticity and only ask that solution be stationary, which implies that we have to relax the condition of spherical symmetry as well and look for stationary and axisymmetric space-times[82, 83]. In the second case, we find:

- the Kerr black holes with angular momentum but does not include charges;
- the Kerr-Newman black holes with angular momentum and electric (and possibly magnetic) charges.

In Einstein's theory of general relativity, the Kerr vacuum or the Kerr metric describes the geometry of spacetime around a rotating massive object[82]. According to this metric, such rotating objects should exhibit frame dragging, a strange prediction of general relativity[62, 1, 2, 7]. In simple terms, this effect predicts that a body coming close to a rotating mass will be entrained to participate in its rotation, not because of any applied force or torque that can be felt, but rather because of curvature of spacetime associated with rotating bodies. At close enough distances, all bodies even light itself, must rotate with the object, the region where this holds is the ergosphere[7].

In the polar spherical coordinates, r, θ, φ, the Kerr metric describes the geometry of spacetime in the vicinity of a mass M rotating with angular momentum J[62, 1, 2, 84]:

$$\begin{aligned} ds^2 = d\tau^2 = \left(1 - \frac{r_s r}{\rho^2}\right) dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 + \\ - \frac{A}{\rho^2} sin^2\theta d\varphi^2 + \frac{2 r_s r \alpha sin^2\theta}{\rho^2} dt d\varphi; \end{aligned} \tag{2.35}$$

where:

- $r_s = 2M$ is the Schwarzschild radius;
- the length scales α, ρ^2 and Δ have been introduced for brevity:

$$\alpha = \frac{J}{Mc} = \frac{J}{M} \tag{2.36}$$

represents the specific angular momentum of the source,

$$\rho^2 = r^2 + \alpha^2 cos^2\theta, \tag{2.37}$$

$$\Delta = r^2 - r_s r + \alpha^2 = r^2 - 2Mr + \alpha^2; \tag{2.38}$$

- A is:

$$A = (r^2 + \alpha^2)^2 - \alpha^2 \Delta sin^2\theta = (r^2 + \alpha^2)\rho^2 + r r_s \alpha^2 sin^2\theta. \tag{2.39}$$

In the non-relativistic limit where r_s goes to zero, the Kerr metric becomes the orthogonal metric for the oblate spheroidal coordinates:

$$ds^2 = dt^2 - \frac{\rho^2}{r^2 + \alpha^2} dr^2 - \rho^2 d\theta^2 - (r^2 + \alpha^2) sin^2\theta d\varphi^2. \tag{2.40}$$

We may rewrite this metric in the following form:

$$ds^2 = \left(g_{tt} - \frac{g_{t\varphi}^2}{g_{\varphi\varphi}} \right) dt^2 - g_{rr} dr^2 - g_{\theta\theta} d\theta^2 - g_{\varphi\varphi} \left(d\varphi - \frac{g_{t\varphi}}{g_{\varphi\varphi}} dt \right); \tag{2.41}$$

where g_{ij} are the components of the metric tensor. This metric is equivalent to a rotating reference frame that is rotating with angular speed Ω that depends on both the radius r and the colatitude θ[85]:

$$\Omega = -\frac{g_{t\varphi}}{g_{\varphi\varphi}} = \frac{r_s \alpha r}{\rho^2 (r^2 + \alpha^2) + r_s \alpha^2 r sin^2\theta}; \tag{2.42}$$

in the plane of equator this simplifies to:

$$\Omega = \frac{r_s \alpha}{r^3 + \alpha^2 r + r_s \alpha^2}. \tag{2.43}$$

Thus, an inertial reference frame is entrained by the rotating central mass to participate in the latter's rotation, this is frame-dragging.

At the center of a Kerr black hole as described by general relativity lies a gravitational singularity, a region where the spacetime loses meaning. The Kretsschmann invariant for a Kerr black hole is given by[84]:

$$\begin{aligned} K &= R_{klij}R^{klij} \\ &= \frac{48M^2(r^2-\alpha^2cos^2\theta)[(r^2+\alpha^2cos^2\theta)^2-16r^2\alpha^2cos^2\theta]}{(r^2+\alpha^2cos^2\theta)^6}. \end{aligned} \tag{2.44}$$

At $r=0$ and $\theta=\frac{\pi}{2}$ the Kretsschmann scalar blows up, becomes infinite, indicating the presence of a singularity.

The Kerr metric has two physical relevant surfaces on which it appears to be singular. The inner surface corresponds to an event horizon similar to that observed in the Schwarzschild black hole; this occurs where the purely radius component g_{rr} of the metric goes to infinity. Solving the equation:

$$\frac{1}{g_{rr}} = 0 \Longrightarrow \Delta = r^2 - r_s r + \alpha^2 = r^2 - 2Mr + \alpha^2 = 0 \tag{2.45}$$

yields the solution:

$$r_{inner\pm} = \frac{r_s \pm \sqrt{(r_s^2 - 4\alpha^2)}}{2} \Longrightarrow r_{inner\pm} = M \pm \sqrt{(M^2-\alpha^2)}. \tag{2.46}$$

Another singularity occurs where the purely temporal component g_{tt} of the metric changes sign from positive to negative. Again solving a quadratic equation:

$$g_{tt} = 0 \Longrightarrow \left(1 - \frac{r_s r}{\rho^2}\right) = 0 \tag{2.47}$$

yields the solution:

$$r_{outer\pm} = \frac{r_s \pm \sqrt{(r_s^2 - 4\alpha^2cos^2\theta)}}{2} \tag{2.48}$$

or

$$r_{outer\pm} = M \pm \sqrt{(M^2-\alpha^2cos^2\theta)}. \tag{2.49}$$

Due to the $cos^2\theta$ quantity in the square root, this outer surface resembles a flattened sphere that touches the inner surface at the poles of the rotation axis, where θ equals 0 or π. The space between r_{outer+} and r_{inner+} surfaces is called the ergosphere.

A moving physical body experiences a positive proper time along its worldline, its path through spacetime. However, this is impossible within the ergosphere, where g_{tt} is negative, unless the physical body is co-rotating with the interior mass M with an angular speed at least of Ω. Thus, no body can rotate opposite to the central mass within the ergosphere.

As with the event horizon in the Schwarzschild black hole the apparent singularities $r_{outer\pm}$ and $r_{inner\pm}$ are coordinate singularities; in fact, the spacetime can be smoothly continued through them by an appropriates choice of coordinates[2, 86, 84].

A black hole in general is surrounded by a surface, called the event horizon, where the escape velocity is equal to the velocity of light. Within this surface, no physical body or observer can maintain itself at a constant radius. It is forced to fall inwards, and so this is sometimes called the static limit.

A rotating black hole has the same static limit at its event horizon but there is an additional surface outside the event horizon called the "ergosurface" given by:

$$(r - M)^2 = (M^2 - J^2 cos^2\theta) \tag{2.50}$$

in Boyer-Lindquist coordinates[7, 87], which can be intuitively characterized as the sphere where "the rotational velocity of the surrounding space" is dragged along with the velocity of light. Physical bodies falling within the ergosphere are forced to rotate faster and thereby gain energy. Because they are still outside the event horizon, they may escape the black hole. The net process is that the rotating black hole emits energetic particles at the cost of its own total energy. The possibility of extracting spin energy from a rotating black hole was first proposed by the mathematician physicist Roger Penrose in 1969 and is thus called the Penrose process[7, 87].

We note that for the Kerr metric, the concentric event horizons becomes degenerate for:

$$M^2 = \alpha^2;$$

which corresponds to an "extreme" condition for the Kerr black hole. In particular the extremal configurations, that is the configurations that saturate the bound $M = |\alpha|$, have some special properties. One is that, in that case the two horizons r_+ and r_- coincide and:

$$r_+ = r_- = \frac{1}{2} r_s \Longrightarrow r_+ = r_- = M,$$

or

$$r_H \equiv r_+ = r_- = M = |\alpha|.$$

2.1.4 Kerr-Newman black hole

The Kerr-Newmann metric, or Kerr-Newmann (K-N) black hole, is a solution of the Einstein-Maxell equations (2.15) in general relativity, describing the spacetime geometry in the region surrounding a charged and rotating mass[87, 2, 7]. In other words, the Kerr-Newmann metric describes the geometry of spacetime in the vicinity of a rotating mass M with charge Q. One way to express this metric is by writing down its line element in a particular set of spherical coordinates (r, θ, φ) (in the Boyer-Lindquist coordinates[4])[7, 87]:

$$ds^2 = \frac{\Delta}{\rho^2}(dt - \alpha sin^2\theta d\varphi)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 +$$
$$-\frac{sin^2\theta}{\rho^2}[(r^2+\alpha^2)d\varphi - \alpha dt]^2; \tag{2.51}$$

where:

$$\alpha = \frac{J}{Mc} = \frac{J}{M} \tag{2.52}$$

and a representing the specific angular momentum of the source,

$$\rho^2 = r^2 + \alpha^2 cos^2\theta, \tag{2.53}$$

$$\Delta = r^2 - r_s r + \alpha^2 + Q^2 = r^2 - 2Mr + \alpha^2 + Q^2. \tag{2.54}$$

The Kerr-Newmann black hole has two coordinate singularities corresponding to the outer and inner horizon:

$$r_\pm = M \pm \sqrt{(M^2 - \alpha^2 - Q^2)}. \tag{2.55}$$

There are therefore three cases to consider:

- $M^2 < \alpha^2 + Q^2$
  If $M^2 < \alpha^2 + Q^2$ the two horizons disappear and we have a naked singularity. Using the cosmic censorship conjecture[78, 25, 27], spacetime singularities should always be hidden inside a horizon. The conjecture implies, in the Kerr-Newmann case, the bound;

$$M^2 \geq \alpha^2 + Q^2; \tag{2.56}$$

[4]The coordinate transformation from Boyer-Lindquist coordinates r, θ, φ to Cartesian coordinates x, y, z is given by:

$$\begin{aligned} x &= (\sqrt{r^2+\alpha^2}) sin\theta cos\varphi \\ y &= (\sqrt{r^2+\alpha^2}) sin\theta sin\varphi \\ z &= r cos\theta. \end{aligned}$$

- $M^2 > \alpha^2 + Q^2$
  In this case we have:

$$r_+ = M + \sqrt{(M^2 - \alpha^2 - Q^2)} \qquad r_- = M - \sqrt{(M^2 - \alpha^2 - Q^2)} \tag{2.57}$$

corresponding to the outer and inner horizon for the K-N black hole. The Kretsschmann invariant is in this case [75]:

$$\begin{aligned} K = R_{klij}R^{klij} = \\ = \frac{8}{(r^2+\alpha^2 cos^2\theta)^6}[Q^4(7r^4 - 34\alpha^2 r^2 cos^2\theta + 7\alpha^4 cos^4\theta) + \\ +6M^2(r^6 - 15\alpha^2 r^4 cos^2\theta + 15\alpha^4 r^2 cos^4\theta - \alpha^6 cos^6\theta) + \\ -12MQ^2 r(r^4 - 10\alpha^2 r^2 cos^2\theta + 5\alpha^4 cos^4\theta)]; \end{aligned} \tag{2.58}$$

which clearly reduces to the expected expression for the Schwarzschild black hole if α and Q are both zero (equation (2.3)). For the Kerr-Newmann black hole at $r = 0$ and $\theta = \pi/2$ the Kretsschmann scalar blows up, becomes infinite, indicating the presence of a singularity; and at $r = r_+$ and $r = r_-$ the Kretsschmann invariant does not become infinite, indicating the absence of singularities.

- $M^2 = \alpha^2 + Q^2$
  These concentric event horizons becomes degenerate for:

$$M^2 = \alpha^2 + Q^2; \tag{2.59}$$

which corresponds to an "extreme" condition for the Kerr-Newman black hole. In particular the extremal configurations, that is the configurations that saturate the bound $M = |\sqrt{(\alpha^2 + Q^2)}|$, have some special properties. One is that, in that case the two horizons r_+ and r_- coincide and:

$$r_+ = r_- = \frac{1}{2}r_s \Longrightarrow r_+ = r_- = M, \tag{2.60}$$

or

$$r_H \equiv r_+ = r_- = M = |\sqrt{(\alpha^2 + Q^2)}|. \tag{2.61}$$

In this metric the static solutions ($\alpha = 0$) the condition (2.56) is guaranteed as long as $r > r_+$, r_+ being the outer horizon:

$$r_+ = M + \sqrt{M^2 - Q^2}\,. \tag{2.62}$$

and:

$$r > M + \sqrt{M^2 - Q^2 - \alpha^2 \cos^2\theta} \equiv r_e \tag{2.63}$$

where $r_e > r_+$ defines the external boundary of the ergosphere, where the component g_{tt} of the metric vanishes, while eq. (2.62) is the radius of the outer event horizon.

The Kerr-Newman metric is a generalization of other exact solutions in general relativity:

- Schwarzschild metric if the charge Q and the angular momentum J (or α) is zero;
- Reissner-Nordström metric if the angular momentum J (or α) is zero;
- Kerr metric if the charge Q is zero;
- Minkowski space if the gravitational constant G is zero.

2.2 Introduction to black hole thermodynamics

In Physics, thermodynamics of black holes is an area of study that seeks to reconcile the laws of thermodynamics with the existence of black holes event horizons[61, 1, 2, 25, 63, 51]. The only way to satisfy the second law of thermodynamics is to admit that black holes have entropy. If black holes carried no entropy, it would be possible to violate the second law by throwing mass into the black hole. The increase of the entropy of the black hole more than compensates for the decrease of the entropy carried by the body that was swallowed. Moreover, the notion of black hole entropy is motivated by two results in general relativity[69, 25, 51]:

1. *Area theorem* The area theorem, states that the area (A) of a black hole event horizon never decreases with time:

$$dA \geq 0; \tag{2.64}$$

2. *No-hair theorem* A stationary black hole is characterized by only three quantities; mass, angular momentum and charge.

Starting from theorems proved by Stephen Hawking, Jacob Bekenstein conjectured that the black hole entropy was proportional to the area of its event horizon divided by the Planck area. Bekenstein suggested $(0.5ln2)/(4\pi)$[88] as the constant of proportionality, asserting that if the constant was not exactly this, it must be very close to it. The next year, Hawking showed that black holes emit thermal Hawking radiation corresponding to a certain temperature, the Hawking temperature. Using the thermodynamic relationship between energy, temperature and entropy, Hawking was able to confirm Bekenstein's conjecture and fix the constant of proportionality at 1/4, or[11, 12]:

$$S_{BH} = \frac{k_B}{\ell_p^2}\frac{A}{4}; \tag{2.65}$$

where $\ell_p^2 = G\hbar/c^3$ is the squared Planck length and k_B is Boltzmann's constant. The subscript BH either stands for "Bekenstein-Hawking" or "black hole". The black hole entropy is proportional to the area of its event horizon A[68, 69, 70, 71].

The four laws of black hole mechanics are physical properties that black holes are believed to satisfy. The laws, analogous to the laws of thermodynamics, were discovered by Brandon Carter, Stephen Hawking and James

Bardeen[61]. The laws of black hole mechanics are expressed in natural units, and they are[61, 63]:

1. *The zeroth law.* The horizon has constant surface gravity k for a stationary black hole[5].

   The zeroth law is analogous to the zeroth law of thermodynamics which states that the temperature is constant throughout a body in thermal equilibrium. It suggests that the surface gravity is analogous to temperature. T constant for thermal equilibrium for a normal system is analogous to k constant over the horizon of a stationary black hole;

2. *The first law.* We have[89, 90, 23, 50]:

   $$dM = \frac{k}{8\pi} dA + \Omega_H dJ + \Phi dq; \tag{2.66}$$

   where: M is the mass, J is the angular momentum, Φ is the electrostatic potential, q is the electric charge, A is the horizon area, k is the surface gravity and Ω_H is the angular velocity of the black hole. The generalization to include a magnetic charge p is:

   $$dM = \frac{k}{8\pi} dA + \Omega_H dJ + \Phi dq + \chi dp; \tag{2.67}$$

   with χ is the magnetic potential.

   The left hand side, dM is the change in energy/mass. Although the first term does not have an immediately obvious physical interpretation, the second and third terms on the right hand side represent changes in energy due to rotation and electromagnetism. Analogously, the first law of thermodynamics is a statement of energy conservation, which contains on its right hand side the term TdS;

3. *The second law.* The horizon area is, assuming the weak energy condition, a non-decreasing function of time,

   $$\frac{dA}{dt} \geq 0. \tag{2.68}$$

[5] See Appendix D.

This "law" was superseded by Hawking's discovery that black holes radiate, which causes both the black hole's mass and the area of its horizon to decrease over time. The second law is the statement of Hawking's area theorem. Analogously, the second law of thermodynamics states that the change in entropy of an isolated system will be greater than or equal to zero for a spontaneous process, suggesting a link between entropy and the area of a black hole horizon. However, this version violates the second law of thermodynamics by matter losing energy as it falls in, giving a decrease in entropy. Generalised second law introduced as:
total entropy = *black hole entropy* + *outside entropy*;

4. *The third law.* It is not possible to form a black hole with vanishing surface gravity. $k = 0$ is not possible to achieve.
Extremal black hole[6] have vanishing surface gravity. Stating that k can not go to zero is analogous to the third law of thermodynamics which states, the entropy of a system at absolute zero is a well-defined constant. This is because a system at zero temperature exists in its ground state. Furthermore, $\triangle S$ will reach zero at zero Kelvins, but S itself will also reach zero, at least for perfect crystalline substances. No experimentally verified violations of the laws of thermodynamic are known.

The four laws of black hole mechanics suggest that one should identify the surface gravity of a black hole with temperature and the area of the event horizon with entropy, at least up to some multiplicative constants. If one only considers black holes classically, then they have zero temperature and, by the no hair theorem, zero entropy, and the laws of black hole mechanics

[6]In theoretical physics, an extremal black hole is a black hole with the minimal possible mass that can be compatible with a given charge and angular momentum. In other words, this is the smallest possible black hole that can exist while rotating at a given fixed constant speed.

remain an analogy. However, when quantum mechanical effects are taken into account, one finds that black holes emit thermal radiation, the Hawking radiation, at temperature:

$$T_H = \frac{k}{2\pi}; \tag{2.69}$$

From the first law of black hole mechanics, this determines the multiplicative constant of the Bekenstein-Hawking entropy which is:

$$S_{BH} = \frac{k_B}{\ell_p^2}\frac{A}{4}. \tag{2.70}$$

Hawking and Page showed that black hole thermodynamics is more general that black holes, that cosmological event horizons also have an entropy and temperature[91, 92].

Chapter 3

Extremal, static and spherically symmetric black holes in supergravity

In our study of the black holes in the previous chapter we have mentioned that some special properties arise when of mass and electric charges are related to satisfy specific relations.

In this chapter, we present the main features of the physics of extremal, static and spherically symmetric black holes embedded in supersymmetric theories of gravitation. In particular, we present a detailed derivation of the effective one-dimensional Lagrangian, which encodes the dynamics of this class of solution.

3.1 Static black holes in four-dimensions

Let us recall the main facts about the description of a static and spherically symmetric black hole in four-dimensions as solution of a Hamiltonian system[24]. We start from the four dimensional bosonic action of a generic supergravity theory, describing m scalar fields Φ^s coupled to n_V of vectors field A_μ^Λ[3] [1]:

$$S = \int \sqrt{-g} d^4x \left(-\frac{1}{2}R + I_{\Lambda\Gamma} F_{\mu\nu}^\Lambda F^{\Gamma|\mu\nu} + \right.$$

$$\left. + \frac{1}{2\sqrt{-g}} R_{\Lambda\Gamma} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^\Lambda F_{\rho\sigma}^\Gamma + \frac{1}{2} g_{rs}(\Phi) \partial_\mu \Phi^r \partial^\mu \Phi^s \right); \tag{3.1}$$

where:

- R is the curvature scalar;
- F^Λ are field strengths;
- $\mathcal{N}_{\Lambda\Gamma}(\Phi)$ is the vector kinetic matrix and it is a complex, symmetric, n_V x n_V matrix depending on the scalar field Φ. The imaginary part $Im\mathcal{N}_{\Lambda\Gamma} = I_{\Lambda\Gamma}$ is negative definite and generalizes the inverse of the squared coupling constant appearing in ordinary gauge theories while its real part $Re\mathcal{N}_{\Lambda\Gamma} = R_{\Lambda\Gamma}$ is instead a generalization of the theta-angle of quantum chromodynamics. In supergravity theories it is in general not a constants, but a function of scalar field. In the presence of scalar fields, the black hole solutions will be modified with respect to the solutions described in the previous chapter;
- $g_{rs}(\Phi)$ with $r, s = 1, \cdots, m$ is the scalar metrix on the σ-model described the scalar manifold $\mathcal{M}_{scalar}$ of real dimension m[3, 93][2].

[1] See Appendix A.

[2] In quantum field theory, a nonlinear σ-model (which is the "generalization" of a σ-model) describes a scalar field Φ which takes on values in a nonlinear manifold called the target manifold T[64, 65, 67, 94].
The tangent manifold is equipped with a Riemannian metric g. Φ is a differentiable map from Minkowski space M (or some other space) to T. In the coordinate notation, with the coordinates Φ^a with $a = 1, \cdots, m$ where m is the dimension of T, the Lagrangian density is given by:

$$\mathcal{L} = +\frac{1}{2} g_{ab}(\Phi) \partial_\mu \Phi^a \partial^\mu \Phi^b - V(\Phi)$$

where here, we have used a $(+, -, -, -)$ metric signature. In more than two dimensions, nonlinear σ-models are nonrenormalizable; this means they can only arise as effective field

The number of scalars and vectors, namely m and n_V, and the geometric properties of the scalar manifold $\mathcal{M}_{scalar}$ depend on the number N of supersymmetries.

Using the Euler-Lagrange equation, for the bosonic action (3.1), one obtains the Einstein equations:

$$-\frac{1}{2}\left(R_{\mu\nu}-\frac{g_{\mu\nu}}{2}R\right)+\frac{1}{2}g_{ab}(\Phi)\partial_\mu\Phi^a\partial_\nu\Phi^b-\frac{g_{\mu\nu}}{2}\frac{g_{ab}}{2}\partial\Phi^a\partial\Phi^b+$$
$$+2F^T_{\mu\rho}Im\mathcal{N}F_\nu{}^\rho-\frac{g_{\mu\nu}}{2}F^TIm\mathcal{N}F=0; \qquad (3.2)$$

with:

$$R=g_{ab}(\Phi)\partial_\mu\Phi^a\partial^\mu\Phi^b, \qquad (3.3)$$

and

$$-\frac{1}{2}R_{\mu\nu}=-\frac{1}{2}g_{ab}(\Phi)\partial_\mu\Phi^a\partial_\nu\Phi^b+$$
$$-2F^T_{\mu\rho}Im\mathcal{N}F_\nu{}^\rho+\frac{g_{\mu\nu}}{2}F^TIm\mathcal{N}F. \qquad (3.4)$$

We make the following Ansatz for a spherically symmetric and stationary metric is[37, 95, 38]:

$$ds^2=a^2(r)dt^2-\frac{1}{a^2(r)}dr^2-b^2(r)d\Omega^2; \qquad (3.5)$$

with $d\Omega^2=(d\theta^2+sin^2\theta d\varphi^2)$.
For this metric we have:

- the only non-zero $R^i{}_j$ Ricci curvature tensor are:

$$R^t{}_t=\frac{d}{dr}\left(a(r)\frac{d}{dr}a(r)\right)+\frac{2a(r)}{b(r)}\left(\frac{d}{dr}a(r)\right)\left(\frac{d}{dr}b(r)\right), \qquad (3.6)$$

theories.

There is a special class of nonlinear σ-models with the internal symmetry group G. If G is a Lie group and H is a Lie subgroup, then the quotient space G/H is a manifold (subject to certain technical restrictions like H being a closed subset) and is also a homogeneous space of G or in other words, a nonlinear realization of G.

$$R^r{}_r = \frac{d}{dr}\left(a(r)\frac{d}{dr}a(r)\right) + \frac{2a(r)}{b(r)}\frac{d}{dr}\left(a(r)\frac{d}{dr}b(r)\right), \tag{3.7}$$

$$R^\theta{}_\theta = -\frac{1}{b^2(r)}\left[1 - \frac{d}{dr}\left(a^2(r)b(r)\frac{d}{dr}b(r)\right)\right], \tag{3.8}$$

$$R^\varphi{}_\varphi = R^\theta{}_\theta = -\frac{1}{b^2(r)}\left[1 - \frac{d}{dr}\left(a^2(r)b(r)\frac{d}{dr}b(r)\right)\right]; \tag{3.9}$$

- the R scalar curvature is:

$$R = +2\frac{d}{dr}\left(a(r)\frac{d}{dr}a(r)\right) - \frac{2}{b^2(r)} + \frac{2a^2(r)}{b^2(r)}\left(\frac{d}{dr}b(r)\right)^2 + \\ + \frac{4}{b(r)}\frac{d}{dr}\left(a^2(r)\frac{d}{dr}b(r)\right). \tag{3.10}$$

In addition to the condition of the equation (3.5), we assume the following Ansatz for the vector field strengths:

$$F^\Lambda = \frac{\tilde{q}^\Lambda}{4\pi b^2}dt \wedge dr + \frac{p^\Lambda}{4\pi}sin(\theta)d\theta \wedge d\varphi. \tag{3.11}$$

where $\tilde{q}_\Lambda = (I^{-1})_\Lambda^\Sigma(q_\Sigma - R_{\Lambda\Gamma}p^\Gamma)$, q_Σ and p^Λ being the quantized electric and magnetic charges. This Ansatz is dictated by the general p-brane solution of supergravity bosons equations in any dimensions[96].

From this last equation we get[3]:

$$^*F^\Lambda = \frac{p^\Lambda}{4\pi b^2}dt \wedge dr - \frac{\tilde{q}^\Lambda}{4\pi}sin(\theta)d\theta \wedge d\varphi. \tag{3.12}$$

and

$$\begin{aligned} G_\Lambda &\equiv \frac{i}{2}\frac{\partial \mathcal{L}}{\partial F^\Lambda} = (-I^*F + RF)_\Lambda = \\ &= \frac{(I\tilde{q} + Rp)_\Lambda}{4\pi}sin(\theta)d\theta \wedge d\varphi + \frac{(R\tilde{q} - Ip)_\Lambda}{4\pi b^2}dt \wedge dr; \end{aligned} \tag{3.13}$$

where:

$$I = Im\mathcal{N}\,, \qquad R = Re\mathcal{N}\,.$$

[3] See Appendix A.

With any field-strength F^Λ we may associate a magnetic charge:

$$\int_{S^2} F^\Lambda = p^\Lambda, \tag{3.14}$$

and an electric charge

$$\int_{S^2} G_\Lambda = q_\Lambda, \tag{3.15}$$

where S^2 is a spatial two-sphere in the spacetime geometry of the dyonic solution, for instance in Minkowski spacetime the two-sphere at radius infinity S^2_∞. From (3.13) and (3.15) we find:

$$(I\tilde{q} + Rp)_\Lambda = q_\Lambda, \tag{3.16}$$

or

$$\tilde{q}_\Lambda = (I^{-1})^\Sigma_\Lambda (q_\Sigma - R_{\Lambda\Gamma} p^\Gamma); \tag{3.17}$$

and we assume all scalar fields Φ to be function of r. In the absence of scalar fields $R_{\Lambda\Gamma} = 0$, $I_{\Lambda\Gamma} = -1$ and we have that $\tilde{q}$ reduces to the electric charge q.

The equation of motion for the scalar fields obtained from the bosonic action (3.1) are:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left(\sqrt{-g} g^{\mu\nu} \partial_\nu \Phi^a \right) + \Gamma^a{}_{bc} \partial_\mu \Phi^b \partial^\mu \Phi^c = -\frac{1}{b^4} g^{ab} \partial_b V_{eff}, \tag{3.18}$$

with:

$$\frac{-1}{b^4} \partial_a V_{eff} = \left[F^T_{\mu\rho} (\partial_a I) F^{\mu\rho} + F^T_{\mu\rho} (\partial_a R)^* F^{\mu\rho} \right], \tag{3.19}$$

and V_{eff} is the effective potential, also called "geodesic potential" or function V_{eff} [24].

From the equations (3.11), (3.12) and (3.16) we get that the function V_{eff} has the following form:

$$V_{eff} = \frac{-1}{2} \left[Q^T M Q \right], \tag{3.20}$$

where:

$$Q = \begin{pmatrix} p^\Lambda \\ \\ q_\Lambda \end{pmatrix} \tag{3.21}$$

and

$$M = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ \\ -I^{-1}R & I^{-1} \end{pmatrix}. \tag{3.22}$$

We have denoted by Q the vector of the quantized magnetic and electric charges.

If we consider the Einstein equations (3.2), with

$$\begin{aligned} -\frac{1}{2}R_{\mu\nu} &= -\frac{1}{2}g_{ab}(\Phi)\partial_\mu\Phi^a\partial_\nu\Phi^b + \\ -2F^T_{\mu\rho}Im\mathcal{N}F_\nu{}^\rho &+ \frac{g_{\mu\nu}}{2}F^T Im\mathcal{N}F \end{aligned} \tag{3.23}$$

we obtain that:

$$-\frac{1}{2}R_{tt} = -\frac{a^2}{2b^4}V_{eff} \tag{3.24}$$

and

$$-\frac{1}{2}R_{\theta\theta} = -\frac{1}{2b^2}V_{eff} \tag{3.25}$$

thus we have:

$$R_{tt} = \frac{a^2}{b^2}R_{\theta\theta} \tag{3.26}$$

or

$$R^t{}_t = -R^\theta{}_\theta. \tag{3.27}$$

But from our Ansatz eq. (3.5), we obtained that:

$$R^t{}_t = \frac{\frac{d}{dr}\left(ab^2\frac{d}{dr}a(r)\right)}{b^2}, \tag{3.28}$$

$$R^{\theta}{}_{\theta} = -\frac{1}{b^2}\left[1 - \frac{d}{dr}\left(a^2 b \frac{d}{dr} b(r)\right)\right]; \tag{3.29}$$

for which the condition $R^{t}{}_{t} = -R^{\theta}{}_{\theta}$ implies:

$$\frac{\frac{d}{dr}\left(ab^2 \frac{d}{dr} a(r)\right)}{b^2} = -\frac{1}{b^2}\left[1 - \frac{d}{dr}\left(a^2 b \frac{d}{dr} b(r)\right)\right]; \tag{3.30}$$

or:

$$\frac{d^2}{dr^2}\left(a^2 b^2\right) = 2. \tag{3.31}$$

A solution for this differential equation, in the Kallosh notation[27], is:

$$a^2(\tau) = e^{2U(\tau)} \tag{3.32}$$

and

$$b^2(\tau) = \left[(r - r_0)^2 - c^2\right] e^{-2U(\tau)} \tag{3.33}$$

where c is the extremality parameter[4] and τ is the evolution coordinate and is related to the radius coordinate r by the following relation:

$$dr = \frac{c^2}{sinh^2(c\tau)} d\tau \tag{3.34}$$

or:

$$r = -c[coth(c\tau)] + r_0 \qquad \longrightarrow \qquad coth(c\tau) = -\frac{r - r_0}{c}, \tag{3.35}$$

so that we have:

$$b^2(\tau) = e^{-2U(\tau)} \frac{c^2}{sinh^2(c\tau)}. \tag{3.36}$$

Note that the dependence of the evolution parameter τ on the radius coordinate r is implicitly given by the equation:

$$\frac{c^2}{sinh^2(c\tau)} = (r - r_0)^2 - c^2, \tag{3.37}$$

[4]The extremality parameter should not be confused with the spend of light.

which in the extremal case $c \to 0$ can be written as:

$$\tau = -\frac{1}{r - r_0}. \tag{3.38}$$

With these results, our Ansatz for a spherically symmetric and stationary metric eq. (3.5) becomes:

$$ds^2 = e^{2U(\tau)}dt^2 - e^{-2U(\tau)}\left[\frac{c^4}{sinh^4(c\tau)}d\tau^2 + \frac{c^2}{sinh^2(c\tau)}d\Omega^2\right]; \tag{3.39}$$

which correspond to the general Ansatz for a spherically symmetric static black hole [51, 50, 3, 97].

Note that the extremality limit at which the two horizons coincide, $r_H = r_+ = r_- = r_0$ is $c \to 0$. In this case the equation (3.39) takes the following simple form using the r coordinate:

$$ds^2 = e^{2U}dt^2 - e^{-2U}\left[dr^2 + (r - r_H)^2 d\Omega^2\right]. \tag{3.40}$$

If we require the horizon to have to have a finite area A, the scalar function U in the near-horizon limit should behave as:

$$e^{2U} \overset{\tau\to\infty}{\longrightarrow} \frac{4\pi}{A}\frac{c^2}{sinh^2(c\tau)} = \frac{4\pi}{A}(r - r_-)(r - r_+), \tag{3.41}$$

such that the near-horizon metric reads:

$$ds^2 = \frac{4\pi}{A}(r - r_-)(r - r_+)dt^2 - \frac{A}{4\pi}\left[\frac{1}{(r - r_-)(r - r_+)}dr^2 + d\Omega^2\right]. \tag{3.42}$$

This metric coincides with the near-horizon metric of a Reissner-Nordström (R-N) black hole with horizons located at $r_\pm$. It is convenient to introduce the radial coordinate δ defined as:

$$\delta = 2e^{(c\tau)}, \tag{3.43}$$

the metric becomes[3]:

$$ds^2 = \left(\frac{\delta c}{r_H}\right)^2 dt^2 - (r_H)^2\left[d\delta^2 + d\Omega^2\right]. \tag{3.44}$$

where $r_H = \sqrt{\frac{A}{4\pi}}$ is the radius of the outer horizon. The coordinate δ measures the physical distance from the horizon in units of r_H; in fact the distance of a point at some finite δ_0 from the horizon is finite:

$$\int_0^{\delta_0} r_H d\delta = r_H \delta_0. \tag{3.45}$$

Using this feature, R. Kallosh, N. Sivanandam and M. Soroush in: "The non-BPS black hole attractor equation"[40], give an intuitive argument in order to justify the absence of a universal behavior for the scalar fields near the horizon of a non-extremal black hole: the distance from the horizon is not "long enough" in order for the scalar fields to "loose memory" of their initial values at infinity[3].

Consider now the extremal case $c = 0$. The relation between r and τ is given by equation (3.38). In order to have a finite horizon area, U should behave near the horizon as:

$$e^{-2U} \sim \left(\frac{r_H}{r - r_H}\right)^2. \tag{3.46}$$

The physical distance from the horizon is now measured in units r_H by the coordinate $\omega = ln(r - r_H)$ in terms of which the near-horizon metric reads:

$$ds^2 = \frac{e^{2\omega}}{r_H^2} dt^2 - r_H^2(d\omega^2 + d\Omega^2). \tag{3.47}$$

Since now the horizon is located at $\omega \to -\infty$, the distance of a point at same finite ω_0 from the horizon is always infinite, as opposite to the non-extremal case:

$$\int_{-\infty}^{\omega_0} r_H d\omega = \infty. \tag{3.48}$$

Therefore, as noted in the: 2006 paper by R. Kallosh, N. Sivanandam and M. Soroush [40], the infinite distance from the horizon in the extremal case justifies the fact the scalar fields at the horizon "loose memory" of their initial values at infinity and therefore exhibit a universal behavior[40, 3].

We end this section by observing that the extremality parameter is related to the inner and out horizons by:

$$r_{\pm} = r_0 \pm c. \tag{3.49}$$

In addiction, the extremality parameter is related to the temperature T and entropy S of the black hole through

$$c = 2ST. \tag{3.50}$$

In fact if we consider the Reissner-Nordström (R-N) black hole[76, 77], the Bekenstein-Hawking entropy-area formula is[27]:

$$S_{BH} = \frac{A}{4} = \pi r_+^2, \tag{3.51}$$

and the temperature is:

$$T = \frac{k}{2\pi}, \tag{3.52}$$

where k is the surface gravity[5]:

$$\begin{aligned} k &= \frac{\sqrt{M^2 - Q^2}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2}} \\ &= \frac{r_0}{r_+^2}, \end{aligned} \tag{3.53}$$

since:

$$r_0 = \frac{r_+ - r_-}{2}, \tag{3.54}$$

we have:

$$2ST = 2\pi r_+^2 \frac{r_0}{r_+^2} \frac{1}{2\pi} = r_0 \tag{3.55}$$

and we find that:

$$c = 2ST. \tag{3.56}$$

[5]See Appendix D.

Analogously for the Kerr-Newmann (K-N) black hole, we have:

$$S_{B-H} = \frac{A}{4} = \pi R_+^2;$$

and

$$T = \frac{k}{2\pi},$$

with:

$$\begin{aligned} k &= \frac{\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}} \\ &= \frac{r_0}{R_+^2}; \end{aligned}$$

and we get:

$$2ST = 2\pi R_+^2 \frac{r_0}{R_+^2}\frac{1}{2\pi} = r_0 = c.$$

3.2 The effective one-dimensional Lagrangian

Let us recall the main facts, presented in the previous two sections, about the description of a static black hole in four-dimensions as solution of a Lagrangian system[24]. We start from the four dimensional bosonic action, equation (3.1), of a generic supergravity theory, describing m scalar fields Φ^s coupled to n_V of vectors field A_μ^Λ[3].

From the Ansatz for a spherically symmetric and static black hole reads, eq.s (3.5) and (3.11), we have obtained the equation of motion for the scalar fields:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\Phi^a\right) + \Gamma^a{}_{bc}\partial_\mu\Phi^b\partial^\mu\Phi^c = -\frac{1}{b^4}g^{ab}\partial_b V_{eff}, \tag{3.57}$$

with,

$$V_{eff} = -\frac{1}{2}\left[Q^T M Q\right].$$

Since the only variable is r, the radial part of that equation gives:

$$\frac{1}{b^2}\partial_r\left(b^2 g^{rr}\partial_r\Phi^a\right)+\Gamma^a{}_{bc}(\partial_r\Phi^b)(\partial^r\Phi^c)=-\frac{1}{b^4}g^{ab}\partial_b V_{eff}, \tag{3.58}$$

with $\sqrt{-g}=\sqrt{b^4 sin^2\theta}=b^2 sin\theta$. Furthermore from:

$$ds=d\tau=\frac{dr}{b^2 g^{rr}}, \tag{3.59}$$

we obtain:

$$\frac{d^2}{d\tau^2}\Phi^a+\Gamma^a{}_{bc}\left(\frac{d}{d\tau}\Phi^b\right)\left(\frac{d}{d\tau}\Phi^c\right)=g^{ab}\frac{\partial}{\partial\Phi^b}V_{eff}; \tag{3.60}$$

or better[6]:

$$\frac{D^2}{D\tau^2}\Phi^a\equiv\ddot{\Phi}^a+\Gamma^a{}_{bc}\dot{\Phi}^b\dot{\Phi}^c=g^{ab}\frac{\partial V(\Phi,p,q)}{\partial\Phi^b}e^{2U(\tau)}, \tag{3.61}$$

with: $V_{eff}\equiv V(\Phi,p,q)e^{2U(\tau)}$ and $\dot{\Phi}^b\equiv\left(\frac{d}{d\tau}\Phi^b\right)$. It may be noted that this equation corresponds to the traditional equation, of motion for the scalar fields, that is found in literature[23, 3, 51, 50]; also note that the evolution variable τ does not describe the temporal evolution but the radial one.

Making use of the ansatz (3.5) and (3.11) the action becomes:

$$\begin{aligned}S=\alpha\int dr\Bigg[&2ab\left(\frac{d}{dr}a\right)\left(\frac{d}{dr}b\right)+a^2\left(\frac{d}{dr}b\right)^2+\\&-a^2b^2g_{ab}\left(\frac{\partial}{\partial r}\Phi^a\right)\left(\frac{\partial}{\partial r}\Phi^b\right)-\frac{1}{b^2}V_{eff}\Bigg],\end{aligned} \tag{3.62}$$

where α is a factor of proportionality and V_{eff} is the effective potential[24]. Let us now write down the equations of motion of $a(r)$ and $b(r)$, obtained from this action:

$$\frac{\partial\mathcal{L}}{\partial a}-\frac{d}{dr}\left(\frac{\partial\mathcal{L}}{\frac{d}{dr}a}\right)=0\rightarrow\frac{\frac{d^2}{dr^2}b}{b}=-\frac{1}{2}g_{ab}\left(\frac{\partial}{\partial r}\Phi^a\right)\left(\frac{\partial}{\partial r}\Phi^b\right) \tag{3.63}$$

[6] Here the dotted quantities are differentiated with respect to the evolution parameter τ.

and

$$\frac{\partial \mathcal{L}}{\partial b} - \frac{d}{dr}\left(\frac{\partial \mathcal{L}}{\frac{d}{dr}b}\right) = 0 \to -1 + \frac{\left(\frac{d}{dr}a^2\right)\left(\frac{d}{dr}b^2\right)}{2} + a^2\left(\frac{d}{dr}b\right)^2 =$$

$$= a^2 b^2 g_{ab}\left(\frac{\partial}{\partial r}\Phi^a\right)\left(\frac{\partial}{\partial r}\Phi^b\right) - \frac{1}{b^2}V_{eff}, \tag{3.64}$$

having used the relation:

$$\frac{d^2}{dr^2}\left(a^2 b^2\right) = 2; \tag{3.65}$$

from which we get that:

$$\frac{\frac{d^2}{dr^2}b}{b} = \frac{1}{a^2b^2}\left[+1 - \frac{\left(\frac{d}{dr}a^2\right)\left(\frac{d}{dr}b^2\right)}{2} - a^2\left(\frac{d}{dr}b\right)^2\right] - \frac{1}{a^2b^4}V_{eff}. \tag{3.66}$$

But from eq. (3.36), we have:

$$b(\tau) = \left[(r-r_0)^2 - c^2\right]^{\frac{1}{2}} e^{-U(\tau)}; \tag{3.67}$$

and

$$\frac{d}{dr}b(\tau) = -b(\tau)\left(\frac{d}{dr}U(\tau)\right) + e^{-U(\tau)}\frac{(r-r_0)}{[(r-r_0)^2-c^2]^{\frac{1}{2}}}, \tag{3.68}$$

so that

$$\begin{aligned}\frac{\frac{d^2}{dr^2}b(\tau)}{b} &= \left[-\left(\frac{d^2}{dr^2}U(\tau)\right) + \left(\frac{d}{dr}U(\tau)\right)^2\right] + \\ &- 2\left(\frac{d}{dr}U(\tau)\right)\frac{(r-r_0)}{[(r-r_0)^2-c^2]} - \frac{c^2}{[(r-r_0)^2-c^2]^2}.\end{aligned} \tag{3.69}$$

Comparing this equation with the equation (3.66), we get that:

$$\frac{1}{a^2b^2}\left[+1 - \frac{\left(\frac{d}{dr}a^2\right)\left(\frac{d}{dr}b^2\right)}{2} - a^2\left(\frac{d}{dr}b\right)^2\right] - \frac{1}{a^2b^4}V_{eff} =$$

$$= \left[-\left(\frac{d^2}{dr^2}U(\tau)\right) + \left(\frac{d}{dr}U(\tau)\right)^2\right] +$$

$$-2\left(\frac{d}{dr}U(\tau)\right)\frac{(r-r_0)}{[(r-r_0)^2-c^2]} - \frac{c^2}{[(r-r_0)^2-c^2]^2}. \tag{3.70}$$

Recalling that the evolution coordinate τ is related to the radius coordinate r by the following relation:

$$dr = \frac{c^2}{sinh^2(c\tau)}d\tau = \left[(r-r_0)^2 - c^2\right] d\tau; \tag{3.71}$$

we can deduce that:

$$\frac{d}{dr}U(\tau) = \left(\frac{d}{d\tau}U(\tau)\right)\frac{d\tau}{dr} = \left(\frac{d}{d\tau}U(\tau)\right)\frac{1}{\left[(r-r_0)^2 - c^2\right]}; \tag{3.72}$$

and

$$\begin{aligned}\frac{d^2}{dr^2}U(\tau) = -2\frac{(r-r_0)}{\left[(r-r_0)^2-c^2\right]}\frac{d}{dr}U(\tau) + \\ + \left(\frac{d^2}{d\tau^2}U(\tau)\right)\frac{1}{\left[(r-r_0)^2-c^2\right]^2}.\end{aligned} \tag{3.73}$$

Using the constraint:

$$\left(\frac{d}{d\tau}U(\tau)\right)^2 + \frac{1}{2}g_{ab}\left(\frac{d}{d\tau}\Phi^a\right)\left(\frac{d}{d\tau}\Phi^b\right) - V(\Phi,p,q)e^{2U(\tau)} = c^2 \tag{3.74}$$

we obtain the following equation of motion for $U(\tau)$:

$$\frac{d^2}{d\tau^2}U(\tau) \equiv \ddot{U}(\tau) = V(\Phi,p,q)e^{2U(\tau)}. \tag{3.75}$$

The equations of motion (3.61) and (3.75) can be associated to a one dimensional theory whose Lagrangian is:

$$\mathcal{L} = \left(\frac{d}{d\tau}U(\tau)\right)^2 + \frac{1}{2}g_{ab}\left(\frac{d}{d\tau}\Phi^a\right)\left(\frac{d}{d\tau}\Phi^b\right) + V(\Phi,p,q)e^{2U(\tau)}. \tag{3.76}$$

It may be noted that if we introduce the metric[24]:

$$G_{ij} = \begin{pmatrix} 2 & 0 \\ 0 & g_{ab} \end{pmatrix} \tag{3.77}$$

together with the Lagrangian variables $q^i(\tau) = (U(\tau), \Phi^a(\tau))$, the Lagrangian (3.76) becomes[7]:

$$\mathcal{L} = \frac{1}{2}G_{ij}\left(\frac{d}{d\tau}q^i\right)\left(\frac{d}{d\tau}q^j\right) + V(\Phi,p,q)e^{2U(\tau)}. \tag{3.78}$$

[7]Note that here and in the following by abuse of language we adopt the terms Hamiltonian and Lagrangian even if the evolution parameter τ does not describe the temporal evolution but the radial one.

Once the Lagrangian is known we can proceed with a Hamiltonian approach using the phase space that stems from the $q^i(\tau) = (U(\tau), \Phi^a(\tau))$ functions, introducing the conjugate momenta to $q^i(\tau)$[24]:

$$p_i(\tau) = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = G_{ij}\dot{q}^j. \tag{3.79}$$

In terms of the variables q^i and p_i the Hamiltonian $H(p, q)$ then reads:

$$H(p, q) = \frac{1}{2} p_i G^{ij} p_j - V(q)e^{2U}, \tag{3.80}$$

or

$$H(p, q) = \frac{1}{2} \dot{q}^i G^{ij} \dot{q}^j - V(q)e^{2U}. \tag{3.81}$$

From these equations we can say that the constraint (3.74) acquires the meaning of "energy conservation"[24]:

$$H(p, q) = c^2 \leftrightarrow \frac{1}{2} \dot{q}^i G^{ij} \dot{q}^j - V(q)e^{2U} = c^2. \tag{3.82}$$

Chapter 4

Hamilton-Jacobi formalism and static black holes

We have learned in the previous chapter that static and spherically symmetric black holes are conveniently described by an effective one-dimensional Lagrangian. The above construction works well in the static, rotationally invariant case where the metric only depends in a non trivial way on the evolution radial variable τ so that the Einstein Lagrangian can be reduced to an effective one-dimensional Lagrangian.

The fields equations of the effective theory can be described in terms of a set of first order equations, the Hamilton formalism. In this chapter we present the application of the Hamilton-Jacobi equation to the first order description of four dimensional static and spherically symmetric black holes. In particular we discuss that there exists a prepotential characterizing the flow which coincides with the Hamiltonian principal function associated with the one-dimensional Lagrangian[24].

In the study of black holes solutions in supergravity theories, of particular relevance is the issue of describing the spatial evolution of the metric and the scalar fields in terms of a first order dynamical system of equations written in terms of a fake superpotential, also called a function $\mathcal{W}$ or prepotential[24, 42, 43, 44, 45]. In the case of four dimensional static and spherically symmetric black holes, if we collectively denote the scalar and metric degrees of freedom characterizing the solution by $q^i(\tau)$, the issue has been of whether it is possible to define a function called fake superpotential $\mathcal{W}(q^i)$, depending on the quantized electric and magnetic charges and of q^i, such that the radial evolution of q^i is solution to a system of equation[24]:

$$\frac{dq^j}{d\tau} \equiv \dot{q}^j = G^{ij} \left(\frac{\partial \mathcal{W}}{\partial q^i} \right)$$

with G^{ij} being a non-degenerate metric. These equations are suitable for studying the attractors mechanism[50, 20, 98, 22, 99, 39] of black holes solutions[41, 100] as well as higher dimensional black brane solutions[101].

4.1 Introduction: Hamilton-Jacobi formalism

In theoretical physics, the Hamilton-Jacobi formalism is a necessary condition describing extremal geometry in generalizations of calculus of variational problems; while in physics, the Hamilton-Jacobi equation is a reformulation of classical mechanics and, thus, equivalent to other formulations such as Newton's laws of motion, Hamiltonian mechanics and Lagrangian mechanics[102, 103]. The Hamilton-Jacobi equation is particularly useful in identifying conserved quantities for mechanical systems, which may be possible even when the mechanical problem itself can not be solved completely[102].

The Hamilton-Jacobi equation is also the only formulation of mechanics in which the motion of particle can be represented as a wave. In this sense, the Hamilton-Jacobi formalism fulfilled a long-held goal of theoretical physics of finding an analogy between the propagation of light and the motion of a particle. The wave equation followed by mechanical systems is similar to, but not identical with, Schödinger's equation; for this reason, the Hamilton-Jacobi formalism is considered the "closed approach" of classical mechanics to quantum mechanics[102, 7].

The Hamilton-Jacobi equation is a first order, non-linear partial differential equation for a function:

$$S = S(q^i; t) \qquad\qquad (i = 1, \ldots, N) \tag{4.1}$$

called Hamilton's principal function; with q^i are the Lagrangian variables (or generalized coordinates) and t is the time[1]. This equation may be derived from Hamiltonian mechanics by treating S as the generating function for a canonical transformation of the classical Hamiltonian:

$$H = H(q^i; p_i; t) \qquad\qquad (i = 1, \ldots, N) \tag{4.2}$$

with p_i are the conjugate momenta (or generalized coordinates) to q^i[102, 103]. The conjugate momenta correspond to the first derivatives of S with respect to the generalized coordinates:

$$p_i = \frac{\partial S}{\partial q^i}. \tag{4.3}$$

Principal function as solved from the equation from $N + 1$ undetermined constants, the last being from integrating $\frac{\partial S}{\partial t}$, and the first N denoted as $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N$. The relationship then between p_i and q^i describes the orbit in phase space in terms of these constants of motion, and

$$\beta^i = \frac{\partial S}{\partial \alpha_i}. \tag{4.4}$$

are also constants of motion and can be inverted to solve q^i.

Any canonical transformation involving a tipe-2 generating function $G_2(q^i; P_i; t)$ leads to the relations[102, 103]:

$$p_i = \frac{\partial G_2}{\partial q^i} \qquad\qquad Q^i = \frac{\partial G_2}{\partial P_i} \qquad\qquad K = H + \frac{\partial G_2}{\partial t}. \tag{4.5}$$

To derive the Hamilton-Jacobi equations, we choose a generating function $S(q^i; P_i; t)$ that makes the new Hamiltonian K identically zero. Hence, all its derivatives are also zero, and Hamilton's equations become trivial:

$$\frac{dP_i}{dt} = \frac{dQ^i}{dt} = 0; \tag{4.6}$$

[1]For a general discussion of canonical transformations see for example[102, 103].

i.e., the new generalized coordinates and momenta are constants of motion; also the new generalized momenta P_i are usually denoted $\alpha_1, \alpha_2, \ldots, \alpha_{N-1}, \alpha_N$ i.e. $\alpha_i = P_i$.
The equation for the transformed Hamiltonian K is:

$$K = H + \frac{\partial S}{\partial t}; \tag{4.7}$$

let:

$$S(q^i; \alpha_i; t) = G_2(q^i; \alpha_i; t) + A, \tag{4.8}$$

where A is a arbitrary constant, then S satisfies Hamilton-Jacobi equation:

$$H\left(q^i; \frac{\partial S}{\partial q^i}; t\right) + \frac{\partial S}{\partial t} = 0, \tag{4.9}$$

since:

$$p_i = \frac{\partial G_2}{\partial q^i} = \frac{\partial S}{\partial q^i},$$

and with $K = 0$, we have:

$$H\left(q^i; \frac{\partial S}{\partial q^i}; t\right) + \frac{\partial G_2}{\partial t} = 0 \longrightarrow H\left(q^i; \frac{\partial S}{\partial q^i}; t\right) + \frac{\partial S}{\partial t} = 0.$$

The new generalized coordinates Q^i are also constants, typically denoted as $\beta^1, \beta^2, \ldots, \beta^{N-1}, \beta^N$. Once we have solved for $S(q^i; \alpha_i; t)$, these also give useful equations:

$$Q^i = \beta^i = \frac{\partial S(q^i; \alpha_i; t)}{\partial \alpha_i}. \tag{4.10}$$

Ideally, these N equations can be inverted to find the original generalized coordinates q^i as a function of the constants α_i and β^i, thus solving the original problem.

Both Hamilton principal function S and characteristic function are closely related to action; in fact, the time derivative of S is:

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q^i}\frac{\partial q^i}{\partial t} = -H + \sum_i \frac{\partial S}{\partial q^i}\frac{\partial q^i}{\partial t} = L; \tag{4.11}$$

therefore:

$$S = \int L(q^i; \frac{\partial q^i}{\partial t}; t)dt, \tag{4.12}$$

so S is actually classical action plus an undetermined constant.
When H does not explicitly depend on time, one can introduce:

$$W(q^i; \alpha_i) = S(q^i; \alpha_i; t) + Et, \tag{4.13}$$

where W is usually called Hamilton's characteristic function.

4.2 Hamilton-Jacobi equation and static black holes

Once the Lagrangian (3.78) is known we can proceed with a Hamiltonian approach using the phase space that stems from the $q^i(\tau) = (U(\tau), \Phi^a(\tau))$ Lagrangian variables, introducing the conjugate momenta to $q^i(\tau)$[24]:

$$p_i(\tau) = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = G_{ij}\dot{q}^j. \tag{4.14}$$

In terms of the variables q^i and p_i the Hamiltonian $H(p, q)$ then reads:

$$H(p, q) = \frac{1}{2}\dot{q}^i G_{ij}\dot{q}^j - V(q)e^{2U} = \frac{1}{2}p_i G^{ij} p_j - V(q)e^{2U}, \tag{4.15}$$

From these equations we can say that the constraint (3.74) acquires the meaning of "energy conservation"[24]:

$$H(p, q) = c^2 \leftrightarrow \frac{1}{2}\dot{q}^i G_{ij}\dot{q}^j - V(q)e^{2U} = c^2. \tag{4.16}$$

Let us recall how the solution of the equations of motion can be obtained by applying the theory of the Hamilton-Jacobi formalism. We consider the principal Hamiltonian function $S(q^i; P_i; \tau)$ depending on: q^i, new constant momenta P_i and evolution parameter τ. It is defined by the set of first order equation[24]:

$$H = -\frac{\partial S}{\partial \tau}, \qquad p_i = \frac{\partial S}{\partial q^i}, \qquad Q^i = \frac{\partial S}{\partial P_i}, \tag{4.17}$$

where Q^i, P_i are new constant canonical variables which can be expressed in terms of the initial values of q^i and p_i. From the general theory of canonical transformations it is known that the above transformation generated by S always exists locally in the q^i and p_i space, in a neighborhood of any point which is not critical, namely in which[102, 103]:

$$\left(\frac{\partial H}{\partial q}, \frac{\partial H}{\partial p}\right) \neq (\mathbf{0} \ , \ \mathbf{0} \ ). \tag{4.18}$$

From the first two relations of eq. (4.17) we have:

$$S(q^i, \tau) = \mathcal{W}(q^i) - c^2\tau, \tag{4.19}$$

$$p_i = \frac{\partial \mathcal{W}}{\partial q^i}, \tag{4.20}$$

and from the Hamiltonian constraint (4.16), we obtain:

$$H(p,q) = \frac{1}{2}\left(\frac{\partial \mathcal{W}}{\partial q^i}\right) G^{ij} \left(\frac{\partial \mathcal{W}}{\partial q^j}\right) - V(q)e^{2U} = c^2. \tag{4.21}$$

this last equation defines the Hamilton-Jacobi equation for the superpotential $\mathcal{W}$[46, 47], usually called Hamiltonian's characteristic function.

From eq. (4.14) and eq. (4.20) we have:

$$\dot{q}^j = G^{ij} p_j = G^{ij}\left(\frac{\partial \mathcal{W}}{\partial q^j}\right). \tag{4.22}$$

This shows that, provided a solution to the equations (4.19) - (4.21) is found, the evolution of the scalar fields and metric can be described in terms of a dynamical system of the form (4.22)[24].

It may be noted that the functions S in equation (4.19) generalizes the expression for the prepotential conjectured in [42] for the general non-extremal case. To make contact with the proposal in[42], let us consider the following expression for the principal function $S(q^i; U; \tau)$:

$$S(q^i; U; \tau) = 2e^{2U(\tau)} W(q^i; \tau) + c^2\tau = \mathcal{W}(U; q^i) - c^2\tau. \tag{4.23}$$

This equation reproduces the first order expressions for the superpotential as given in[42, 43]:

$$\frac{d}{d\tau}U(\tau) \equiv \dot{U}(\tau) = We^{U}, \tag{4.24}$$

and

$$\frac{d\Phi^a}{d\tau} \equiv \dot{\Phi}^a = 2e^U G^{ab}\left(\frac{\partial \mathcal{W}}{\partial q^b}\right), \tag{4.25}$$

together with the condition found in[42] for the non-extremal case:

$$\frac{\partial W}{\partial \tau} = -c^2 e^{-U}. \tag{4.26}$$

We observe that the solution of the set of differential equations (4.17) - (4.21) in terms of Hamiltonian's principal functions S is formally given by[104]:

$$S(q,\tau) = S_0 + \int_{q_0,\tau_0}^{q,\tau} L(q,\dot{q})d\tau, \tag{4.27}$$

where the integral is performed along the solution of Hamiltonian's equations, i.e. the characteristic trajectory $\gamma = q^i(\tau)$, such that:

$$q^i(\tau) = q^i \qquad\qquad q^i(\tau_0) = q^i_0. \tag{4.28}$$

The above formula provides, in the most general case, only a local definition of S: local in τ, to avoid multivaluedness of S[104], and local in the configuration space with coordinates q^i's, being S defined only on the points q^i, for fixed (q^i_0, τ_0), for which the interpolating characteristic trajectory, satisfying the equations (4.28), exists. In fact locally in the neighborhood of a non-critical point in the phase space, there always exist a complete solution $S(q^i; P_i, \tau)$ to the Hamilton-Jacobi equation and it has the form (4.27), where P_i can be seen as a complete set of integration constants[104]. In what follows, we shall use equation (4.27) bearing its local validity in mind.

In our conditions, we can use the Hamiltonian constraint to find the expression of the principal function in terms of the potential as follows:

$$S(q,\tau) = S_0 + \int_{q_0,\tau_0}^{q,\tau} \left(2V(q)e^{2U} + c^2\right) d\tau, \tag{4.29}$$

so that, using the equations (3.78) and (4.16), the function $\mathcal{W}$ is given by:

$$\mathcal{W}(q,\tau) = \mathcal{W}_0 + \int_{q_0,\tau_0}^{q,\tau} \left(\mathcal{L}(q,\dot{q}) + c^2\right) d\tau, \tag{4.30}$$

or

$$\mathcal{W}(q,\tau) = \mathcal{W}_0 + 2\int_{q_0,\tau_0}^{q,\tau} \left(V(q)e^{2U} + c^2\right) d\tau. \tag{4.31}$$

Actually the above equation can also be derived from direct integration of formula (4.21)[24]. Indeed the eq. (4.21) has the form of the eikonal equation for a wave front $\mathcal{W} = const.$ propagating in a medium of reflective index n

$$n = \sqrt{2\left(V(q)e^{2U} + c^2\right)} : \tag{4.32}$$

$$n^2 = \left(\frac{\partial\mathcal{W}}{\partial q^a}\right) G^{ab} \left(\frac{\partial\mathcal{W}}{\partial q^b}\right). \tag{4.33}$$

From equation (4.22) we get that $\partial_a\mathcal{W}$ is tangent to the "light rays" namely the characteristics $\gamma = q^i(\tau)$. Introducing the proper distance s along a characteristic:

$$ds = \sqrt{\dot{q}^a G_{ab}\dot{q}^b} d\tau = \sqrt{2\left(V(q)e^{2U} + c^2\right)} d\tau, \tag{4.34}$$

using the eq. (4.22), we have got:

$$\begin{aligned} \frac{d\mathcal{W}}{ds} &= \frac{\partial\mathcal{W}}{\partial q^a}\frac{dq^a}{ds}\frac{d\tau}{ds} \\ &= \left(\frac{\partial\mathcal{W}}{\partial q^a}\right) G^{ab} \left(\frac{\partial\mathcal{W}}{\partial q^b}\right)\frac{d\tau}{ds} \\ &= \sqrt{2\left(V(q)e^{2U} + c^2\right)}, \end{aligned} \tag{4.35}$$

that is:

$$d\mathcal{W} = \sqrt{2\left(V(q)e^{2U} + c^2\right)} ds = 2\left(V(q)e^{2U} + c^2\right) d\tau. \tag{4.36}$$

We can observe, from the above equation follows it that $d\mathcal{W}/d\tau$ along γ is a monotonic increasing function of τ along a solution[24] and the same is true for the principal functions S, since the Lagrangian is non-negative.

Before we finish this section, let us review the construction of function $\mathcal{W}$ for the Reissner-Nordström (R-N) black holes[105]. The q^a variables now consist of the function U alone. This is for instance a solution to $N = 2$ pure supergravity. With respect to the only vector field of the theory, the graviphoton, the solution can have in general a magnetic and an electric charge p, e. The geodesic potential reads[24]:

$$V = Q^2 e^{2U}; \tag{4.37}$$

and

$$Q^2 = \frac{1}{2}(e^2 + p^2). \tag{4.38}$$

The Hamilton-Jacobi equation and the Hamilton constraint read:

$$(\dot{U}(\tau))^2 = \frac{\partial \mathcal{W}}{\partial U} = \left(Q^2 e^{2U} + c^2\right). \tag{4.39}$$

We can then apply equation (4.31) to find, upon changing variables from τ to U:

$$\begin{aligned} \mathcal{W}(q,\tau) &= \mathcal{W}_0 + 2\int_{U_0,\tau_0}^{U,\tau} \left(Q^2 e^{2U} + c^2\right) d\tau \\ &= \mathcal{W}_0 + 2\int_{U_0}^{U} \left(Q^2 e^{2U} + c^2\right) \frac{1}{\dot{U}} dU \\ &= \mathcal{W}_0 + 2\int_{U_0}^{U} \sqrt{(Q^2 e^{2U} + c^2)} dU \end{aligned} \tag{4.40}$$

whose solution is:

$$\mathcal{W}(q,\tau) = \mathcal{W}_0 + 2\left\{\sqrt{(Q^2 e^{2U} + c^2)} - \frac{c}{2} ln\left(\frac{\sqrt{(Q^2 e^{2U} + c^2)} + c}{\sqrt{(Q^2 e^{2U} + c^2)} - c}\right)\right\}. \tag{4.41}$$

4.3 The $\mathcal{W}$ superpotential and duality

Let us now consider an extended supergravity theory in four dimensions. It is known that the global symmetries of the Bianchi identities and the equations of motion are encoded in the isometry of group G of the scalar manifold, whose action on the scalar fields is associated with a simultaneous linear symplectic action on the field strengths F^Λ and their duals G_Λ[106, 30, 107]. The duality action of group G is defined by an embedding D of G inside the group $Sp(2n_V, R)$[24, 46]:

$$g \in G : \begin{cases} \Phi^a \to \Phi^{a'} = g \star \Phi^a \\ \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \to D(g) \cdot \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix} \end{cases} \tag{4.42}$$

where $D(g)$ is the $2n_V$ x $2n_V$ symplectic matrix associated with g and $g\star$ is the non-linear action of g on the scalar fields.

We are going to prove that the function $\mathcal{W}(q)$ is invariant under the duality action of G. Recalling that the metric, and therefore the function U, is dual invariant field[46, 24], we define:

$$(g \star q^i) \equiv (U, g \star \Phi^a). \tag{4.43}$$

The on-shell global invariance of the four dimensional theory under G means that, if $\gamma = (q^j(\tau))$ is a characteristic trajectory of the Lagrangian system:

$$\mathcal{L} = \frac{1}{2} G_{ij} \left(\frac{d}{d\tau} q^i \right) \left(\frac{d}{d\tau} q^j \right) + V(\Phi, p, q) e^{2U(\tau)} \tag{4.44}$$

with electric e magnetic charge parameters $Q = (p^\Lambda, q_\Lambda)$ then

$$g \star \gamma = (g \star q^i(\tau)) \tag{4.45}$$

is a trajectory of the Lagrangian system (4.45) with charge parameters $D(g) \cdot Q$.

Now we evaluate the dependence of the geodesic potential Ve^{2U} on the electric and magnetic charges explicit by writing $V(q, Q)e^{2U}$. From general properties of the symplectic matrix $M(\Phi)$, defined in equation (3.22), we have got:

$$M(g \star q) = D(g)^{-T} M(q) D(g)^{-1}. \tag{4.46}$$

From this is follows that the effective potential Ve^{2U}, or equivalently V, is duality invariant, in the sense that:

$$Ve^{2U}(g \star q, D(g) \cdot Q) = Ve^{2U}(q, Q); \tag{4.47}$$

or equivalently

$$V(g \star q, D(g) \cdot Q) = V(q, Q). \tag{4.48}$$

The group G is then a global symmetry group of the one-dimensional Lagrangian $\mathcal{L}$ in equation (4.44) and thus of both Hamilton's principal and characteristic functions $S(q, \tau; Q)$ and function $\mathcal{W}(q; Q)$[24]. In fact, from the equations (4.31) and (4.47) we have:

$$\begin{aligned}
\mathcal{W}(q; Q) &= \mathcal{W}_0 + 2 \int_{q_0, \tau_0}^{q, \tau} \left(V(q)e^{2U} + c^2\right) d\tau \\
&= \mathcal{W}_0 + 2 \int_{g \star q_0, \tau_0}^{g \star q, \tau} \left\{V(g \star q, D(g) \cdot Q)e^{2U} + c^2\right\} d\tau \\
&= \mathcal{W}(g \star q, D(g) \cdot Q).
\end{aligned} \tag{4.49}$$

We can assert that a duality transformation $g \in G$ maps a black hole solution $(U(\tau), \Phi^a(\tau))$ with electric and magnetic charges $Q = (p^\Lambda, q_\Lambda)$ into a new solution $(U^{'}(\tau) = U(\tau), \Phi^{'a}(\tau) = g \star \Phi^a(\tau))$ with charges $Q^{'} = D(g) \cdot Q$. More accurately, if $U(\tau), \Phi^a(\tau)$ is defined by the boundary condition Φ_0 for the scalar fields $U^{'}(\tau) = U(\tau)$, $\Phi^{'a}(\tau)$ is the unique solution, within our class, with charges $Q^{'} = D(g) \cdot Q$ defined by the boundary condition $\Phi^{'}_0(\tau) = g \star \Phi_0$[46]:

$$g \in G : \left\{ \begin{array}{c} U(\tau; \Phi_0) \\ \\ \Phi(\tau, \Phi_0) \\ \\ Q \end{array} \right. \quad \rightarrow \left\{ \begin{array}{c} U^{'}(\tau; g \star \Phi_0) = U(\tau, \Phi_0) \\ \\ \Phi^{'}(\tau; g \star \Phi_0) = g \star \Phi(\tau, \Phi_0) \\ \\ Q^{'} = D(g) \cdot Q \end{array} \right. \tag{4.50}$$

We now show that the superpotential W shares with $\mathcal{W}$ the same symmetry property (4.48), namely that it is G-invariant as well:

$$W(q; Q) = W(g \star \Phi, D(g) \cdot Q). \tag{4.51}$$

This is shown using the equations (4.50) and the general form of the function W:

$$\begin{aligned} W(g \star \Phi_0, D(g) \cdot Q) &= \int_{-\infty}^{0} \left[e^{2U'(\tau; g\star\Phi_0)} V(\Phi'(\tau; g \star \Phi_0); D(g) \cdot Q) \right] d\tau = \\ &= \int_{-\infty}^{0} \left[e^{2U(\tau,\Phi_0)} V(\Phi(\tau, \Phi_0); Q) \right] d\tau = \\ &= W(\Phi_0, Q). \end{aligned} \tag{4.52}$$

Recalling that the ADM mass can be expressed in terms of W function, it is a G-invariant quantity as well:

$$M_{ADM}(\Phi_0; Q) = M_{ADM}(g \star \Phi_0, D(g) \cdot Q). \tag{4.53}$$

Extremal black holes can be grouped into orbits with respect to the duality action, eq.s (4.50), of G. These orbits are characterized in terms of G-invariant functions of the quantized charges and scalar fields, which are expressed in terms of H-invariant functions of the matter and central charges[46]. One of these is the scalar-independent quartic invariance $I_4(Q)$ of G which defines the area of the horizon for large black holes. Small black holes are characterized by vanishing horizon area, in other words, belong to the orbits in which $I_4(Q) = 0$.

Chapter 5

Rotating black holes and first order formalism

As we have learned in the previous three chapters, a formalism was developed to interpret the first-order description of static and spherically symmetric black holes in terms of Hamilton-Jacobi theory. In particular, the Hamilton characteristic function was shown to coincide with the fake superpotential $\mathcal{W} = e^{2U} W$ [42, 24] where W is the Hamilton characteristic function.

In this chapter we consider axisymmetric black holes in supergravity and address the general issue of defining a first order description for them. The natural setting where to formulate the problem is the De Donder-Weyl-Hamilton-Jacobi theory associated with the effective two-dimensional sigma-model action describing the axisymmetric solutions[59]. We write the general form of the two functions S_m defining the first-order equations for the fields. It is invariant under the global symmetry group $G_{(3)}$ of the sigma-model. We also discuss the general properties of the solutions with respect to these global symmetries, showing that they can be encoded in two constant matrices belonging to the Lie algebra of $G_{(3)}$, one being the Nöether matrix of the sigma model, while the other is non-zero only for rotating solutions. These two matrices allow a $G_{(3)}$-invariant characterization of the rotational properties of the solution and of the extremality condition. We also comment on extremal, under-rotating solutions from this point of view[1].

5.1 Introduction to Rotating black holes, global symmetry and first order formalism

There has been a considerable progress in the knowledge of static black holes in supergravity, both from the point of view of finding solutions and of their classification[9, 10, 108, 109, 3], in four and higher dimensions.

A relevant role in these developments was played by the use of a first order formalism, corresponding to the introduction of a fake-superpotential[43, 44, 42, 100, 24, 47, 48, 49, 46] that was recognized to be strictly related to the Hamilton characteristic function in a mechanical problem where the evolution is in the radial variable τ [42, 24, 46]. The latter approach naturally applies to both extremal and non-extremal static, single center black holes.

As far as more general solutions, such as stationary and/or multicenter black holes[110, 111, 112, 113, 114, 115], are concerned, a similar comprehensive study is still missing. In particular, the use of a first order formalism has not been much exploited except in very particular cases[116, 117, 118].

A peculiarity of static, spherically symmetric solutions is that one can exploit the symmetries to reduce the Lagrangian to a one-dimensional effective one, where the evolution variable is the radial one [23, 50]. However, when considering four dimensional solutions with less symmetries, in particular stationary solutions where only the time-like Killing vector ∂_t is present, an effective three-dimensional Lagrangian can be obtained upon compactification along the time coordinate [51, 52, 53, 54, 21, 55, 56, 57]. The fields in the effective Lagrangian now depend on the three space variables x^i, $(i = 1, 2, 3)$.

[1]The basis reference, for this chapter, is the paper "Rotating black holes, global symmetry and first order formalism" by Laura Andrianopoli, Riccardo D'Auria, Paolo Giaccone and Mario Trigiante[59].

In particular, for stationary axisymmetric solutions, the presence of an azimuthal angular Killing vector ∂_φ allows a further dimensional reduction to two dimensions.

The problem of extending the Hamilton-Jacobi formalism from mechanical models, whose degrees of freedom depend on just one variable, to field theories where the degrees of freedom depend on two or more variables, was addressed and developed in generality from several points of view (a useful review is given by [58, 59]).

Our main aim in the present chapter is to apply such extended formalism in the study of black holes. We will adhere to the so-called De Donder-Weyl-Hamilton-Jacobi theory, hereafter referred to as DWHJ, which is the simplest extension of the classical Hamilton-Jacobi approach in mechanics[58, 59]. One important difference with respect to the case of classical mechanics consists in the replacement of the Hamilton principal function S, directly related to the fake-superpotential of static black holes, with a *Hamilton principal* 1-*form*, that is with a covariant vector S_i.

As it is usual in the three dimensional approach, by using Hodge-duality in three dimensions all the fields of the parent four dimensional theory are described by three dimensional scalars [51] and their interaction is given by gravity coupled to a σ-model. Correspondingly, the equations of motion give a set of conserved currents. A particularly interesting case is when the σ-model is a symmetric space $G_{(3)}/H^*$, where H^* denotes a suitable non-compact maximal subgroup of $G_{(3)}$ [51]. Note that the effective geodesic Lagrangian is invariant under the three-dimensional isometry group $G_{(3)}$ (we will also refer to it as the three-dimensional *duality group*). One of the main results of our paper is to give a manifestly duality invariant expression for the Hamilton principal vector S_i, thus extending the results obtained for the Hamilton characteristic function $\mathcal{W}$ in the static case [24].

For pure Einstein-Maxwell (E–M) stationary configurations, the three-dimensional σ-model turns out to be $\mathrm{SU}(1,2)/\mathrm{U}(1,1)$. As is well known in General Relativity, in the presence of a time-like Killing vector Einstein-

Maxwell theory is very efficiently described in terms of the so-called Ernst potentials $\mathcal{E}$, Ψ[119, 120], which are complex functions of the $\mathrm{SU}(1,2)$ complex triplet of fields $\mathbb{U} = (W_E, V_E, U_E)$. We found particularly useful, outside the ergosphere, to parametrize the coset $\mathrm{SU}(1,2)/\mathrm{U}(1,1)$ with the homogeneous fields U_E, V_E, W_E, or more precisely with their inhomogeneous counterpart $(u = U_E/W_E, v = V_E/W_E)$, corresponding to four real scalar degrees of freedom[59].

In the present chapter we will give general results on stationary axisymmetric solutions of four dimensional supergravity and then focus on the first-order formulation of the Kerr-Newman solution and its extension in the presence of a NUT charge. Besides finding a duality invariant S_i, we will also express the conserved charges of the black hole [2] in terms of the conserved charges of the σ-model $G_{(3)}/H^*$. Actually, the Nöether charges associated with $G_{(3)}$ global symmetry do not include the angular momentum M_φ. The latter can nevertheless be expressed in terms of quantities which are intrinsic to the σ-model. This is achieved by introducing a new $G_{(3)}$-covariant constant matrix, besides the Nöether charge one $\tilde{Q}$, defined as follows:

$$Q_\psi = -\frac{3}{8\pi} \int_{S_2^\infty} \psi_{[i}\, J_{j]}\, dx^i \wedge dx^j \,,$$

J_i being the Nöether current with value in the Lie algebra of $G_{(3)}$ and $\psi = \partial_\varphi$ is the azimuthal angle Killing vector. From straightforward application of the general four-dimensional expression for the angular momentum one finds that its squared value, for the Kerr-Newmann solution, can be written as the ratio of two $G_{(3)}$ invariants $\mathrm{Tr}(Q_\psi^2)$ and $\mathrm{Tr}(\tilde{Q}^2)$, and thus can be given a description which is invariant with respect to the global symmetry of the σ-model and is straightforwardly generalizable to more general models with $D = 4$ scalar fields[59]. This analysis also provides a $G_{(3)}$-invariant characterization of the extremality parameter (and thus of the extremality condition), see eq.s (5.127), (5.128), so that the cosmic-censor condition for Kerr black holes, $M_{ADM}^4 \geq M_\varphi^2$, can be recast for the generic regular axisymmetric solution, in a $G_{(3)}$-invariant way as:

$$[\mathrm{Tr}(\tilde{Q}^2)]^2 \geq \frac{2}{k}\,\mathrm{Tr}(Q_\psi^2)\,,$$

k being a $G_{(3)}$ representation-dependent constant. In particular we show that in the extremal "ergo-free" solutions [121, 122, 123, 124, 114], both matrices $\tilde{Q}$, Q_ψ are *nilpotent*, the former having a larger degree of nilpotency of the latter. The first-order formalism and the functions S_m for for under-rotating solutions were derived in paper "Multi-Centered Black Hole Flows" by A. Yeranyan[117].

A description of the global symmetry properties of axisymmetric solutions should then include at least the *two independent, mutually orthogonal matrices* $\tilde{Q}$, Q_ψ belonging to the Lie algebra of the global symmetry group[59].

5.2 Hamilton-Jacobi formalism for field theory

We have learned in previous two chapter a formalism was developed to interpret the first-order description of static and spherically symmetric black holes in terms of Hamilton-Jacobi theory. In particular, the Hamilton characteristic function W was shown to be related, for extremal solutions, to the "fake" superpotential $\mathcal{W} = e^{2U}W$ [42, 24]. The above construction works well in the static, rotationally invariant case where the metric only depends in a non-trivial way on the evolution radial variable τ so that the Einstein Lagrangian can be reduced to an effective one-dimensional Lagrangian. For more general black holes with a lower number of isometries we have to extend the Hamilton-Jacobi formalism to a more general setting. In particular, for stationary black holes corresponding to the existence of a Killing vector associated to time translations $\frac{\partial}{\partial t}$[51], the metric can be reduced to the following the general form[59]:

$$ds^2 = e^{2U}(dt - B_i dx^i)^2 - e^{-2U} g^3_{ij} dx^i dx^j \tag{5.1}$$

where the field U, B_i and the 3D metric tensor g^3_{ij} depend on the space coordinates x^i, $i = 1, 2, 3$.

In the static, spherically symmetric case, the Hamilton-Jacobi equations arise in a classical mechanical effective model where the evolution variable τ plays the role of time. A first-order formulation for a more general black-hole solution requires the extension of the Hamilton-Jacobi description from classical mechanics to a field theory depending on two or more variables, see, for example, [58] and references therein. In this setting the Hamilton-Jacobi description has to be generalized to the so-called De Donder-Weyl-Hamilton-Jacobi theory, hereafter referred to as DWHJ, which amounts to the following. Let $\mathcal{L}(z^a, v^a_i, x^i)$ be the Lagrangian density of the system, where z^a $(a = 1, \cdots, n)$ are the field variables which become functions of the x^i, $z^a = \xi^a(x)$, on the extremals, while $v^a_i = \partial_i \xi^a$ on the extremals.[2] The canonical momenta are defined by $\pi^i_a = \frac{\partial L}{\partial v^a_i}$, and the invariant Hamilton density function is:

$$\mathcal{H} = \pi^m_a v^a_m - \mathcal{L}. \tag{5.2}$$

The DWHJ equation is a first-order partial differential equation for the functions $S^i(z, x)$[58, 59]:

$$\partial_i S^i(z, x) + \mathcal{H}(z, x, \pi) = 0\,, \tag{5.3}$$

where

$$\pi^i_a = \partial_a S^i(z, x). \tag{5.4}$$

The functions

$$S_i = \frac{1}{\sqrt{g}} g_{ij} S^j$$

may be thought of as the components of a one-form $S^{(1)} \equiv S_i dx^i$.[3]

[2] With an abuse of notation, we will often use $\partial_i z^a$ to denote the v^a_i.

[3] We observe that, in the presence of a gravitational field, which is the case we will deal

In the field-theory case the issue of integrability is more involved than in mechanics since, even if a complete integral S^i can be found, solutions to the Euler-Lagrange equations can be constructed if the integrability conditions, which are trivial in mechanics:

$$\partial_{[i} v^a_{j]} = 0 \tag{5.5}$$

are satisfied. Taking into account that:

$$v^a_i(\pi, z, x) = v^a_i\left(\frac{\partial S}{\partial z}, z, x\right)$$

this imposes severe constraints on the solutions $S^i(z, x)$. From now on we will mainly focus on the two dimensional case, which is relevant when discussing axisymmetric black holes for which two Killing vectors exist, associated with time translations $\frac{\partial}{\partial t}$ and rotations about an axis $\psi = \frac{\partial}{\partial \varphi}$. Note however that the extension of the formalism from systems depending on two independent variables to systems with three or more independent variables is straightforward and does not bring anything conceptually new[58]. We will denote the independent variables for the two-dimensional case by x^m, $m = 1, 2$. The 3D metric in this case takes the form:

$$g_{ij} dx^i dx^j = \gamma_{mn} dx^m dx^n + \hat{\rho}^2 d\varphi^2 \tag{5.6}$$

where φ denotes the azimuthal angle about the rotation axis, and the fields $\gamma_{mn}, \hat{\rho}$ depend on x^m.

If one introduces the two-form Lagrangian

$$\Omega_0 = -\mathcal{H} dx^m \wedge dx^n + \pi^m_a d\xi^a \wedge dx^n \epsilon_{mn} \tag{5.7}$$

then the Hamilton–Jacobi equations are given by the condition

$$d\Omega_0 = 0, \tag{5.8}$$

with, (5.3) should be modified to contain the covariant divergence $\nabla_i S^i$. However, defining the contravariant vector density $S^i \equiv \sqrt{g} g^{ij} S_j$, S_j being a true covariant vector, makes it possible to trade the covariant derivatives for ordinary ones, so that the equations are formally the same as in flat space. In this case, however, by $\mathcal{H}$ we mean the Hamiltonian density including the factor $\sqrt{|g|}$.

which implies that, locally, there exist two functions S^m in terms of which Ω_0 can be written in the following form:

$$\Omega_0 = dS^m \wedge dx^n \epsilon_{mn}, \tag{5.9}$$

so that[4]:

$$\partial_m S^m = -\mathcal{H}, \tag{5.10}$$

$$\frac{\partial S^m}{\partial z^a} = \pi^m_a. \tag{5.11}$$

5.2.1 Solving DWHJ equations

In the present subsection we discuss in a general setting a possible way to solve the DWHJ equations. Then, in the next sections we will apply this procedure to the study of axisymmetric black holes and their Taub-NUT extensions[5]. We will give here a constructive recipe to find solutions to the field equations by solving the DWHJ equations, following a general procedure given in the literature[58] [6].

As already anticipated, in field theory the expression for S^m is strongly restricted by the integrability constraints (5.5). In particular, as opposed to the one-dimensional classical-mechanics case, it is not always possible to find an expression for S^m valid in an open neighborhood of the extremals $z^a = \xi^a(x)$ in the space of fields *and* coordinates. When this is possible, one says that the extremals $z^a = \xi^a(x)$ are *strongly* embedded in the wave fronts $S^m(z, x)$. In many cases, however, the solution S^m satisfies equations

[4]We denote with ∂_m the derivative with respect to explicit x^m dependence, while total derivative with respect to x^m is denoted by $\frac{d}{dx^m}$:

$$\frac{d}{dx^m} f(\xi, x) \equiv \partial_m \xi^a \frac{\partial f}{\partial \xi^a} + \partial_m f$$

[5]See Appendix C.

[6]See for example [58] and references therein.

(5.10) and (5.11) only on the extremals $z^a = \xi^a(x)$. One then says that the extremals are *weakly* embedded in $S^m(z,x)$.

A possible solution which is weakly embedded in S^m is found by choosing one of the x^m, say x^1, as the evolution variable[59]:

$$\begin{aligned} S^m = (z^a - \xi^a(x))\, \pi_a^m(\xi, x) + \\ +\delta_1^m \int^{x^1} dx^{1'} \mathcal{L}(\xi(x'), \partial_m \xi, x') + O[(z^a - \xi^a(x))^2]\,. \end{aligned} \tag{5.12}$$

Indeed, from (5.12) we find, using (5.2)

$$\partial_a S^m|_{z=\xi} = \pi_a^m \tag{5.13}$$

$$\partial_m S^m|_{z=\xi} = -\partial_m \xi^a \pi_a^m + \mathcal{L}(\xi(x'), \partial_m \xi, x') = -\mathcal{H}(\xi(x'), \partial_m \xi, x')\,. \tag{5.14}$$

Note that equation (5.12) can be understood as a linear approximation of the Taylor expansion of S^m in the neighborhood of the extremal.

5.3 Dimensional "reduction" of $D = 4$, $N = 2$ supergravity

In this section we describe the dimensional reduction of $D = 4$, $N = 2$ supergravity theory to three dimensions relevant for stationary black hole configurations, leading to a non-linear sigma models for coset manifold $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)}$ and discuss some group theoretical points connected with its particular structure. Furthermore we construct the two dimensional theories describing stationary and axially symmetric solutions. In order to find these solutions, we use the techniques developed in the 1988 paper by P. Breitenhner, D. Maison and G. W. Gibbons[51].

5.3.1 Dimensional "reduction" of four dimensional from four to three dimensions (time-reductions)

Four-dimensional rotating black holes depend on a number of degrees of freedom which includes, besides the metric and scalars, also the degrees of freedom corresponding to the gauge potential. It is then natural to extend the phase space to include as further degrees of freedom the electric and magnetic potentials together with their conjugate momenta. This approach was pioneered in[125] in the case of double extremal black holes.

As it is well known, this approach is equivalent to a time reduction of the four dimensional field theory[51]. In this subsection, we describe the dimensional "reduction" of $D = 4$, $N = 2$ supergravity to a non-linear sigma model coupled to gravity in $D = 3$.
The metric of Kerr-Newmann (KN) black hole, with mass M, electric charge Q and magnetic charge P, can be written in the form:

$$ds^2 = g_{tt}[dt - B(r,\theta)d\varphi]^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2; \tag{5.15}$$

with:

$$B(r,\theta) = \frac{\alpha\left(\frac{(Q^2+P^2)}{2} - 2Mr\right) sin^2(\theta)}{\tilde{\Delta}}$$

$$B(r,\theta) = \frac{\alpha(\tilde{\Delta} - \rho^2)sin^2(\theta)}{\tilde{\Delta}}, \tag{5.16}$$

and

$$\Delta = r^2 - r_s r + \frac{(Q^2+P^2)}{2} + \alpha^2; \tag{5.17}$$

$$\tilde{\Delta} = r^2 - r_s r + \frac{(Q^2+P^2)}{2} + \alpha^2 cos^2(\theta); \tag{5.18}$$

$$\rho^2 = r^2 + \alpha^2 cos^2\theta; \tag{5.19}$$

$$\alpha = \frac{J}{Mc} = \frac{J}{M}$$

where α representing the specific angular momentum J of the source and g_{ij} are the components of the metric tensor; and we have:

$$ds^2 = -e^{2U}[dt - B(r,\theta)d\varphi]^2 + e^{-2U}ds_3^2 \tag{5.20}$$

with:

$$e^{2U} = -g_{tt}; \tag{5.21}$$

and the 3-metric ds_3^2 is:

$$ds_3^2 = -g_{tt}\left(g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2\right) = g^{(3)}{}_{ij}dx^i dx^j; \tag{5.22}$$

where x^i, $i = 1, 2, 3$ are the coordinates of the final Euclidean space.
Similarly we decompose the vector fields:

$$A = A_\mu dx^\mu = A_0 dt + A_\varphi d\varphi = A_0 V + A_3; \tag{5.23}$$

with:

$$A_0 = -\frac{[Qr - \alpha P cos(\theta)]}{\rho^2}, \tag{5.24}$$

and

$$A_\varphi = \frac{[\alpha Q r sin^2(\theta) - (r^2 + \alpha^2) P cos(\theta)]}{\rho^2}, \tag{5.25}$$

into A_0 and A_3, from which the $3 - D$ field strengths can be computed.
We start from the four dimensional bosonic action of a generic supergravity theory, describing m scalar fields Φ^s coupled to n_V of vectors field $\hat{A}^\Lambda_\mu$[3]:

$$S_4 = \int \sqrt{-g^{(4)}} d^4x \left(+\frac{1}{2}\hat{R} + \frac{1}{4} I_{\Lambda\Gamma} \hat{F}^\Lambda_{\hat{\mu}\hat{\nu}} \hat{F}^{\Gamma|\hat{\mu}\hat{\nu}} + \right.$$

$$\left. + \frac{1}{8\sqrt{-g^{(4)}}} R_{\Lambda\Gamma} \varepsilon^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \hat{F}^\Lambda_{\hat{\mu}\hat{\nu}} \hat{F}^\Gamma_{\hat{\rho}\hat{\sigma}} - \frac{1}{2} g_{rs}(\phi) \partial_{\hat{\mu}} \phi^r \partial^{\hat{\mu}} \phi^s \right); \tag{5.26}$$

where the gauge field-strength two-form is defined as $\hat{F}^\Gamma = d\hat{A}^\Gamma$ and $R_{\Lambda\Gamma}$, $I_{\Lambda\Gamma}$ are the real and imaginary part of the complex kinetic matrix $\mathcal{N}_{\Lambda\Gamma}(\Phi)$, with the convention that $I_{\Lambda\Gamma} < 0$. Note that we have adopted a notation in which four-dimensional indices are denoted with a hat. For the fields themselves a similar notation is used, except for the scalar fields, as their reduction is trivial.
For the vector fields on the other hand, the following ansatz is used:

$$\hat{A}^\Gamma_\mu = A^\Gamma_\mu + A^\Gamma_0 V; \tag{5.27}$$

$$\hat{A}_0^\Gamma = V. \tag{5.28}$$

Using the above equations, one obtains the following Lagrangian in three dimensions:

$$\mathcal{L}_{(3d)} = e_{(3)} \left[\frac{1}{2} R - (dU)^2 + \frac{e^{4U}}{8} F^{(0)}_{\mu\nu} F^{(0)\mu\nu} - \frac{1}{2} g_{rs}(\phi) \partial_i \phi^r \partial^i \phi^s + \right.$$
$$\left. + \frac{e^{2U}}{4} I_{\Lambda\Gamma} F^\Lambda_{ij} F^{\Gamma|ij} - \frac{e^{-2U}}{2} \partial_i A^\Lambda_0 I_{\Lambda\Gamma} \partial^i A^\Gamma_0 + \frac{1}{2e_{(3)}} \varepsilon^{ijk} R_{\Lambda\Gamma} F^\Lambda_{ij} \partial_k A^\Gamma_0 \right], \tag{5.29}$$

with $R \equiv R_{(3)}$. The above Lagrangian still contains the three-dimensional vector fields. In three dimensions however, vectors are dual to scalar fields; thus one can obtain a Lagrangian where only metric and the scalar fields are present. In order to dualize the vectors, we add the Lagrange multipliers A^Γ and $\tilde{a}$ to $\mathcal{L}_{(3d)}$ by writing:

$$\mathcal{L}_{(mult)} = \frac{\varepsilon^{ijk}}{2} \left[-F^\Gamma_{ij} \partial_k A_\Gamma + \frac{1}{2} F^{(0)}_{ij} \partial_k \tilde{a} \right]. \tag{5.30}$$

Considering:

$$\mathcal{L} = \mathcal{L}_{(3d)} + \mathcal{L}_{(mult)} \tag{5.31}$$

and imposing:

$$\frac{\partial \mathcal{L}}{\partial F^\Lambda_{\mu\nu}} = 0, \tag{5.32}$$

$$\frac{\partial \mathcal{L}}{\partial F^{(0)}_{\mu\nu}} = 0 \tag{5.33}$$

the following duality relations are obtained:

$$F^\Lambda_{ij} = e_{(3)} e^{-2U} \varepsilon_{ijk} I^{-1|\Lambda\Gamma} \left(\partial^k A_\Gamma - R_{\Gamma\Sigma} \partial^k A^\Sigma_0 \right), \tag{5.34}$$

and

$$F^{(0)}_{ij} = -e_{(3)} e^{-4U} \varepsilon_{ijk} \left(\partial^k \tilde{a} + 2 A^{\Gamma|T}_0 \partial^k A_\Gamma \right) \tag{5.35}$$

with:

$$\tilde{a} = a - A_0^{\Gamma|T} A_\Gamma \tag{5.36}$$

and

$$\partial_k \tilde{a} = \partial_k a - \partial_k A_0^{\Gamma|T} A_\Gamma - A_0^{\Gamma|T} \partial_k A_\Gamma \,. \tag{5.37}$$

We have:

$$F_{ij}^{(0)} = -e_{(3)} e^{-4U} \varepsilon_{ijk} \left(\partial^k a + \mathcal{Z}^T \mathbb{C} \partial^k \mathcal{Z} \right) \tag{5.38}$$

where $\mathbb{C}$ being the antisimmetric $2n_V$ x $2n_V$ $Sp(2n_V, R)$-invariant metric, for which we shall use the following invariant metric[24, 46]:

$$\mathbb{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{5.39}$$

In this theory all the vectors are dualized to scalar fields so as to obtain a sigma model coupled to gravity. Let us introduce the n_v three dimensional scalars $\zeta^\Lambda = A_0^\Lambda$ which, together with the scalars $\tilde{\zeta}^\Lambda$, dual in $D = 3$ to the vectors A_i^Λ, form the symplectic vector of electric and magnetic potentials $\mathcal{Z}^M = (\zeta^\Lambda, \tilde{\zeta}_\Lambda)$. Finally we shall denote by a the axion dual in $D = 3$ to the Kaluza-Klein vector A_i^0[24]. The final $D = 3$ action reads:

$$S_3 = \int \sqrt{|g^{(3)}|} d^3x \left(+\frac{1}{2} R_3 - \frac{1}{2} G_{IJ}(\Phi) \partial_\mu \Phi^I \partial^\mu \Phi^J \right) ; \tag{5.40}$$

where: $g_{(3)} \equiv \det(g_{(3)})$, $\Phi^I = (U, \phi^r, a, \mathcal{Z}^M)$, and the sigma model metric reads:

$$\begin{aligned} &\frac{1}{2} G_{IJ}(\Phi) d\Phi^I d\Phi^J = \\ (dU)^2 + \frac{1}{2} g_{rs}(\phi) d\phi^r d\phi^s &+ \frac{e^{-4U}}{4} (\omega)^2 + \frac{e^{-2U}}{2} d\mathcal{Z}^T \mathcal{M} d\mathcal{Z} ; \end{aligned} \tag{5.41}$$

with

$$\mathcal{M} \equiv \mathcal{M}_{(4)} = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix} . \tag{5.42}$$

and the one-form ω is defined as:

$$\omega = da + \mathcal{Z}^T \mathbb{C} d\mathcal{Z}; \tag{5.43}$$

where $\mathbb{C}$ is the antisimmetric $2n_V$ x $2n_V$ $Sp(2n_V, R)$-invariant metric.
In the Kerr-Newmann case, we have that:

- $I = -1$ and $R = 0$;
- A and A_0 are two scalars;
- the axion dual a is:

$$a = \frac{\alpha(2M + CP)cos(\theta) - CQr}{\rho^2} + cost; \tag{5.44}$$

where C is a constant, and taking $C = 0$ we have:

$$a = \frac{2\alpha M cos(\theta)}{\rho^2} + cost. \tag{5.45}$$

Here, all the propagating degrees of freedom have been reduced to scalars by 3D Hodge-dualization[51]. In particular, a is the Hodge-dual of the 3D graviphoton B_i and the scalars $\mathbf{Z}^M = (\mathcal{Z}^\Lambda, \mathcal{Z}_\Lambda)$ include the electric components A_0^Λ of the 4D vector fields together with the Hodge dual of their magnetic components A_i^Λ $(i = 1, 2, 3)$. Finally, $\mathcal{M}_{(4)}(\phi)$ is the negative-definite symmetric, symplectic matrix depending on 4D scalar fields introduced in [126, 127].

The isometry group $G_{(3)}$ of the σ-model metric $G_{ab}(\xi)$ contains as non trivial subgroups the 4-dimensional U-duality group $G_{(4)}$ times the group $SU(1,2)$ under which the degrees of freedom of the 4d metric transform. The latter factor is universal and actually it is the 3D isometry group of any 4D Einstein–Maxwell gravitational theory with a single Killing vector. In this case the 3D σ-model is in fact the coset manifold $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)}$, describing a non-compact version of the universal hypermultiplet, the universal pseudo-hypermultiplet.

5.3.2 Dimensional "reduction" from three to two dimensions

Stationary and axisymmetric black holes of the four-dimensional theory are characterized by their invariance under two commuting Killing vectors, corresponding to time translations $\frac{\partial}{\partial t}$ and rotations $\frac{\partial}{\partial \varphi}$[51]. Since we want to make use of the sigma model obtained for the three-dimensional theory after suitable dualizations we prefer to employ a two step procedure. First we use one Killing vector to reduce from four to three dimensions and then the second one to do the step to two dimensions.

For Kerr-Newmann (K-N) black holes the metric can be reduced to the following the general form:

$$ds_{(4)}{}^2 = e^{2U}(dt - Bd\varphi)^2 - e^{-2U}\left(\gamma_{mn}dx^m dx^n + \tilde{\rho}^2 d\varphi^2\right) \tag{5.46}$$

where φ denotes the azimuthal angle about the rotation axis, while the scalar fields $U, B, \tilde{\rho}$ and the 2D metric tensor γ_{mn} depend on the space-time coordinates x^m, $m = 1, 2$. Hence an effective description can be given in terms of a two dimensional theory, where the evolution variables are now x^m. According to a general procedure in General Relativity one can perform a coordinate transformation such that the field $\tilde{\rho}$ is chosen as one of the new harmonic coordinates, the second coordinate z being defined by $dz = -{}^\star d\tilde{\rho}$ [7]. In these new variables $x^m = (\tilde{\rho}, z)$, named Weyl-coordinates, the 2D metric is conformally flat[120, 51]:

$$\gamma_{mn} = \lambda^2 \delta_{mn}\,, \tag{5.47}$$

with:

$$\lambda^2 = \tilde{\Delta}\,, \tag{5.48}$$

and

$$\tilde{\rho}^2 = \Delta sin^2(\theta), \tag{5.49}$$

[7] Here * denotes Hodge-dualization in two dimensions.

$$z = (r - M)cos(\theta). \tag{5.50}$$

Similarly we decompose the vector fields $A^{(3)|\Gamma}$ into pieces A_m^Γ and A_φ^Γ, with $m = 1, 2$, perpendicular and parallel to $\frac{\partial}{\partial\varphi}$, respectively. We have that, for the two-dimensional electromagnetic potential:

$$A_m^{(2)} = 0, \tag{5.51}$$

and

$$A_\varphi^{(2)} = -\frac{[P\Delta cos(\theta) - \alpha Q r sin^2(\theta)]}{\tilde{\Delta}}; \tag{5.52}$$

the corresponding field strengths is:

$$\begin{aligned} F^{(2)} &= F_{mn}^{(2)} dx^m \wedge dx^n \\ &= F_{r\varphi}^{(2)} dr \wedge d\varphi + F_{\theta\varphi}^{(2)} d\theta \wedge d\varphi \end{aligned} \tag{5.53}$$

with:

$$F_{r\varphi}^{(2)} = \frac{2r}{\rho^4} + \frac{2(M - r)}{\left[\rho^2 - 2Mr + \frac{(Q^2+P^2)}{2}\right]^2}, \tag{5.54}$$

and

$$F_{\theta\varphi}^{(2)} = \frac{\alpha^2 sin(2\theta)[2\rho^2 - 2Mr + (Q^2 + P^2)]^2}{\rho^4\left[\rho^2 - 2Mr + \frac{(Q^2+P^2)}{2}\right]^2}. \tag{5.55}$$

In this case one may further reduce the 3D Lagrangian to two dimensions by compactification on φ. The resulting 2D Lagrangian takes the form:

$$\mathcal{L}_{(2)} = \sqrt{|g^{(2)}|}\tilde{\rho}\left(\frac{R_{(2)}}{2} - \frac{1}{2}G_{ab}(z)\partial_m z^a \partial^m z^b + \frac{\partial_m \tilde{\rho}\, \partial^m \lambda}{\lambda\tilde{\rho}}\right), \tag{5.56}$$

with $g^{(2)} = det(\gamma_{mn})$. The dynamics of the fields $z^a(x^m)$ is totally captured by the σ-model effective action:

$$S_{eff} = \int d^2x \sqrt{|g^{(2)}|}\tfrac{\tilde{\rho}}{2} G_{ab}(z)\partial_m z^a \partial^m z^b, \tag{5.57}$$

where $\tilde{\rho}(x^m)$ is a harmonic function in the subspace spanned by x^m. The metric on this space can be made conformally flat by a suitable choice of the x^m and the conformal factor absorbed in the definition of λ, so that the equations for z^a and $\tilde{\rho}$ can be written in a flat 2D space, with $R_{(2)} = 0$ spanned by x^m[51].

5.4 The 2D Effective Lagrangian and its Field-Theoretical DWHJ description

In the presence of a time-like Killing vector ∂_t, the vielbein V^a $(a = 0, 1, 2, 3)$ of space-time can be put in the form:

$$V^0 = e^U(dt + \tilde{B}) = e^U D^0\,; \quad V^i = e^{-U} D^i \tag{5.58}$$

where D^i $(i = 1, 2, 3)$ are 3D vielbein. The time-reduced 3-dimensional Lagrangian describing a stationary 4D black hole in the presence of a given number of scalars ϕ^r and gauge fields A^Λ has the following form[8][59]:

$$\begin{aligned} \frac{1}{\sqrt{|g^{(3)}|}}\mathcal{L}_{(3)} &= \frac{1}{2}R - \frac{1}{2}G_{ab}(z)\partial_i z^a \partial^i z^b = \\ &= \frac{1}{2}R - \frac{1}{2}G_{IJ}(\Phi)d\Phi^I d\Phi^J = \\ = \frac{1}{2}R - &\left[(dU)^2 + \frac{1}{2}g_{rs}(\phi)d\phi^r d\phi^s + \frac{e^{-4U}}{4}(\omega)^2 + \frac{e^{-2U}}{2}d\mathcal{Z}^T\mathcal{M}_{(4)}d\mathcal{Z}\right], \end{aligned} \tag{5.59}$$

where $g_{(3)} \equiv \det(g_{(3)})$. Here, all the propagating degrees of freedom have been reduced to scalars by 3D Hodge-dualization[51]. In particular, the scalars $\mathbf{Z} = (\mathcal{Z}^\Lambda, \mathcal{Z}_\Lambda) = \{\mathcal{Z}^M\}$ include the electric components A_0^Λ of the 4D vector fields together with the Hodge dual of their magnetic components A_i^Λ $(i = 1, 2, 3)$ and a is related to the Hodge-dual of the 3D graviphoton ω_i. More precisely,

$$A^\Lambda_{(4)} = A_0^\Lambda D^0 + A^\Lambda_{(3)}\,, \quad A^\Lambda_{(3)} \equiv A_i^\Lambda D^i\,, \tag{5.60}$$

$$\mathbf{F}^M_{(4)} = \begin{pmatrix} F^\Lambda_{(4)} \\ G_{\Lambda(4)} \end{pmatrix} = d\mathcal{Z}^M \wedge D^0 + e^{-2U}\mathbb{C}^{MN}\mathcal{M}_{(4)NP}{}^* d\mathcal{Z}^P\,, \tag{5.61}$$

$$da \;=\; -e^{4U*}d\omega - \mathbf{Z}^T\mathbb{C}d\mathbf{Z}\,, \tag{5.62}$$

[8]For the $D = 4$ supergravity theory we use the units $\hbar = c = 8\pi G = 1$ and the normalization of the vector fields as in[3, 59].

where:

$$F^{\Lambda}_{(4)} = dA^{\Lambda}_{(4)}\,, \qquad G_{\Lambda(4)} = -\frac{1}{2}{}^{*}\left(\frac{\partial L}{\partial F^{\Lambda}_{(4)}}\right) \tag{5.63}$$

and $\mathcal{M}_{(4)}(\phi)$ is the negative-definite symmetric, symplectic matrix depending on 4D scalar fields introduced in [126, 127].

The isometry group $G_{(3)}$ of the σ-model metric $G_{ab}(z)$ contains as non trivial subgroups the 4-dimensional U-duality group $G_{(4)}$ times the group $SL(2,\mathbb{R})$, the Ehlers group, under which the degrees of freedom of the 4D metric transform. The simplest 3D model is the one originating form a pure 4D Einstein–Maxwell gravitational theory with a single time-like Killing vector. In this case $G_{(4)} = \mathrm{U}(1)$ and the 3D σ-model has the homogeneous-symmetric target space $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)}$. Its field content consists of four scalars belonging to a pseudo-Riemannian version of the universal hypermultiplet, dubbed the universal *pseudo-hypermultiplet*. We will discuss in more detail the properties of this theory in the following subsection 5.5.

We will mainly focus our attention on stationary axisymmetric solutions admitting the two Killing vectors ∂_t and ∂_φ. In this case, as pointred one may further reduce the 3D Lagrangian to two dimensions by compactification along φ. The fields now depend on the space coordinates x^m, $m = 1, 2$, and we assume that the three-dimensional space metric can be expressed in block-diagonal form as:

$$g_{(3)} = \begin{pmatrix} \lambda^2 h_{mn} & 0 \\ 0 & \hat{\rho}^2 \end{pmatrix}. \tag{5.64}$$

The resulting 2D Lagrangian takes the form[51, 59]:

$$\mathcal{L}_{(2)} \;=\; \sqrt{h}\,\hat{\rho}\left(\frac{R_{(2)}}{2} - \tfrac{1}{2}G_{ab}(z)\partial_m z^a \partial^m z^b + \frac{\partial_m\hat{\rho}\,\partial^m\lambda}{\lambda\hat{\rho}}\right), \tag{5.65}$$

with $h \equiv \det(h_{mn})$. As shown in[51, 59], the dynamics of the fields z^a is totally captured by the σ-model effective action:

$$S_{eff} = \int d^2x\,\sqrt{h}\,\tfrac{\hat{\rho}}{2}G_{ab}(z)\partial_m z^a \partial^m z^b\,, \tag{5.66}$$

where $\hat{\rho}(x^m)$ is a harmonic function in the subspace spanned by x^m.[9] The metric on this space can be made conformally flat by a suitable choice of the x^m and the conformal factor absorbed in the definition of λ, so that the equations for z^a and $\hat{\rho}$ can be written in a flat 2D space, with $R_{(2)} = 0$, spanned by x^m, with metric h_{mn}. As we shall show in Sect. 5.5.1, in suitable coordinates,

$$\sqrt{h}\,\hat{\rho} = \sin\theta\,. \tag{5.67}$$

The equation for λ can then be solved once the solutions to the σ-model are known [51].

We shall restrict our analysis to symmetric supergravities in which the scalar manifold $\mathcal{M}_{scal}$ of the $D = 3$ theory, spanned by the z^a, is homogeneous symmetric, i.e. of the form:

$$\mathcal{M}_{scal} = \frac{G_{(3)}}{H^*}\,. \tag{5.68}$$

We shall use for this manifold the solvable Lie algebra parametrization by identifying the scalar fields z^a with parameters of a suitable solvable Lie algebra. Let us recall the main points [60]. The isometry group $G_{(3)}$ of the target space is the global symmetry group of the S_{eff} and H^* is a suitable non-compact semisimple maximal subgroup of it. The scalars $z^a = \{U,\, a,\, \phi^r,\, \mathbf{Z}\}$ correspond to a *local* solvable parametrization, i.e. the corresponding patch, to be dubbed *physical patch* $\mathcal{U}$, is isometric to a solvable Lie group generated by a solvable Lie algebra $Solv$:

$$\mathcal{M}_{scal} \supset \mathcal{U} \equiv e^{Solv}\,, \tag{5.69}$$

$Solv$ is defined by the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of $G_{(3)}$ with respect to its maximal compact subalgebra $\mathfrak{H}$. The solvable parametrization

[9] According to a general procedure in General Relativity one can perform a coordinate transformation such that the field $\hat{\rho}$ is chosen as one of the new harmonic coordinates, the second coordinate z being defined by $dz = -{}^\star d\hat{\rho}$. Here * denotes Hodge-dualization in two dimensions. In these new variables $x^m = (\hat{\rho}, z)$, named Weyl-coordinates, the 2D metric is conformally flat $\gamma_{mn} = \lambda^2\delta_{mn}$ [120, 51].

z^a can be defined by the following exponential map:

$$\mathbb{L}(z^a) = \exp(-aT_\bullet)\,\exp(\sqrt{2}\mathcal{Z}^M\,T_M)\,\exp(\phi^r\,T_r)\,\exp(2UT_0)\,, \tag{5.70}$$

where the generators T_0, $T_\bullet$, T_r, T_M satisfy the equations (5.81). We can use for the generators of $\mathfrak{g}$ a representation in which the generators of $\mathfrak{H}^*$, the Lie algebra of H^*, are invariant under the involution $\sigma : M \to -\eta M^\dagger\eta$, where $\eta \equiv (-1)^{2T_0}$. The vielbein P and connection $\tilde{W}$ 1-forms on the manifold are computed as the odd and even components, respectively, of the left-invariant one-form with respect to σ:

$$\mathbb{L}^{-1}d\mathbb{L} = P + \tilde{W}\,, \tag{5.71}$$

$P = \eta P^\dagger\eta = -\sigma(P)$, $\tilde{W} = -\eta\tilde{W}^\dagger\eta = \sigma(\tilde{W})$. In terms of P the metric on the manifold reads:

$$dS^2_{(3)} = G_{ab}(z)dz^a\,dz^b = k\,\mathrm{Tr}(P^2)\,, \tag{5.72}$$

where $k = 1/(2\mathrm{Tr}(T_0^2))$ is a representation-dependent constant. It is also useful to introduce the hermitian, H^*-invariant matrix $\mathcal{M}$:

$$\mathcal{M}(z) \equiv \mathbb{L}\eta\mathbb{L}^\dagger = \mathcal{M}^\dagger \tag{5.73}$$

in terms of which we can write the geodesic Lagrangian as:

$$\begin{aligned}\mathcal{L}_{(2)eff} &= \frac{1}{2}\hat{\rho}\,\sqrt{h}\,G_{ab}(z)\partial_m z^a\,\partial^m z^b \\ &= \frac{k}{8}\hat{\rho}\,\sqrt{h}\,\mathrm{Tr}\left[\mathcal{M}^{-1}\partial_m\mathcal{M}\mathcal{M}^{-1}\partial^m\mathcal{M}\right]\end{aligned} \tag{5.74}$$

with a canonically conjugate momentum

$$\begin{aligned}\pi_a^m &= \frac{\partial\mathcal{L}}{\partial\partial_m z^a} \\ &= \frac{k}{4}\hat{\rho}\,\sqrt{h}\,\mathrm{Tr}\left[\mathcal{M}^{-1}(z)\partial_a\mathcal{M}(z)\mathcal{M}^{-1}(z)\partial_b\mathcal{M}(z)\right]\,\partial^m z^b\,.\end{aligned} \tag{5.75}$$

The corresponding equations of motion are:

$$\partial_m\left(\sqrt{h}\,\hat{\rho}h^{mn}J_n\right) = 0\,, \tag{5.76}$$

where:

$$J_m \equiv \frac{1}{2}\partial_m\xi^a\,\mathcal{M}^{-1}\partial_a\mathcal{M}\,. \tag{5.77}$$

5.4.1 Conserved quantities

We shall restrict our analysis to symmetric supergravities in which the scalar manifold $\mathcal{M}_{scal}$ of the $D = 3$ theory, spanned by the ξ^a, is homogeneous symmetric, i.e. of the form:

$$\mathcal{M}_{scal} = \frac{G_{(3)}}{H^*} \,. \tag{5.78}$$

The isometry group $G_{(3)}$ of the target space is the global symmetry group of the theory and H^* is a non-compact semisimple maximal subgroup of it. The scalars $z^a = \{U,\, a,\, \phi^r,\, \mathbf{Z}\}$ correspond to a *local* solvable parametrization, i.e. the corresponding patch, to be dubbed *physical patch* $\mathcal{U}$, is isometric to a solvable Lie group generated by a solvable Lie algebra $Solv$:

$$\mathcal{M}_{scal} \supset \mathcal{U} \equiv e^{Solv} \,, \tag{5.79}$$

$Solv$ is defined by the Iwasawa decomposition of the Lie algebra $\mathfrak{g}$ of $G_{(3)}$ with respect to its maximal compact subalgebra $\mathfrak{H}$. The solvable parametrization z^a in can be defined by the following exponential map:

$$\mathbb{L}(\phi^I) = \exp(-aT_\bullet)\, \exp(\sqrt{2}\mathcal{Z}^M\, T_M)\, \exp(\phi^r\, T_r)\, \exp(2UT_0) \,, \tag{5.80}$$

where the generators T_0, $T_\bullet$, T_r, T_M satisfy the following commutation relations:

$$\begin{aligned} &[T_0,\, T_M] = \frac{1}{2}\, T_M \;\; ; \;\; [T_0,\, T_\bullet] = T_\bullet \;\; ; \;\; [T_M\, T_N] = \mathbb{C}_{MN}\, T_\bullet \,, \\ &[T_0, T_r] = [T_\bullet, T_r] = 0 \;\; ; \;\; [T_r, T_M] = {T_r}^N{}_M\, T_N \;\; ; \;\; [T_r, T_s] = -{T_{rs}}^{s'} T_{s'} \end{aligned} \tag{5.81}$$

and ${T_r}^N{}_M$ representing the symplectic representation of T_r on contravariant symplectic vectors $d\mathcal{Z}^M$. We can use for the generators of $\mathfrak{g}$ a representation in which the generators of $\mathfrak{H}^*$, Lie algebra of H^*, are invariant under the involution:

$$\sigma : M \to -\eta M^\dagger \eta \,, \tag{5.82}$$

where $\eta \equiv (-1)^{2T_0}$. The vielbein P and connection $\tilde{W}$ 1-forms on the manifold are computed as the odd and even components, respectively, of the left-invariant one-form with respect to σ:

$$\mathbb{L}^{-1}d\mathbb{L} = P + \tilde{W}\,, \tag{5.83}$$

$$P = \eta P^{\dagger}\eta = -\sigma(P)\,, \tag{5.84}$$

$$\tilde{W} = -\eta\tilde{W}^{\dagger}\eta = \sigma(\tilde{W})\,. \tag{5.85}$$

In terms of P the metric on the manifold reads:

$$dS^2_{(3)} = G_{ab}(z)dz^a\,dz^b = k\,\mathrm{Tr}(P^2)\,, \tag{5.86}$$

where

$$k = 1/(2\mathrm{Tr}(T_0^2)) \tag{5.87}$$

is a representation-dependent constant. It is also useful to introduce the hermitian, H^*-invariant matrix $\mathcal{M}$:

$$\mathcal{M}(z) \equiv \mathbb{L}\eta\mathbb{L}^{\dagger} = \mathcal{M}^{\dagger}\,, \tag{5.88}$$

in terms of which we can write the Nöether currents:

$$J_m \equiv \frac{1}{2}\partial_m\xi^a\,\mathcal{M}^{-1}\partial_a\mathcal{M}\,. \tag{5.89}$$

The quantity $J = J_m\,dx^m$ is a 1-form with value in $\mathfrak{g}$ and the equations of motion can be cast in the form:

$$\partial_m\left(\sqrt{\det(\gamma_{mn})}\,\tilde{\rho}\gamma^{mn}J_n\right) = 0\,, \tag{5.90}$$

which imply that the integral:

$$\tilde{Q} = \frac{1}{2}\int\sqrt{\det(\gamma_{mn})}\,\tilde{\rho}\gamma^{rr}J_r d\theta\,, \tag{5.91}$$

in an r-independent matrix in $\mathfrak{g}$.

Using the notation of[60, 56] and from it we may derive the set of Nöether currents $J_{A\,m}$ and the corresponding *constants of motion* $\tilde{Q}_A$ characterizing the solution at radial infinity:

$$J_{A\,m} \equiv k\,\mathrm{Tr}\left(T_A^\dagger\, J_m\right)\,, \tag{5.92}$$

$$\begin{aligned}\tilde{Q}_A = k\,\mathrm{Tr}\left(T_A^\dagger\,\tilde{Q}\right) &= \frac{1}{4\pi}\int_{S_2} {}^{*3}J_A \\ &= \frac{1}{2}\int \sqrt{h}\,\hat{\rho} h^{rr}\,J_{A\,r} d\theta \end{aligned} \tag{5.93}$$

which consist in the ADM mass M ($T_A = T_0$), the NUT charge ℓ ($T_A = T_\bullet$), the $D = 4$ scalar charges Σ_r ($T_A = T_r$) and the electric-magnetic charges Γ^M ($T_A = T_M$). The currents $J_{A\,m}$ read:

$$J_{\bullet m} \;= \tfrac{k}{2}\,\mathrm{Tr}(T_\bullet^\dagger \mathcal{M}^{-1}\partial_m\mathcal{M}) = -\tfrac{1}{2}\,e^{-4U}\,(\partial_m a + \mathbf{Z}^T\mathbb{C}\partial_m\mathbf{Z})\,, \tag{5.94}$$

$$J_{0\,m} \;= \tfrac{k}{2}\,\mathrm{Tr}(T_0^\dagger \mathcal{M}^{-1}\partial_m\mathcal{M}) = \partial_m U + \tfrac{1}{2}\,e^{-2U}\,\mathbf{Z}^T\mathcal{M}\partial_m\mathbf{Z} - a\,J_{\bullet m}\,, \tag{5.95}$$

$$\begin{aligned} J_{M\,m} &= \frac{k}{2}\,\mathrm{Tr}(T_M^\dagger \mathcal{M}^{-1}\partial_m\mathcal{M}) = \frac{1}{\sqrt{2}}\,e^{-2U}\,\mathcal{M}_{(4)\,MN}\,\partial_m\mathcal{Z}^N + \\ &+ \sqrt{2}\,\mathbb{C}_{MN}\,\mathcal{Z}^N\,J_{\bullet m}\,, \end{aligned} \tag{5.96}$$

$$\begin{aligned} J_{s\,m} &= \frac{k}{2}\,\mathrm{Tr}(T_s^\dagger \mathcal{M}^{-1}\partial_m\mathcal{M}) \\ &= \frac{1}{\sqrt{2}}\,\mathbb{L}_{4\,s}{}^{\hat{s}'}\,V_{4\,s''}{}^{\hat{s}'}\partial_m\phi^{s''} + e^{-2U}\,\mathbf{Z}^T T_s\mathcal{M}\,\partial_m\mathbf{Z} + \\ &- T_{sMN}\mathcal{Z}^M\mathcal{Z}^N\,J_{\bullet m}\,, \end{aligned} \tag{5.97}$$

where $\mathbb{L}_{4\,s}{}^{\hat{s}'}$ is the coset representative of the symmetric scalar manifold in four-dimensions in the solvable parametrization, as a matrix in the adjoint representation of the solvable group, $V_{4\,s}{}^{\hat{s}'}$ is the vielbein of the same manifold and the hat denotes rigid indices.

The conserved quantities are then obtained as the flux of the currents across the 2-sphere at infinity, according to eq.s (5.92) and (5.93):

$$M = \frac{1}{4\pi}\int_{S_2} {}^{*_3}J_0 \ ; \tag{5.98}$$

$$\ell = -\frac{1}{4\pi}\int_{S_2} {}^{*_3}J_\bullet \ ; \tag{5.99}$$

$$\Gamma^M = \frac{\sqrt{2}}{4\pi}\,\mathbb{C}^{MN}\int_{S_2} {}^{*_3}J_N \ , \tag{5.100}$$

$$\Sigma_s \ = \tfrac{1}{4\pi}\int_{S_2} {}^{*_3}J_s \ . \tag{5.101}$$

The other conserved quantity characterizing the axisymmetric solution is the angular momentum M_φ along the rotation axis Z. The expression of the angular momentum in terms of a conserved current can be found in standard textbooks (see for instance [2, 128] and [59]). Here we would like to give an expression of it in terms of quantities which are intrinsic to the $D = 3$ effective action: the Killing vector field $\psi = \partial_\varphi$ and $J_\bullet$. To this end we start from the representation of M_φ as the integral over the sphere at infinity S_2^∞ of a suitable 2-form, as given in [2]:

$$M_\varphi \ = \tfrac{1}{16\pi}\int_{S_2^\infty} J^{(2)} \ ; \tag{5.102}$$

with:

$$J^{(2)} \equiv \sqrt{g}\,\epsilon_{\mu\nu\rho\sigma}\,\nabla^\rho\psi^\sigma\,dx^\mu\wedge dx^\nu \,. \tag{5.103}$$

The above integral can also be written in the form:

$$\begin{aligned} M_\varphi &= \frac{1}{8\pi}\int_{S_2^\infty}\sqrt{g}\,g^{\mu\,[t}\,\Gamma^{r]}_{\mu\varphi}\,d\theta d\varphi \\ &= \frac{1}{8\pi}\int_{S_2^\infty}\sqrt{g}\,g^{\mu\,[t}\,g^{r]\nu}\partial_{[\mu}g_{\nu]\varphi}\,d\theta d\varphi = \\ &= \frac{1}{8\pi}\int_{S_2^\infty}\sqrt{g_{(3)}}\,\Big[\frac{1}{2}\,g^{rr}_{(3)}g^{\varphi\varphi}_{(3)}\left(\partial_r\omega_\varphi g^{(3)}_{\varphi\varphi} - \omega_\varphi\,\partial_r g^{(3)}_{\varphi\varphi} + \right. \\ &+ \left. e^{4U}\,\omega_\varphi^2\partial_r\omega_\varphi + 4\omega_\varphi\,g^{(3)}_{\varphi\varphi}\,\partial_r U\right)\Big]\ d\theta d\varphi\,. \end{aligned} \tag{5.104}$$

Using the asymptotic behavior of the metric for axisymmetric solutions [128]:

$$\omega_\varphi = \frac{2M_\varphi}{r}\sin^2(\theta) + O\left(\frac{1}{r^2}\right) \ ; \tag{5.105}$$

$$g^{(3)}_{rr} = 1 + O\left(\frac{1}{r^2}\right) \ ; \tag{5.106}$$

$$g^{(3)}_{\theta\theta} = r^2\left(1 + O\left(\frac{1}{r}\right)\right) \ ; \tag{5.107}$$

$$g^{(3)}_{\varphi\varphi} = r^2 \sin^2(\theta)\left(1 + O\left(\frac{1}{r}\right)\right) \ ; \tag{5.108}$$

$$e^{2U} = 1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right) \ ; \tag{5.109}$$

we see that only the first two terms in the integral (5.104) survive the asymptotic limit and yield contributions which are both proportional to M_φ, the second term contributing twice the first to the asymptotic limit. The first contribution in particular can be expressed in terms of ψ, $J_\bullet$, so that we can write:

$$\begin{aligned} M_\varphi &= -\frac{3}{8\pi}\int_{S_2^\infty} \psi_{[i}\, J_{\bullet j]}\, dx^i \wedge dx^j \\ &= -\frac{3}{4\pi}\int_{S_2^\infty} \psi_{[\theta}\, J_{\bullet\varphi]}\, d\theta\, d\varphi = \\ &= \frac{3}{8\pi}\int_{S_2^\infty} \psi_\varphi\, J_{\bullet\theta}\, d\theta\, d\varphi \,, \end{aligned} \tag{5.110}$$

where $\psi_\varphi = g^{(3)}_{\varphi\varphi}$.

$G_{(3)}$-invariant characterization of the angular momentum

Let us define a new constant $\mathfrak{g}$-matrix as follows[59]:

$$Q_\psi \ = -\tfrac{3}{8\pi}\int_{S_2^\infty} \psi_{[i}\, J_{j]}\, dx^i \wedge dx^j = \tfrac{3}{8\pi}\int_{S_2^\infty} \psi_\varphi\, J_\theta\, d\theta\, d\varphi \in \mathfrak{g}\,. \tag{5.111}$$

In the asymptotic limit $r \to \infty$ the components of J_m have the following behavior:

$$J_r = \frac{\tilde{Q}}{r^2} + O\left(\frac{1}{r^3}\right) \;; \tag{5.112}$$

$$J_\theta = \frac{Q_\psi}{r^2}\,\sin\theta + O\left(\frac{1}{r^3}\right)\,. \tag{5.113}$$

According to the general formula (5.110), the angular momentum can be written as:

$$M_\varphi = k\,\mathrm{Tr}(T^\dagger_\bullet\,Q_\psi)\,. \tag{5.114}$$

As pointed out earlier, $G_{(3)}$ is the global symmetry group of the three-dimensional effective theory[59]. As an isometry group, its elements have a non-linear action on the coordinates:

$$g \in G_{(3)}\;:\; z^a \;\longrightarrow\; z^a_g = z^a_g(z)\,, \tag{5.115}$$

where $z^a_g(z)$ are non-linear functions of the z^a, depending on the parameters of the transformation g. The same transformation, being a global symmetry, maps a solution $\xi^a(x)$ into an other one of the same theory $\xi^a_g(x)$. The asymptotic limit $r \to \infty$, for the scalar fields, defines a single point $\xi_0 = (\xi^a_0)$ on the scalar manifold[59]:

$$\lim_{r\to\infty} \xi^a(x) = \xi^a_0\,. \tag{5.116}$$

Since the action of $G_{(3)}$ on the scalar manifold is transitive, we can always map the point at infinity to the origin $O(\xi^a_0 \equiv 0)$. Once we fix $\xi_0 = O$, we can only act on the solutions by means of the stability group H^* of the origin.

From the definition (5.73) we deduce the transformation property of the matrix $\mathcal{M}(z)$ under an isometry g:

$$\mathcal{M}(z) \;\longrightarrow\; \mathcal{M}(z_g) = g\,\mathcal{M}(z)\,g^\dagger\,, \tag{5.117}$$

where, with an abuse of notation, we have used the same symbol g to denote the matrix form of g in the representation of $\mathcal{M}$. The $\mathfrak{g}$-valued current $J_m = J_m(\xi(x))$ therefore transforms under an isometry g by conjugation:

$$J_m(\xi) \;\longrightarrow\; J_m(\xi_g) = (g^\dagger)^{-1}\, J_m(\xi)\, g^\dagger\,, \tag{5.118}$$

and so do the $\mathfrak{g}$-valued constant matrices $\tilde{Q}$ and Q_ψ:

$$\tilde{Q}(\xi) \quad\longrightarrow\quad \tilde{Q}(\xi_g) = (g^\dagger)^{-1}\, \tilde{Q}(\xi)\, g^\dagger \;\;; \tag{5.119}$$

$$Q_\psi(\xi) \;\longrightarrow\; Q_\psi(\xi_g) = (g^\dagger)^{-1}\, Q_\psi(\xi)\, g^\dagger\,. \tag{5.120}$$

Generic axisymmetric stationary solutions are distinguished from the static ones by the following $G_{(3)}$-invariant property[59]:

$$\text{axisymmetric solutions} \quad\Rightarrow\quad Q_\psi \neq 0. \tag{5.121}$$

In particular for solutions in the same $G_{(3)}$-orbit as the Kerr-Newmann-Taub-NUT (KN-Taub-NU) one, $\mathrm{Tr}(Q_\psi^2) \neq 0$. In the universal model originating from Einstein-Maxwell supergravity in four dimensions, see Sect. 5.5, $G_{(3)} = \mathrm{SU}(1,2)$, and we can evaluate on the KN-Taub-NUT solutions $\tilde{Q}$ and Q_ψ explicitly. Using the covariant expression for the matrix $\mathcal{M}$ in terms of U_E, V_E, W_E, given in Appendix B and eq.s (5.176) - (5.178) introduced in Section 5.5 we find[59]:

$$\tilde{Q} \;=\; \begin{pmatrix} 0 & 0 & (M - i\,\ell) \\ 0 & 0 & -\,\frac{Q+iP}{\sqrt{2}} \\ (M + i\,\ell) & \frac{Q-iP}{\sqrt{2}} & 0 \end{pmatrix}, \tag{5.122}$$

$$Q_\psi \;=\alpha \begin{pmatrix} 0 & 0 & (\ell + i\,M) \\ 0 & 0 & -i\,(Q+iP)/\sqrt{2} \\ (\ell - i\,M) & -i\,(Q-iP)/\sqrt{2} & 0 \end{pmatrix}. \tag{5.123}$$

Then:

$$\mathrm{Tr}(\tilde{Q}^2) = \frac{2}{k}\,(M^2 + \ell^2 - \frac{P^2+Q^2}{2})\ , \tag{5.124}$$

$$\mathrm{Tr}(Q_\psi^2) = \frac{2\alpha^2}{k}\,(M^2 + \ell^2 - \frac{P^2+Q^2}{2})\ , \tag{5.125}$$

where $\alpha \equiv M_\varphi/M$ and $k = 1$ in the fundamental representation of $\mathrm{SU}(1,2)$, so that[59]:

$$\left(\frac{M_\varphi}{M}\right)^2 = \alpha^2 = \frac{\mathrm{Tr}(Q_\psi^2)}{\mathrm{Tr}(\tilde{Q}^2)}\,. \tag{5.126}$$

We wish to stress here that the above formula, although derived in the universal model, holds in all supergravity theories admitting the KN-Taub-NUT solution. This is a $G_{(3)}$-invariant characterization of the angular momentum, which holds for all solutions in the same $G_{(3)}$-orbit as the KN-Taub-NUT one. Using this result, we can write the extremality parameter in a $G_{(3)}$-invariant fashion[59]:

$$c^2 = M^2 + \ell^2 - \frac{P^2+Q^2}{2} - \alpha^2 = \frac{k}{2}\mathrm{Tr}(\tilde{Q}^2) - \frac{\mathrm{Tr}(Q_\psi^2)}{\mathrm{Tr}(\tilde{Q}^2)}\,, \tag{5.127}$$

so that the extremality condition becomes:

$$c^2 = 0 \;\Leftrightarrow\; \mathrm{Tr}(\tilde{Q}^2) = \frac{2}{k}\,\frac{\mathrm{Tr}(Q_\psi^2)}{\mathrm{Tr}(\tilde{Q}^2)}\,, \tag{5.128}$$

from which it is apparent that, as opposed to the static case, extremality does not imply nilpotency of $\tilde{Q}$, as noted in [57, 59]. Equation (5.128) provides a $G_{(3)}$-invariant characterization of extremality[59]. There is a class of extremal rotating solutions for which both sides of this equation vanish separately. These are the "ergo-free" (under-rotating) solutions constructed in [59, 122, 123, 124] and further generalized in [114] within cubic supergravity models. Below we shall comment on some general $G_{(3)}$-invariant properties of these solutions in terms of the matrices $\tilde{Q}$ and Q_ψ[59].

Under-rotating solutions.

In [59, 122, 123, 124] under-rotating solutions were constructed within the Kaluza-Klein theory originating from pure gravity in $D = 5$, as a limit of a dilatonic rotating black hole. In order to perform a similar limit in the context of supergravity, we need to consider a model which is larger than the universal one, but which contains it as a consistent truncation. The simplest choice is the $\mathcal{N} = 2$ t^3-model in four dimensions, which consists of supergravity coupled to one vector multiplet, whose complex scalar field t parametrizes a special Kähler manifold with prepotential $\mathcal{F}(t) = t^3$. Upon time-like reduction to $D = 3$ we end up with an Euclidean sigma-model with target space $\mathrm{G}_{2(2)}/[\mathrm{SL}(2) \times \mathrm{SL}(2)]$ and global symmetry group $G_{(3)} = \mathrm{G}_{2(2)}$. Extremal solutions to this model were studied in [53, 129, 57].

We shall not enter into the mathematical details of model but limit ourselves to illustrate the procedure for generating an extremal under-rotating solution from a non-extremal rotating one. The scalar fields originating from the $D = 4$ vector fields are four $(\mathcal{Z}^M) = (\mathcal{Z}^0, \mathcal{Z}^1, \mathcal{Z}_0, \mathcal{Z}_1)$, parametrizing the solvable generators $(T_M) = (T_0, T_1, T^0, T^1)$. Adopting a suitable representation of $\mathrm{G}_{2(2)}$ for the generators (for example the fundamental real **7** representation), we can consider two commuting generators of Harrison transformations[59]:

$$K_0 \equiv \frac{1}{2}\,(T_0 + T_0^\dagger)\ ; \tag{5.129}$$

$$K_1 \equiv \frac{1}{2}\,(T^1 + T^{1\,\dagger})\,, \tag{5.130}$$

and "boost" the Kerr solution with parameters M, α using the Harrison transformation[59]:

$$\mathcal{O} \equiv e^{\log(\beta_1 M)\,K_0 + \log(\beta_2 M)\,K_1}\ . \tag{5.131}$$

The resulting solution is a non-extremal axion-dilaton rotating black hole with ADM-mass, electric-magnetic and scalar charges and angular momentum depending on the Kerr parameters M, α and encoded in the $\mathfrak{g}_{2(2)}$-valued matrices[59]:

$$\tilde{Q} = \mathcal{O}^{-1}\, Q^{(K)}\, \mathcal{O} \;; \tag{5.132}$$

$$Q_\psi = \mathcal{O}^{-1}\, Q_\psi^{(K)}\, \mathcal{O} \;; \tag{5.133}$$

$Q^{(K)}$ and $Q_\psi^{(K)}$ being the matrices corresponding to the original Kerr solution. We shall give the complete solution elsewhere, focussing here only on the characteristic quantities at radial infinity. Redefining $\alpha = \Omega\, M = M_\varphi/M$, these quantities read[59]:

$$M_{ADM} = \frac{1}{8}\left(M^2(\beta_1 + 3\beta_2) + \frac{1}{\beta_1} + \frac{3}{\beta_2}\right) \;; \tag{5.134}$$

$$p^1 = \sqrt{3}\frac{M^2\beta_2^2 - 1}{2\sqrt{2}\beta_2} \;; \tag{5.135}$$

$$q_0 = -\frac{M^2\beta_1^2 - 1}{2\sqrt{2}\beta_1} \;; \tag{5.136}$$

$$\Sigma = i\, \frac{\sqrt{3}\,(-M^2\beta_2\beta_1^2 + M^2\beta_2^2\beta_1 + \beta_1 - \beta_2)}{8\beta_1\beta_2} \;; \tag{5.137}$$

$$M_\varphi = \frac{(\beta_1\beta_2^3 M^4 + 3\beta_2(\beta_1+\beta_2)M^2 + 1)\,\Omega}{8\sqrt{\beta_1}\beta_2^{3/2}} \;; \tag{5.138}$$

while $p^0 = q_1 = \ell = 0$. Taking the the $M \to 0$ limit while keeping β_1, β_2 and Ω fixed, the above quantities remain finite:

$$M_{ADM} = \frac{1}{8}\left(\frac{1}{\beta_1} + \frac{3}{\beta_2}\right) \;; \tag{5.139}$$

$$p^1 = -\frac{\sqrt{3}}{2\sqrt{2}\beta_2} \;; \tag{5.140}$$

$$q_0 = \frac{1}{2\sqrt{2}\beta_1} \;; \tag{5.141}$$

$$\Sigma = i\,\frac{\sqrt{3}\,(\beta_1-\beta_2)}{8\beta_1\beta_2}\ ; \tag{5.142}$$

$$M_\varphi = \frac{\Omega}{8\sqrt{\beta_1}\beta_2^{3/2}}\ . \tag{5.143}$$

Inspection of the full solution shows that, as $M \to 0$, the ergo-sphere disappears and the three dimensional spatial part of the metric becomes conformally flat.

This limit corresponds to taking a singular Harrison transformation $\mathcal{O}$ ($\log(\beta_1\,M)$, $\log(\beta_2\,M) \to -\infty$) and at the same time a singular limit of the Kerr parameters (M, $\alpha \to 0$). As a result the matrices $\tilde{Q}$, Q_ψ remain finite but become *nilpotent*. In particular $\tilde{Q}$ is a step-3 nilpotent matrix while $\tilde{Q}_\psi$ is step 2. The fact that Q_ψ has a *lower* degree of nilpotency than $\tilde{Q}$ is consistent with the fact that[59]:

$$\lim_{M\to 0}\mathrm{Tr}(\tilde{Q}^2) = 0\ ; \tag{5.144}$$

$$\lim_{M\to 0}\frac{\mathrm{Tr}(Q_\psi^2)}{\mathrm{Tr}(\tilde{Q}^2)} = 0\ ; \tag{5.145}$$

and the extremality condition (5.128) is satisfied. This is consistent with the classification of extremal solutions of [55, 57] in terms of suitable nilpotent subalgebras $\mathfrak{N}$ of $\mathfrak{g}$. In this case the matrices $\tilde{Q}$ and Q_ψ would correspond to characteristic generators of $\mathfrak{N}$.

5.4.2 A duality invariant expression for the DWHJ vector S_m

Let us now apply the construction of section 5.2 to our specific effective Lagrangian (5.74). The direct application of eq. (5.12) to our specific geodesic model is possible but lacks the property of being manifestly invariant under the isometry group $G_{(3)}$. However, the use of the $G_{(3)}$-valued matrix $\mathcal{M}$ introduced in (5.73) makes it possible to write an alternative expression for S^m

which does exhibit manifest duality invariance (provided we transform both the off-shell fields z^a and their on-shell expression on a given background $\xi^a(x)$). The expression is the following:

$$S^m = -\frac{k}{4}\hat{\rho}\,\sqrt{h}\,\mathrm{Tr}\left[\mathcal{M}^{-1}(z)\partial^m\mathcal{M}(\xi)\right] + \delta^m_r\int^r dr'\mathcal{L}(\xi(x'),\partial_m\xi,x')\,. \quad (5.146)$$

Indeed, from (5.146) we find:

$$\frac{\partial S^m}{\partial z^a} = \frac{k}{4}\hat{\rho}\,\sqrt{h}\,\mathrm{Tr}\left[\mathcal{M}^{-1}(z)\frac{\partial\mathcal{M}}{\partial z^a}\mathcal{M}^{-1}(z)\partial^m\mathcal{M}(\xi)\right]\,. \quad (5.147)$$

so that, for a weakly embedded solution $z = \xi$, we reproduce the on-shell expression of the conjugate momentum (5.75). Correspondingly we also find, using the field equations:

$$\begin{aligned}\partial_m S^m|_{z=\xi} &= \left(\mathcal{L} - \frac{k}{4}\hat{\rho}\,\sqrt{h}\,\mathrm{Tr}\left[\mathcal{M}^{-1}(z)\partial_m\mathcal{M}(\xi)\mathcal{M}^{-1}(\xi)\partial^m\mathcal{M}(\xi)\right]\right)_{z=\xi} \\ &= -\mathcal{H}|_{z=\xi}\,. \end{aligned} \quad (5.148)$$

One may ask what the relation between the solution (5.146) and the general relation (5.12) is. The answer can be found by realizing that a Taylor-expansion of S^m given in (5.146) in powers of $z-\xi$, taking into account (5.70) and (5.73), exactly reproduces (5.12). It is important to stress that S_m, as defined above, is $G_{(3)}$-invariant provided we simultaneously transform z^a and $\xi^a(x)$ in its expression, as it follows from the transformation property (5.117) of the matrix $\mathcal{M}$:

$$g\in G_{(3)}\;:\;\; S_m(z,\xi)\;\longrightarrow\; S_m(z_g,\xi_g) = S_m(z,\xi)\,, \quad (5.149)$$

An important property of the DWHJ construction is that one can compute the conserved currents of the theory by varying S^m with respect to the parameters which it depends on [58]. In particular, we can reproduce the conserved Nöether currents $\hat{\rho}J_m$ of (5.77) by performing an infinitesimal isometry transformation on S^m, at fixed background $\xi^a(x)$, and then by varying S^m with the corresponding symmetry parameters. If we set:

$$g = \mathbf{1} + \epsilon^\alpha\mathbf{T}_\alpha\,, \quad (5.150)$$

the isometry transformed matrix is:

$$\mathcal{M}(z_g) = g \cdot \mathcal{M}(z) \cdot g^\dagger \simeq \mathbf{1} + \epsilon^\alpha \left(\mathbf{T}_\alpha \cdot \mathcal{M} + \mathcal{M} \cdot \mathbf{T}_\alpha^\dagger\right) , \tag{5.151}$$

On the g-transformed S^m we get:

$$\begin{aligned}\left.\frac{\partial S^m(z_g)}{\partial \epsilon^\alpha}\right|_{z=\xi} &= -\frac{k}{4}\hat{\rho}\sqrt{h}\,\left[\left(\mathcal{M}^{-1}(z)\partial^m\mathcal{M}(\xi)\right)_i{}^j (T_\alpha)_j{}^i + \right.\\ &\quad \left. + \left(\mathcal{M}^{-1}(z)\partial^m\mathcal{M}(\xi)\right)^j{}_i (T_\alpha)^i{}_j\right] \\ &= -2\,\hat{\rho}\sqrt{h}\;\mathrm{Tr}[T_\alpha^\dagger \cdot J^m] \,. \end{aligned} \tag{5.152}$$

5.5 Application to Einstein–Maxwell axisymmetric solutions

In the absence of four dimensional scalar fields ($\partial_i\phi \to 0$, $\mathcal{M}_{(4)} \to 1_{(4)}$), the geodesic part of the Lagrangian (5.59) reduces to[59]:

$$\begin{aligned}\frac{1}{\sqrt{|g^{(3)}|}}\mathcal{L}_{(3)} &= (dU)^2 + \frac{e^{-4U}}{4}(\omega)^2 + \frac{e^{-2U}}{2} d\mathcal{Z}^T\mathcal{M}d\mathcal{Z} \\ &= \frac{1}{2}G_{ab}(z)dz^a dz^b, \end{aligned} \tag{5.153}$$

where $G_{ab}(z)$ is now the metric of the manifold:

$$\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)} \tag{5.154}$$

which is a pseudo-Kähler manifold, that is a non compact version of the Kähler manifold $CP(2)$.

As it is well known in General Relativity, the simplest and most useful way to describe such theory is the use of the so-called Ernst potentials $\mathcal{E}$, ψ[120, 119, 59] defined as:

$$\mathcal{E} = e^{2U} - |\Psi|^2 + i\,a \ ; \tag{5.155}$$

$$\Psi = \frac{1}{\sqrt{2}}(\mathcal{Z}^0 + i\,\mathcal{Z}_0) \ . \tag{5.156}$$

In terms of the Ernst potentials the metric (5.72) reads[59]:

$$ds^2_{(3)} = \frac{e^{-4U}}{2}\,|d\mathcal{E} + \bar{\Psi} d\Psi|^2 - e^{-2U}\,|d\Psi|^2 \tag{5.157}$$

The group $\mathrm{SU}(1,2)$ acts non-linearly on the potentials $\mathcal{E}, \psi$. However, one can introduce homogeneous complex coordinate fields (W_E, V_E, U_E) transforming in the $\mathbf{3}$ of $\mathrm{SU}(1,2)$, in terms of which the Ernst potentials can be written as follows[59];

$$\mathcal{E} = \frac{U_E - W_E}{U_E + W_E}\,; \quad \psi = \frac{V_E}{U_E + W_E}. \tag{5.158}$$

Going to inhomogeneous variables $u = U_E/W_E$, $v = V_E/W_E$, they take the form

$$\mathcal{E} = \frac{u-1}{u+1}\ ; \quad \Psi = \frac{v}{u+1}\,. \tag{5.159}$$

The scalar manifold $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times \mathrm{SU}(1,1)}$ can then be described in terms of the complex fields $z^a = (u, v)$ (where $a = 1, 2$).

We notice that the manifold (5.154) is a non-compact version of the minimal model $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times \mathrm{SU}(1,1)}$, which describes a particular case of a symmetric space of $N = 2$ special geometry in four dimensional supergravity. Accordingly, we can say that the variables (u, v) are "special coordinates" in terms of which the upper components of the corresponding holomorphic symplectic section (X^Λ, F_Λ) read[59][10]:

$$X^\Lambda = \begin{pmatrix} W_E \\ U_E \\ V_E \end{pmatrix} = W_E \begin{pmatrix} 1 \\ u \\ v \end{pmatrix}, \tag{5.160}$$

while the lower components F_Λ are given in terms of the holomorphic homogeneous degree two prepotential $F(X^\Lambda)$, as $F_\Lambda = \frac{\partial F}{\partial X^\Lambda}$. The holomorphic prepotential in terms of the inhomogeneous coordinates reads:

$$\mathcal{F} = \frac{F(X^\Lambda)}{W_E^2} = \frac{\imath}{4}\,(1 - u^2 - v^2)\,, \tag{5.161}$$

[10] See Appendix E.

and the Kähler potential $\mathcal{K}$ has the following form:

$$\begin{aligned}\mathcal{K} &= -\log\left[\imath\left(2\,(\mathcal{F}-\bar{\mathcal{F}})-(z^a-\bar{z}^a)(\partial_a\mathcal{F}+\partial_{\bar{a}}\bar{\mathcal{F}})\right)\right]\\ &= -\log\left[|u|^2+|v|^2-1\right]\,,\end{aligned}\tag{5.162}$$

where $z^a=(u,v)$.

The coordinate patch u,v is defined by the condition:

$$|u|^2+|v|^2>1\,,\tag{5.163}$$

whose physical meaning will be given in the next subsection.

The σ-model metric in the special coordinates has the form[59]:

$$dS^2_{(3)} = 2\,g_{a\bar{b}}\,dz^a\,d\bar{z}^b\ \ ;\tag{5.164}$$

$$g_{a\bar{b}} = \partial_a\partial_{\bar{b}}\mathcal{K} = e^{2\mathcal{K}}\begin{pmatrix}(1-|v|^2) & \bar{u}\,v\\ \bar{v}\,u & (1-|u|^2)\end{pmatrix} = e^{2\mathcal{K}}\,(\delta_{a\bar{b}}-z_a\bar{z}_{\bar{b}})\,,\tag{5.165}$$

$$g^{\bar{a}b} = -e^{-\mathcal{K}}\,(\delta^{\bar{a}b}-\bar{z}^{\bar{a}}z^b)\,;\tag{5.166}$$

where $z_a\equiv\epsilon_{ab}\,z^b$ The eigenvalues of $g_{a\bar{b}}$ are: $-1/(|u|^2+|v|^2-1)\,,1/(|u|^2+|v|^2-1)^2$ and, if $|u|^2+|v|^2>1$, $g_{a\bar{b}}$ has the correct signature $(-,-,+,+)$.

5.5.1 Relation to known black-hole solutions

For stationary, axisymmetric, asymptotically flat solutions admitting the two Killing vectors ∂_t and ∂_φ, the most general case of complex scalar fields u,v corresponds to a Kerr–Newman solution with NUT-charge[11], whose metric reads [120, 130, 131, 132, 59]:

$$ds^2 = \frac{\tilde{\Delta}}{|\rho|^2}(dt+\bar{B})^2-\frac{|\rho|^2}{\tilde{\Delta}}\left(\frac{\tilde{\Delta}}{\Delta}dr^2+\tilde{\Delta}d\theta^2+\Delta\sin^2\theta d\varphi^2\right)\,,\tag{5.167}$$

where:

$$\Delta=(r-M)^2-c^2\tag{5.168}$$

[11]See Appendix C.

$$\tilde{\Delta} = \Delta - \alpha^2 \sin^2\theta \tag{5.169}$$

$$\rho = r + \imath\,(\ell + \alpha\cos\theta)\,, \tag{5.170}$$

$$\bar{B} = \left(\alpha\sin^2\theta\frac{|\rho|^2 - \tilde{\Delta}}{\tilde{\Delta}} + 2\ell\cos(\theta)\right)d\varphi \tag{5.171}$$

with:

$$c^2 = M^2 + \ell^2 - \frac{1}{2}(Q^2 + P^2) - \alpha^2 \tag{5.172}$$

as given in (5.127) in terms of the Boyer–Lindquist coordinates (r,θ), of the electric and magnetic charges (Q,P) and of the ADM-mass and NUT charge (M,ℓ). The parameter α is related to the angular momentum $J = M_\varphi$ of the solution by $\alpha = M_\varphi/M$. Here the metric field $U(r,\theta)$ is given by $e^{2U} = \frac{\tilde{\Delta}}{|\rho|^2}$[59][12]. For this solution the fields λ, $\hat{\rho}$ and the flat 2D metric h_{mn} read:

$$\lambda^2 \;\; = \tilde{\Delta}\;;\quad \hat{\rho} = \sqrt{\Delta}\,\sin\theta\;;\quad h_{mn}\begin{pmatrix} 1/\Delta & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{5.173}$$

so that $\sqrt{h}\,\hat{\rho} = \sin(\theta)$. The latter expression holds, in suitable coordinates, for all axisymmetric solutions. The Ernst potentials are then[59]:

$$\mathcal{E} = \frac{r - 2M + \imath\,(\alpha\cos\theta - \ell)}{r + \imath(\alpha\cos\theta + \ell)} \tag{5.174}$$

$$\Psi = \frac{-Q + \imath\,P}{r + \imath\,(\alpha\cos\theta + \ell)} \tag{5.175}$$

and the corresponding homogeneous coordinates can be chosen as[59]:

$$U_E = r - M + \imath\,\alpha\cos\theta; \tag{5.176}$$

[12]The Kerr-Newman solution with NUT-charge, eq. (5.98), include for vanishing NUT charge $\ell = 0$ the Kerr-Newman black hole in Boyer–Lindquist coordinates, eq. (5.17).

$$V_E = \frac{1}{\sqrt{2}}(-Q + \imath\, P); \tag{5.177}$$

$$W_E = M - \imath\, \ell. \tag{5.178}$$

Let us observe that only an $SU(1,1)$ subset of the $SU(1,2)$ invariance is realized on the four dimensional fields, under which the "charges" (W_E, V_E) form a doublet while U is a singlet. The Kerr-Newmann solution is retrieved by setting $\ell = 0$ in eq.s (5.174) and (5.175), the Reissner-Nordström electric-magnetic solution by further setting $\alpha = 0$ and finally the Schwarzschild solution is obtained from Reissner-Nordström when $Q = P = 0$[59].

Let us relate the explicit expressions for the Ernst potentials here with the σ-model description given above. The metric function $\tilde{\Delta}$ in (5.169) appears to be related to the $SU(1,2)$-invariant Kähler potential $\mathcal{K}$ in (5.162)[59]:

$$\tilde{\Delta} = |U_E|^2 + |V_E|^2 - |W_E|^2 = |W_E|^2 e^{-\mathcal{K}}. \tag{5.179}$$

According to the identification (5.176)-(5.178) the condition (5.163) acquires a precise physical meaning. In the static solutions ($\alpha = 0$) the condition (5.163) is guaranteed as long as $r > r_+$, r_+ being the outer horizon:

$$r_+ = M + \sqrt{M^2 + \ell^2 - \frac{P^2 + Q^2}{2}}\,. \tag{5.180}$$

On the other hand, the Kerr-Newmann case ($\ell = 0$) it gives:

$$r > M + \sqrt{M^2 - \frac{Q^2 + P^2}{2} - \alpha^2 \cos^2\theta} \equiv r_e \tag{5.181}$$

where $r_e > r_+$ defines the external boundary of the ergosphere, where the component g_{00} of the metric vanishes, while:

$$r_+ = M + \sqrt{M^2 - \frac{Q^2 + P^2}{2} - \alpha^2} \tag{5.182}$$

is the radius of the outer event horizon. Then we see that the special-coordinate patch described by u, v breaks down on the ergosphere.

If we cross the ergosphere surface $\tilde{\Delta} = 0$ we are bound to change the coordinate patch. The new patch can be described by the CP(2) riemannian space $SU(1,2)/U(2)$, with Kaehler potential:

$$\mathcal{K} = -\log(1 - |u|^2 - |v|^2). \tag{5.183}$$

The universal model considered here, and the KN-Taub-NUT solution thereof, can be embedded in more general supergravity models (for instance in all $\mathcal{N} = 2$ symmetric supergravity models, dimensionally reduced to $D = 3$) and thus it is interesting to consider the $G_{(3)}$-invariant properties of this solution[59]. In light of the discussion at the end of Sect 5.4, the description of such properties should take into account, aside from the Nöether charge matrix $\tilde{Q}$, also the constant matrix Q_ψ.

5.6 Kerr-Newmann Solution from Schwarzschild

In this section we give an alternative way to generate the Hamilton principal 1-form $S^{(1)}$ corresponding to the Kerr-Newmann solution. It makes use of duality symmetry and general coordinate transformations starting from the Schwarzschild solution.

We will proceed in two steps. We first need an explicitly $SU(1,2)$-duality invariant expression for the $\mathcal{W}_3$ of the Reissner-Nordström solution in 3D. This can be achieved by using the generating technique of $SU(1,2)$ to generate solutions in 3D. In particular, starting from Schwarzschild field variables:

$$U_E = r - M, \tag{5.184}$$

$$V_E = 0, \tag{5.185}$$

$$W_E = M, \tag{5.186}$$

the action of the $SU(1,2)$ Harrison and Ehlers transformations generate electric, magnetic and in general also a NUT charge, thus leading to a Reissner-Nordström-NUT (RN-NUT) solution. Next, as a second step we use a procedure first introduced by Clément[133, 134, 59] allowing the generation of a Kerr-Newmann solution from Reissner-Nordström by an appropriate sequence of $SU(1,2)$ and coordinate transformations.

5.6.1 $\mathcal{W}_3$ for the RN-NUT Solution

Let us recall that in the static case the prepotential $\mathcal{W}_3$ provides a first order description of $D = 3$ static solutions[59, 24]:

$$\frac{d\bar{z}^{\bar{a}}}{d\tau} = g^{\bar{a}b}\,\partial_b \mathcal{W}_3 \tag{5.187}$$

satisfying the Hamilton-Jacobi equation:

$$\partial_{\bar{a}}\mathcal{W}_3\, g^{\bar{a}b}\,\partial_b \mathcal{W}_3 = c^2 \tag{5.188}$$

c being the extremality parameter.
Quite generally a static solution is completely defined by a point P of the scalar manifold representing the values of the scalars at radial infinity $\tau = 0$, and the tangent vector to the geodesic, which is an object transforming under H^*. Here H^* is the isotropy group of the coset G/H^*, G being the 3D isometry group. Since the action of G/H^* on P is transitive over the scalar manifold, we can always fix P to be the origin O at which all fields vanish, and study the geodesic solutions corresponding to various choices of the velocity vector at infinity. In this way we break G to the little group H^* of the origin and we expect the $\mathcal{W}_3$ describing the family of solutions with $P = O$ to be an H^*-invariant function[59].
In our case we have:

$$\frac{G}{H^*} = \frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)} \tag{5.189}$$

and we shall prove that the Reissner-Nordström-NUT (RN-NUT) solutions are described by a solution to the Hamilton-Jacobi equation of the form:

$$\begin{aligned}\mathcal{W}_3 &= -c\log\left(\frac{|U_E|+\sqrt{|W_E|^2-|V_E|^2}}{|U_E|-\sqrt{|W_E|^2-|V_E|^2}}\right)\\ &= -c\log\left(\frac{|u|+\sqrt{1-|v|^2}}{|u|-\sqrt{1-|v|^2}}\right)\end{aligned} \tag{5.190}$$

The above function is clearly $H^* = \mathrm{U}(1,1)$-invariant since both $|U_E|$ and $|W_E|^2 - |V_E|^2$ are[59].

Let us recover the expression (5.190) for the $\mathcal{W}_3$ describing the most general static (non-extremal) black hole in our model, from the one-parameter $\mathcal{W}_3^{(S)}$ of the Schwarzschild solution by a *duality (isometric) continuation* of it on the whole σ-model. By duality continuation we mean *defining* the value of $\mathcal{W}_3$ out of the one-dimensional submanifold on which $\mathcal{W}_3^{(S)}$ is defined by means of an isometry transformation on the σ-model. Of course here we are restricting to H^* transformations only and the resulting prepotential will be, by construction, H^*-invariant and still a solution to (5.188) being the latter duality invariant.

The geodesic corresponding to the Schwarzschild black hole is defined by the following prepotential:

$$\mathcal{W}_3^{(S)}(s) = -c\log\left(\frac{s+1}{s-1}\right), \tag{5.191}$$

defined on the submanifold:

$$u = \bar{u} = s\ ; \qquad v = 0\,. \tag{5.192}$$

It is straightforward to check that $\mathcal{W}_3^{(S)}(s)$ satisfies the Hamilton-Jacobi equation:

$$\partial_s\mathcal{W}_3^{(S)}\frac{\partial s}{\partial\bar{z}^{\bar{a}}}\,g^{\bar{a}b}\,\frac{\partial s}{\partial z^b}\partial_s\mathcal{W}_3^{(S)} = \frac{(s^2-1)^2}{4}\left(\partial_s\mathcal{W}_3^{(S)}\right)^2 = c^2\,, \tag{5.193}$$

where we have written:

$$s = \frac{(u+\bar{u})}{2} \qquad \text{and} \qquad z^a = (u,v)\,. \tag{5.194}$$

Next we apply to the Schwarzschild fields a generic H^*-transformation h^*. The latter can be written as the product of a Harrison transformation, a Ehlers $\mathrm{U}(1)_E$-transformation and a second U(1)-transformation (which corresponds to the $D = 4$ duality group). Referring to the notations of Appendix B we have:

$$h^* = H_{arrison}\, h_E\, h\,, \tag{5.195}$$

$$H_{arrison} \quad a = e^{a_1\, J_1 + a_2\, J_2} = \begin{pmatrix} \cosh(\mathbf{a}) & -e^{i\sigma}\, \sinh(\mathbf{a}) & 0 \\ -e^{-i\sigma}\, \sinh(\mathbf{a}) & \cosh(\mathbf{a}) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5.196}$$

$$h_E \quad = e^{\alpha\, J_\bullet} = \mathrm{diag}(e^{-i\alpha}, 1, e^{i\alpha})\ ; \tag{5.197}$$

$$h = e^{\beta\, J} = \mathrm{diag}(e^{-i\beta}, e^{2i\beta}, e^{-i\beta}) \tag{5.198}$$

where we have written:

$$\mathbf{a}\, e^{i\sigma} = a_1 + \imath\, a_2 \tag{5.199}$$

If we apply h^* to the Schwarzschild fields described by:

$$(W_E(s), V_E(s), U_E(s)) = (1, 0, s) \tag{5.200}$$

we find:

$$\begin{pmatrix} W_E \\ V_E \\ U_E \end{pmatrix} = h^* \begin{pmatrix} 1 \\ 0 \\ s \end{pmatrix}\,, \tag{5.201}$$

that is:

$$u = \frac{U_E}{W_E} = e^{2i\alpha}\, \frac{s}{\cosh(\mathbf{a})}\ ; \qquad v = \frac{U_E}{W_E} = -e^{-i\sigma}\, \tanh(\mathbf{a})\,. \tag{5.202}$$

From the above relations we find s in terms of the duality-transformed variables u, v:

$$s = \frac{|u|}{\sqrt{1-|v|^2}}\,. \tag{5.203}$$

Then we define $\mathcal{W}_3$ by duality continuation of $\mathcal{W}_3^{(S)}$:

$$\mathcal{W}_3^{(RN)}(u,v,\bar{u},\bar{v}) = \mathcal{W}_3^{(S)}(s(u,v,\bar{u},\bar{v})) = -c\,\log\left(\frac{|u|+\sqrt{1-|v|^2}}{|u|-\sqrt{1-|v|^2}}\right), \tag{5.204}$$

thus obtaining (5.190).

We may check our result by solving the corresponding first order equations (5.187)[59]:

$$\frac{d\bar{u}}{d\tau} = c\bar{u}\left(\frac{|u^2|-k^2}{|u|\,k}\right), \tag{5.205}$$

$$k^2 = 1-|v|^2 > 0\,, \tag{5.206}$$

$$\frac{dv}{d\tau} = 0\,. \tag{5.207}$$

From the first we derive:

$$\frac{d|u|}{d\tau} = c\,\frac{|u^2|-k^2}{k} \quad\Rightarrow\quad |u| = k\,\frac{Ae^{2c\tau}+1}{1-Ae^{2c\tau}}\,. \tag{5.208}$$

where A is an arbitrary constant that the take equal to one. The second equation is telling us that v is an arbitrary complex constant which we can set to:

$$v = -\frac{Q-iP}{\sqrt{2}M}\,e^{i\alpha} \quad\Rightarrow\quad k = c/M\,. \tag{5.209}$$

Being the phase of u a constant, the general solution can be written as follows:

$$u = k\,\frac{e^{2c\tau}+1}{1-e^{2c\tau}}\,e^{2i\alpha}\,. \tag{5.210}$$

Setting the arbitrary constant $A=0$ and using the relation between τ and r:

$$\tau = \frac{1}{2c}\log\left(\frac{r-M-c}{r-M+c}\right), \tag{5.211}$$

we find:

$$u = \frac{c}{M}\,\frac{2r-2M}{2c}\,e^{2\imath\alpha} = \frac{r-M}{M}\,e^{2\imath\alpha}\,; \tag{5.212}$$

$$v = -\frac{Q-\imath P}{\sqrt{2}M}\,e^{\imath\alpha}\,, \tag{5.213}$$

which defines the Reissner-Nordström-NUT solution where M, P, Q are the parameters of a Reissner-Nordström solution and α is the effect of a Ehler U(1)-transformation. The Nöether charge matrix reads:

$$\tilde{Q} = \mathcal{M}^{-1}\frac{d}{d\tau}\mathcal{M} = \begin{pmatrix} 0 & 0 & 2e^{\imath\alpha}M \\ 0 & 0 & -\imath\sqrt{2}(P-\imath Q) \\ 2e^{-\imath\alpha}m & \sqrt{2}(Q-\imath P) & 0 \end{pmatrix}\,. \tag{5.214}$$

The fields are obtained by the general formulas:

$$U = \frac{1}{2}\log\left(\frac{|u|^2+|v|^2-1}{|1+u|^2}\right)\;; \tag{5.215}$$

$$\Psi = \frac{v}{1+u}\;; \tag{5.216}$$

$$a = -i\frac{u-\bar{u}}{|1+u|^2}\,. \tag{5.217}$$

Using the generators of the solvable algebra of $\frac{\mathrm{SU}(1,2)}{\mathrm{U}(1)\times\mathrm{SU}(1,1)}$[13] we can compute the physical charges in terms of the parameters of the solution. The ADM mass $\hat{M}$ and NUT charge read[59]:

$$\hat{M} = \mathrm{Tr}(H_0^\dagger\,\tilde{Q}) = M\,\cos(2\alpha)\;; \tag{5.218}$$

$$\ell = -\,\mathrm{Tr}(G^\dagger\,\tilde{Q}) = -M\,\sin(2\alpha)\,. \tag{5.219}$$

[13]See Appendix B.

while the complex charge $\frac{\hat{Q}+i\,\hat{P}}{\sqrt{2}}$ is:

$$\frac{\hat{Q}+i\,\hat{P}}{\sqrt{2}} = -\,\mathrm{Tr}((T_1+iT_2)^\dagger\,\tilde{Q}) = \frac{Q+i\,P}{\sqrt{2}}e^{i\alpha}\,. \tag{5.220}$$

Using the above identifications, the matrix $\tilde{Q}$ in (5.214) reduces to the Nöether charge matrix in the first of eq.s (5.123), identifying hatted with un-hatted quantities. This represents the fact that the Nöether charge matrix $\tilde{Q}$ is the same for the KN-Taub-NUT and the RN-Taub-NUT solutions. The difference resides in the matrix Q_ψ which vanishes in the latter solution.

Since the Maxwell-Einstein theory is a consistent truncation of a generic $\mathcal{N}=2$ model, the above procedure for constructing a manifestly H^*-invariant $\mathcal{W}_3$ for the generic solution in the same $G_{(3)}$-orbit as the Schwarzschild one, from a *duality completion* of $\mathcal{W}_3^{(S)}$, applies to a generic $\mathcal{N}=2$, $D=4$ supergravity. In this case the Nöether charge $\tilde{Q}$ of a generic representative of the Schwarzschild orbit, is a diagonalizable matrix in the space $\mathfrak{K}$, orthogonal complement of $\mathfrak{H}^*$ in $\mathfrak{g}$ (the point at infinity ξ_0 is always set to coincide with the origin O), and transforms under the adjoint action of H^* in a characteristic H^*-representation. In particular $\tilde{Q}$ can be diagonalized using an H^*-transformation. The modulus s in $\mathcal{W}_3^{(S)}$ is a function of the eigenvalues of $\tilde{Q}$, and thus is an H^*-invariant function of the parameters Q_A of $\tilde{Q}$: $s=f(Q_A)$. These parameters also provide a parametrization of the coset $G_{(3)}/H^* \equiv e^{\mathfrak{K}}$ and, in the physical patch U, can be expressed in terms of the scalar fields z^a, so that we can locally express s as a H^*-invariant function of z^a: $s=f(Q_A(z^a))=s(z^a)$. A duality completion procedure, analogous to the one illustrated above, allows then to determine the following H^*-invariant expression for $\mathcal{W}_3$ for the Schwarzschild orbit[59]:

$$\mathcal{W}_3 = -c\log\left(\frac{s(z^a)+1}{s(z^a)-1}\right)\,. \tag{5.221}$$

In the case of the universal model $s(z^a)$ was given in eq. (5.203).

5.6.2 The Clément Generating Technique

Having at our disposal a duality invariant $\mathcal{W}_3$ for the Reissner-Nordström solution, we may now apply a procedure, introduced in [133, 134], to relate static and rotating black-hole solutions. In this way we shall arrive at the explicit expression of the U_E, V_E, W_E variables (5.176) - (5.178) for the Kerr-Newmann solution[133, 134, 59]. We shall apply to the Reissner-Nordström set of homogeneous variables associated to (5.212) and (5.213), which for definiteness we choose to be[59]:

$$U_E = r - M; \tag{5.222}$$

$$V_E = \frac{1}{\sqrt{2}}(Q - \imath\, P); \tag{5.223}$$

$$W_E = M + \imath\, \ell. \tag{5.224}$$

the transformation $\Pi \cdot R \cdot \Pi$, where:

$$\Pi = \{U_E \to V_E, V_E \to U_E, W_E \to -W_E\} \tag{5.225}$$

is a $SU(1,2)$ involution, and R is the following 4D space-time coordinate transformation:

$$R : \begin{cases} d\varphi &= d\varphi' + \gamma\Omega dt' \\ dt &= \gamma dt' \end{cases} \tag{5.226}$$

relating the original reference frame to one rotating with constant angular velocity Ω. The constant time-rescaling factor γ will be fixed in the following to have the standard expression for the Ernst potentials of the Kerr-Newmann solution[59].

The first involution Π gives rise to the following new potential:

$$\mathcal{E}' = \frac{U'_E - W'_E}{U'_E + W'_E} = \frac{-\frac{1}{\sqrt{2}}(Q - \imath P) + M - \imath\ell}{-\frac{1}{\sqrt{2}}(Q - \imath P) - M + \imath\ell}\,, \tag{5.227}$$

$$\psi' = \frac{V'_E}{U'_E + W'_E} = \frac{r - M}{-\frac{1}{\sqrt{2}}(Q - \imath P) - M + \imath\ell} \tag{5.228}$$

One can readily see that the new solution corresponds to a Bertotti-Robinson space time (BR metric)[80, 81, 59], with radius:

$$R_{BR} \equiv |V_E - W_E| = \sqrt{(\frac{Q}{\sqrt{2}} + M)^2 + (\frac{P}{\sqrt{2}} + \ell)^2}\,. \tag{5.229}$$

The coordinate transformation R induces the following transformation of the 4D static metric and gauge fields:

$$R : \begin{cases} e^{2\tilde{U}'} &= \gamma^2\left(e^{2U'} - e^{-2U'}\hat{\rho}^2\Omega^2\right) \\ \tilde{\omega} &= \frac{\hat{\rho}^2\Omega}{\gamma\left(e^{4U'} - \hat{\rho}^2\Omega^2\right)} \\ \tilde{\hat{\rho}} &= \gamma\hat{\rho} \end{cases} \tag{5.230}$$

where

$$e^{2U'} = \frac{|U_E|^2 + |V_E|^2 - |W_E|^2}{R_{BR}^2} \equiv \frac{\Delta}{R_{BR}^2} \tag{5.231}$$

$$\hat{a}' = a' = \frac{(\bar{V}_E W_E - V_E \bar{W}_E)}{R_{BR}^2} = \frac{2(\frac{Q}{\sqrt{2}}\ell - \frac{P}{\sqrt{2}}M)}{R_{BR}^2} \tag{5.232}$$

We have introduced here the $SU(1,2)$ invariant $\tilde{\Delta}$, which, in the coordinates (5.222) - (5.224), is:

$$\tilde{\Delta} = (r - M)^2 - c_{RT}^2, \tag{5.233}$$

where:

$$c_{RT}^2 \equiv |W_E|^2 - |V_E|^2 = M^2 + \ell^2 - \frac{1}{\sqrt{2}}(Q^2 + P^2), \tag{5.234}$$

is the extremality parameter of the dyonic Reissner-Nordström-NUT solution. Note that $c_{RT}^2 = \frac{k}{2}\,\mathrm{Tr}[\tilde{Q}^2]$ (see eq.s (5.124) and (5.125)).

The redefinition of the metric implies a transformation of the gauge field-strengths, that corresponds to the following transformation on the gradient of the Ernst potential ψ (here $x^m = (r, \theta)$)[59]:

$$\partial_m\hat{\psi}' = \gamma\left[\partial_m\Psi' - \hat{\rho}\Omega e^{-2U'}\left({}^{\star(2)}\partial_m\overline{\Psi'}\right)\right]\,. \tag{5.235}$$

The integration of equation (5.235) is easily performed by observing that:

$$^{\star(2)}\partial_r\hat{\overline{\psi'}} = 0 \tag{5.236}$$

since $\psi' = \psi'(r)$ is only function of the radial variable. Further observing that:

$$\partial_r\psi' = -\frac{\gamma}{R_{BR}^2}\left[(\frac{Q}{\sqrt{2}} + M) + \imath(\ell + \frac{P}{\sqrt{2}})\right] \tag{5.237}$$

the final result is[59]:

$$\begin{aligned}\hat{\psi}' &= \gamma\{\psi'(r) + \imath(V_E - W_E)\Omega\cos\theta\} \\ &= \frac{\gamma}{R_{BR}^2}\left\{(r-M)(\bar{V}_E - \bar{W}_E) + \imath\alpha\cos\theta\right\}\end{aligned} \tag{5.238}$$

together with

$$\begin{aligned}\hat{\mathcal{E}}' &= e^{2\hat{U}'} - |\hat{\psi}'|^2 + \imath\hat{a}' \\ &= -\frac{\gamma^2}{R_{BR}^2}\left(c_{RT}^2 + \alpha^2\right) + \frac{\imath\,(\bar{V}_E W_E - V_E\bar{W}_E)}{R_{BR}^2}\end{aligned} \tag{5.239}$$

where we have defined $\alpha \equiv (\Omega R_{BR}^2)$.

We may give a simpler expression to the Ernst potentials by fixing the time rescaling γ as:

$$\gamma^2 = \frac{c_{RT}^2}{c_{RT}^2 + \alpha^2}\,. \tag{5.240}$$

With this redefinition we obtain[59]:

$$\hat{\mathcal{E}}' = \frac{\hat{U}'_E - \hat{W}'_E}{\hat{U}'_E + \hat{W}'_E} = \frac{V_E + W_E}{V_E - W_E} \tag{5.241}$$

$$\hat{\psi}' = \frac{\hat{V}'_E}{\hat{U}'_E + \hat{W}'_E} = \frac{\gamma(U_E + \imath\alpha\cos\theta)}{V_E - W_E}\,. \tag{5.242}$$

implying the following transformation on the homogeneous variables:

$$R\cdot\Pi : \begin{cases} \hat{U}'_E &= V_E \\ \hat{V}'_E &= \gamma(U_E + \imath\alpha\cos\theta) \\ \hat{W}'_E &= -W_E \end{cases} \tag{5.243}$$

Performing again the transformation Π as given in eq. (5.225), we finally obtain the Kerr-Newmann (TaubNUT) fields in terms of the corresponding variables of the Reissner-Nordström (TaubNUT) solution[59]:

$$\Pi \cdot R \cdot \Pi : \begin{cases} \hat{U}''_E &= \gamma(U_E + \imath\alpha\cos\theta) \\ \hat{V}''_E &= V_E \\ \hat{W}''_E &= W_E \end{cases} \tag{5.244}$$

corresponding to the potentials:

$$\hat{\mathcal{E}}'' = \frac{\gamma(U_E + \imath\alpha\cos\theta) - W_E}{\gamma(U_E + \imath\alpha\cos\theta) + W_E}\,, \tag{5.245}$$

$$\hat{\psi}'' = \frac{V_E}{\gamma(U_E + \imath\alpha\cos\theta) + W_E}\,. \tag{5.246}$$

They correspond to the standard Kerr-Newmann potentials [119, 59]:

$$\mathcal{E}_{KN} = 1 - \frac{2M}{r + \imath\alpha\cos\theta} \tag{5.247}$$

$$\psi_{KN} = \frac{\frac{-1}{\sqrt{2}}(Q - \imath P)}{r + \imath\alpha\cos\theta}\,. \tag{5.248}$$

if we set, besides $\ell = 0$;

$$r \to \gamma(r - M) + M\,, \quad \alpha \to \gamma\alpha\,. \tag{5.249}$$

For the Kerr-Newmann solution, the field a appearing in (5.59) is given by the imaginary part of $\mathcal{E}$[59],

$$a = 2\frac{M\alpha\cos\theta}{|\rho|^2}\,. \tag{5.250}$$

5.7 The DWHJ principal 1-form for the Kerr-Newmann solution

Let us explicitly compute the DWHJ principal functions S^r, S^θ for the Kerr-Newmann solution[59].

We recall, from section 5.5.1, that the two-dimensional metric is:

$$h_{mn} = \begin{pmatrix} 1/\Delta & 0 \\ 0 & 1 \end{pmatrix} . \tag{5.251}$$

We have:

$$\partial_a S^m = \pi_a^m = \sin(\theta)\; g_{a\bar{b}}(z)\, h^{mn}\, \partial_n \bar{z}^{\bar{b}} \tag{5.252}$$

that is:

$$\pi_a^r = \sin(\theta)\; g_{a\bar{b}}(z)\, \Delta\, \partial_r \bar{z}^{\bar{b}}\;\; , \tag{5.253}$$

$$\pi_a^\theta = \sin(\theta)\; g_{a\bar{b}}(z)\, \partial_\theta \bar{z}^{\bar{b}}\; . \tag{5.254}$$

Equation (5.252), recalling (5.12), admits the weakly embedded solution[59]:

$$S^m = 2\,\Re\left[(z^a - \xi^a(x)\pi_a^m(x))\right] + \delta_r^m \int^r d\hat{r}\mathcal{L}(\xi, \partial\xi, \hat{x}) \tag{5.255}$$

Using (5.165), if we denote by ξ^u, ξ^v the on-shell values of the field u, v[59]:

$$\xi^u = \frac{r - M + \imath\, \alpha \cos\theta}{M + \imath\, \ell}; \tag{5.256}$$

$$\xi^v = \frac{-Q + \imath\, P}{\sqrt{2}(M + \imath\, \ell)}; \tag{5.257}$$

we find:

$$\begin{aligned} & S^r(z, x) = \\ & +2\, \sin(\theta)(M^2 + \ell^2)^2 \frac{\Delta(x)}{\tilde{\Delta}^2(x)} \Re\left[(u - \xi^u)(v - |\xi^v|^2) + (v - \xi^v)\xi^u \bar{\xi}^v\right] + \\ & \quad + \int^r d\hat{r}\mathcal{L}(\xi, \partial\xi, \hat{x})\, ; \end{aligned} \tag{5.258}$$

$$\begin{aligned} & S^\theta(z, x) = \\ & \quad -2\, \alpha\, \sin^2(\theta) \frac{(M^2 + \ell^2)^2}{\tilde{\Delta}^2(x)} \Im\left[(u - \xi^u)(v - |\xi^v|^2) + (v - \xi^v)\xi^u \bar{\xi}^v\right] . \end{aligned} \tag{5.259}$$

In this chapter we have addressed the issue of the first order description of generic, not necessary extremal, asymmetric solutions. This was done by working out the general form of the principal functions S^m associated with the corresponding effective 2D sigma-model in the DWHJ setting.

Chapter 6

Conclusions

In this thesis we have addressed the issue of the first order description of generic axisymmetric black holes in supergravity. An important issue in my research work was to extend the Hamilton-Jacobi formalism from mechanical models, whose degrees of freedom depend on just one variable, to field theories where the degrees of freedom depend on two or more variables. This problem was addressed and developed in generality in field theory[58, 59], but not much was known in the context of gravitational field theories. An important issue in the this thesis was to apply such extended formalism to the study of black holes.

We have worked with the so-called De Donder-Weyl-Hamilton-Jacobi (DWHJ) theory, which is the simplest extension of the classical Hamilton-Jacobi approach in mechanics[58, 59]. One important difference with respect to the case of classical mechanics consists in the replacement of the Hamilton principal function S, directly related to the fake-superpotential of static black holes, with a *Hamilton principal* 1*-form*, that is with a covariant vector S_i.

In the first part of in my thesis I reported the description of static and spherically symmetric black holes in a Lagrangian and Hamiltonian framework, where the prepotential characterizing the flow has a natural interpretation as Hamilton principal function.

A first achievement in my thesis is to formulate the physics of rotating black holes in terms of an effective two dimensional Lagrangian, whose independent variables are the radial variable r and the angular variable θ. It was particularly useful to formulate the theory in such a way that all the propagating degrees of freedom have been reduced to scalars, by use of 3D Hodge-dualization [51]. In this way, the effective 3-dimensional Lagrangian has the form of a non linear sigma model, whose scalars include the degrees of freedom of the space-time metric and of the electric and magnetic components of the gauge vectors.

In chapter five, of this thesis, we have addressed the issue of the first order description of generic (not necessarily extremal) axisymmetric solutions. This was done by working out the general form of the principal functions S_m associated with the corresponding effective 2D sigma-model in the DWHJ setting. We have also given a characterization of the general properties of such solutions with respect to the global symmetry group of the effective 2D sigma-model which describes them. This was done by introducing, aside from the Nöether charge matrix, a further characteristic constant matrix Q_ψ, in the Lie algebra of $G_{(3)}$, associated with the rotational motion of the black hole.

As a direction for further investigation it would be interesting to generalize this analysis to more general stationary solutions, including (non necessarily extremal) multicenter black holes, which requires the extension from 2D to 3D. In this respect there is virtually no conceptual obstruction in generalizing the DWHJ construction and the general formula for S_m, which we have mainly used here within a 2D effective sigma-model description, to the full 3D effective description of stationary solutions. It would moreover be interesting to analyze the axisymmetric solutions to symmetric supergravities from the point of view of the *integrability* of the corresponding effective 2D sigma-model, which we have not exploited here. This latter property being related to the presence in a gravity/supergravity theory, once dimensionally reduced to $D = 2$, of an infinite dimensional global symmetry group, generalizing the Geroch group of pure Einstein gravity (see for instance [135, 136]).

Appendix A

Duality in supersymmetric theories

In this appendix we present the concept of duality in supersymmetric theories. Duality is an invertible map between two theories sending states into states, while preserving the interactions, amplitudes and symmetries. Two theories that are dual to one and another can in some sense be viewed as being physically identical[64, 65, 8, 66, 67].

A.1 Duality in field theory

In this section we give some information on duality in field theory and supergravity. In particular, we start by briefly reviewing the early idea of duality in field theory, from the Dirac work on monopoles to its extensions in the context of spontaneously broken symmetry, and then introduce the concept of duality in extended supersymmetric theories.

A.1.1 The duality of electricity and magnetism

A magnetic monopole is a hypothetical particle in particle physics that is magnet with only one magnetic pole. A magnetic monopole would then have a net "magnetic charge". Modern interest in the concept stems from particles theories, notably the grand unification and superstring theories, which predict their existence[137, 67, 138].

The magnetic monopole was first hypothesized by Pierre Curie in 1894, but the quantum theory of magnetic charge started with a paper by the physicist Paul A. M. Dirac in 1931[139]. In this paper, Dirac showed that the existence of magnetic monopoles was consistent with Maxwell's equations only if electric charges are quantized, which is always observed. Since then, several systematic monopole searches have been performed. Experiments in 1975[140] and 1982[141] produced candidate events that were initially interpreted as monopoles, but are now regarded as inconclusive[142].

The equations governing the electromagnetic field are Maxwell's equations[64, 65, 8, 66, 67],

$$\nabla \cdot \mathbf{E} = \rho \qquad\qquad \nabla \wedge \mathbf{B} - \partial_t \mathbf{E} = \mathbf{j}, \tag{A.1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad\qquad \nabla \wedge \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}, \tag{A.2}$$

where: $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field and

$$j^\mu = (\rho, \mathbf{j}), \tag{A.3}$$

is the electric current four-vector, $\mathbf{j}$, is the current density. These equations can be written in the compact relativistic notation[64, 143, 65, 67],

$$\partial_\nu F^{\mu\nu} = -j^\mu, \tag{A.4}$$

$$\partial_\nu {}^*F^{\mu\nu} = 0, \tag{A.5}$$

where $F^{\mu\nu}$ is the electromagnetic field tensor:

$$F^{0i} = -E^i \qquad\qquad F^{0i} = -\varepsilon_{ijk} B^k, \tag{A.6}$$

and ${}^*F^{\mu\nu}$ is the dual tensor of $F^{\mu\nu}$,

$${}^*F^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \tag{A.7}$$

which may obtained formally from $F^{\mu\nu}$ by replacing $\mathbf{E}$ by $\mathbf{B}$ and $\mathbf{B}$ by $-\mathbf{E}$[64, 65, 67][1].

In vacua, where j^μ vanishes, the Maxwell equations are symmetric under the "duality" transformation:

$$F^{\mu\nu} \to {}^*F^{\mu\nu}, \qquad {}^*F^{\mu\nu} \to -F^{\mu\nu}, \tag{A.8}$$

or, equivalently,

$$\mathbf{E} \to \mathbf{B}, \qquad \mathbf{B} \to -\mathbf{E}, \tag{A.9}$$

which, roughly speaking, interchanges electricity with magnetism. Could such a symmetry be valid even in the presence of matter. In such a theory we would have to introduce a magnetic current

$$j^\mu_{(m)} = (p, \mathbf{k}), \tag{A.10}$$

on the right hand side of equations (A.2) and (A.5), giving the new field equations:

$$\partial_\nu F^{\mu\nu} = -j^\mu, \tag{A.11}$$

$$\partial_\nu {}^*F^{\mu\nu} = -j^\mu_{(m)}. \tag{A.12}$$

The equations (A.11) and (A.12) are symmetric under the duality transformation of equations (A.8) augmented by:

$$j^\mu \to j^\mu_{(m)}, \qquad j^\mu_{(m)} \to -j^\mu. \tag{A.13}$$

If the electric and magnetic currents result from point particles at space-time points x_i, as we shall suppose,

$$j^\mu = \sum_i q_i \int dx_i^\mu \delta_4(x - x_i) \tag{A.14}$$

[1]We use the conventions that $\varepsilon^{\mu\nu\rho\sigma}$ is totally antisymmetric with $\varepsilon^{0123} = 1$ and Greek indices take the values 0, 1, 2, 3 whilst Latin indices only take the values 1, 2, 3.

and

$$j^{\mu}_{(m)} = \sum_i p_i \int dx_i^{\mu} \delta_4(x - x_i) \tag{A.15}$$

where the integral over x_i is taken along the world line of the i-th particle whose electric and magnetic charges are q_i and p_i respectively. In conventional electrodynamics the Lorentz force law for particle of (electric) charge q and rest mass leads to the equation of motion,

$$m\frac{d^2x^{\mu}}{d\tau^2} = qF^{\mu\nu}\frac{dx_{\nu}}{d\tau}. \tag{A.16}$$

In a duality theory this equation would be generalised to

$$m\frac{d^2x^{\mu}}{d\tau^2} = \left(qF^{\mu\nu} + p^*F^{\mu\nu}\right)\frac{dx_{\nu}}{d\tau}. \tag{A.17}$$

where p is the particle's magnetic charge. The equations (A.11), (A.12), (A.14), (A.15) and (A.17) completely specify the dynamics of a classical, i.e. non-quantum mechanical, system of electrically and magnetically charged particles interacting with the electromagnetic field in such a way that it possess the dual symmetry of equations (A.8) and (A.13).

In discussing further whether nature might indeed possess such a duality it is natural to ask at this point whether it is consistent with quantum theory. Actually Dirac was lead naturally to a theory possessing this symmetry by considering a quantum mechanics in which the wave function had a non-integrable, or path-dependent, phase factor. Dirac's work pointed out the profound theoretical consequences of the existence of magnetic monopoles at the quantum level. One can see immediately that quantization may not be straightforward since this procedure usually exploits the canonical, Hamiltonian, formalism. Now the canonical variables for the electromagnetic field are not the components of $F^{\mu\nu}$ but rather the components of the four vector potential $(A^{\mu}) = (\Phi, \mathbf{A})$, whose defining property is

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}. \tag{A.18}$$

This equation implies the vanishing of

$$\partial_\mu {}^* F^{\mu\nu} = 0 \tag{A.19}$$

and, consequently, of the magnetic current, $j^\mu_{(m)}$, destroying the dual symmetry.

Dirac was able to circumvent this difficulty, showing that a dually symmetric electromagnetic theory could be quantized, provided that for any electric charge q and magnetic charge p in theory, the condition

$$\frac{qp}{4\pi\hbar} = \frac{n}{2}, \qquad n \text{ an integer} \tag{A.20}$$

was satisfied[143, 144]. This is the celebrated Dirac quantization condition. The occurrence of the modified Planck constant, $\hbar$, emphasises that, in Dirac's approach, it is quantum mechanical in origin.

A.1.2 The 't Hooft-Polyakov monopole

In theoretical physics, the 't Hooft-Polyakov monopole is a topological solution similar to the Dirac monopole but without any singularities. It was first found independently by Gerard 't Hooft[145] and Alexander M. Polyakov[146].

Dirac introduced the notation of magnetic charge in field theory. The Dirac monopole is a localized source corresponding to a singularity of the theory; later 't Hooft-Polyakov extended that notation by shounning the existence of non-singular solitonic[2] monopole, like solutions in the effective action of non Abelian gauge theories coupled to scalar fields[145, 146, 143, 144]. The monopole solutions appears looking for finite energy configurations, the

[2]In mathematics and physics, a soliton is a self-reinforcing solitary wave (a wave packet or pulse) that maintains its shape while it travels at constant speed. Solitons are caused by a cancellation of nonlinear and dispersive effects in the medium. "Dispersive effects" refer to dispersion relations between the frequency and the speed of the waves. Solitons arise as the solutions of a widespread class of weakly nonlinear despersive partial differential equations describing physical systems[145, 146, 143, 144].

magnetic charge being a topological charge, p satisfying the Dirac quantitation condition:

$$p = -\frac{4\pi}{q} n_m, \tag{A.21}$$

Due to this fact, in the 't Hooft-Polyakov monopole, the relativistic mass M of the soliton is fixed in terms of fields in terms of the topological charge and is a Bogomol'nyi bound[147, 148];

$$M \geq ap, \ , \tag{A.22}$$

where a is a parameter characterizing the configuration and has an interpretation in supergravity theory[147, 148].

We now consider not only particles carrying either an electric or magnetic charge, but particles that carry both types of charges (e, p), they are called dyons[3] . For a system composed by two dyons of charges (e_1, p_1) and (e_2, p_2), it is possible to show that the charge quantization gets generalized to the "Dirac-Schwinger-Zwanziger" relation[149]:

$$e_1 p_2 - e_2 p_1 = 2\pi n \qquad n \in N; \tag{A.23}$$

and that the general solution is:

$$e = e_0 \left(n_e + \frac{\theta}{2\pi} n_m \right); \tag{A.24}$$

$$p = n_m p_0 = n_m \frac{2\pi n_0}{e_0}; \tag{A.25}$$

where $n_m, n_e \in Z$ and $n_0 \in N$. This solution is equivalent to:

$$e + ip = e_0(n_e + \mathcal{N} n_m); \tag{A.26}$$

with:

$$\mathcal{N} \equiv \left(\frac{\theta}{2\pi} + i \frac{2\pi n_0}{e_0^2} \right). \tag{A.27}$$

[3]In physics, a dyon is a hypothetical particle in four-dimensional theories with both electric and magnetic charges. A dyon with a zero electric charge is a magnetic monopole.

From the shape of the solution (A.26) we say that the physical states of charges $(e,p) = e + ip$ are located on a discrete two dimensional lattice of periods e_0 and $\mathcal{N} e_0$ and represented by a vector of the lattice (n_e, n_m); and given a lattice, related to each other by the action of $SL(2,Z)$. Hence, the duality transformations:

$$F \leftrightarrow {}^*F; \tag{A.28}$$

$$e \leftrightarrow p; \tag{A.29}$$

should belong to the discrete group $SL(2,Z)$.

For dyons the Bogomolnyi bound (A.22) can be generalized in this way:

$$M(e,p) \geq a|e + ip|; \tag{A.30}$$

and the mass of BPS states:

$$M(e,p) = a|e + ip|. \tag{A.31}$$

Note that from this relation one recovers the right expression for the masses of all particles in the spontaneously broken gauge theory. Indeed it is exact both for elementary excitations, like gauge bosons of change $(e,0)$, whose mass is given by the Higgs mechanism, and for solitons, like the monopoles discussed above, of change $(0,p)$, in which case it coincides with (A.22). This given evidence for the Montonen-Olive conjecture is an exact symmetry.

A.2 Supersymmetry algebras that include topological charges

Let us now turn to duality in gauge theories with extended supersymmetry where this concept can be implemented in a natural way. Indeed a crucial ingredient for the existence of 'tHooft-Polyakov monopoles like solution in electric-magnetic duality is the presence in the theory of Higgs field transforming in the adjoint representation of the gauge group U.

This requirement is always satisfied in N-extended supersymmetric theories [6, 7, 8]. Indeed the supersymmetry algebra for $N \geq 2$ prescribes that vector supermultiplet includes stolei fields. This fact has the effect that the conditions for the existence of 'tHooft-Polyakov monopoles are always present, so that duality arises in a very natural way in these theories.

The solitonic configurations, if present, are directly related to the structure of the supersymmetry algebra, with profound implications on the spectrum of states and on the quantum validity of the solution, at least for BPS states[147, 148, 150, 6, 151].

In this section, we shall show that in supersymmetric theories with solitons, the usual supersymmetry algebra include the topological quantum numbers as central charges[150, 6, 151].

A.2.1 Extended supersymmetry

Supersymmetry is, by definition, a symmetry between fermions and bosons[6]. A supersymmetric field theoretical model consists of a set of quantum field and a Lagrangian for them which exhibit such a symmetry. A supersymmetric model which is covariant under general coordinate transformations or a model which posses local ("gauged") supersymmetry is called a supergravity model[6, 7, 8].

In theoretical physics, extended supersymmetry is supersymmetry whose supersymmetry generators $Q_{\alpha i}$ carry not only a spinor index α, but also an additional index $i = 1, 2, \cdots, N$ where $N \geq 2$ is integer[152, 150, 6, 153][4].

Extended supersymmetry is also called $N = 2$, $N = 4$ supersymmetry for example. The more extended supersymmetry is, the more it constrains physical observables and parameters. The minimal (un-extended) supersymmetry is a realistic conjecture for particle physics, but extended supersymmetry is very important for analysis of mathematical properties of quantum field theory and superstring theory.

[4] Any $Q_{\alpha i}$ is a generator of supersymmetry.

Two particularly important examples of these spectra are the $N = 4$ Yang-Mills multiplet with $\lambda_0 = -1$ and the $N = 8$ supergravity multiplet with $\lambda_0 = -2$, where λ_0 being the minimal value of helicity of the representation:

$N = 4$

helicity:	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
states:	1	4	6	4	1

;

$N = 8$

helicity:	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2
states:	1	8	28	56	70	56	28	8	1

.

The bosonic generators are thus the four momenta P_μ and the six Lorentz generators $M_{\mu\nu}$, plus a certain number of Hermitian internal symmetry generators B_r. The following equations summarise the supersymmetry algebra:

$$[P_\mu, P_\nu] = 0; \tag{A.32}$$

$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho); \tag{A.33}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\mu\sigma}M_{\nu\rho}); \tag{A.34}$$

$$[B_r, B_s] = ic_{rs}{}^t B_t; \tag{A.35}$$

$$[B_r, P_\rho] = 0; \tag{A.36}$$

$$[B_r, M_{\rho\sigma}] = 0; \tag{A.37}$$

$$[Q_{\alpha i}, P_\mu] = 0; \tag{A.38}$$

$$[\bar{Q}^i{}_{\dot{\alpha}}, P_\mu] = 0; \tag{A.39}$$

$$[Q_{\alpha i}, M_{\mu\nu}] = \frac{1}{2}(\sigma_{\mu\rho})_\alpha{}^\beta Q_{\beta i}; \tag{A.40}$$

$$[\bar{Q}^i{}_{\dot{\alpha}}, M_{\mu\nu}] = -\frac{1}{2}\bar{Q}^i{}_{\dot{\beta}}(\bar{\sigma}_{\mu\nu})^{\dot{\beta}}{}_{\dot{\alpha}}; \tag{A.41}$$

$$[Q_{\alpha i}, B_r] = (b_r)_i{}^j Q_{\alpha j}; \tag{A.42}$$

$$[\bar{Q}^i{}_{\dot{\alpha}}, B_r] = -\bar{Q}^j{}_{\dot{\alpha}}(b_r)_j{}^i; \tag{A.43}$$

$$\{Q_{\alpha i}, \bar{Q}^j{}_{\dot{\beta}}\} = 2\delta_i^j(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu; \tag{A.44}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = 2\varepsilon_{\alpha\beta}Z_{ij}; \tag{A.45}$$

with

$$Z_{ij} = a_{ij}^r B_r; \tag{A.46}$$

$$\{\bar{Q}^i{}_{\dot{\alpha}}, \bar{Q}^j{}_{\dot{\beta}}\} = -2\varepsilon_{\dot{\alpha}\dot{\beta}}Z^{ij} \qquad \text{with} \qquad Z^{ij} = (Z_{ij})^\dagger; \tag{A.47}$$

$$[Z_{ij}, \text{anything}] = 0. \tag{A.48}$$

The Z^{ij} are the central charges, and the B_r are the internal symmetry generators.

We can choose a basis in our representation space where the Z^{ij} are skew-diagonal and represented by complex numbers z_{ij}. These form an antisymmetric N x N matrix which can be brought into a standard form with the help of a unitary matrix U:

$$\bar{z}_{ij} = U_i^k U_j^l z_{kl}. \tag{A.49}$$

The standard form is, for even N,

$$\bar{z} = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \tag{A.50}$$

where D is a real, diagonal matrix with non-negative eigenvalues:

$$0 \leq z_{(r)}; \qquad r = 1, \ldots, \frac{N}{2}. \tag{A.51}$$

If N is odd, there is an additional row and column in (A.84) with all zeros:

$$\bar{z} = \begin{pmatrix} 0 & D & 0 \\ -D & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We use the unitary matrix U to redefine our Q's,

$$U_i^j Q_{\alpha j} \to Q_{\alpha j}; \qquad \bar{Q}^j_{\dot{\alpha}} (U^{-1})^i_j \to \bar{Q}^i_{\dot{\alpha}},$$

and introduce double-indices $i = (a, r)$ compatible with the obvious from (A.84), i.e. $a = 1, 2$ and $r = 1, 2, \ldots, \frac{N}{2}$. Again, for odd N, the last charge $Q_{\alpha N}$.is not touched by this.

The algebra of the Q's is now:

$$\{Q_{\alpha ar}, \bar{Q}^{bs}_{\dot{\beta}}\} = 2\delta_a^b \delta_r^s (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu; \tag{A.52}$$

$$\{Q_{\alpha ar}, Q_{\beta bs}\} = 2\varepsilon_{\alpha\beta}\varepsilon_{ab}\delta_{rs} z_r; \tag{A.53}$$

$$\{\bar{Q}^{ar}_{\dot{\alpha}}, \bar{Q}^{bs}_{\dot{\beta}}\} = 2\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon^{ab}\delta^{rs} z_r. \tag{A.54}$$

For odd N, we also have:

$$\{Q_{\alpha N}, Q_{\beta i}\} = 0; \tag{A.55}$$

$$\{Q_{\alpha N}, \bar{Q}^j{}_{\dot{\beta}}\} = 2\delta^i_N (\sigma^\mu)_{\alpha\dot{\beta}} P_\mu. \tag{A.56}$$

Let us first consider the massless case. We find in the standard frame:

$$P_\mu = (E, 0, 0, E) \tag{A.57}$$

that $Q_{2i} = 0$. This implies, through (A.87), that all $z_r = 0$ and we conclude that:

massless particle representations represent central charges trivially.

For the massive case, we introduce linear combinations:

$$A^{\pm}_{\alpha r} = \frac{1}{2}(Q_{\alpha 1 r} \pm \bar{Q}^{\dot{\alpha} 2 r}); \tag{A.58}$$

and their Hermitian adjoints. We notice how dotted and undotted Lorentz indices are mixed in such a way that covariance under the rotation subgroup is maintained since Q_α and $Q^{\dot{\alpha}}$ transforms in the same way under it, see technical appendix.

In terms of $A^{\pm}$, the rest-frame algebra, from (A.52) to (A.56), now reads:

$$\{A^{\pm}, A^{\pm}\} = \{A^{\pm}, A^{\mp}\} = \{A^{\pm}, (A^{\mp})^{\dagger}\} = 0; \tag{A.59}$$

$$\{A^{\pm}_{\alpha r}, (A^{\mp}_{\beta s})^{\dagger}\} = 2\delta_{\alpha\beta}\delta_{rs}(m \pm z_{(r)}); \tag{A.60}$$

and we conclude immediately, from the positivity of the left-hand side of the last equation, that:

$$|z_{(r)}| \leq m. \tag{A.61}$$

Let us assume that this bound is satisfied for a number n_0 of eigenvalues $z_{(r)}$ of the central charges. Then the corresponding A^- are represented trivially, and after rescaling the remaining generators:

$$q^{\pm}_{\alpha r} = \frac{A^{\pm}_{\alpha r}}{\sqrt{(m \pm z_{(r)})}}. \tag{A.62}$$

$$q_{\alpha N} = \frac{Q_{\alpha N}}{\sqrt{m}} \qquad \text{(if } N \text{ odd)}, \tag{A.63}$$

we have the Clifford algebra for $2(N - n_0)$ fermionic degrees of freedom. As far as the spectrum is concerned, we have the same situation as without central charges, except that:
N is effectively reduced by n_0, the number of central charges that satisfy the bound $m = z$.
The simplest representation with central charge, the $N = 2$ hypermultiplet[152, 150, 6] has one central charge which saturates the bound, and the spectrum is a doubled version of the massive Wess-Zumino model.

A.2.2 Supersymmetry and topological chages

Let us now to duality in gauge theories with includes supersymmetry where, as we will see, this concept can be implement in natural way. Indeed, as it was pointed out above, a crucial ingredient for the existence of 't Hooft-Polyakov monopole is a topological solutions in non Abelian gauge theories, and therefore for having electric-magnetic duality is the presence in the theory of Higgs fields transforming in the adjoint representation of the gauge group U[6, 7, 8].

We show that in supersymmmetric theories with solitons, the usual supersymmetry algebra include the topological quantum numbers as central charges[151].

It is the electric and magnetic charges and their generalizations that will appear as central charges[151]. We do not usually thing of the electric charge as a boundary term, but using Gauss's law it can be written as one:

$$q = \int d^3x \partial_i F_{0i}; \tag{A.64}$$

for the magnetic charge we would write:

$$p = \int d^3x \partial_i \varepsilon^{0ijk} F_{jk}. \tag{A.65}$$

The supersymmetric algebra in four dimensions is:

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij} (\gamma^\mu)_{\alpha\beta} P_\mu; \tag{A.66}$$

The Haag-Lopuszanski-Sohnius theorem[154] showed that this equation can be modified to include central charge; the most general form is:

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}(\gamma^\mu)_{\alpha\beta}P_\mu + \delta_{\alpha\beta}U_{ij} + (\gamma^5)_{\alpha\beta}V_{ij}; \tag{A.67}$$

where the central charges U and V satisfy:

$$U_{ij} = -U_{ji} \qquad \text{and} \qquad V_{ij} = -V_{ji}. \tag{A.68}$$

We will consider here the four dimensional model in which boundary terms enter as central charges. It is the $N = 2$ Yang-Mills theory, with Lagrangian:

$$\begin{aligned} L = \int d^4x \Bigg[-\frac{1}{4}[F^a_{\mu\nu}F^a_{\mu\nu}] + i\frac{1}{2}(\bar{\psi}^\alpha_i \gamma^\mu D_\mu \psi^\alpha_i) + \frac{1}{2}(D_\mu A^a D_\mu A^a) + \\ + \frac{1}{2}(D_\mu B^a D_\mu B^a) + \frac{1}{2}(g^2 Tr[A,B][A,B]) + \\ + i\frac{1}{2} g\varepsilon_{ij} Tr([\bar{\psi}^i, \psi^j]A + [\bar{\psi}^i, \gamma_5\psi^j]B) \Bigg]; \end{aligned} \tag{A.69}$$

where:

- ψ_i, $i = 1, 2$ are two Majorana fermion;
- g is a coupling constant;
- A is a scalar field;
- B is a pseudoscalar field,

all in the adjoint representation of gauge group.

The most important property of this Lagrangian is that the vacuum energy is independent of the values of B and A in certain directions in field space. As long as B and A commute, the vacuum energy is classically zero. This consideration persists quantum mechanically because of the supersymmetry spontaneously broking some of the gauge symmetries, and therefore B and A may have nonzero vacuum expectation values.

In the gauge group $O(3)$, B and A to commute and:

- the vacuum expectation value of B may be set to zero by chiral rotation;
- a nonzero vacuum expectation value of A will spontaneously break $O(3)$ down to $U(1)$.

The supersymmetry current for the Lgrangian (A.69) is:

$$S_{\mu i} = Tr\left(\sigma^{\alpha\beta}F_{\alpha\beta}\gamma_\mu\psi_i + \varepsilon_{ij}D_a A\gamma^a\gamma_\mu\psi_j + \right.$$
$$\left. + \varepsilon_{ij}D_a B\gamma^a\gamma_5\psi_j + g\gamma_\mu\gamma_5[A,B]\psi_i\right). \tag{A.70}$$

Witten and Olive have calculated from (A.70) the supersymmetry charges and their anticommutators[151] and have founded that the following operators appear in the supersymmetry algebra:

$$U = \int d^3x\partial_i\left(A^a F^a_{0i} + \frac{1}{2}B^a\varepsilon_{ijk}F^a_{jk}\right); \tag{A.71}$$

$$V = \int d^3x\partial_i\left(\frac{1}{2}A^a\varepsilon_{ijk}F^a_{jk} + B^a F^a_{0i}\right). \tag{A.72}$$

The supersymmetry algebra becomes

$$\{Q_{\alpha i}, \bar{Q}_{\beta j}\} = \delta_{ij}(\gamma^\mu)_{\alpha\beta}P_\mu + \varepsilon_{ij}\left(\delta_{\alpha\beta}U + (\gamma^5)_{\alpha\beta}V\right); \tag{A.73}$$

U and V can be nonvanishing iff the vacuum expectation value $\langle A\rangle$ or $\langle B\rangle$ is nonzero.

From the equations (A.70)-(A.73) we can say that:

- One can derive inequality for the masses; the eq. (A.73) implies that, for each particle state, the values of U and V and the mass M are related by:

$$M^2 \geq U^2 + V^2. \tag{A.74}$$

  The proof of the validity of that report was made by Witten and Olive[151];

- To make the meaning of eq. (A.74) more clear, let us consider the special case of an $O(3)$ gauge theory, the "Georgi-Glashow" model[5][155]. $\langle B\rangle$ may by assumed to vanish while a nonzero $\langle A\rangle$ spontaneously breaks $O(3)$ down to $U(1)$.

  The expression of electric charges q in Higgs theory is:

$$q = \frac{1}{\langle A\rangle}\int d^3x\partial_i(A^a F^a_{0i}); \tag{A.75}$$

  for the magnetic charge g we have:

$$p = \frac{1}{\langle A\rangle}\int d^3x\partial_i\varepsilon^{0ijk}(A^a F^a_{jk}). \tag{A.76}$$

  Comparing equation (A.71) and (A.75), we see that:

$$U = \langle A\rangle q; \tag{A.77}$$

  and comparing (A.72) and (A.76), we have:

$$V = \langle A\rangle p. \tag{A.78}$$

  Equation (A.74) becomes:

$$M \geq \langle A\rangle\sqrt{q^2+p^2}; \tag{A.79}$$

[5]In particle physics, the "Georgi-Glashow" model is a particular grand unification theory (GUT) proposed by Howard Georgi and Sheldon Glashow in 1974. In this model the standard model gauge groups $SU(3)$ x $SU(2)$ x $U(1)$ are combined into a single simple gauge group $SU(5)$. The unified group $SU(5)$ is then thought to be spontaneously broken to the standard model subgroup at some high energy scale called the grand unification scale[155].

Since the "Georgi-Glashow" model combines leptons and quarks into single irreducible representations, there exist interactions which do not conserve baryon number, although they still conserved $B-L$. This yields a mechanism for proton decay, and the rate of proton decay can be predicted from the dynamics of the model. However, proton decay has not yet been observed experimentally, and the resulting lower limit on the lifetime of the proton contradicts the predictions of this model. However, the elegance of the model has led particle physicists to use it as the foundation for more complex models which yield longer proton lifetimes[155].

- Photon, Higgs particles, fermions, W and Z_0 and magnetic monopoles thay all satisfy $M = \sqrt{U^2 + V^2}$ or for $O(3)$ $M = \langle A \rangle \sqrt{q^2 + p^2}$. For example, in $O(3)$ theory, the W^+ boson has magnetic charge $p = 0$ and electric charge $q = +e$. So equation (A.79) if exactly realized $M_W = e\langle A \rangle$ and this is the well-known Higgs formula for the W^+ mass;

- An irreducible representation of equation (A.73) has 2^{2N} states for nonzero mass, but 2^N (helicity) states for zero mass. For example with $N = 2$ an irreducible representation of eq. (A.73) has sixteen states if mass is nonzero, but four states if mass is zero.

Looking at eq. (A.73), written in the rest frame:

$$\{Q_\alpha, \bar{Q}_\beta\} = \delta_{\alpha\beta}\gamma^0 M + CU_{\alpha\beta} + \gamma^5 V_{\alpha\beta}; \tag{A.80}$$

one can extract the crucial relation:

$$M^2 \geq \frac{1}{N}\left(U^{\alpha\beta}U_{\alpha\beta} + V^{\alpha\beta}V_{\alpha\beta}\right) = \frac{1}{N}|U_{\alpha\beta} + iV_{\alpha\beta}|^2 \geq |Z_M|^2; \tag{A.81}$$

where Z_M denoted the maximum skew-eigenvalue of the complex central charge:

$$Z_{\alpha\beta} = U_{\alpha\beta} + iV_{\alpha\beta}. \tag{A.82}$$

From equation (A.81), when written for an $N = 2$ system, where:

$$U_{\alpha\beta} = \varepsilon_{\alpha\beta}U, \qquad V_{\alpha\beta} = \varepsilon_{\alpha\beta}V; \tag{A.83}$$

it is easy to recognize the Bogomol'nyi bound (A.22):

$$M \geq |Z_M|^2 = \langle A \rangle |q + ip|^2; \tag{A.84}$$

One striking consequence is that, in four dimensional theories, we can determine the exact quantum mechanical mass spectrum; for example, in a certain supersymmetric form of the 'Georgi-Glashow"[155], with $O(3)$ broken down $U(1)$ by the Higgs phenomenon: the mass of any particle is the vacuum expectation value of the Higgs field times:

$$\sqrt{q^2 + p^2}; \tag{A.85}$$

where q and p being the electric and magnetic charges of that particle.

A.3 Supergravity action in four-dimensions

In this section we will apply the concept of duality discussed in the previous two sections to the study of the general structure of an Abelian theory of scalars and vectors displaying covariance under a group of duality rotations, in $D = 4$. To this aim we first have to present the main features of four dimensional N-extended supergravities. These theories contain in the bosonic sector: the metric, a number n_V of vectors and m of real scalar fields. The relevant bosonic action has the following form[3, 22, 20, 98, 99]:

$$S = \int \sqrt{-g} d^4x \left(-\frac{1}{2} R + Im\mathcal{N}_{\Lambda\Gamma} F^{\Lambda}_{\mu\nu} F^{\Gamma|\mu\nu} + \right.$$
$$\left. + \frac{1}{2\sqrt{-g}} Re\mathcal{N}_{\Lambda\Gamma} \varepsilon^{\mu\nu\rho\sigma} F^{\Lambda}_{\mu\nu} F^{\Gamma}_{\rho\sigma} + \frac{1}{2} g_{rs}(\Phi) \partial_\mu \Phi^r \partial^\mu \Phi^s \right); \tag{A.86}$$

where:

- R is the curvature scalar;
- F^{Λ} are field strengths;
- $\mathcal{N}_{\Lambda\Gamma}(\Phi)$ is the vector kinetic matrix and it is a complex, symmetric, n_V x n_V matrix depending on the scalar fields Φ^s. The imaginary part $Im\mathcal{N}$ is negative definite and generalizes the inverse of the squared coupling constant appearing in ordinary gauge theories while its real part $Re\mathcal{N}$ is instead a generalization of the theta-angle of quantum chromodynamics;

- $g_{rs}(\Phi)$ with $r, s = 1, \cdots, m$ is the scalar metrix on the σ-model described the scalar manifold $\mathcal{M}_{scalar}$ of real dimension m[3, 93][6].

The number of scalars and vectors, namely m and n_V, and the geometric properties of the scalar manifold $\mathcal{M}_{scalar}$ depend on the number N of supersymmetries and are resumed in the following table:

N	Duality group G	isotropy H	$\mathcal{M}_{scalar}$	n_V	m
3	$SU(3,n)$	$SU(3,n)xU(n)$	$\frac{SU(3,n)}{S(U(3,n)xU(n))}$	$3+n$	$6n$
4	$SU(1,1)\bigotimes SO(6,n)$	$U(4)xSO(n)$	$\frac{SU(1,1)}{U(1)}\bigotimes\frac{SO(6,n)}{SO(6)xSO(n)}$	$6+n$	$6n+2$
5	$SU(1,5)$	$U(5)$	$\frac{SU(1,5)}{S(U(1)xU(5))}$	10	10
6	$SO^*(12)$	$U(6)$	$\frac{SO^*(12)}{U(1)xSU(6)}$	16	30
7	$E_{7(7)}$	$SU(8)$	$\frac{E_{7(7)}}{SU(8)}$	28	70
8	$E_{7(7)}$	$SU(8)$	$\frac{E_{7(7)}}{SU(8)}$	28	70

In this table, n_V stands for the number of vectors and m for the number

[6]In quantum field theory, a nonlinear σ-model (which is the "generalization" of a σ-model) describes a scalar field Φ which takes on values in a nonlinear manifold called the target manifold T[64, 65, 67, 94].
The tangent manifold is equipped with a Riemannian metric g. Φ is a differentiable map from Minkowski space M (or some other space) to T. In the coordinate notation, with the coordinates Φ^a with $a = 1, \cdots, m$ where m is the dimension of T, the Lagrangian density is given by:

$$\mathcal{L} = +\frac{1}{2} g_{ab}(\Phi)\partial_\mu \Phi^a \partial^\mu \Phi^b - V(\Phi)$$

where here, we have used a $(+, -, -, -)$ metric signature. In more than two dimensions, nonlinear σ-models are nonrenormalizable; this means they can only arise as effective field theories.

There is a special class of nonlinear σ-models with the internal symmetry group G. If G is a Lie group and H is a Lie subgroup, then the quotient space G/H is a manifold (subject to certain technical restrictions like H being a closed subset) and is also a homogeneous space of G or in other words, a nonlinear realization of G.

of real scalar fields. In all the cases the duality group G is embedded in $Sp(2n_V, R)$.

In supergravity theories, the vector kinetic matrix $\mathcal{N}$ is in general not a constant, its components being functions of the scalar fields. However, in extended supergravity, $N \geq 2$, the relation between the kinetic matrix $\mathcal{N}$ and scalar geometry has a very general and universal form. Such a lift is necessary because of supersymmetry since vectors and scalars generically belong to the same supermultiplet and must rotate coherently under symmetry operations. This problem has been solved in a general non supersymmetric framework[156] by considering the possible extension of the Dirac electric-magnetic duality to more general theories involving scalars. In the second part of this section we review this approach and in particular we show how enforcing covariance with respect to such duality rotations leads to a determination of the kinetic matrix $\mathcal{N}$.

Note that the Euler-Lagrange equation:

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) = 0$$

for the bosonic action (A.86), one gets the Einstein equations:

$$-\frac{1}{2}\left(R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R\right) + \frac{1}{2}g_{ab}(\Phi)\partial_\mu\Phi^a\partial_\nu\Phi^b - \frac{g_{\mu\nu}}{2}\frac{g_{ab}}{2}\partial\Phi^a\partial\Phi^b + \\ +2F^T_{\mu t} Im\mathcal{N} F_\nu{}^t - \frac{g_{\mu\nu}}{2}F^T Im\mathcal{N}F = 0; \quad \text{(A.87)}$$

with:

$$R = g_{ab}(\Phi)\partial_\mu\Phi^a\partial^\mu\Phi^b, \quad \text{(A.88)}$$

and

$$-\frac{1}{2}R_{\mu\nu} = -\frac{1}{2}g_{ab}(\Phi)\partial_\mu\Phi^a\partial_\nu\Phi^b + \\ -2F^T_{\mu t} Im\mathcal{N} F_\nu{}^t + \frac{g_{\mu\nu}}{2}F^T Im\mathcal{N}F. \quad \text{(A.89)}$$

Let us review the general structure of an abelian theory of scalars and vectors displaying covariance under a group of duality rotations, in $D = 4$. The basis references are: the 1981 paper by Gaillard and Zumino[156] and the book "Supergravity and Superstrings. A Geometric Perspective. Vol. 1 Mathematical foundations" by L. Castellani, R. D'Auria and P. Frè[152].

We consider a theory of n_V abelian gauge fields A^Λ_μ in a four dimension spacetime with Lorentz signature. They correspond to a set of n_V differential $1-$forms:

$$A^\Lambda = A^\Lambda_\mu dx^\mu \qquad (\Lambda = 1, \cdots, n_V). \tag{A.90}$$

The corresponding field strengths are:

$$F^\Lambda \equiv dA^\Lambda \equiv F^\Lambda_{\mu\nu} dx^\mu \wedge dx^\nu \tag{A.91}$$

with

$$F^\Lambda_{\mu\nu} = \frac{1}{2}\left(\partial_\mu A^\Lambda_\nu - \partial_\nu A^\Lambda_\mu\right) \tag{A.92}$$

and their Hodge duals are defined by:

$$({}^*F^\Lambda)_{\mu\nu} = \frac{\sqrt{-g}}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\Lambda|\rho\sigma} \tag{A.93}$$

The dynamics of a system of abelian gauge fields coupled to scalars in a gravity theory is encoded in the equation (A.86) for the Lagrangian density.

Introducing self-dual and antiself-dual combinations:

$$F^\pm = \frac{1}{2}\left(F \pm i^*F\right), \tag{A.94}$$

$${}^*F^\pm = \mp i F^\pm, \tag{A.95}$$

the vector part of the bosonic action defined by eq. (A.86) can be rewritten in the form:

$$S = \int \sqrt{-g} d^4x \mathcal{L}_{vec}, \tag{A.96}$$

with:

$$\mathcal{L}_{vec} = i\left[F^{-T}\bar{\mathcal{N}}F^{-} - F^{+T}\mathcal{N}F^{+}\right]. \tag{A.97}$$

Introducing further the new tensors:

$$\begin{aligned} {}^*G_{\Lambda|\mu\nu} &\equiv \frac{1}{2}\frac{\partial\mathcal{L}}{\partial F^{\Lambda}_{\mu\nu}} \\ &= Im\mathcal{N}_{\Lambda\Sigma}F^{\Sigma}_{\mu\nu} + Re\mathcal{N}_{\Lambda\Sigma}{}^*F_{\Lambda|\mu\nu}; \end{aligned} \tag{A.98}$$

and introducing self-dual combinations:

$$G^{\pm}_{\Lambda|\mu\nu} \equiv \pm\frac{i}{2}\frac{\partial\mathcal{L}}{\partial F^{\pm\Lambda}_{\mu\nu}}. \tag{A.99}$$

In terms of F^{Λ} and G_{Λ} the Maxwell equations read:

$$\nabla^{\mu *}F^{\Lambda}_{\mu\nu} = 0, \tag{A.100}$$

$$\nabla^{\mu *}G_{\Lambda|\mu\nu} = 0; \tag{A.101}$$

or equivalent

$$\nabla^{\mu}ImF^{\pm\Lambda}_{\mu\nu} = 0, \tag{A.102}$$

$$\nabla^{\mu}ImG^{\pm}_{\Lambda|\mu\nu} = 0; \tag{A.103}$$

i.e. these are the Bianchi identities and field equations associated with the Lagrangian (A.86). This suggests that we introduce the $2n_V$ column vector:

$$\mathbf{V} = \begin{pmatrix} {}^*F \\ {}^*G \end{pmatrix}. \tag{A.104}$$

and that we consider general linear transformations on such a vector:

$$\begin{pmatrix} {}^*F \\ {}^*G \end{pmatrix}' = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} {}^*F \\ {}^*G \end{pmatrix}. \tag{A.105}$$

For any constant matrix:

$$\mathbf{S} \; = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in GL(2n_V, R) \tag{A.106}$$

the new vector of electric and magnetic field-strengths

$$\mathbf{V}^{\;\prime} = \; \mathbf{S} \; \cdot \; \mathbf{V} \tag{A.107}$$

satisfies the some equations (A.100) and (A.101) as the old one. In a condensed notation we can write:

$$\partial \; \mathbf{V} \;\; = 0 \leftrightarrow \partial \; \mathbf{V}^{\;\prime} = 0. \tag{A.108}$$

Separating the self-dual and antiself-dual parts:

$$F = \left(F^{+} + F^{-}\right), \tag{A.109}$$

$$G = \left(G^{+} + G^{-}\right), \tag{A.110}$$

and taking into account that we have:

$$G^{+} = \mathcal{N} F^{+} \tag{A.111}$$

and

$$G^{-} = \bar{\mathcal{N}} F^{-} \tag{A.112}$$

the duality rotation of equation (A.105) can we rewritten as:

$$\begin{pmatrix} F^{+} \\ G^{+} \end{pmatrix}^{\prime} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^{+} \\ \mathcal{N} F^{+} \end{pmatrix} \tag{A.113}$$

and

$$\begin{pmatrix} F^{-} \\ G^{-} \end{pmatrix}^{\prime} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^{-} \\ \bar{\mathcal{N}} F^{-} \end{pmatrix}. \tag{A.114}$$

Now, let us note that, since in the bosonic action (A.86) we are considering the gauge fields are coupled to the scalar sector via the scalar dependent kinematic matrix $\mathcal{N}(\Phi)$, when a duality rotation is performed on the vector field strengths and their duals, we have to assume that the scalars get transformed correspondingly, through the action of some diffeomorphism on the scalar manifold $\mathcal{M}_{scalar}$. In particular, the kinetic matrix $\mathcal{N}$ transforms under a duality rotation. Then, a duality transformation ξ acts in the following way on a supersymmetric system[3]:

$$\xi : \begin{cases} V & \to V'^{\mp} = S_\xi V^{\mp} \\ \Phi & \to \Phi' = \xi(\Phi) \\ \mathcal{N}(\Phi) & \to \mathcal{N}'(\xi(\Phi)) \end{cases} . \tag{A.115}$$

Thus, the transformation laws of equations of motion of matrix S_ξ, and of kinetic matrix $\mathcal{N}(\Phi)$, will be introduced by a diffeomorphism of the scalar fields.

Consider in particular on the first relation in the equation (A.115), that explicitly reads:

$$\begin{pmatrix} F^{\pm'} \\ G^{\pm'} \end{pmatrix} = \begin{pmatrix} A_\xi F^{\pm} + B_\xi G^{\pm} \\ C_\xi F^{\pm} + D_\xi G^{\pm} \end{pmatrix}, \tag{A.116}$$

we note that contains the magnetic field strength $G^{\pm}$ introduced in eq. (A.98) which is defined as a variation of the Kinetic Lagrangian. Under the transformations (A.115) the lagrangian transforms in the following way:

$$\begin{aligned} \mathcal{L}' = i \Big[& (A_\xi + B_\xi \mathcal{N})_\Gamma{}^\Lambda (A_\xi + B_\xi \mathcal{N})^\Sigma{}_\Delta \mathcal{N}'_{\Lambda\Sigma} F^{+\Gamma} F^{+\Delta} + \\ & -(A_\xi + B_\xi \mathcal{N})_\Gamma{}^\Lambda (A_\xi + B_\xi \mathcal{N})^\Sigma{}_\Delta \mathcal{N}'_{\Lambda\Sigma} F^{-\Gamma} F^{-\Delta} - \Big] . \end{aligned} \tag{A.117}$$

We observe that the equation (A.115) must be consistent with the definition of $G^{\pm}$ as a variation of the Lagrangian (A.117):

$$\begin{aligned} G'^{+}_\Lambda &= (C_\xi + D_\xi \mathcal{N})_{\Gamma\Lambda} F^{+\Gamma} \\ &\equiv -\frac{1}{2} \frac{\partial \mathcal{L}'}{\partial F'^{+\Lambda}} = (A_\xi + B_\xi \mathcal{N})^\Delta{}_\Sigma \mathcal{N}'_{\Lambda\Sigma} F^{+\Sigma}; \end{aligned} \tag{A.118}$$

that implies:

$$\mathcal{N}'_{\Lambda\Sigma}(\Phi') = \left[(C_\xi + D_\xi \mathcal{N}) \cdot (A_\xi + B_\xi \mathcal{N})^{-1}\right]_{\Lambda\Sigma}. \tag{A.119}$$

The condition that the matrix $\mathcal{N}$ is symmetric, and that this property must be true in the duality transformed system, gives the constraint:

$$\mathbf{S} \in Sp(2n_V, R), \tag{A.120}$$

that is:

$$\mathbf{S}^T C \, \mathbf{S} = C, \tag{A.121}$$

where C is the symplectic invariant $2n_V$ x $2n_V$ matrix:

$$C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{A.122}$$

It is useful to rewrite the symplectic condition (A.121) in terms of the n_V x n_V blocks defining $\mathbf{S}$:

$$\begin{cases} A^T C - C^T A &= 0 \\ B^T D - D^T B &= 0 \\ A^T D - C^T B &= 1 \end{cases} \tag{A.123}$$

The above observation has important implications on the scalar manifold $\mathcal{M}_{scalar}$. Indeed, it implies that on the scalar manifold the following homomorphism is defined:

$$Diff(\mathcal{M}_{scalar}) \to Sp(2n_V, R). \tag{A.124}$$

In particular, the presence on the manifold of a function of scalars transforming with a fractional linear transformation under a duality rotation on the scalars, induces the existence on $\mathcal{M}_{scalar}$ of a linear structure inherited from the vectors; and this may be rephrased by saying that the scalar manifold is endowed with symplectic bundle[3]. As the transformation functions of this bundle are given in terms of the constant matrix $\mathbf{S}$, the symplectic bundle is flat. In particular for $N = 2$ four dimensional theory this implies that the scalar manifold be a special manifold, that is Kähler-Hodge manifold endowed with a flat symplectic bundle[3, 157].

If one is interested in the global symmetries of the theory, i.e. global symmetries of the field equations and Bianchi identities, we will need to restrict the homomorphysm in (A.124) to the isometries of the scalar manifold, which leave the scalar sector of the action invariant. The transformations in equation (A.115), which are duality symmetries of the system Bianchi-identities/field-equations, cannot be extended in general to the symmetries of the Lagrangian. The vector part of the boson action (A.86) is in general not invariant under the action of the isometry group of the metric g_{rs}, but the scalar part is invariant. The transformed lagrangian under the action of $\mathbf{S} \in Sp(2n_V, R)$ can be rewritten:

$$\begin{aligned} Im(F^{-\Lambda}G^{-}_{\Lambda}) &\rightarrow Im(F'^{-\Lambda}G'^{-}_{\Lambda}) \\ &= Im\left[F^{-\Lambda}G^{-}_{\Lambda} + 2(C^TB)_{\Lambda}{}^{\Sigma}F^{-\Lambda}G^{-}_{\Sigma} + \right. \\ &+ \left. (C^TA)_{\Lambda\Sigma}F^{-\Lambda}F^{-\Sigma} + (D^TB)^{\Lambda\Sigma}G^{-}_{\Lambda}G^{-}_{\Sigma}\right]. \end{aligned} \tag{A.125}$$

One can conclude that:

- It is evident from the latter relation that only the transformation with $B = C = 0$ are symmetries;
- If $C \neq 0$ and $B = 0$ the Lagrangian varies for a topological term:

  $$(C^TA)_{\Lambda\Sigma}F^{\Lambda}_{\mu\nu}{}^{*}F^{\Sigma|\mu\nu} \tag{A.126}$$

  corresponding to a redefinition of the function $Re\mathcal{N}_{\Lambda\Sigma}$; such a transformation being a total derivative it leaves classical physics invariant, but it is relevant in the quantum theory;
- For $B \neq 0$ neither the action nor the perturbative partition function are invariant. Let us observe that in this case the transformation law

(A.119) of the kinetic matrix $\mathcal{N}(\Phi)$ contains the transformation:

$$\mathcal{N}(\Phi) \to \frac{1}{\mathcal{N}(\Phi)} \tag{A.127}$$

that is it exchanges the strong and weak coupling regimes of the theory. One may then think of such a quantum field theory as being described by a collection of local Lagrangians, each defined in a local patch. They are all equivalent once one defines for each of them what is electric and what is magnetic. Duality transformations map this set of Lagrangians one into the other.

At this point we observe that the supergravity bosonic action (A.86) is exactly of the form considered in this section as far as the matter content is concerned, so that we may apply the above considerations about duality rotations to the supergravity case. In particular, the duality acts in all theory with $N \geq 2$ supersymmetries, where the vector supermultiplets contain both scalars and vectors. For $N = 1$ supergravity, instead, scalars and vectors are still present but they are not related by supersymmetry, and as a consequence they are not related by U-duality rotations, so that the previous formalism does not necessarily apply. There are however $N = 1$ models where the scalar moduli space is given by a special-Kähler model. This is the case for the example for the compactification of the heterotic theory on Calabi-Yan manifolds[3, 157].

Appendix B

The $\mathfrak{su}(2,1)$-Algebra

Let us choose the $\mathrm{SU}(2,1)$-invariant and the $H^* = \mathrm{U}(1,1)$-invariant metrics η and $\bar{\eta}$, respectively, to be:

$$\eta = \mathrm{diag}(-1,1,1) \ ; \ \ \bar{\eta} = \mathrm{diag}(-1,1,-1)\,, \tag{B.1}$$

where the latter defines the coset generators. The solvable Lie algebra $Solv$ defining the Iwasawa decomposition of $\mathfrak{su}(2,1)$ with respect to $\mathfrak{u}(2)$ is generated by:

$$\begin{aligned}
&Solv = \mathrm{span}(H_0, T_1, T_2, T_\bullet)\,,\\
&H_0 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix} \ ; \ T_1 = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \ ; \ T_2 = \begin{pmatrix} 0 & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & -\frac{i}{2} \\ 0 & -\frac{i}{2} & 0 \end{pmatrix}\,,\\
&T_\bullet = \begin{pmatrix} -\frac{i}{2} & 0 & \frac{i}{2} \\ 0 & 0 & 0 \\ -\frac{i}{2} & 0 & \frac{i}{2} \end{pmatrix}\,.
\end{aligned} \tag{B.2}$$

The H^* algebra $\mathfrak{u}(1,1)$ is generated by the compact component $K_\bullet$ of $T_\bullet$, the non-compact components K_1, K_2 of T_1, T_2, respectively, and the compact

$D = 4$ duality generator J:

$$\mathfrak{u}(1,1) = \mathrm{span}(K_1, K_2, K_\bullet, K)\,,$$

$$K_\bullet = T_\bullet - T_\bullet^\dagger = \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \;;\; K_1 = T_1 + T_1^\dagger = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,,$$

$$K_2 = T_2 + T_2^\dagger = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;;\; K = \begin{pmatrix} -i & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -i \end{pmatrix}\,. \tag{B.3}$$

The $\mathrm{SU}(2,1)/\mathrm{U}(1,1)$-coset representative describing the physical patch of the manifold is:

$$\mathbb{L} = e^{-aT_\bullet}\, e^{\sqrt{2}(\mathcal{Z}^0\, T_1 + \mathcal{Z}_0\, T_2)}\, e^{2UH_0}\,. \tag{B.4}$$

The matrix $\mathcal{M} = \mathbb{L}\bar{\eta}\mathbb{L}^\dagger$ has the following simple form:

$$\mathcal{M} = \mathbb{L}\bar{\eta}\mathbb{L}^\dagger = \eta - \frac{2}{I_2}\,\eta\overline{\mathbb{U}}\,\mathbb{U}^T\eta\,, \tag{B.5}$$

where

$$\mathbb{U} \equiv \begin{pmatrix} W_E \\ V_E \\ U_E \end{pmatrix}\,,\quad I_2 \equiv \mathbb{U}^T\eta\overline{\mathbb{U}} = |U_E|^2 + |V_E|^2 - |W_E|^2\,. \tag{B.6}$$

Appendix C

The Taub-NUT solution

The asymptotically flat, static, spherically Schwarzschild and Reissner- Nordström black holes solutions that we have studied in this thesis. To find more solutions, we have to relax these conditions or couple to gravity more general "types of matter". If we stay with the Einstein-Maxwell theory, one possibility is to look for static, axially symmetric solutions and another possibility is to relax the condition of staticity and only ask that the solution be stationary, which implies that we have to relax the condition of spherical symmetry as well and look for stationary and axisymmetric space-times[83]. In the first case one finds solutions like those in Weyl's family[83]. In the second case, we find the Kerr-Newman black holes with: electric and/or magnetic charge, angular momentum and also the Taub-Newman-Unti-Tambourino (Taub-NUT) solution, which may but need not include charges[158, 159]. The Taub-NUT solutions does not describe a black holes because it is not asymptotically flat. In fact, the only stationary, axially symmetric black holes of the Einstein-Maxwell theory belong to the Kerr-Newman family of solutions[160, 161].

The Taub-NUT solutions has a number of features that are particularly interesting for us, which we are going to discuss in this appendix. In particular, it carries a new "type of mass", NUT charge, with is of topological nature and can be viewed as "gravitational magnetic charge", so the solution is a sort “mass dyon” and its Euclidean continuation is the solution call as a Kaluza-Klein monopole. This is a important solution with interesting properties such as the self-duality of its curvature and its relation to the Belaving-Polyakov-Schwarz-Tyupkon (BPST) instanton[83] and the ’t Hooft-Polyakov monopole[145, 146].

C.1 Properties of the Taub-NUT solution

In general, for stationary axisymmetric, metrics corresponding to the existence of two Killing vectors, associated to time translations $\frac{\partial}{\partial t}$ and rotations $\frac{\partial}{\partial \varphi}$[51]; a general ansatz, in $D = 4$, for these space-times has the form:

$$ds^2 = g_{tt}dr^2 + 2g_{t\varphi}dtd\varphi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\varphi\varphi}d\varphi^2, \tag{C.1}$$

where all the components may dependent on θ and r. The new interesting terms is the component $g_{t\varphi}$. If the metric is asymptotically flat for $r \to \infty$ and $g_{t\varphi}$ has the asymptotic behavior:

$$g_{t\varphi} \sim 2J\frac{sin^2(\theta)}{r} \tag{C.2}$$

then this equation describes a space-time with angular momentum J in the direction of the z axis. The only vacuum solution of the kind is the Kerr’s, which in Boyer–Lindquist coordinates takes the form:

$$\begin{aligned} ds^2 = d\tau^2 = \left(1 - \frac{r_s r}{\Sigma}\right)dt^2 - \frac{\Sigma}{\Delta}dr^2 - \Sigma d\theta^2 + \\ -\frac{A}{\Sigma}sin^2\theta d\varphi^2 + \frac{2r_s r\alpha sin^2\theta}{\Sigma}dtd\varphi; \end{aligned} \tag{C.3}$$

where:

$$\alpha = \frac{J}{Mc} = \frac{J}{M} \qquad \Sigma = r^2 + \alpha^2 cos^2\theta, \tag{C.4}$$

$$\Delta = r^2 - r_s r + \alpha^2 = r^2 - 2Mr + \alpha^2; \tag{C.5}$$

$$A = (r^2 + \alpha^2)^2 - \alpha^2\Delta sin^2(\theta) = (r^2 + \alpha^2)\Sigma + rr_s\alpha^2 sin^2\theta. \tag{C.6}$$

If $M^2 \geq \alpha^2$ this solution describes rotating black holes with mass M and angular momentum $J = M\alpha$.

If, the metric (C.1) asymptotically,

$$g_{t\varphi} \sim 2\ell cos(\theta)\,, \tag{C.7}$$

the solution describes a body with NUT charge ℓ. We will discuss soon the meaning of this new charge. The simplest vacuum solution with this kind of charge is the Taub-NUT solution[158, 159]:

$$ds^2 = f(r)(dt + 2\ell cos(\theta)d\varphi)^2 - \frac{1}{f(r)}dr^2 + (r^2 + \ell^2)d\Omega^2, \tag{C.8}$$

where:

$$f(r) = \frac{(r - r_+)(r - r_-)}{(r^2 + \ell^2)} \qquad r_\pm = M \pm r_0, \tag{C.9}$$

$$r_0 = M^2 + \ell^2 \qquad d\Omega^2 = d\theta^2 + sin^2(\theta)d\varphi^2, \tag{C.10}$$

which is a generalization of the Schwarzschild metric with NUT charge and reduces to it when $\ell = 0$.

Let us list some properties of this solution:

- the space-time is non-trivial in the $M \to 0$ limit, in which it may be interpreted as the gravitational field of pure "spike" of spin[162, 163];

- the Newtonian gravitational potential is given in this approximation by;

$$\Phi_N \sim \frac{(g_{tt} - 1)}{2} = -\frac{M}{r}\,. \tag{C.11}$$

  The Taub-NUT metric has other non-zero components of the metric the $g_{t\varphi}$ term; or we see that the Taub-NUT gravitational field has, as non-zero element of the gravitomagnetic potential:

$$\Phi_\varphi = g_{t\varphi} = 2\ell cos(\theta)\,. \tag{C.12}$$

This is essentially the electromagnetic field of magnetic monopole of charge proportional to ℓ. Thus, the NUT charge ℓ can be considered as a sort of "magnetic mass"[164] and so the Taub-NUT metric can be interpreted as a gravitational dyon[165];

- this metric, eq. (C.8), is not asymptotically flat but defines its own class of asymptotic behavier, the asymptotically Taub-NUT space-times, labeled by ℓ, which is associated with the non-zero at infinity of the off-diagonal $g_{t\varphi}$ term of the metric and, as we are going to see, with the periodicity of the time coordinate;

- this metric, eq. (C.8), does not have curvature singularities and is perfecttly regular at $r = 0$; however, it has the so-called "wire singularities" at $\theta = 0$ and $\theta = \pi$ where the metric fails to be invertible. These point singularities can not be cured simultaneously[166]. Misner, in[166], found a way to make the metric regular everywhere by introducing two coordinate paches;

- the metric function $f(r)$ has two zero at r_+ and r_-, furthermore the metric has coordinate singularities there. For $r < r_-$ and $r > r_+$ the metric has closed time-like curves. Thus, although the form of the metric is equivalent to the Reissner-Nordström metric; and the extremality parameter $c = r_0$ vanishes only for $M = \ell = 0$;

- in the region $r_- < r < r_+$ the coordinate r is time-like and t is space-like. This region describes a non-singular, anisotropic and closed cosmological model. It can be thought of as a closed universe containing gravitational radiation having the longest possible wavelength[167];

- there is no known generalization to higher dimensions. The NUT charge seems to be an intrinsically 4-dimensional charge[168].

C.2 Charged Taub-NUT solution and IWP solutions

Let us consider stationary, axially symmetric solutions of the Einstein-Maxwell system; some of them are the result of adding electric and/or magnetic charges to vacuum solutions.

The metric of Kerr-Newmann (KN) black hole, with mass M, electric charge Q and magnetic charge P, could be written in the form:

$$ds^2 = \frac{\Delta}{\rho^2}(dt - \alpha sin^2\theta d\varphi)^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 + \\ - \frac{sin^2\theta}{\rho^2}[(r^2 + \alpha^2)d\varphi - \alpha dt]^2;; \tag{C.13}$$

with:

$$\Delta = r^2 - r_s r + \frac{(Q^2 + P^2)}{2} + \alpha^2; \tag{C.14}$$

$$\rho^2 = r^2 + \alpha^2 cos^2\theta; \qquad \alpha = \frac{J}{Mc} = \frac{J}{M}; \tag{C.15}$$

where α representing the specific angular momentum J of the source.

The electrically and/or magnetically Taub-NUT solution was found by Brill in [167] and is:

$$ds^2 = f(r)(dt + 2\ell cos(\theta)d\varphi)^2 - \frac{1}{f(r)}dr^2 + (r^2 + \ell^2)d\Omega^2, \tag{C.16}$$

where:

$$f(r) = \frac{(r - r_+)(r - r_-)}{(r^2 + \ell^2)} \qquad r_\pm = M \pm r_0, \tag{C.17}$$

$$r_0 = M^2 + \ell^2 - \frac{Q^2 + P^2}{2} \qquad d\Omega^2 = d\theta^2 + sin^2(\theta)d\varphi^2, \tag{C.18}$$

which it reduces to the Reissner-Nordström solution when we the NUT charge to zero ($\ell = 0$).

In contrast to the Taub-NUT solution, the charged Taub-NUT metric does have an extremal limit:

$$M^2 + \ell^2 = \frac{Q^2 + P^2}{2}, \tag{C.19}$$

in which the extremality parameter $c = r_0$ vanishes and the two zeros of the term $f(r)$ coincide. In this case, by shifting the radial coordinate to:

$$r' = r + M, \tag{C.20}$$

and defining Cartesian coordinates such that $r' = |\overrightarrow{x}_3|$, we find a simple form of the solution[120][1]:

$$ds^2 = \frac{1}{|H_1|^2}(dt + A)^2 - |H_1|^2 d\overrightarrow{x}_3^2, \tag{C.21}$$

with:

$$H = 1 + \frac{M + \imath\ell}{|\overrightarrow{x}_3|}, \tag{C.22}$$

and the 1−form A is defined by patches so it is regular everywhere:

$$A = A_{\underline{i}} dx^i \,, \qquad \varepsilon_{ijk}\partial^i A^j = \pm Im(\bar{H}_1 \partial_k H_1)\,, \tag{C.23}$$

$$A_t = 2Re(e^{\imath\alpha} H_1)\,, \qquad \tilde{A}_t = 2Im(e^{\imath\alpha} H_1) \tag{C.24}$$

As in some of the other extreme solution that we have found so for[2], it turns out that we obtain a equation for any complex harmonic function $H_1(\overrightarrow{x}_3)$. By including the complex phase $e^{\imath\alpha}$ into H_1, we can write the solution as follows:

$$ds^2 = \frac{1}{|H_1|^2}(dt + A)^2 - |H_1|^2 d\overrightarrow{x}_3^2, \tag{C.25}$$

[1]Here we are actually taking the extreme limit of the dyonic solution, which indeed has a simple form. The information on the electric and/or magnetic charges is contained in the $SO(2)$ electric-magnetic-duality phase $e^{i\alpha}$.

[2]But not in all of them; in particular, not in Kerr black hole.

where:

$$A = A_{\underline{i}} dx^i \,, \qquad \varepsilon_{ijk}\partial^i A^j = \pm Im(\bar{H}_1 \partial_k H_1)\,, \tag{C.26}$$

$$A_t = 2Re(H_1)\,, \qquad \tilde{A}_t = -2Re(\imath H_1)\,, \tag{C.27}$$

and

$$\partial_{\underline{i}}\partial_{\underline{i}} H_1 = 0. \tag{C.28}$$

Metrics of the above form are known as conformastationary[119], and the integrability condition of the equation for the one-form A is the Laplace equation for H_1. This class of solutions is known as the Israel-Wilson-Perjés solutions (IWP solutions)[169, 170]. This class contains all the "extreme" solutions that we have found so far, plus many others that may have electric and magnetic charges, mass, NUT charge and also angular momentum. In particolar, the:

$$M^2 = \frac{Q^2 + P^2}{2}, \tag{C.29}$$

Kerr-Newman solutions, for generic angular momentum, belong to this class; their complex harmonic function is:

$$H_1 = 1 + \frac{M}{\sqrt{x^2 + y^2 + (z - \imath\alpha)^2}}. \tag{C.30}$$

In terms of the spheroidal coordinates,

$$x - \imath y = \sqrt{(r-M)^2 + \alpha^2} sin(\theta) e^{+\imath\varphi} \;, \tag{C.31}$$

$$z = (r - M) cos(\theta) \;, \tag{C.32}$$

the harmonic function takes the form:

$$H_1 = 1 + \frac{M}{r - M - \imath\alpha cos(\theta)} \;, \tag{C.33}$$

and the three-dimensional metric becomes:

$$\begin{aligned} d\overrightarrow{x}_3^2 &= [(r-M)^2+\alpha^2 cos^2(\theta)]\left(\frac{dr^2}{(r-M)^2+\alpha^2}+d\theta^2\right)+ \\ &+ [(r-M)^2+\alpha^2]sin^2(\theta)d\varphi^2 \ ; \end{aligned} \tag{C.34}$$

moreover, the 1−form A is given by:

$$A=\frac{(2rM-M^2)\alpha sin^2(\theta)}{(r-M)^2+\alpha^2 cos^2(\theta)}d\varphi \ , \tag{C.35}$$

and:

$$|H_1|^2=\frac{(r-M)^2-\alpha^2 cos^2(\theta)}{r^2+\alpha^2 cos^2(\theta)} \ , \tag{C.36}$$

and we recover the Kerr-Newman solutions with $M^2=\frac{Q^2+P^2}{2}$. These solutions are not black holes because they the bound:

$$M^2\geq\alpha^2+\frac{Q^2+P^2}{2} \ ; \tag{C.37}$$

in fact, it has been argued by Hawking and Hartle that the only black hole type solution in the IWP family of metrics are multi-ERN (multi-extreme Reissner-Nordström) solutions.

In physics, one of the main interests of this family of solution is that it is electric-magnetic duality invariant and it is the most general family that we can have with the above charges always satisfying the condition $M^2=\frac{Q^2+P^2}{2}$. An electric-magnetic duality transformation is nothing but a change in the phase of H_1. Non-extreme solution can be constructed from the Israel-Wilson-Perjés class (IWP class), by adding a "non-extremality function" W_1, as in the Reissner-Nordström case[171].

C.3 Dilaton and dilaton/axion black holes

The a-model describes a real scalar coupled to gravity and to a vector-field strength. The coupling depends on a parameter a, hence the name "a-model", and is exponential. Since the scalar can be identified in some cases with the string dilaton, or with the Kaluza-Klein scalar, which is called also the dilaton sometimes, there models are also generically referred to as dilaton gravity[119]. We will be able to describe the black holes type solution for general value of a and in four-dimension.

It is very convenient to have the most general solution, of $D = 4$ and $a = 1$-model action, written explicitly in terms of the physical charges. Moreover, the most general static solution can be generalized in natural way by adding angular momentum and NUT charge ℓ, becoming the truly most general stationary black holes type solution that we will call the SWIP solution[171]. It will be T-duality and S-duality invariant by defined, and its physical properties will be given in terms of duality invariant combinations of charges.

The solution is determined by two complex harmonic functions, H_1 and H_2, the non-extremality function W_1, the spatial background metric γ_{mn} and ℓ complex constants k^l[130, 131, 132]:

$$ds^2 = e^{2U} W_1 (dt + A_\varphi d\varphi)^2 - \frac{e^{-2U}}{W_1} (\gamma_{mn} dx^m dx^n) \ ; \tag{C.38}$$

with:

$$e^{-2U} = 2Im(H_1 \bar{H}_2) \tag{C.39}$$

and

$$A_\varphi = 2\ell\alpha cos(\theta) + \alpha sin^2(\theta) \left(\frac{e^{-2U}}{W_1} - 1 \right) . \tag{C.40}$$

The complex harmonic functions, H_1 and H_2 take the form:

$$H_1 = \frac{1}{\sqrt{2}} e^{\varphi_0} e^{\imath\beta} \left(\tau_0 + \frac{\tau_0 \mathcal{M}_1 + \bar{\tau}_0 \Upsilon}{r + \imath\alpha cos(\theta)} \right) , \tag{C.41}$$

$$H_2 = \frac{1}{\sqrt{2}} e^{\varphi_0} e^{\imath\beta} \left(1 + \frac{\mathcal{M}_1 + \Upsilon}{r + \imath\alpha cos(\theta)} \right) ; \tag{C.42}$$

and W_1 and the spatial background metric γ_{mn} take the forms:

$$W_1 = 1 - \frac{r_0^2}{r^2 + \alpha^2 cos^2(\theta)} , \tag{C.43}$$

$$\begin{aligned}\gamma_{mn}dx^m dx^n &= \frac{r^2+\alpha^2 cos^2(\theta)-r_0^2}{r^2+\alpha^2-r_0^2}dr^2+(r^2+\alpha^2 cos^2(\theta)-r_0^2)d\theta^2+\\ &+ (r^2+\alpha^2-r_0^2)sin^2(\theta)d\varphi^2; \end{aligned} \tag{C.44}$$

the complex constants k^l are given by:

$$k^l = -\frac{1}{\sqrt{2}}e^{-\imath\beta}\left(\frac{\Gamma^l\mathcal{M}_1+\bar{\Upsilon}\bar{\Gamma}^l}{|\mathcal{M}_1|^2-|\Upsilon|^2}\right). \tag{C.45}$$

The metric can also be written in a more standard form:

$$\begin{aligned} ds^2 = &\left(\frac{\Delta-\alpha^2 sin^2\theta}{\Sigma}\right)dt^2+2\alpha sin^2\theta\frac{\Sigma+\alpha^2 sin^2\theta-\Delta}{\Sigma}dtd\varphi+\\ &-\frac{\Sigma}{\Delta}dr^2-\Sigma d\theta^2-\frac{(\Sigma+\alpha^2 sin^2\theta)^2-\Delta\alpha^2 sin^2\theta}{\Sigma}sin^2\theta d\varphi^2; \end{aligned} \tag{C.46}$$

where:

$$\Sigma = (r+M)^2+(\ell+\alpha cos\theta)^2-|\Upsilon|^2, \tag{C.47}$$

$$\Delta = r^2-r_0^2+\alpha^2. \tag{C.48}$$

We have expressed the functions that enter the metric in terms of physical constants: $\alpha = J/M$ is the angular momentum J per unit mass M, the have combined the electric and magnetic charges into:

$$\Gamma^l = Q^l+\imath P^l, \tag{C.49}$$

and the mass and NUT charge ℓ into the complex mass:

$$\mathcal{M}_1 = M+\imath\ell. \tag{C.50}$$

The complex dilaton/axion charge Υ and τ_0, its asymptotic value, are defined by:

$$\tau \sim \tau_0-\imath e^{(-2\varphi_0)}\frac{2\Upsilon}{r}. \tag{C.51}$$

In these solutions the charge Υ depends on the conserved charges in this fixed way:

$$\Upsilon = -\frac{1}{2}\sum_l\frac{(\bar{\Gamma}^l)^2}{\mathcal{M}_1}; \tag{C.52}$$

and the non-extremality parameter r_0 is given by:

$$r_0^2 = |\mathcal{M}_1|^2 + |\Upsilon|^2 - \sum_l |\bar{\Gamma}^l|^2 \,. \tag{C.53}$$

In non-static cases when r_0 is zero, the solution is supersymmetric, but for a not zero it is not an extreme black holes. A more appropriate name is supersymmetry parameter. The extremality parameter will be:

$$R_0^2 = r_0^2 - \alpha^2 \,; \tag{C.54}$$

when it is positive, we have two horizons placed at:

$$r_\pm^2 = M \pm R_0 \,; \tag{C.55}$$

and the area of the event horizon is given, for black hole solutions with zero NUT charge, by:

$$A = 4\pi(r_+^2 + \alpha^2 - |\Upsilon|^2) \,. \tag{C.56}$$

We observe that, when $r_0 = 0$, that is $W_1 = 1$, the general SWIP solution has special properties; the principal one is that the back-ground metric γ_{mn} is nothing but the metric of Euclidean three-dimensional space in oblate spheroidal coordinates, which are related to the ordinary Cartesian ones by:

$$\begin{aligned} x &= (\sqrt{r^2 + \alpha^2}) sin\theta cos\varphi \\ y &= (\sqrt{r^2 + \alpha^2}) sin\theta sin\varphi \\ z &= r cos\theta \,; \end{aligned} \tag{C.57}$$

on rewriting the equation (C.38) in Cartesian coordinates, we find the solutions:

$$ds^2 = e^{2U}(dt + A_\varphi d\varphi)^2 - e^{-2U}\left(\gamma_{mn} dx^m dx^n\right) \,; \tag{C.58}$$

with:

$$e^{-2U} = 2Im(H_1 \bar{H}_2) \,, \qquad \tau = \frac{H_1}{H_2} \,, \tag{C.59}$$

$$A_\varphi = 2\ell\alpha cos(\theta) + \alpha sin^2(\theta)\left(e^{-2U} - 1\right) . \tag{C.60}$$

and

$$\sum_{n=1}^{n=l}(k^n)^2 = 0 , \qquad \sum_{n=1}^{n=l}|k^n|^2 = \frac{1}{2} . \tag{C.61}$$

That is, for any arbitrary pair of complex harmonic functions H_1 and H_2 in the three-dimensional Euclidean space, it is clear that we can construct multi-black holes solutions and that $r_0 = 0$ can be reinterpreted as a no-force condition between the black holes. These solutions include the IWP metric, equation (C.25), when

$$H_1 = \imath H_2 \tag{C.62}$$

and this H_1 is equal to $\frac{H_1}{\sqrt{2}}$ in IWP metric, which trivializes the axidilaton τ. These are the only black holes type solutions in the IWP family: the addition of angular momentum eliminates the event horizon and the addition of NUT charge eliminates the asymptotic flatness. Something similar is true for the supersymmetric SWIP solutions above: the only supersymmetric black holes in this family are the static ones with $A_i = 0$, which imposes a non-trivial constraint on the complex harmonic functions[172].

The solution include for vanishing dilaton/axion charge $\Upsilon = 0$, which corresponds to special choices of electric and magnetic charges, the Kerr-Newman black hole in Boyer–Lindquist coordinates (C.13).

The coupling of n vector multiplets to $N = 2$ supergravity theory can, in some cases, be completely described by a prepotential function $F(X)$ of the projective coordinates X^Λ, with $\Lambda = 0, 1, \ldots, n$, that parametrize the scalar manifold. From prepotential $F(X)$ one can derive the Kähler potential $\mathcal{K}$[130]:

$$\mathcal{K} = -\log\left(N_{\Lambda\Sigma}X^\Lambda X^\Sigma\right) , \tag{C.63}$$

with:

$$N_{\Lambda\Sigma} = \frac{1}{2}Re[\partial_\Lambda\partial_\Sigma F(X)] . \tag{C.64}$$

from which the Kähler metric of the scalar sigma-model,

$$g_{a\bar{b}} = \frac{\partial}{\partial_{z^a}}\frac{\partial}{\partial_{\bar{z}^b}}\mathcal{K}\,, \qquad z^a \equiv \frac{X^a}{X^0}\,, \tag{C.65}$$

the chiral connection A_μ

$$A_\mu = \frac{\imath}{2}N_{\Lambda\Sigma}[\bar{X}^\Lambda\partial_\mu X^\Sigma - (\partial_\mu\bar{X}^\Lambda)X^\Sigma]\,, \tag{C.66}$$

and also the couplings of the scalars to the vector fields can be derived.

The most general black holes solution of an $N = 2$ theory has to be duality invariant and thus has to be built out of the only invariant that the special geometry formalism contains: the chiral connection A_μ and the Kähler potential $\mathcal{K}$. The metric for extreme black holes, in $N = 2$ supergravity, can always be written in the form[22]:

$$ds^2 = e^{\mathcal{K}}dt^2 - e^{-\mathcal{K}}d\overrightarrow{x}^2, \tag{C.67}$$

where the coordinates X^Λ are identified with real harmonic function H^Λ that are also related to the $n+1$ $U(1)$ vector potentials of the theory. In [130] it was realized that one could also use complex harmonic functions, and then the one-form A_i that appears in non-static SWIP black holes solutions:

$$ds^2 = e^{\mathcal{K}}(dt + A_i dx^i)^2 - e^{-\mathcal{K}}d\overrightarrow{x}^2, \tag{C.68}$$

is realized to the chiral one-form of the $N = 2$ supergravity theory by:

$$A_{\underline{i}} = \varepsilon_{ijk}\partial^{\underline{j}}A^{\underline{k}}\,. \tag{C.69}$$

More precisely, in $N = 4$ and $D = 4$ supergravity, with only two vector fields corresponds to an $N = 2$ and $D = 4$ theory with prepotential $F(X) = 2X^0X^1$; the axidilaton is just $\tau = X^1/X^0$ and with harmonic functions:

$$X^0 = \imath H_2\,, \qquad X^1 = H_1\,, \tag{C.70}$$

gives the SWIP solutions.

It is natural to conjecture that a similar recipe should work in general cases since the basic principle of correspondence between elements of the metric and special geometry invariant should be valid. However the SWIP solutions remain the only equations whose complete explicit form is known. Also it is to be expected that general non-supersymmetric black hole solutions of extended supergravity can also be constructed by introducing a background metric and non-extremality functions[83].

Appendix D

The surface gravity

The surface gravity, g, of an astronomical or other object is the gravitational acceleration experienced at its surface. The surface gravity may be thought of as the acceleration due to gravity experienced by a hypothetical test particle which is very close to the object's surface and which, in order not to disturb the system, has negligible mass.
In relativity, the Newtonian concept of acceleration turns out not be clear cut. For a black hole, which can only be truly treated relativistically, one can not define a surface gravity as the acceleration experienced by a test body at the object's surface. This is because the acceleration of a test body at the event horizon of a black hole turns out to be infinite in relativity. Because of this, a renormalized value is used that corresponds to the Newtonian value in the non-relativistic limit. The value used is generally the local proper acceleration multiplied by the gravitational redshift factor; that is, one can define the surface gravity for a black hole whose event horizon is a Killing horizon[61, 2]. The surface gravity k of a static Killing horizon is the acceleration, as exerted at infinity, needed to keep an object at the horizon. Mathematically, if k^a is a suitably normalized Killing vector, then the surface gravity is defined by:

$$k^a \bigtriangledown_a k^b = kk^b, \tag{D.1}$$

where the equation is evaluated at the horizon. For a static and asymptotically flat spacetime, the normalization should be chosen so that $k^a k_a \to 1$ as $r \to \infty$, and so that $k \geq 0$. For the Schwarzschild solution, we take k^a to be the time translation Killing vector:

$$k^a \partial_a = \frac{\partial}{\partial t}, \tag{D.2}$$

and more generally for the Kerr-Newman solution we take

$$k^a \partial_a = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \phi}, \tag{D.3}$$

the linear combination of the time translation and axisymmetry Killing vectors which is null at the horizon, where Ω_H is the angular velocity of the black hole.

The surface gravity for the Schwarzschild solution is:

$$k = \frac{1}{4M}, \tag{D.4}$$

where M is the mass, and the surface gravity for the Kerr-Newman solution is:

$$k = \frac{\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}}{2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - \frac{J^2}{M^2}}}, \tag{D.5}$$

where: M is the mass, Q is the electric charge and J is the angular momentum.

Appendix E

The Special Kähler geometry of $D = 4$ Model

In the present appendix we shall discuss the main geometric quantities related to the special Kähler geometry of the model under consideration. Recall that a special Kähler manifold $\mathcal{M}_{SK}$[32, 33, 34, 35, 36], of the complex dimension n, is a Hodge-Kähler manifold on which a flat, holomorphic and symplectic vector structure is defined, with structure group $Sp(2n+2, R)$. If $\Omega(z^a)$ is a holomorphic section of this bundle:

$$\Omega(z^a) = (\Omega^M(z^a)) = \begin{pmatrix} X^\Lambda(z^a) \\ F_\Lambda(z^a) \end{pmatrix}, \tag{E.1}$$

$$\Lambda = 0, \ldots, n; \qquad a = 1, \ldots, n; \qquad M = 1, \ldots, 2n+2; \tag{E.2}$$

the Kähler potential $\mathcal{K}$ is expressed as follows:

$$\begin{aligned} \mathcal{K}(z^a, \bar{z}^a) &= -\log\left(-\imath\, \Omega \mathbb{C} \bar{\Omega}\right) \\ &= -\log\left[-\imath\, \left(X^\Lambda \bar{F}_\Lambda - F_\Lambda \bar{X}_\Lambda\right)\right], \end{aligned} \tag{E.3}$$

where $\mathbb{C}$ being the $Sp(2n+2, R)$-invariant metric:

$$\mathbb{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{E.4}$$

The complex vector field $\Omega(z^a)$ also belongs to a holomorphic line bundle, namely it transforms by multiplication times a holomorphic function[173, 56]:

$$\Omega(z^a) \rightarrow e^{-f(z)}\Omega(z^a). \tag{E.5}$$

This implies, according to equation (E.3), a Kähler transformation on the potential

$$\mathcal{K} : \mathcal{K} \rightarrow \mathcal{K} + f(z) + \bar{f}(\bar{z}). \tag{E.6}$$

It is useful to introduce a section of a $U(1)$-bundle over the scalar manifold,

$$V(z^a, \bar{z}^a) \equiv e^{\frac{\mathcal{K}}{2}}\Omega(z^a), \tag{E.7}$$

which as $\Omega(z^a) \longrightarrow e^{-f(z)}\Omega(z^a)$, transforms under a $U(1)$-transformation:

$$V(z^a, \bar{z}^a) \rightarrow e^{-i\theta}V(z^a, \bar{z}^a), \tag{E.8}$$

where $\theta = \theta(z, \bar{z}) = Im(f)$. This vector satisfies the property of being covariantly holomorphic with respect to the $U(1)$-connection:

$$\nabla_{\bar{a}} V \equiv \left(\partial_{\bar{a}} - \frac{1}{2}\partial_{\bar{a}}\mathcal{K}\right) V(z^a, \bar{z}^a) = 0, \tag{E.9}$$

where:

$$\partial_a \equiv \frac{\partial}{\partial_{z^a}}; \qquad \partial_{\bar{a}} \equiv \frac{\partial}{\partial_{\bar{z}^a}}. \tag{E.10}$$

If we define:

$$U_a = ({U_a}^M) \equiv \nabla_a V = \left(\partial_a + \frac{1}{2}\partial_a\mathcal{K}\right) V(z^a, \bar{z}^a), \tag{E.11}$$

the following properties hold:

$$V\mathbb{C}\bar{V} = \imath; \qquad U_a\mathbb{C}\bar{V} = \bar{U}_{\bar{a}}\mathbb{C}\bar{V} = 0; \qquad U_a\mathbb{C}\bar{U}_{\bar{b}} = -\imath g_{a\bar{b}}. \tag{E.12}$$

If ${E_a}^I$, with $I = 1, \ldots, n$, is the complex vielbein matrix of the manifold,

$$g_{a\bar{b}} = \sum_I {E_a}^I \bar{E}^I_{\bar{b}} \tag{E.13}$$

and ${E_I}^a$ its inverse, we introduce the quantities:

$$U_I \equiv {E_I}^a U_a, \tag{E.14}$$

in terms of which the following $(2n+2) \times (2n+2)$ matrix $\hat{\mathbb{L}}_4$ is defined:

$$\hat{\mathbb{L}}_4 = \sqrt{2}\left(Re(V), Re(U_I), -Im(V), Im(U_I)\right), \tag{E.15}$$

which, by virtue of equations (E.12), is symplectic:

$$\hat{\mathbb{L}}_4^T \mathbb{C} \hat{\mathbb{L}}_4 = \mathbb{C}. \tag{E.16}$$

In terms of this matrix one can construct the symmetric, symplectic and negative definite matrix $\mathcal{M}_4 = (\mathcal{M}_{4MN})$:

$$\mathcal{M}_4 = \mathbb{C}\hat{\mathbb{L}}_4 \hat{\mathbb{L}}_4^T \mathbb{C}. \tag{E.17}$$

This matrix is related to $R_{\Lambda\Gamma}$ and $I_{\Lambda\Gamma}$ as follows:

$$\mathcal{M}_4 = \begin{pmatrix} I + RI^{-1}R & -RI^{-1} \\ -I^{-1}R & I^{-1} \end{pmatrix}. \tag{E.18}$$

For symmetric homogeneous special Kähler manifolds, the symplectic bundle defines an embedding of the isometric group G_4 into $Sp(2n{+}2, R)$, realized by the symplectic representation $\mathbf{R}$ by which G_4 acts on the symplectic section V as part of the structure group. The global symmetric of the $D = 4$ model, duality symmetries, consist in the simultaneous action of G_4 on the scalar fields and on the symplectic vector of the electric field strengths and their magnetic duals of the representation $\mathbf{R}$.

One can always, by suitably fixing the symplectic gauge, choose a section $\Omega(z^a)$ in which $X^\Lambda(z^a)$ can be regarded as projective coordinates for the manifold. In particular, in a local patch in which $X^0 \neq 0$, X^a/X^0 are independent functions of z^a and can be thus used as coordinates, known as *special coordinates*[173, 56]. In the special coordinate patch we can then choose:

$$z^a \equiv \frac{X^a}{X^0} \tag{E.19}$$

in the first place. Moreover the lower components can be expressed in terms of a prepotential $F(X)$:

$$F(X) : F_\Lambda = \frac{\partial F(X)}{\partial X^\Lambda}, \tag{E.20}$$

$F(X)$ being a homogeneous function of degree two in the X^Λ. In the special coordinates the whole geometric structure can be derived by a single holomorphic prepotential:

$$\mathcal{F}(z) = \frac{F(X)}{(X^0)^2}. \tag{E.21}$$

In particular, the equation (E.3) the Kähler potential $\mathcal{K}$ becomes:

$$\mathcal{K} = -\log\left\{\imath\,\left[2\,(\mathcal{F}-\bar{\mathcal{F}}) - (z^a - \bar{z}^a)(\partial_a \mathcal{F} + \partial_{\bar{a}}\bar{\mathcal{F}})\right]\right\}. \tag{E.22}$$

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List of Figures

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