Stochastic Orders Between Used Systems and Systems with Used Components *

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ABSTRACT

Consider an n–component coherent system having random lifetime $T_{\boldsymbol{X}}$, where $\boldsymbol{X} = (X_1, \ldots, X_n)$ is the vector of the non-independent components' lifetimes. Stochastic comparisons of the residual life of $T_{\boldsymbol{X}}$ at a fixed time $t \geq 0$, conditioned on $\{T_{\boldsymbol{X}} > t\}$ or on $\{X_i > t, \forall i = 1, \ldots, n\}$, are investigated. Sufficient conditions on the vector $\boldsymbol{X}$ that imply this comparison in the usual stochastic order are provided, together with sufficient conditions under which the lifetime $T_{\boldsymbol{X}}$ satisfies the NBU aging property.

Key words Aging notions; Coherent systems; Path sets; Positive dependence concepts, Stochastic orders.

1 Introduction

Coherent systems are often considered in reliability theory to describe the structure and the performance of complex systems. Consider an item formed by a number n of components, i.e., an *n-component system*. Its *structure function* $\phi : \{0,1\}^n \to \{0,1\}$ is a function that maps the state vector $\widehat{\boldsymbol{x}} = (\widehat{x}_1, \ldots, \widehat{x}_n)$ of its components (where $\widehat{x}_i = 1$ if component i is working and $\widehat{x}_i = 0$ if it is failed) to the state $\widehat{y} \in \{0,1\}$ of the system itself. The system is said to be *coherent* whenever every component is relevant (i.e., it affects the working or failure of the system) and the structure function is monotone in every component (i.e., replacing a failed component by a working component cannot cause a working system to fail). For example, k-out-of-n systems, and series and parallel systems in particular, are coherent systems. See Esary and Marshall (1970) or Barlow and Proschan (1981) for a detailed introduction to coherent systems and related properties and applications.

Several problems and results dealing with aging properties for lifetimes of coherent systems, or with stochastic comparisons among coherent systems, have been considered in reliability literature. In particular, the closure property of some aging notions with respect to construction of coherent systems has been investigated, in most of the cases assuming independence among the lifetimes of the system's components (see, e.g, Barlow and Proschan, 1981, Samaniego, 1985, Deshpande et al., 1986, Franco et al., 2001, Li and Chen, 2004).

Among others, a natural question dealing with coherent systems is on the comparison between the reliability of a used coherent system and the reliability of a systems with used components. Precisely, denoted with $\boldsymbol{X}$ the vector of the component's lifetimes and with $T_{\boldsymbol{X}}$ the lifetime of the system, one can consider stochastic comparisons between the residual lifetimes $[T_{\boldsymbol{X}} - t \mid T_{\boldsymbol{X}} > t]$ and $[T_{\boldsymbol{X}} - t \mid X_i > t, \forall i = 1, \ldots, n]$, for $t \geq 0$. In fact, it is commonly assumed that the former is smaller, in some stochastic sense, than the latter. The intuitive explanation of this fact is that the reliability of a system with all components being in working state is higher with respect to the case with some of them being in failure state, even if the system is not in failure state. This assertion, which is actually true under assumption of independence among components (see, e.g., Pellerey and Petakos, 2002, or Li and Lu, 2003), is not always verified for non-independent components, as shown for example in Section 2.

This problem, and similar problems, have been recently investigated for example in Khaledi and Shaked (2007), Navarro et al. (2008) or Samaniego et al. (2009) under the assumption of independence among components' lifetimes, or in Zhang (2010), under assumption of exchangeability of components' lifetimes. The purpose of this paper is to generalize some of the results appearing in the above mentioned references, in particular providing conditions on the vector $\boldsymbol{X}$ such that

$$[T_{\boldsymbol{X}} - t \mid T_{\boldsymbol{X}} > t] \leq_{st} [T_{\boldsymbol{X}} - t \mid X_i > t, \, \forall i = 1, \cdots, n], \tag{1.1}$$

even under the case of components having non independent or exchangeable lifetimes, where $\leq_{st}$ denotes the usual stochastic order (whose definition is recalled below). These conditions are described in Section 2. As a corollary of the main result, a few statements that describe conditions on $\boldsymbol{X}$ such that the system's lifetime $T_{\boldsymbol{X}}$ satisfies some of the most well-know aging properties are presented in Section 3.

For ease of reference, some notations are introduced, and the definitions of several stochastic orders and dependence concepts which will be used in sequel are recalled.

Throughout this note, the terms *increasing* and *decreasing* stand for non-decreasing and non-increasing, respectively. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be *increasing* when $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$ for $\boldsymbol{x} \leq \boldsymbol{y}$, which denotes $x_i \leq y_i$ for all $i = 1, \cdots, n$. All random variables under investigation are non-negative, and expectations are implicitly assumed to be finite once they appear. The notation $[X \mid A]$ stands for the random object whose distribution is the conditional distribution of X given the event A. The dimension of a random vector is clear from the context and unless otherwise stated it is assumed to be n. We will denote with $I = \{1, \cdots, n\}$ the set of component's indices, and with $I_i = \{1, \cdots, i\}$, for $i = 1, \cdots, n$, their subsets. For any nonempty $A \subset I$, $\boldsymbol{X}_A$ and $\boldsymbol{x}_A$ denote the random vector of those X_i's with $i \in A$ and the corresponding constant vector, respectively. Besides, for any $s \geq 0$, notation $\boldsymbol{s}$ denotes the constant vector $(s, \cdots, s)$ with the dimension conforming to its circumstance. Finally, the following notation is adopted: $\boldsymbol{x} \wedge \boldsymbol{y} = (x_1 \wedge y_1, \cdots, x_1 \wedge y_1)$, $\boldsymbol{x} \vee \boldsymbol{y} = (x_1 \vee y_1, \cdots, x_1 \vee y_1)$, and $u \wedge v = \min\{u, v\}$, $u \vee v = \max\{u, v\}$.

Some well-known stochastic orders are recalled in the following definition. Further details, properties and applications of these orders may be found in Shaked and Shanthikumar (2007).

Definition 1.1. Given two random vectors (or variables) $\boldsymbol{X}$ and $\boldsymbol{Y}$, $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the:

(i) *likelihood ratio order* (denoted by $\boldsymbol{X} \leq_{lr} \boldsymbol{Y}$) if their joint densities f and g satisfies $f(\boldsymbol{x})g(\boldsymbol{y}) \leq f(\boldsymbol{x} \wedge \boldsymbol{y})g(\boldsymbol{x} \vee \boldsymbol{y})$ for any $\boldsymbol{x}$ and $\boldsymbol{y}$;

(ii) *stochastic order* (denoted by $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$) if $\mathrm{E}[\phi(\boldsymbol{X})] \leq \mathrm{E}[\phi(\boldsymbol{Y})]$ for any increasing function ϕ with finite expectations;

(iii) *increasing convex order* (denoted by $\boldsymbol{X} \leq_{icx} \boldsymbol{Y}$) if $\mathrm{E}[\phi(\boldsymbol{X})] \leq \mathrm{E}[\phi(\boldsymbol{Y})]$ for any increasing and convex function ϕ with finite expectations;

(iv) *increasing concave order* (denoted by $\boldsymbol{X} \leq_{icv} \boldsymbol{Y}$) if $\mathrm{E}[\phi(\boldsymbol{X})] \leq \mathrm{E}[\phi(\boldsymbol{Y})]$ for any increasing and concave function ϕ with finite expectation;

(v) *upper orthant order* (denoted by $\boldsymbol{X} \leq_{uo} \boldsymbol{Y}$) if $\mathrm{E}[\prod_{i=1}^n \phi_i(X_i)] \leq \mathrm{E}[\prod_{i=1}^n \phi_i(Y_i)]$ for any set of non–negative increasing functions $\phi_i, i = 1 \ldots, n$ such that expectations exist.

Recall that, in the univariate case, $X \leq_{st} Y$ if, and only if, $\mathrm{P}(X > t) \leq \mathrm{P}(Y > t)$ for all

$t \in \mathbb{R}$. The following two positive dependence notions also are well-known (see, e.g., Joe, 1997, or Shaked and Shanthikumar, 2007).

Definition 1.2. A random vector $\boldsymbol{X}$ is said to be *multivariate total positive of order 2* (MTP2) if its joint density f satisfies $f(\boldsymbol{x})f(\boldsymbol{y}) \leq f(\boldsymbol{x} \vee \boldsymbol{y})f(\boldsymbol{x} \wedge \boldsymbol{y})$ for any $\boldsymbol{x}$, $\boldsymbol{y}$.

Definition 1.3. For a bivariate vector $\boldsymbol{X} = (X_1, X_2)$, X_2 is said to be *right tail increasing* (RTI) in X_1 if $[X_2 \mid X_1 > x_1]$ is stochastically increasing in x_1 (and similarly X_1 is said to be RTI in X_2 if $[X_1 \mid X_2 > x_2]$ is stochastically increasing in x_2).

It should be mentioned that MTP2 property implies RTI property in both directions, while the reverse may not be true (see, e.g., Joe, 1997, or Müller and Scarsini, 2005, and references therein).

Finally, we recall that for a coherent system having structure function ϕ the relationship between the vector $\boldsymbol{X}$ of component's lifetimes and system's lifetime $T_{\boldsymbol{X}}$ is described by the relation $T_{\boldsymbol{X}} = \tau(\boldsymbol{X})$, where the *coherent life function* $\tau : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\tau(x_1, \dots, x_n) = \sup\{t \geq 0 : \phi(\widehat{x}_{1,t}, \dots, \widehat{x}_{n,t}) = 1\},$$

where $\widehat{x}_{i,t} = 1$ if $x_i > t$, or $\widehat{x}_{i,t} = 0$ if $x_i \leq t$, for $i \in I$. It should recall that coherent life functions are increasing and such that

$$\tau(t_1 - s, \dots, t_n - s) = \tau(t_1, \dots, t_n) - s, \tag{1.2}$$

for every $s \geq 0$ and $t_i \geq s$, $i \in I$ (see Esary and Marshall, 1970). Also, a subset $J = \{i_1, \dots, i_J\} \subseteq \{1, \dots, n\}$ of the components indices is said to be a *path set* if the system is working whenever the components indexed in J are working.

2 Main results

First, we show that stochastic inequality (1.1) does not necessarily hold. In fact, let $\boldsymbol{X} = (X_1, X_2)$ be such that

$$\begin{aligned} \mathrm{P}((X_1, X_2) = (2,1)) &= 1/4 \\ \mathrm{P}((X_1, X_2) = (2,2)) &= 3/8 \\ \mathrm{P}((X_1, X_2) = (3,1)) &= 1/4 \\ \mathrm{P}((X_1, X_2) = (3,2)) &= 1/8 \end{aligned}$$

and let $T_{\boldsymbol{X}} = \max\{X_1, X_2\}$. Letting $t = 1.5$ and $s = 1$ it holds that

$$\mathrm{P}(T_{\boldsymbol{X}} - t > s | T_{\boldsymbol{X}} > t) = \frac{\mathrm{P}(\max\{X_1, X_2\} > 2.5)}{\mathrm{P}(\max\{X_1, X_2\} > 1.5)} = 3/8,$$

while

$$\mathrm{P}(T_{\boldsymbol{X}} - t > s | X_i > t,\ \forall i) = \frac{\mathrm{P}(\max\{X_1, X_2\} > 2.5, X_1 > 1.5, X_2 > 1.5)}{\mathrm{P}(X_1 > 1.5, X_2 > 1.5)} = 1/4,$$

so that (1.1) can not be satisfied.

The following statement provides the first sufficient condition under which the stochastic comparison between $[T_{\boldsymbol{X}} - t \mid T_{\boldsymbol{X}} > t]$ and $[T_{\boldsymbol{X}} - t \mid X_i > t, \forall i = 1, \ldots, n]$ does hold.

Theorem 2.1. Let $\boldsymbol{X}$ be a vector of component's lifetimes such that, for any nonempty $A \subset I$ and $\boldsymbol{s} = (s, \ldots, s)$ with $s \geq 0$,

$$[\boldsymbol{X}_{\bar{A}} - \boldsymbol{s} \mid \boldsymbol{X} > \boldsymbol{s}] \geq_{st} [\boldsymbol{X}_{\bar{A}} - \boldsymbol{s} \mid \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}]. \tag{2.1}$$

Then, (1.1) holds for any coherent system with lifetime $T_{\boldsymbol{X}} = \tau(\boldsymbol{X})$, i.e.,

$$[T_{\boldsymbol{X}} - s \mid T_{\boldsymbol{X}} > s] \leq_{st} [T_{\boldsymbol{X}} - s \mid \boldsymbol{X} > \boldsymbol{s}], \quad s \geq 0.$$

Proof: Denote with $J_1, J_2, \ldots, J_\ell = I$ all possible path sets of the coherent system which has lifetime $T_{\boldsymbol{X}}$. Then it holds that, for any $s \geq 0$,

$$\{T_{\boldsymbol{X}} > s\} = \bigcup_{i=1}^{\ell} \left\{\boldsymbol{X}_{J_i} > \boldsymbol{s},\ \boldsymbol{X}_{\bar{J}_i} \leq \boldsymbol{s}\right\}.$$

For any s, $t \geq 0$, let

$$\begin{aligned} a_i &= \mathrm{P}(\boldsymbol{X}_{J_i} > \boldsymbol{s},\ \boldsymbol{X}_{\bar{J}_i} \leq \boldsymbol{s}), \qquad i = 1, \cdots, \ell, \\ b_i &= \mathrm{P}(T_{\boldsymbol{X}} > s + t,\ \boldsymbol{X}_{J_i} > \boldsymbol{s},\ \boldsymbol{X}_{\bar{J}_i} \leq \boldsymbol{s}), \quad i = 1, \cdots, \ell. \end{aligned}$$

We have

$$\begin{aligned} & \mathrm{P}(T_{\boldsymbol{X}} > s + t \mid T_{\boldsymbol{X}} > s) \\ = \ & \frac{\mathrm{P}\left(T_{\boldsymbol{X}} > s + t,\ T_{\boldsymbol{X}} > s\right)}{\mathrm{P}(T_{\boldsymbol{X}} > s)} \\ = \ & \frac{\sum_{i=1}^{\ell} \mathrm{P}\left(T_{\boldsymbol{X}} > s + t,\ \boldsymbol{X}_{J_i} > \boldsymbol{s},\ \boldsymbol{X}_{\bar{J}_i} \leq \boldsymbol{s}\right)}{\sum_{i=1}^{\ell} \mathrm{P}\left(\boldsymbol{X}_{J_i} > \boldsymbol{s},\ \boldsymbol{X}_{\bar{J}_i} \leq \boldsymbol{s}\right)} \\ = \ & \frac{\sum_{i=1}^{\ell} b_i}{\sum_{i=1}^{\ell} a_i}. \end{aligned}$$

Now, for any path set J_i, denoted with n_i its cardinality, consider the system corresponding to the structure function $\phi_{J_i} : \{0,1\}^{n_i} \to \{0,1\}$ defined as $\phi_{J_i}(\widehat{\boldsymbol{x}}_{J_i}) = \phi(\widehat{\boldsymbol{x}}_{J_i}, 0_{\bar{J}_i})$, i.e., letting in failed state all the components outside the path set. Let $T^i_{\boldsymbol{X}_{J_i}} = \tau_i(\boldsymbol{X}_{J_i})$ denote the lifetime of the subsystem whose structure function is ϕ_{J_i}. Clearly, for any $\widehat{\boldsymbol{x}} \in \{0,1\}^n$ we have $\phi_{J_i}(\widehat{\boldsymbol{x}}_{J_i}) =$

$\phi(\widehat{\boldsymbol{x}}_{J_i}, \mathbf{0}_{\bar{J}_i}) \le \phi(\widehat{\boldsymbol{x}}_{J_i}, \widehat{\boldsymbol{x}}_{\bar{J}_i}) = \phi(\widehat{\boldsymbol{x}})$, so that $\{T^i_{\boldsymbol{X}_{J_i}} > t\} \subseteq \{T_{\boldsymbol{X}} > t\}$. Moreover, since coherent life functions are increasing, by (1.2) and (2.1) it holds that

$$
\begin{aligned}
\frac{b_i}{a_i} &= \mathrm{P}(T_{\boldsymbol{X}} > s + t \mid \boldsymbol{X}_{J_i} > \boldsymbol{s},\, \boldsymbol{X}_{\bar{J}_i} \le \boldsymbol{s}) \\
&= \mathrm{P}(\tau(\boldsymbol{X}) > s + t \mid \boldsymbol{X}_{J_i} > \boldsymbol{s},\, \boldsymbol{X}_{\bar{J}_i} \le \boldsymbol{s}) \\
&= \mathrm{P}(\tau(\boldsymbol{X} - \boldsymbol{s}) > t \mid \boldsymbol{X}_{J_i} > \boldsymbol{s},\, \boldsymbol{X}_{\bar{J}_i} \le \boldsymbol{s}) \\
&= \mathrm{P}(\tau_i(\boldsymbol{X}_{J_i} - \boldsymbol{s}) > t \mid \boldsymbol{X}_{J_i} > \boldsymbol{s},\, \boldsymbol{X}_{\bar{J}_i} \le \boldsymbol{s}) \\
&\le \mathrm{P}(\tau_i(\boldsymbol{X}_{J_i} - \boldsymbol{s}) > t \mid \boldsymbol{X}_{J_\ell} > \boldsymbol{s}) \\
&\le \mathrm{P}(\tau(\boldsymbol{X} - \boldsymbol{s}) > t \mid \boldsymbol{X}_{J_\ell} > \boldsymbol{s}) \\
&= \mathrm{P}(\tau(\boldsymbol{X}) > s + t \mid \boldsymbol{X}_{J_\ell} > \boldsymbol{s}) \\
&= \mathrm{P}(T_{\boldsymbol{X}} > s + t \mid \boldsymbol{X}_{J_\ell} > \boldsymbol{s}) \\
&= \frac{b_\ell}{a_\ell}, \qquad \text{for any } i = 1, \cdots, \ell.
\end{aligned}
$$

Thus, $b_i a_\ell \le a_i b_\ell$ for $i = 1, \cdots, \ell$. This invokes

$$a_\ell b_1 + \cdots + a_\ell b_\ell \le a_1 b_\ell + \cdots + a_\ell b_\ell$$

and hence

$$\frac{\sum_{i=1}^{\ell} b_i}{\sum_{i=1}^{\ell} a_i} \le \frac{b_\ell}{a_\ell},$$

which is just

$$\mathrm{P}(T_{\boldsymbol{X}} - s > t \mid T_{\boldsymbol{X}} > s) \le \mathrm{P}(T_{\boldsymbol{X}} - s > t \mid \boldsymbol{X} > \boldsymbol{s}),$$

i.e., the assertion. ■

Theorem 2.1 has a very nice physical implication and describes conditions under which a coherent system of used components is better than an used coherent system, in the sense of having stochastically larger life length. This essentially claims that the positive dependence, or the independence, among the components of the coherent system is a sufficient condition for this property. Herewith, we address some other sufficient conditions for the assumption (2.1) to hold.

Theorem 2.2. If the joint density of $\boldsymbol{X} = (X_1, \cdots, X_n)$ is MTP2, then (2.1) holds for any nonempty $A \subseteq I$ and $s \ge 0$.

Proof: Recall that the MTP2 property of $(X_1, \cdots, X_n)$ is equivalent to $\boldsymbol{X} \le_{lr} \boldsymbol{X}$. Taking A and B as $\{\boldsymbol{X}_{\bar{A}} > \boldsymbol{s}, \boldsymbol{X}_A \le \boldsymbol{s}\}$ and $\{X_i > s,\, i = 1, \cdots, n\}$ respectively in Theorem 6.E.2 of Shaked and Shanthikumar (2007), we immediately obtain

$$[\boldsymbol{X} \mid \boldsymbol{X} > \boldsymbol{s}] \ge_{lr} [\boldsymbol{X} \mid \boldsymbol{X}_A \le \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}].$$

Now, by Theorem 6.E.4(b) of Shaked and Shanthikumar (2007) it follows that

$$[\boldsymbol{X}_{\bar{A}} \mid \boldsymbol{X} > \boldsymbol{s}] \geq_{lr} [\boldsymbol{X}_{\bar{A}} \mid \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}],$$

and, by Theorem 6.E.8 in the same reference, we have

$$[\boldsymbol{X}_{\bar{A}} \mid \boldsymbol{X} > \boldsymbol{s}] \geq_{st} [\boldsymbol{X}_{\bar{A}} \mid \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}],$$

for any $s \geq 0$. ■

A long list of multivariate distributions are MTP2. For example, a large number of vectors of lifetimes having an archimedean survival copula, or described by means of multivariate frailty models, satisfy this property (see, on this aim, Bassan and Spizzichino, 2005, or Durante et al., 2008, and references therein). Other examples may be found in Marshall and Olkin (1979) or Joe (1997). However, there are also many cases where this property is not satisfied, like, for example, when $\boldsymbol{X}$ does not admit a density. In this case, property (2.1) may be verified under alternative conditions, described in the following two statements.

Before giving the next statements, observe that inequality (2.1) is verified by all joint distributions that satisfy the dynamic multivariate positive aging notions defined in Shaked and Shanthikumar (1991) and references therein. Among them, the weaker one is the property introduced in Norros (1985), called *weakened by failures* (WBF): a vector $\boldsymbol{X}$ is said to be WBF if

$$[\boldsymbol{X}_{\bar{A}} - \boldsymbol{s} \mid \boldsymbol{X}_A = \boldsymbol{x}_A, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}] \geq_{st} [\boldsymbol{X}_{\bar{A}} - \boldsymbol{s} \mid \boldsymbol{X}_A = \boldsymbol{x}_A, X_i = x_i, \boldsymbol{X}_{\bar{A}-\{i\}} > \boldsymbol{s}]$$

for all $A \subseteq I, i \in I$, $\boldsymbol{x}_A \leq \boldsymbol{s}$ and $x_i \leq s$. Clearly, the assumptions of Theorem 2.1 are satisfied whenever $\boldsymbol{X}$ is WBF. The next result shows that inequality (2.1) is satisfied even under weaker assumptions.

Theorem 2.3. If, for any $B \subset \bar{A} \subseteq I$, any $\boldsymbol{x}_B \geq \boldsymbol{0}$ and any $\boldsymbol{y}_{\bar{B}} \geq \boldsymbol{x}_{\bar{B}}$,

$$[\boldsymbol{X}_B - \boldsymbol{x}_B \mid \boldsymbol{X}_B > \boldsymbol{x}_B, \boldsymbol{X}_{\bar{B}} = \boldsymbol{y}_{\bar{B}}] \geq_{uo} [\boldsymbol{X}_B - \boldsymbol{x}_B \mid \boldsymbol{X}_B > \boldsymbol{x}_B, \boldsymbol{X}_{\bar{B}} = \boldsymbol{x}_{\bar{B}}], \tag{2.2}$$

then the inequality (2.1) holds.

Proof: Without loss of generality, let $\bar{A} = \{1, \cdots, k\}$, and fix $\boldsymbol{s} = (s, \ldots, s), s \geq 0$. For $i = 2, \cdots, k$, set $B = I_{i-1} = \{1, \ldots, i-1\}$ in (2.2). Let us denote $\bar{I}_i = \{i+1, \ldots, n\}$ and $\bar{I}_{i-1} = \{i, \ldots, n\}$. Thus, for any $\boldsymbol{y}_{I_{i-1}} \geq \boldsymbol{x}_{I_{i-1}} \geq \boldsymbol{s}$,

$$\begin{aligned} &\mathrm{P}(X_i > s+t, \boldsymbol{X}_{\bar{I}_i} > \boldsymbol{s} \mid \boldsymbol{X}_{\bar{I}_{i-1}} > \boldsymbol{s}, \boldsymbol{X}_{I_{i-1}} = \boldsymbol{y}_{I_{i-1}}) \\ &\quad \geq \ \mathrm{P}(X_i > s+t, \boldsymbol{X}_{\bar{I}_i} > \boldsymbol{s} \mid \boldsymbol{X}_{\bar{I}_{i-1}} > \boldsymbol{s}, \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}), \end{aligned}$$

which implies

$$\begin{aligned}
&\lim_{\boldsymbol{\Delta}\to\mathbf{0}+}\frac{\mathrm{P}(X_i>s+t,\boldsymbol{X}>\boldsymbol{s},\,\boldsymbol{y}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{y}_{I_{i-1}}+\boldsymbol{\Delta})}{\mathrm{P}(\boldsymbol{X}>\boldsymbol{s},\,\boldsymbol{y}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{y}_{I_{i-1}}+\boldsymbol{\Delta})}\\
=\ &\lim_{\boldsymbol{\Delta}\to\mathbf{0}+}\frac{\mathrm{P}(X_i>s+t,\boldsymbol{X}_{\bar{I}_i}>\boldsymbol{s},\,\boldsymbol{y}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{y}_{I_{i-1}}+\boldsymbol{\Delta})}{\mathrm{P}(X_i>s,\boldsymbol{X}_{\bar{I}_i}>\boldsymbol{s},\,\boldsymbol{y}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{y}_{I_{i-1}}+\boldsymbol{\Delta})}\\
\ge\ &\lim_{\boldsymbol{\Delta}\to\mathbf{0}+}\frac{\mathrm{P}(X_i>s+t,\boldsymbol{X}_{\bar{I}_i}>\boldsymbol{s},\,\boldsymbol{x}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{x}_{I_{i-1}}+\boldsymbol{\Delta})}{\mathrm{P}(X_i>s,\boldsymbol{X}_{\bar{I}_i}>\boldsymbol{s},\,\boldsymbol{x}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{x}_{I_{i-1}}+\boldsymbol{\Delta})}\\
=\ &\lim_{\boldsymbol{\Delta}\to\mathbf{0}+}\frac{\mathrm{P}(X_i>s+t,\boldsymbol{X}>\boldsymbol{s},\,\boldsymbol{x}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{x}_{I_{i-1}}+\boldsymbol{\Delta})}{\mathrm{P}(\boldsymbol{X}>\boldsymbol{s},\,\boldsymbol{x}_{I_{i-1}}\le\boldsymbol{X}_{I_{i-1}}<\boldsymbol{x}_{I_{i-1}}+\boldsymbol{\Delta})}.
\end{aligned}$$

This yields, for any $i=2,\cdots,k$ and $\boldsymbol{y}_{\bar{B}}\ge\boldsymbol{x}_{\bar{B}}\ge\boldsymbol{s}$,

$$\mathrm{P}(X_i>s+t\mid\boldsymbol{X}>\boldsymbol{s},\boldsymbol{X}_{I_{i-1}}=\boldsymbol{y}_{I_{i-1}})\ge\mathrm{P}(X_i>s+t\mid\boldsymbol{X}>\boldsymbol{s},\boldsymbol{X}_{I_{i-1}}=\boldsymbol{x}_{I_{i-1}}).\qquad(2.3)$$

Moreover, the inequality (2.2) implies, for $\boldsymbol{y}_{\bar{B}}\ge\boldsymbol{x}_{\bar{B}}$ and $\boldsymbol{y}_B\ge\boldsymbol{x}_B$,

$$\frac{\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_{\bar{B}}=\boldsymbol{y}_{\bar{B}})}{\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_{\bar{B}}=\boldsymbol{y}_{\bar{B}})}\ge\frac{\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_{\bar{B}}=\boldsymbol{x}_{\bar{B}})}{\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_{\bar{B}}=\boldsymbol{x}_{\bar{B}})}.$$

Denote C the complimentary set of B with respect to $\bar{A}$, i.e., $B\cup C=\bar{A}$ and $B\cap C=\emptyset$. Then, $\bar{B}=A\cup C$. Setting $\boldsymbol{y}_C=\boldsymbol{x}_C$, it follows that

$$\begin{aligned}
&\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{t}_A)\cdot\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{v}_A)\\
&\ge\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{t}_A)\cdot\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{v}_A),
\end{aligned}$$

for every $\boldsymbol{t}_A\ge\boldsymbol{v}_A$.

Fix any $\boldsymbol{x}_A$, and denote $D_1=\{\boldsymbol{v}_A:\mathbf{0}\le\boldsymbol{v}_A\le\boldsymbol{x}_A\}$, $D_2=\{\boldsymbol{t}_A:\boldsymbol{t}_A\ge\boldsymbol{x}_A\}$. By the previous inequality we have

$$\begin{aligned}
&\int_{D_2}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{t}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{t}_A\mid\boldsymbol{x}_C)\\
&\quad\cdot\int_{D_1}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{v}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{v}_A\mid\boldsymbol{x}_C)\\
\ge\ &\int_{D_2}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{t}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{t}_A\mid\boldsymbol{x}_C)\\
&\quad\cdot\int_{D_1}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{v}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{v}_A\mid\boldsymbol{x}_C),
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{\int_{D_2}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{t}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{t}_A\mid\boldsymbol{x}_C)}{\int_{D_2}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{t}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{t}_A\mid\boldsymbol{x}_C)}\\
\ge\ &\frac{\int_{D_1}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{y}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{v}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{v}_A\mid\boldsymbol{x}_C)}{\int_{D_1}\mathrm{P}(\boldsymbol{X}_B>\boldsymbol{x}_B\mid\boldsymbol{X}_C=\boldsymbol{x}_C,\boldsymbol{X}_A=\boldsymbol{v}_A)\mathrm{d}F_{\boldsymbol{X}_A|\boldsymbol{X}_C}(\boldsymbol{v}_A\mid\boldsymbol{x}_C)},
\end{aligned}$$

i.e.,

$$\frac{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{y}_B, \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A > \boldsymbol{x}_A)}{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{x}_B, \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A > \boldsymbol{x}_A)} \geq \frac{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{y}_B, \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A \leq \boldsymbol{x}_A)}{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{x}_B, \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A \leq \boldsymbol{x}_A)} .$$

The last inequality is equivalent to

$$\frac{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{y}_B \mid \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A > \boldsymbol{x}_A)}{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{x}_B \mid \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A > \boldsymbol{x}_A)} \geq \frac{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{y}_B \mid \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A \leq \boldsymbol{x}_A)}{\mathrm{P}(\boldsymbol{X}_B > \boldsymbol{x}_B \mid \boldsymbol{X}_C = \boldsymbol{x}_C, \boldsymbol{X}_A \leq \boldsymbol{x}_A)}, \tag{2.4}$$

whenever $\boldsymbol{y}_B \geq \boldsymbol{x}_B$.

Now, setting $B = \bar{A}$, $C = \emptyset$, $\boldsymbol{x}_B = \boldsymbol{s} = (s, \ldots, s)$ and $\boldsymbol{y}_B = (s+t, s, \ldots, s)$ in (2.4) yields

$$\begin{aligned}
& \mathrm{P}(X_1 > t+s \mid \boldsymbol{X} > \boldsymbol{s}) \\
= & \frac{\mathrm{P}(X_1 > t+s, \boldsymbol{X}_A > \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s})}{\mathrm{P}(\boldsymbol{X}_A > \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s})} \\
= & \frac{\mathrm{P}(X_1 > t+s, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s} \mid \boldsymbol{X}_A > \boldsymbol{s})}{\mathrm{P}(\boldsymbol{X}_{\bar{A}} > \boldsymbol{s} \mid \boldsymbol{X}_A > \boldsymbol{s})} \\
\geq & \frac{\mathrm{P}(X_1 > t+s, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s} \mid \boldsymbol{X}_A \leq \boldsymbol{s})}{\mathrm{P}(\boldsymbol{X}_{\bar{A}} > \boldsymbol{s} \mid \boldsymbol{X}_A \leq \boldsymbol{s})} \\
= & \mathrm{P}(X_1 > t+s \mid \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}, \boldsymbol{X}_A \leq \boldsymbol{s}), \quad \text{for any } s, t \geq 0.
\end{aligned}$$

That is,

$$[X_1 - s \mid \boldsymbol{X} > \boldsymbol{s}] \geq_{st} [X_1 - s \mid \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}], \quad \text{for any } s \geq 0. \tag{2.5}$$

By (2.4) again, letting $i = 2, \cdots, k$ and $C = I_{i-1}$, it holds that, for s, $t \geq 0$ and $\boldsymbol{x}_{I_{i-1}} \geq \boldsymbol{s}$,

$$\begin{aligned}
& \mathrm{P}(X_i > t+s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X} > \boldsymbol{s}) \\
= & \frac{\mathrm{P}(X_i > t+s, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s} \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A > \boldsymbol{s})}{\mathrm{P}(\boldsymbol{X}_{\bar{A}_{i-1}} > \boldsymbol{s} \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A > \boldsymbol{s})} \\
\geq & \frac{\mathrm{P}(X_i > t+s, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s} \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A \leq \boldsymbol{s})}{\mathrm{P}(\boldsymbol{X}_{\bar{A}_{i-1}} > \boldsymbol{s} \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A \leq \boldsymbol{s})} \\
= & \mathrm{P}(X_i > t+s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}).
\end{aligned}$$

That is, for $i = 2, \cdots, k$,

$$[X_i - s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X} > \boldsymbol{s}] \geq_{st} [X_i - s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}].$$

On the other hand, by (2.3), we have, for $\boldsymbol{y}_{I_{i-1}} \geq \boldsymbol{x}_{I_{i-1}} \geq \boldsymbol{s}$,

$$[X_i - s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{y}_{I_{i-1}}, \boldsymbol{X} > \boldsymbol{s}] \geq_{st} [X_i - s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X} > \boldsymbol{s}],$$

and thus,

$$[X_i - s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{y}_{I_{i-1}}, \boldsymbol{X} > \boldsymbol{s}] \geq_{st} [X_i - s \mid \boldsymbol{X}_{I_{i-1}} = \boldsymbol{x}_{I_{i-1}}, \boldsymbol{X}_A \leq \boldsymbol{s}, \boldsymbol{X}_{\bar{A}} > \boldsymbol{s}]. \tag{2.6}$$

Finally, by applying Theorem 6.B.3 of Shaked and Shanthikumar (2007) to (2.5) and (2.6), we reach the desired result (2.1). ■

The following statement provides alternative conditions for (2.1) in the bivariate case.

Theorem 2.4. If X_2 is RTI in X_1 and X_1 is RTI in X_2, then, for any $s \geq 0$,

$$[X_1 - s \mid X_1 > s, X_2 > s] \geq_{st} [X_1 - s \mid X_1 > s, X_2 \leq s]$$

and

$$[X_2 - s \mid X_2 > s, X_1 > s] \geq_{st} [X_2 - s \mid X_2 > s, X_1 \leq s].$$

That is, the inequality (2.1) holds.

Proof: Let $s, t \geq 0$ and denote

$$\begin{aligned} A &= \{X_1 > s+t, X_2 > s\} , \\ B &= \{s+t \geq X_1 > s, X_2 > s\} , \\ C &= \{X_1 > s+t, X_2 \leq s\} , \\ D &= \{s+t \geq X_1 > s, X_2 \leq s\} . \end{aligned}$$

Since X_2 is RTI in X_1, it holds that

$$\frac{\mathrm{P}(A)}{\mathrm{P}(A \cup C)} = \frac{\mathrm{P}(X_1 > s+t, X_2 > s)}{\mathrm{P}(X_1 > s+t)} \geq \frac{\mathrm{P}(X_1 > s, X_2 > s)}{\mathrm{P}(X_1 > s)} = \frac{\mathrm{P}(A \cup B)}{\mathrm{P}(A \cup B \cup C \cup D)}.$$

Note that A, B, C and D are mutually exclusive, the above inequality may be rephrased as

$$\frac{\mathrm{P}(A)}{\mathrm{P}(A) + \mathrm{P}(C)} \geq \frac{\mathrm{P}(A) + \mathrm{P}(B)}{\mathrm{P}(A) + \mathrm{P}(C) + \mathrm{P}(B) + \mathrm{P}(D)}.$$

Equivalently,

$$\frac{\mathrm{P}(A)}{\mathrm{P}(A) + \mathrm{P}(C)} \geq \frac{\mathrm{P}(B)}{\mathrm{P}(B) + \mathrm{P}(D)},$$

which implies

$$\mathrm{P}(A) \cdot \mathrm{P}(D) \geq \mathrm{P}(B) \cdot \mathrm{P}(C),$$

and hence

$$\mathrm{P}(A) \cdot \mathrm{P}(D) + \mathrm{P}(A) \cdot \mathrm{P}(C) \geq \mathrm{P}(B) \cdot \mathrm{P}(C) + \mathrm{P}(A) \cdot \mathrm{P}(C).$$

This is just

$$\frac{\mathrm{P}(A)}{\mathrm{P}(A \cup B)} \geq \frac{\mathrm{P}(C)}{\mathrm{P}(C \cup D)}.$$

Consequently, we have, for any $s, t \geq 0$,

$$\begin{aligned} & \mathrm{P}(X_1 > s+t \mid X_1 > s, X_2 > s) \\ = & \frac{\mathrm{P}(X_1 > s+t, X_2 > s)}{\mathrm{P}(X_1 > s, X_2 > s)} \\ = & \frac{\mathrm{P}(A)}{\mathrm{P}(A \cup B)} \\ \geq & \frac{\mathrm{P}(C)}{\mathrm{P}(C \cup D)} \\ = & \frac{\mathrm{P}(X_1 > s+t, X_2 \leq s)}{\mathrm{P}(X_1 > s, X_2 \leq s)} \\ = & \mathrm{P}(X_1 > s+t \mid X_1 > s, X_2 \leq s). \end{aligned}$$

That is, $[X_1 - s \mid X_1 > s, X_2 > s] \geq_{st} [X_1 - s \mid X_1 > s, X_2 \leq s]$.

In a completely similar manner, we also have, for any $s \geq 0$

$$[X_2 - s \mid X_2 > s, X_1 > s] \geq_{st} [X_2 - s \mid X_2 > s, X_1 \leq s].$$

Thus, (2.1) is validated. ■

3 Sufficient conditions for positive aging

Conditions under which lifetimes of coherent systems satisfy aging properties have been studied extensively in the literature (see, e.g., Barlow and Proschan, 1981, or Lai and Xie, 2006), in most of the cases under the assumption of independence among component's lifetimes. Some interesting results dealing with the case of dependent components have been recently shown for example in Hu and Li (2007) and Navarro and Shaked (2010), where conditions on the joint density of the vector of component's lifetimes such that parallel and series systems have monotonic hazard and reverse hazard rates are described. Some results in the same spirit, but for more general coherent systems and weaker aging notions, are provided in this section.

Denote with $X_t = (X - t \mid X > t)$ the residual life of a random lifetime X at time $t \geq 0$. The following are among the most important univariate aging concepts

Definition 3.1. A nonnegative random variable X is said to be

(i) *new better than used* (NBU) if $X \geq_{st} X_t$ for all $t \geq 0$;

(ii) *new better than used in the 2nd stochastic dominance* (NBU(2)) if $X \geq_{icv} X_t$ for all $t \geq 0$;

(iii) *new better than used in the increasing convex order* (NBUC) if $X \geq_{icx} X_t$ for all $t \geq 0$.

The aging notions defined above can be generalized to the multivariate setting as follows. Denote with

$$\boldsymbol{X}_t = [(X_1 - t, \cdots, X_n - t) \mid X_1 > t, \cdots, X_n > t]$$

the residual life vector of $\boldsymbol{X}$ at time $t \geq 0$.

Definition 3.2. A nonnegative random vector $\boldsymbol{X}$ is said to be

(i) *multivariate new better than used* (M-NBU) if $\boldsymbol{X} \geq_{st} \boldsymbol{X}_t$ for all $t \geq 0$;

(ii) *multivariate new better than used in the 2nd stochastic dominance* (M-NBU(2)) if $\boldsymbol{X} \geq_{icv} \boldsymbol{X}_t$ for all $t \geq 0$;

(iii) *multivariate new better than used in the increasing convex order* (M-NBUC) if $\boldsymbol{X} \geq_{icx} \boldsymbol{X}_t$ for all $t \geq 0$.

Readers may refer to Pellerey (2008) or Li and Pellerey (2011) for examples of bivariate distributions with the M-NBU property.

According to Theorem 5.1 of Barlow and Proschan (1981), a coherent system may inherit the NBU property of its independent components. Theorem 3.1 below builds this preservation property for coherent systems of dependent components. Note that the assumption in (2.1) holds when all concerned components are mutually independent, thus Theorem 3.1 forms an interesting extension for Theorem 5.1 of Barlow and Proschan (1981).

Theorem 3.1. Under the assumption of (2.1), any coherent system is NBU whenever the components' lifetimes vector $\boldsymbol{X}$ is M-NBU.

Proof: By Theorem 2.1 and inequality (1.2), we have

$$[T_{\boldsymbol{X}} - s \mid T_{\boldsymbol{X}} > s] \leq_{st} [T_{\boldsymbol{X}} - s \mid \boldsymbol{X} > \boldsymbol{s}] \stackrel{st}{=} T_{\boldsymbol{X}_s}, \quad \text{for any } s \geq 0.$$

The M-NBU property of $\boldsymbol{X}$ implies $\boldsymbol{X}_s \leq_{st} \boldsymbol{X}$ for any $s \geq 0$. Due to the monotonicity of the coherent life functions, we have

$$T_{\boldsymbol{X}_s} \leq_{st} T_{\boldsymbol{X}}, \quad \text{for any } s \geq 0.$$

Thus, it holds that

$$[T_{\boldsymbol{X}} - s \mid T_{\boldsymbol{X}} > s] \leq_{st} T_{\boldsymbol{X}}, \quad \text{for any } s \geq 0.$$

This completes the proof. ■

Example 3.1. Consider a random vector $\boldsymbol{X}$ having the joint survival function

$$\bar{F}(x_1, \cdots, x_n) = \left(\frac{e^{bx_1} + e^{bx_2} + \cdots + e^{bx_n}}{n}\right)^{-\theta}, \qquad \theta, b > 0.$$

One may easily verify that the series system of these components has the reliability function $e^{-b\theta x}$ of an exponential distribution and thus is NBU. In fact, it can be verified that $\boldsymbol{X}$ has MTP2 density and satisfies the M-NBU property (Pellerey, 2008). According to Theorem 3.1, any coherent system with its components having lifetimes $\boldsymbol{X}$ is also NBU. ■

Example 3.2. Consider the random vector $\boldsymbol{X}$ having a Marshall-Olkin bivariate exponential distribution, i.e., having joint survival function

$$\overline{F}(x_1, x_2) = \mathrm{P}(X_1 > x_1, X_2 > x_2) = \exp\big\{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 (x_1 \vee x_2)\big\},$$

with $x_1, x_2 \geq 0$ and $\lambda_i \geq 0$, $i = 1, 2, 3$. As show in Corollary 4.2 in Li and Pellerey (2011), such a vector $\boldsymbol{X}$ satisfies the M-NBU property. Moreover, even if it does not satisfy the MTP2 property because of the singularity due to $\mathrm{P}(X_1 = X_2) > 0$, it satisfies the RTI propery, as can be easily verified. Thus, according to Theorem 3.1 and Theorem 2.4, the lifetime $T_{\boldsymbol{X}}$ of any coherent system whose components' lifetimes are described by $\boldsymbol{X}$ is NBU. ■

In a similar fashion, we may build the following result, which serves as a generalization of Theorem 1 in Pellerey and Petakos (2002).

Theorem 3.2. Under the assumption (2.1), any coherent system with convex [concave] coherent life function has a lifetime $T_{\boldsymbol{X}}$ which is NBUC [NBU(2)] whenever the components vector $\boldsymbol{X}$ is M-NBUC [M-NBU(2)].

As an immediate consequence, we get Corollary 3.1 below, which generalizes the preservation properties of NBUC and NBU(2) aging notions under parallel (series) systems with independent components due to Li et al (2000) and Li and Kochar (2001).

Corollary 3.1. Under the assumption (2.1), the lifetime of a parallel [series] system is NBUC [NBU(2)] whenever the vector of components' lifetimes $\boldsymbol{X}$ is M-NBUC [(M-NBU(2)].

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