Steklov problems in perforated domains with a coefficient of indefinite sign

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Abstract

We consider homogenization of Steklov spectral problem for a divergence form elliptic operator in periodically perforated domain under the assumption that the spectral weight function changes sign. We show that the limit behaviour of the spectrum depends essentially on wether the average of the weight function over the boundary of holes is positive, or negative or equal to zero. In all these cases we construct the asymptotics of the eigenpairs.

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Introduction

The paper studies Steklov spectral problem in a periodically perforated domain for the Laplace operator or for more general divergence form elliptic operator with periodic coefficients, under the assumptions that the Steklov condition is imposed on the perforation boundary and that the corresponding periodic weight function changes sign.

Previously, periodic homogenization of a bulk spectral problem with sign-changing density for an elliptic operator or an elliptic system was carried out in recent works [12], [11]. It was shown that the asymptotic behaviour of spectrum depends crucially on whether the mean value of the weight function is positive, or negative, or equal to zero.

The idea of studying Steklov and other spectral problems with sign-changing weight function arose during the conference "Differential Equations and Related Topics" in Moscow in 2007. It occur after the talk "Homogenization in perforated domains with Fourier boundary conditions" that focused on homogenization of elliptic problems with Fourier boundary condition on the perforation surface under the assumption that the coefficient of the boundary operator changes sign. It turned out that the limit behaviour of solutions depend crucially on whether the average of this coefficient over the perforation surface is positive, or negative, or equal to zero (see [4] for further details).

Steklov spectral problem, although has many common features with the bulk problem, differs essentially from the bulk problem due to the facts that the surface volume of the perforation tends to infinity, as the period vanishes, and that the perforation geometry is asymptotically singular.

The detailed formulation of the studied Steklov problem is

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} ; \tag{0.1}$$

here Ω is a smooth bounded domain, Ω_ε is the corresponding perforated domain, Γ_ε is the surface of a smooth periodic perforation consisting of disjoint inclusions, ν_ε is the exterior unit normal on Γ_ε, and ε is a small positive parameter. We assume that the function ρ is periodic and changes sign (see Section 1 for further details).

We also study a slightly more general problem of the form

$$\begin{cases} -\operatorname{div}\big(a(\frac{x}{\varepsilon})\nabla u_\varepsilon\big) = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon^a} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{0.2}$$

with a periodic symmetric matrix $a(y)$ that satisfies the uniform ellipticity conditions, $\nu_\varepsilon^a = a(x/\varepsilon)\nu_e$.

We first prove that the spectrum of the considered Steklov problem is discrete and, since the weight function ρ defines an indefinite metric on the perforation border (see [**?**])), the spectrum consists of two infinite sequences, one converges to $+\infty$ and another to $-\infty$.

We show that the asymptotic behaviour of spectrum in (0.1), as $\varepsilon \to 0$, depends essentially on whether the average of ρ over the surface of the hole is greater than zero, or less than zero, or equal to zero.

If the average of ρ is positive (negative), then the positive (negative) part of the spectrum behaves in a regular way and admits homogenization like in the classical case when $\rho > 0$. In particular, for any $k \in \mathbb{N}$, the k-th positive eigenvalue is of order ε, and the corresponding eigenfunction has a bounded H^1 norm. The convergence result in this case is presented in Theorem 1.4.

If ρ has zero average then both positive and negative eigenvalues have finite limits and the limit behaviour of the corresponding eigenpairs can be described in terms of the effective quadratic operator pencil. This operator pencil has a very simple structure and can be reduced to a standard eigenvalue problem for an elliptic operator in Ω. Notice that in this case the k-th negative and positive eigenfunctions are bounded in H^1-norm. The asymptotic behaviour of the spectrum in the case of zero average ρ is described in Theorem 1.7.

Finally, if the average of ρ is positive then the negative part of the spectrum of (0.1) (or (0.2)) shows a singular behaviour. Namely, for any $k \in \mathbb{N}$ the k-th negative eigenvalue is of order $1/\varepsilon$ and the corresponding eigenfunctions are rapidly oscillating.

We show that studying the negative part of the spectrum can be reduced to studying the negative part of the spectrum of an auxiliary problem that exhibits more regular behaviour. This reduction is done by means of factorization with the first negative eigenfunction of the corresponding cell periodic spectral problem. Further details can be found in Theorem 1.5 and its proof.

1 Setting of the problem and main results

In this section we provide a detailed set up of the studied Steklov spectral problem, introduce necessary notation and auxiliary problems, and then formulate the main results of the paper.

Let Ω be a smooth bounded domain in $\mathbb{R}^n$. We denote by $Y=(0,1)^n$ the unit cube of $\mathbb{R}^n$, and by $\omega = Y\setminus B$ the perforated reference cell, for a given closed set $B\subset Y$ with sufficiently smooth boundary $\partial B=\Gamma$. Setting

$$J_\varepsilon = \{z\in\mathbb{Z}^n \,:\, \varepsilon(Y+z)\subset\Omega\}, \tag{1.3}$$

we define $B_\varepsilon = \bigcup\limits_{z\in J_\varepsilon}\varepsilon(z+B)$, $\Gamma_\varepsilon = \bigcup\limits_{z\in J_\varepsilon}\varepsilon(z+\Gamma)$. Then a perforated domain is introduced as

$$\Omega_\varepsilon = \Omega\setminus B_\varepsilon.$$

Notice that, according to (1.3), B_ε does not intersect the external boundary $\partial\Omega$.

Remark 1.1. *Another possibility is not to remove the perforation in the vicinity of $\partial\Omega$. Instead, we can keep this part of perforation and impose the homogeneous Dirichlet boundary condition on it. We denote*

$$\widetilde{\Omega}_\varepsilon = \Omega\setminus\bigcup_{z\in\mathbb{Z}^n}\varepsilon(z+B). \tag{1.4}$$

Throughout this paper we assume that the exterior boundary $\partial\Omega$ has the regularity $C^{2,\alpha}$. In many our statements this regularity can be replaced with just Lipschitz continuity of the boundary. However, in this case we obtain only convergence results without estimating the rate of convergence.

In what follows the symbol $\Gamma_\#$ stands for the periodic extension of Γ in $\mathbb{R}^n$. Also, the lower index # in the functional space notation indicates that the corresponding functions are periodic.

Given a function $\rho\in L^\infty_\#(\Gamma)$, we study the asymptotic behaviour of the eigenvalue problems

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon,\\ \frac{\partial u_\varepsilon}{\partial\nu_\varepsilon} = \lambda_\varepsilon\rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon,\\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

as $\varepsilon \to 0$. The corresponding weak formulation reads

$$\begin{cases} u_\varepsilon \in H_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v d\sigma_x \quad \forall v \in H_\varepsilon, \end{cases} \tag{1.6}$$

where

$$H_\varepsilon = \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}$$

is a Hilbert space equipped with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx,$$

and σ_x denotes the $(n-1)$-dimensional surface measure.

We also consider a similar problem in $\widetilde{\Omega}_\varepsilon$

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \widetilde{\Omega_\varepsilon}, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\widetilde{\Omega}_\varepsilon \setminus \Gamma_\varepsilon. \end{cases} \tag{1.7}$$

Every solution u_ε of problem (1.5) or (1.7) can be extended to the whole domain Ω as a function $\tilde{u}_\varepsilon \in H_0^1(\Omega)$, with uniform estimates

$$\int_\Omega |\nabla \tilde{u}_\varepsilon|^2 dx \le c_0 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx, \qquad \int_\Omega |\tilde{u}_\varepsilon|^2 dx \le c_0 \int_{\Omega_\varepsilon} |u_\varepsilon|^2 \, dx$$

for all $\varepsilon > 0$ and for some $c_0 > 0$ that does not depend on ε (see, for instance, [1]). In the sequel, abusing slightly the notation, we still denote this extension by u_ε. Let us notice that, thanks to the above inequality, the usual Friedrichs inequality in H_ε holds true with a constant c_1 independent of ε, i.e.,

$$\int_{\Omega_\varepsilon} u^2 \, dx \le c_1 \int_{\Omega_\varepsilon} |\nabla u|^2 \, dx \qquad \forall u \in H_\varepsilon. \tag{1.8}$$

Throughout this paper we assume that the coefficient ρ satisfies the condition of **indefinite sign**

$$\sigma_y(\{y \in \Gamma : \rho(y) > 0\}) > 0 \quad \text{and} \quad \sigma_y(\{y \in \Gamma : \rho(y) < 0\}) > 0. \tag{1.9}$$

The limit behaviour of problems (1.5) appears to be different if the mean value $\overline{\rho}$ of ρ,

$$\overline{\rho} = \frac{1}{\sigma_y(\Gamma)} \int_\Gamma \rho(y) d\sigma_y(y), \tag{1.10}$$

is zero or non zero.

We begin by considering problem (1.5) for a fixed positive ε.

Proposition 1.2. *For each $\varepsilon > 0$ the spectrum of problem (1.5) consists of two sequences of eigenvalues*

$$0 < \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \ldots \le \lambda_j^\varepsilon \to +\infty \tag{1.11}$$

$$0 > \lambda_{-1}^\varepsilon \ge \lambda_{-2}^\varepsilon \ge \ldots \ge \lambda_{-j}^\varepsilon \to -\infty \quad \text{as } j \to +\infty \tag{1.12}$$

Moreover, for all $\varepsilon > 0$ there exists an orthonormal basis in H_ε of eigenfunctions $u_j^\varepsilon \in H_\varepsilon$ which are solutions to problem (1.5) corresponding to $\lambda_\varepsilon = \lambda_j^\varepsilon$, and for all $i, j \in \mathbb{Z} \setminus \{0\}$

$$\int_{\Omega_\varepsilon} \nabla u_i^\varepsilon \cdot \nabla u_j^\varepsilon \, dx = \delta_{ij}. \tag{1.13}$$

Furthermore,

$$\lambda_1^\varepsilon \quad \text{and} \quad \lambda_{-1}^\varepsilon \qquad \text{are simple.} \tag{1.14}$$

The proof of this proposition will be given in Section 2.

Similar statement holds true for problem (1.7). Orthogonality condition in this case reads

$$\int_{\widetilde{\Omega}_\varepsilon} \nabla u_i^\varepsilon \cdot \nabla u_j^\varepsilon \, dx = \delta_{ij}. \tag{1.15}$$

If $\overline{\rho} > 0$, the asymptotic analysis of the positive eigenvalues (1.11) as $\varepsilon \to 0$ involves the spectral properties of the Dirichlet problem

$$\begin{cases} -\mathrm{div}(a^{\mathrm{eff}} \nabla u) = \lambda \overline{\rho} \sigma_x(\Gamma) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.16}$$

where a^{eff} is a symmetric positive definite constant $(n \times n)$-matrix whose associated quadratic form is defined by

$$a^{\mathrm{eff}} \xi \cdot \xi = \inf \left\{ \int_\omega |\xi + \nabla w(y)|^2 dy : w \in H^1_\#(Y) \right\} \quad \forall \xi \in \mathbb{R}^n, \tag{1.17}$$

and $H^1_\#(Y)$ denotes the space of Y-periodic functions $\varphi(y)$ with finite norm

$$\|\varphi\|_{H^1_\#(y)} = \left(\int_Y (|\varphi|^2 + |\nabla \varphi|^2) dy \right)^{1/2}.$$

The function w_ξ that provides a minimum in (1.17) has the form $w_\xi = \xi \cdot \chi$ with the vector-function χ being a periodic solution to the classical cell problem

$$\begin{cases} \Delta \chi = 0 & \text{in } \omega, \\ \nabla \chi \cdot \nu = -\nu(y) & \text{on } \Gamma. \end{cases} \tag{1.18}$$

From the classical theory of elliptic operatotrs it follows that the spectrum of (1.16) is discrete and consists of a sequence $\{\lambda_j\}_{j\in\mathbb{N}}$ of positive eigenvalues,

$$0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \to +\infty \quad \text{as } j \to +\infty, \tag{1.19}$$

and that the corresponding eigenfunctions $\{u_j\}_{j\in\mathbb{N}} \in H_0^1(\Omega)$ form, under proper normalization, an orthonormal basis in $L^2(\Omega)$. For our purposes it is convenient to normalize u_j, $j \in \mathbb{N}$, as follows

$$\int_\Omega a^{\rm eff}\nabla u_i \cdot \nabla u_j dx = \delta_{ij}. \tag{1.20}$$

Then

$$\int_\Omega u_i u_j\, dx = (\overline{\rho}\lambda_i\sigma_x(\Gamma))^{-1}\delta_{ij}. \tag{1.21}$$

In what follows we use the notation

$$\Lambda = \{\lambda_j : j \in \mathbb{N}\}.$$

The asymptotic analysis of negative eigenvalues in (1.12) as $\varepsilon \to 0$ requires two more auxiliary spectral problems. The first one is stated in the periodicity cell with periodic boundary conditions:

$$\begin{cases} -\Delta p = 0 & \text{in } \omega, \\ \frac{\partial p}{\partial \nu} = \alpha\rho p, & \text{on } \Gamma, \\ p \quad \text{is } Y\text{-periodic.} \end{cases} \tag{1.22}$$

The corresponding weak formulation reads

$$\begin{cases} \displaystyle\int_\omega \nabla p \cdot \nabla w\, dy = \alpha \int_\Gamma \rho p w\, d\sigma_y \quad \forall w \in H^1_\#(Y), \\ p \in H^1_\#(Y). \end{cases} \tag{1.23}$$

Here, α is the spectral parameter. The statement below describes the behaviour of spectrum of problem (1.22). This statement will be proved in Section 2. The proof is more involved than that of Proposition 1.2 because the quadratic form related to (1.23) is not coercive.

Proposition 1.3. *Let $\overline{\rho} > 0$. Then the spectrum of problem problem (1.22) is discrete and consists of two sequences of eigenvalues*

$$0 = \alpha_1 < \alpha_2 \le \ldots \le \alpha_j \to +\infty \quad \textit{as } j \to +\infty, \tag{1.24}$$

$$0 > \alpha_{-1} > \alpha_{-2} \ge \ldots \ge \alpha_{-j} \to -\infty \quad \textit{as } j \to +\infty. \tag{1.25}$$

Moreover α_1, α_{-1} are simple and the associated eigenfunctions $p_1, p_{-1} \in H^1_\#(Y) \cap L^\infty(\omega)$ can be normalized as follows

$$p_{\pm 1} > 0 \quad \textit{in } \omega, \qquad \int_\Gamma \rho(p_{\pm 1})^2\, d\sigma_y = \pm 1. \tag{1.26}$$

Finally, if $\partial\omega \in \mathcal{C}^{2,\alpha}$ and $\rho \in C^\alpha(\partial B)$, then $p_\pm \in \mathcal{C}^2(\overline{\omega})$, and $0 < C_- \leq p_\pm \leq C^+$ for some constants C_- and C^+.

Now, we introduce the second spectral problem, which is stated in the whole set Ω and involves a new weight function $\rho^* = \rho^*(y)$ and its mean value $\overline{\rho}^*$:

$$\rho^* = \rho\, p_{-1}^2, \tag{1.27}$$

$$\overline{\rho}^* = \frac{1}{\sigma_y(\Gamma)} \int_\Gamma \rho^*(y) d\sigma_y. \tag{1.28}$$

Due to Proposition 1.3,

$$\int_\omega |\nabla p_{-1}|^2\, dy = \alpha_{-1} \int_\Gamma p_{-1}^2 \rho\, d\sigma_y > 0,$$

and hence

$$\overline{\rho}^* = \frac{1}{\sigma_y(\Gamma)} \int_\Gamma \rho^*\, d\sigma_y = \int_\Gamma p_{-1}^2 \rho\, d\sigma_y < 0. \tag{1.29}$$

Define by $\tilde{a}^{\mathrm{eff}}$ the constant positive definite $(n \times n)$-matrix whose associated quadratic form is defined by

$$\tilde{a}^{\mathrm{eff}} \xi \cdot \xi = \inf \left\{ \int_\omega |\xi + \nabla w(y)|^2 (p_{-1}(y))^2 dy : w \in H^1_\#(Y) \right\} \quad \forall \xi \in \mathbb{R}^n, \tag{1.30}$$

Notice that a minimum in (1.30) is attained at the function $\tilde{w}_\xi = \xi \cdot \tilde{\chi}$ with the vector-function $\tilde{\chi}$ being a periodic solution to the following cell problem

$$\begin{cases} \operatorname{div}\big((p_{-1})^2 (\mathbf{I} + \nabla \tilde{\chi})\big) = 0 & \text{in } \omega, \\ \nabla \tilde{\chi} \cdot \nu = -\nu(y) & \text{on } \Gamma, \end{cases} \tag{1.31}$$

here $\mathbf{I}$ stands for the unit matrix.

We now introduce the effective spectral problem:

$$\begin{cases} -\operatorname{div}(\tilde{a}^{\mathrm{eff}} \nabla v) = \varkappa \overline{\rho}^* \sigma_y(\Gamma) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.32}$$

where $\varkappa$ is a spectral parameter.

Problem (1.32) is classical. Since $\overline{\rho}^* < 0$, the spectrum of this problem consists of a sequence

$$0 > \varkappa_{-1} > \varkappa_{-2} \geq \varkappa_{-3} \geq \cdots \geq \varkappa_{-j} \longrightarrow -\infty, \qquad \text{as } j \to \infty. \tag{1.33}$$

The corresponding eigenfunctions $\{v_{-j}\}_{j\in\mathbb{N}}$, under proper normalization, form an orthonormal basis in $L^2(\Omega)$. However, we normalize them in a different way. Namely, we assume that

$$\int_\Omega \tilde{a}^{\mathrm{eff}} \nabla v_{-i} \cdot \nabla v_{-j} dx = \delta_{ij}. \tag{1.34}$$

The following results concern the case of $\overline{\rho} > 0$. It should be noted that, in this case, the positive and the negative parts of the spectrum show totally different behaviour. We first deal with the positive part of the spectrum.

Theorem 1.4. *Let $\overline{\rho} > 0$, and let $(\lambda_j^\varepsilon, u_j^\varepsilon)$ be the j-th eigenpair of problem (1.5), (1.13), or problem (1.7) with $j > 0$. Then*

(i) For all $j \in \mathbb{N}$

$$\frac{\lambda_j^\varepsilon}{\varepsilon} \to \lambda_j \qquad as\ \varepsilon \to 0, \tag{1.35}$$

where λ_j is the j-th eigenvalue of problem (1.16).

(ii) Under the additional assumption that Ω is a $\mathcal{C}^{2,\delta}$ domain with some $\delta > 0$ the rate of convergence in (1.40) can be estimated as follows: for every $j \in \mathbb{N}$ there exist constants $\varepsilon_j, C_j > 0$ such that

$$\left| \frac{\lambda_j^\varepsilon}{\varepsilon} - \lambda_j \right| \le C_j \sqrt{\varepsilon} \quad \text{for all } \varepsilon \in (0, \varepsilon_j). \tag{1.36}$$

(iii) If, for $j \in \mathbb{N}$, λ_j is an eigenvalue of problem(1.16) of multiplicity m_j, $\lambda_{j-1} < \lambda_j = \lambda_{j+1} = \ldots = \lambda_{j+m_j-1} < \lambda_{j+m_j}$, then there exist orthogonal $m_j \times m_j$ matrices $\mathcal{U}^\varepsilon$ and constants $\varepsilon_j > 0$ and $C_j > 0$ such that, for all $\varepsilon \in (0, \varepsilon_j]$,

$$\left\| u_{j+l-1}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon \, u_{j+k-1} \right\|_{L^2(\Omega)} \le C_j \sqrt{\varepsilon}, \quad l = 1, \cdots, m_j, \tag{1.37}$$

$$\left\| u_{j+l-1}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon \, U_{j+k-1}^\varepsilon \right\|_{H^\varepsilon(\Omega)} \le C_j \sqrt{\varepsilon}, \quad l = 1, \cdots, m_j \tag{1.38}$$

with $U_j^\varepsilon(x) = u_j(x) + \varepsilon \chi(x/\varepsilon) \nabla u_j(x)$, here χ is a solution of problem (1.18).

(iv) The function $\{U_j^\varepsilon\}$ are almost orthogonal and normalized in H_ε that is

$$\left| \langle U_k^\varepsilon, U_l^\varepsilon \rangle_{H_\varepsilon} - \delta_{k,l} \right| \le C \sqrt{\varepsilon}. \tag{1.39}$$

The same results hold true for problem (1.7)

We turn to the negative part of the spectrum. Here, in addition to the above assumptions, we suppose that the boundary of B has regularity $C^{2,\alpha}$ and that ρ is Hölder continuous, $\rho \in C^\alpha(\partial B)$. Here we only consider problem (1.7).

Theorem 1.5. *Let $\overline{\rho} > 0$, and let $(\lambda_{-j}^\varepsilon, u_{-j}^\varepsilon)$ be the j-th negative eigenpair of problem (1.7), (1.15). Then* *(i) For all $j \in \mathbb{N}$*

$$\frac{1}{\varepsilon}\Big(\lambda_{-j}^{\varepsilon} - \frac{\alpha_{-1}}{\varepsilon}\Big) \to \varkappa_{-j} \qquad \text{as } \varepsilon \to 0, \tag{1.40}$$

where α_{-1} is defined in (1.25), and $\varkappa_{-j}$ is the j-th (negative) eigenvalue of problem (1.32).

(ii) If Ω is a $\mathcal{C}^{2,\delta}$ domain for some $\delta > 0$ then for every $j \in \mathbb{N}$ there exist constants $\varepsilon_j, C_j > 0$ such that

$$\left|\frac{1}{\varepsilon}\Big(\lambda_{-j}^{\varepsilon} - \frac{\alpha_{-1}}{\varepsilon}\Big) - \varkappa_{-j}\right| \le C_j\sqrt{\varepsilon} \quad \text{for all } \varepsilon \in (0, \varepsilon_j). \tag{1.41}$$

(iii) If, for $j \in \mathbb{N}$, $\varkappa_{-j}$ is an eigenvalue of problem (1.32) of multiplicity m_{-j}, $\varkappa_{-j} = \varkappa_{-(j+1)} = \ldots = \varkappa_{-(j+m_j-1)}$, then there exist orthogonal $m_{-j} \times m_{-j}$ matrices $\mathcal{U}^\varepsilon$ and constants $\varepsilon_{-j} > 0$ and $C_{-j} > 0$ such that, for all $\varepsilon \in (0, \varepsilon_{-j}]$,

$$\left\| \frac{u_{-(j+l-1)}^{\varepsilon}}{\|u_{-(j+l-1)}^{\varepsilon}\|_{L^2(\Omega)}} - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^{\varepsilon}\, v_{-(j+k-1)}^{\varepsilon} \right\|_{L^2(\Omega)} \le C_{-j}\sqrt{\varepsilon}, \quad l = 1, \cdots, m_j, \tag{1.42}$$

with $v_{-j}^{\varepsilon}(x) = (\|v_{-j}\|_{L^2(\Omega)})^{-1} v_{-j}(x)\hat{p}_{-1}(x/\varepsilon)$; here $\hat{p}_{-1}$ is the eigenfunction of problem (1.22) that corresponds to α_{-1} and is normalized by

$$\int_\omega (\hat{p}_{-1}(y))^2 dy = 1.$$

(iv) The functions $\{U_{-j}^{\varepsilon}\}$, $U_{-j}^{\varepsilon}(x) = v_{-j}(x) + \varepsilon\tilde{\chi}(x/\varepsilon)\nabla v_{-j}(x)$, are almost orthogonal and normalized in H_ε that is

$$\Big|\langle U_{-k}^{\varepsilon}, U_{-l}^{\varepsilon}\rangle_{H_\varepsilon} - \delta_{k,l}\Big| \le C\sqrt{\varepsilon}. \tag{1.43}$$

Remark 1.6. *In contrast with problem (1.7) we cannot assure that the interval $\big(\alpha_{-1}/\varepsilon, 0\big)$ belongs to the resolvent set of spectral problem (1.5). If there are eigenvalues of problem (1.5) on this interval, then the corresponding eigenfunctions concentrate in the vicinity of $\partial\Omega$ that is they are of boundary layer type.*

In order to write down the limit problem in the case $\overline{\rho} = 0$ we introduce one more cell problem:

$$\begin{cases} -\Delta\theta = 0 & \text{in } \omega, \\ \dfrac{\partial\theta}{\partial\nu} = \rho, & \text{on } \Gamma, \\ \theta \quad \text{is } Y\text{-periodic}, \end{cases} \tag{1.44}$$

Since $\overline{\rho} = 0$, this problem is solvable, its solution is unique up to an additive constant. Denote

$$\Xi = \int_\Gamma \rho(y)\theta(y)d\sigma_y = \int_\omega \nabla\theta(y)\cdot\nabla\theta(y)dy > 0,$$

and consider the following operator pencil

$$\begin{cases} -\mathrm{div}(a^{\mathrm{eff}}\nabla u) = \lambda^2 \Xi u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.45}$$

and a spectral problem

$$\begin{cases} -\mathrm{div}(a^{\mathrm{eff}}\nabla u) = \nu \Xi u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.46}$$

with a^{eff} defined in (1.17).

Since (1.46) has a discrete spectrum $0 < \nu_1 < \nu_2 \le \nu_3 \le \cdots \le \nu_j \to \infty$, and all the eigenvalues ν_j are positive, the spectrum of (1.45) is discrete, real and consists of two series

$$\lambda_j^+ = \sqrt{\nu_j}, \qquad \lambda_j^- = -\sqrt{\nu_j}, \quad j = 1, 2, \dots \tag{1.47}$$

Here, for the corresponding eigenfunctions, we impose the following normalization conditions

$$\int_\Omega a^{\mathrm{eff}}\nabla u_i \cdot \nabla u_j dx + \Xi\sqrt{\nu_i\nu_j}\int_\Omega u_i u_j dx = \delta_{ij}. \tag{1.48}$$

Theorem 1.7. *Let $\overline{\rho} = 0$, and let $(\lambda_j^\varepsilon, u_j^\varepsilon)$, $j \in \mathbb{Z}\setminus\{0\}$, be the j-th eigenpair of problem (**??**), (1.13). Then*

(i) For all $j \in \mathbb{N}$

$$\lambda_{\pm j}^\varepsilon \to \lambda_j^\pm, \qquad \textit{as } \varepsilon \to 0, \tag{1.49}$$

where $\lambda_j^\pm$ are defined in (1.47).

(ii) Under the additional assumption that Ω is a $\mathcal{C}^{2,\delta}$ domain with some $\delta > 0$, for every $j \in \mathbb{N}$ there exist constants $\varepsilon_j, C_j > 0$ such that

$$\left|\lambda_{\pm j}^\varepsilon - \lambda_j^\pm\right| \le C_j\sqrt{\varepsilon} \quad \text{for all } \varepsilon \in (0, \varepsilon_j). \tag{1.50}$$

(iii) If, for $j \in \mathbb{N}$, ν_j is an eigenvalue of problem(1.46) of multiplicity m_j, $\nu_{j-1} < \nu_j = \nu_{j+1} = \ldots = \nu_{j+m_j-1} < \nu_{j+m_j}$, then there exist orthogonal $m_j \times m_j$ matrices $\mathcal{U}^\varepsilon$ and constants $\varepsilon_j > 0$ and $C_j > 0$ such that, for all $\varepsilon \in (0, \varepsilon_j]$,

$$\left\| u_{\pm(j+l-1)}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon\, u_{j+k-1} \right\|_{L^2(\Omega)} \le C_j\sqrt{\varepsilon}, \quad l = 1, \cdots, m_j, \tag{1.51}$$

$$\left\| u^\varepsilon_{\pm(j+l-1)} - \sum_{k=1}^{m_j} \mathcal{U}^\varepsilon_{lk}\, U^\varepsilon_{\pm(j+k-1)} \right\|_{H^\varepsilon(\Omega)} \le C_j \sqrt{\varepsilon}, \quad l = 1, \cdots, m_j \tag{1.52}$$

with $U^\varepsilon_{\pm j}(x) = u_j(x) + \varepsilon\chi(x/\varepsilon)\nabla u_j(x) + \lambda_j^{\pm}\theta(x/\varepsilon)u_j(x)$*, here* χ *and* θ *are solutions of problems (1.18) and (1.44), respectively.*

(iv) The function $\{U^\varepsilon_j\}$ *are almost orthogonal and normalized in* H_ε *that is*

$$\left| \langle U^\varepsilon_k, U^\varepsilon_l \rangle_{H_\varepsilon} - \delta_{k,l} \right| \le C \sqrt{\varepsilon}, \qquad k,\, j \in \mathbb{Z} \setminus \{0\}. \tag{1.53}$$

2 Preliminary statements

We begin this section by recalling some inequalities valid in H_ε. In what follows we denote

$$\omega^i_\varepsilon = \varepsilon(\omega + i), \qquad \Gamma^i_\varepsilon = \varepsilon(\Gamma + i), \quad i \in \mathbb{Z}^n.$$

Poincaré-Wirtinger inequality. Under our assumptions on Ω_ε and Γ_ε, there exist a positive constant k such that for each $u \in H_\varepsilon$ the following inequality holds:

$$\int_{\Gamma_\varepsilon} |u - \overline{u}_\varepsilon|^2 d\sigma_x \le k\, \varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2\, dx, \tag{2.54}$$

where we denote by $\overline{u}_\varepsilon(\cdot)$ the piece-wise constant function obtained by taking the mean value of u over each perforated cell ω^i_ε, i.e.,

$$\overline{u}_\varepsilon(x) = \frac{1}{|\omega^i_\varepsilon|} \int_{\omega^i_\varepsilon} u(y)dy, \qquad \text{if } x \in \omega^i_\varepsilon; \tag{2.55}$$

here $|\omega^i_\varepsilon|$ stands for the Lebesgue measure of ω^i_ε. The above inequality remains valid if $\overline{u}_\varepsilon$ is replaced with the piece-wise constant function being equal in each ω^i_ε to the surface average of u over Γ^i_ε.

Trace inequality

$$\int_{\Gamma_\varepsilon} |u|^2 d\sigma_x \le k_t \Big(\varepsilon^{-1} \int_{\Omega_\varepsilon} |u|^2\, dx + \varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2)\, dx \Big), \tag{2.56}$$

Both inequalities can be easily obtained from the standard Poincaré-Wirtinger and trace inequalities, (see [2], [14]) by means of scaling arguments.

Given $g \in L^2(\Gamma_\varepsilon)$, consider the following boundary value problem with non-homogeneous Neumann boundary conditions on Γ_ε

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n_\varepsilon} = g, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.57}$$

The corresponding weak formulation reads

$$\begin{cases} u_\varepsilon \in H_\varepsilon, \\ \displaystyle\int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Gamma_\varepsilon} gv d\sigma_x \quad \forall v \in H_\varepsilon, \end{cases} \tag{2.58}$$

where

$$H_\varepsilon = \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}$$

is a Hilbert space equipped with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx.$$

Proposition 2.1. *For every $g \in L^2(\Gamma_\varepsilon)$ there exists a unique solution $u_\varepsilon \in H_\varepsilon$ to problem (2.57). Moreover u_ε satisfies the following a-priori estimate*

$$||u_\varepsilon||_{H_\varepsilon} \leq c\varepsilon^{-1/2}||g||_{L^2(\Gamma_\varepsilon)}, \tag{2.59}$$

where the constant $c > 0$ is independent of ε.

Proof. The existence and uniqueness of u_ε is a straightforward consequence of the Reisz representation theorem for the problem

$$a(u, v) = F(v) \quad \forall v \in H,$$

where

$$a(u, v) = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx, \quad F(v) = \int_{\Gamma_\varepsilon} gv \, d\sigma_x, \quad H = H_\varepsilon.$$

Moreover, replacing $v = u_\varepsilon$ in the weak formulation (2.58), and using Friedrichs and trace inequalities (1.8), (2.56),we obtain that

$$\begin{aligned} ||u_\varepsilon||^2_{H_\varepsilon} &= \int_{\Omega_\varepsilon} |\nabla u_e|^2 \, dx = \int_{\Gamma_\varepsilon} gu_\varepsilon \, d\sigma_x \leq ||g||_{L^2(\Gamma_\varepsilon)}||u_\varepsilon||_{L^2(\Gamma_\varepsilon)} \leq \\ &\leq ||g||_{L^2(\Gamma_\varepsilon)} \left(k_t\Big(\varepsilon^{-1}\int_{\Omega_\varepsilon} |u|^2 \, dx + \varepsilon\int_{\Omega_\varepsilon} |\nabla u|^2) \, dx\Big)\right)^{1/2} \leq \\ &\leq c\varepsilon^{-1/2}||g||_{L^2(\Gamma_\varepsilon)}||u_\varepsilon||_{H_\varepsilon}. \end{aligned}$$

Dividing by $||u_\varepsilon||_{H_\varepsilon}$ we obtain the desired inequality (2.59). □

We introduce the operator $K_\varepsilon : H_\varepsilon \to H_\varepsilon$ in the following way. For every $u \in H_\varepsilon$, we define $K_\varepsilon u$ as the unique solution to the problem

$$\int_{\Omega_\varepsilon} \nabla(K_\varepsilon u) \cdot \nabla v \, dx = \int_{\Gamma_\varepsilon} \rho_\varepsilon uv \, d\sigma_x, \qquad \forall v \in H_\varepsilon. \tag{2.60}$$

The existence and uniqueness of $K_\varepsilon u$ follows directly from Proposition 2.1.

Proposition 2.2. *The operator $K_\varepsilon : H_\varepsilon \to H_\varepsilon$ is linear, compact and self-adjoint.*

Proof. The linearity and self-adjointness of K_ε follows directly from its definition (see (2.60)). In order to prove the compactness of K_ε notice that formula (2.60) defines a bounded linear operator $\tilde{K}_\varepsilon$ that maps $L^2(\Gamma_\varepsilon)$ in H_ε. Since K_ε is the composition of the trace operator $H_\varepsilon \mapsto L^2(\Gamma_\varepsilon)$ and $\tilde{K}_\varepsilon$, the desired compactness follows from the compactness of the mentioned trace operator (see, for instance, [8]). □

Assume that $\mu_\varepsilon \neq 0$ is an eigenvalue of the operator K_ε and u_ε is a corresponding eigenfunction. It means that

$$K_\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon$$

i.e.

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n_\varepsilon} = \frac{1}{\mu_\varepsilon}\rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

Thus, $\lambda_\varepsilon = \frac{1}{\mu_\varepsilon}$ is an eigenvalue of problem (1.5). Now, we recall the spectral properties of K_ε.

From general spectral theory, the spectrum of the operator K_ε is at most countable, it consists of two sequences (possibly finite or empty) of positive and negative eigenvalues, and of zero. The latter implies the essential spectrum of K_ε. Every non-zero eigenvalue has finite multiplicity. We denote by μ_j^ε, μ_{-j}^ε the positive and negative eigenvalues, for every $j \in \mathbb{N} \setminus \{0\}$, with the convention that the positive eigenvalues are enumerated in decreasing order, the negative ones in increasing order, and each eigenvalue is repeated a number of times equal to its multiplicity. Moreover, we denote by u_j^ε, and u_{-j}^ε a sequence of corresponding H_ε-normalized eigenfunctions. The following variational characterizations hold true

$$\mu_1^\varepsilon = \max_{\substack{u \in H_\varepsilon, \\ u \neq 0}} \frac{\displaystyle\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx}, \tag{2.61}$$

$$\mu_{-1}^\varepsilon = \min_{\substack{u \in H_\varepsilon, \\ u \neq 0}} \frac{\displaystyle\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx}. \tag{2.62}$$

For each $j \in \mathbb{N}$, $j \geq 2$ one has also

$$\mu_j^\varepsilon = \max_{\substack{(u,u_i^\varepsilon)_{H_\varepsilon}=0, \\ i=1,\dots,j-1}} \frac{\displaystyle\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx} = \min_{\dim V = j-1} \max_{u \in V^\perp} \frac{\displaystyle\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon, d\sigma_x}{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx}, \tag{2.63}$$

$$\mu^\varepsilon_{-j} = \min_{\substack{(u,u^\varepsilon_{-i})_{H_\varepsilon}=0,\\ i=1,\dots,j-1}} \frac{\displaystyle\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx} = \max_{\dim V = j-1} \; \min_{u \in V^\perp} \frac{\displaystyle\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\displaystyle\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx}, \tag{2.64}$$

where $V^\perp$ stands for the orthogonal complement of V in H_ε.

Remark 2.3. From (2.56) and the fact that $\rho \in L^\infty(\Gamma)$, it follows that there exists a positive constant k_0 such that

$$\varepsilon \mu^\varepsilon_j \le k_0 \left(\varepsilon^2 + \frac{1}{\beta^\varepsilon_j} \right) \qquad \text{for all } \varepsilon > 0, \; j \in \mathbb{N}, \tag{2.65}$$

where β^ε_j is the j-th eigenvalue of the Laplacian with homogeneous Neumann boundary conditions at the boundary of the perforation. More precisely, $\{\beta^\varepsilon_j\}_{j=1}^\infty$, $0 < \beta^\varepsilon_1 \le \beta^\varepsilon_2 \le \dots$, is the spectrum of the problem

$$\begin{cases} -\Delta v^\varepsilon_j = \beta^\varepsilon_j v^\varepsilon_j & \text{in } \Omega_\varepsilon, \\ \frac{\partial v^\varepsilon_j e}{\partial \nu_\varepsilon} = 0, & \text{on } \Gamma_\varepsilon, \\ v_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.66}$$

It is known (see, for instance, [13]) that for all $j \in \mathbb{N}$

$$\beta^\varepsilon_j \to \beta_j \qquad \text{as } \varepsilon \to 0, \tag{2.67}$$

with β_j eigenvalue of the corresponding homogenized problem

$$\begin{cases} -\mathrm{div}(a^{\mathrm{eff}} \nabla v_j) = \beta_j |\omega| v_j & \text{in } \Omega, \\ v_j = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.68}$$

and

$$\beta_j \to +\infty, \qquad \text{as } j \to +\infty. \tag{2.69}$$

Proposition 2.4. *If ρ satisfies condition (1.9), then for each $\varepsilon > 0$ the sets*

$$\{j \in \mathbb{N} \,:\, \mu^\varepsilon_j > 0\} \quad \textit{and} \quad \{j \in \mathbb{N} \,:\, \mu^\varepsilon_{-j} < 0\}$$

have infinitely many elements.

Proof.
Step 1. We first prove that

$$\mu^\varepsilon_{-1} < 0 < \mu^\varepsilon_1.$$

Letting

$$\rho^+_\varepsilon = \max\{\rho_\varepsilon, 0\} \quad \rho^-_\varepsilon = \min\{\rho_\varepsilon, 0\},$$

under our assumption (1.9) on ρ we have

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon \rho_\varepsilon^+ \, d\sigma_x > 0.$$

Denote by $\{u_\eta\}_{\eta>0}$ a family of functions $u_\eta \in H_\varepsilon$ such that $\|\sqrt{\rho_\varepsilon^+} - u_\eta\|_{L^2(\Gamma_\varepsilon)} \longrightarrow 0$, as $\eta \to 0$. Such functions u_η can be easily constructed by means of smoothing $\sqrt{\rho_\varepsilon^+}$ on Γ_ε. Since

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon u_\eta^2 \, d\sigma_x \longrightarrow \int_{\Gamma_\varepsilon} \rho_\varepsilon \rho_\varepsilon^+ \, d\sigma_x,$$

as $\eta \to 0$, then for all sufficiently small $\eta > 0$ it holds

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon u_\eta^2 \, d\sigma_x > 0. \tag{2.70}$$

It remains to combine the last inequality with (2.61) in order to conclude that $\mu_1^\varepsilon > 0$.

In a similar way, one can prove that $\mu_{-1}^\varepsilon < 0$.

Step 2. Our next goal is to show that for any $j \in \mathbb{N}$ the inequalities $\mu_{-j}^\varepsilon < 0$ and $\mu_j^\varepsilon > 0$ hold.

Assume that $\mu_1^\varepsilon > 0, \dots, \mu_{j-1}^\varepsilon > 0$, and let $u_1^\varepsilon, \dots, u_{j-1}^\varepsilon$ be the corresponding normalized eigenfunctions, $\langle u_i^\varepsilon, u_k^\varepsilon \rangle_{H_\varepsilon} = \delta_{ik}$ with $i, k = 1, 2, \dots, j-1$.

Consider a collection of sets $\{S_i^\varepsilon\}_{i=1}^j$ with $S_i^\varepsilon \subset \{x \in \Gamma_\varepsilon : \rho(x) > 0\}$, $\sigma_x(S_i^\varepsilon) > 0$, $S_i^\varepsilon \cap S_k^\varepsilon = \emptyset$, $i \neq k$, and denote χ_i^ε the characteristic functions of these sets.

Let $\chi_1^{\delta,\varepsilon}, \dots, \chi_j^{\delta,e}$ be elements of H_ε such that $\|\chi_i^\varepsilon - \chi_i^{\delta,\varepsilon}\|_{L^2(\Gamma_\varepsilon)} \le \delta$, $i = 1, \dots, j$. It is clear that for sufficiently small $\delta > 0$ the functions $\chi_1^{\delta,\varepsilon}, \dots, \chi_j^{\delta,\varepsilon}$ are linearly independent. Therefore, there is a non-trivial linear combination $\Xi = \beta_1^{\delta,\varepsilon}\chi_1^{\delta,\varepsilon} + \dots + \beta_j^{\delta,\varepsilon}\chi_j^{\delta,\varepsilon}$ such that $\langle \Xi, u_i^\varepsilon \rangle_{H_\varepsilon} = 0$, $i = 1, \dots, j-1$.

It is also clear that for sufficiently small $\delta > 0$ we have

$$\int_{\Gamma_\varepsilon} \Xi^2 \rho^+ \, d\sigma_x > 0.$$

Using Ξ as a test function in (2.63) we conclude that $\mu_j^\varepsilon > 0$. In the same way one can show that $\mu_{-j}^\varepsilon < 0$.

It remains to use the induction.

Proof of Proposition 1.2. All the statements of Proposition 1.2 except for (1.14) follow from the spectral properties of the operator K_ε, the fact that $\lambda_j^\varepsilon = (\mu_j^\varepsilon)^{-1}$, and from Proposition 2.4.

It remains to prove (1.14): we will do it for λ_1^ε, the proof for λ_{-1}^ε being analogous. We first show that each eigenfunction u related to λ_1^ε does not change sign in Ω_ε.

Assume the contrary. Then there is an eigenfunction u related to λ_1^ε such that $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$ are non-trivial functions. Clearly,

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon (u^+)^2 d\sigma_x > 0 \qquad \text{and} \qquad \int_{\Gamma_\varepsilon} \rho_\varepsilon (u^-)^2 d\sigma_x > 0.$$

Indeed, if $\int_{\Gamma_\varepsilon} \rho_\varepsilon (u^+)^2 d\sigma_x \leq 0$, then $\int_{\Gamma_\varepsilon} \rho_\varepsilon (u^-)^2 d\sigma_x \geq 1$. Since $\int_{\Omega_\varepsilon} |\nabla u^-|^2 dx < \int_{\Omega_\varepsilon} |\nabla u|^2 dx$, this contradicts the variational principle (2.63). Therefore, $\int_{\Gamma_\varepsilon} \rho_\varepsilon (u^+)^2 d\sigma_x > 0$. Similarly, $\int_{\Gamma_\varepsilon} \rho_\varepsilon (u^-)^2 d\sigma_x > 0$.

By (2.63) we have

$$\int_{\Omega_\varepsilon} |\nabla u^-|^2 dx \leq \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u^-)^2 d\sigma_x, \qquad \int_{\Omega_\varepsilon} |\nabla u^+|^2 dx \leq \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u^+)^2 d\sigma_x.$$

Summing up these inequalities and considering the relation

$$\int_{\Omega_\varepsilon} |\nabla u|^2 dx = \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u)^2 d\sigma_x$$

we conclude that

$$\int_{\Omega_\varepsilon} |\nabla u^+|^2 dx = \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u^+)^2 d\sigma_x.$$

Thus, u^+ is an eigenfunction related to λ_1^ε. Then u^+ is a non-negative solution of the equation $\Delta u^+ = 0$ in Ω_ε, and the fact that u^+ is equal to zero at interior points of Ω_ε contradicts the maximum principle.

If we assume that there are two linearly independent positive eigenfunctions $u, v \in H_\varepsilon$ related to λ_1^ε, then

$$\int_{\Omega_\varepsilon} (u - cv) dx = 0, \qquad \text{for } c = \Big(\int_{\Omega_\varepsilon} v dx\Big)^{-1} \int_{\Omega_\varepsilon} u dx.$$

Therefore, $u - cv$ is an eigenfunction that changes sign. This contradiction shows that λ_1^ε is simple. □

Proof of Proposition 1.3. Our goal is to show that for sufficiently small $\delta > 0$ the quadratic form

$$\mathcal{J}(u) = \int_\omega |\nabla u(y)|^2 dy + \delta \int_\Gamma \rho(y)(u(y))^2 d\sigma_y$$

is coercive that is

$$\mathcal{J}(u) \geq C(\delta) \|u\|^2_{H^1(\omega)} \qquad \text{for all } u \in H^1_\#(Y) \tag{2.71}$$

with $C(\delta) > 0$. The spectral problem for the operator associated with $\mathcal{J}$ reads

$$\begin{cases} \displaystyle\int_\omega \nabla p \cdot \nabla w \, dy + \delta \int_\Gamma \rho p w \, d\sigma_y = \tilde{\alpha} \int_\Gamma \rho p w \, d\sigma_y \quad \forall w \in H^1_\#(Y), \\ p \in H^1_\#(Y). \end{cases} \tag{2.72}$$

The spectrum of this problem coincides with the spectrum of problem (1.23) shifted by δ. Exploiting (2.71) by the same arguments as in the proof of Proposition 1.2 one can deduce that the spectrum of (2.72), and thus of (1.23), is discrete and consists of two infinite sequences of eigenvalues, one of these sequences tends to $-\infty$, another to $+\infty$.

Other statements of Proposition 1.3 can be justified in the same way as in the proof of Proposition 1.2.

To prove (2.71) we represent ρ as $\rho = \overline{\rho} + \hat{\rho}$ with $\overline{\rho} > 0$ defined in (1.10). For an arbitrary function $u \in H^1_\#(Y)$ denote $\overline{u} = (\sigma_y(\Gamma))^{-1} \int_\Gamma u(y)d\sigma_y$, $\hat{u} = u - \overline{u}$. Then

$$\int_\Gamma \rho u^2 d\sigma_y = \int_\Gamma \left(\overline{\rho} u^2 + \hat{\rho}\big(\overline{u} + \hat{u}\big)^2\right) d\sigma_y = \int_\Gamma \left(\overline{\rho} u^2 + 2\hat{\rho}\overline{u}\hat{u} + \hat{\rho}\hat{u}^2)\right) d\sigma_y$$

$$\geq \int_\Gamma \left(\overline{\rho} u^2 - C_\rho(|\overline{u}\hat{u}| + \hat{u}^2)\right) d\sigma_y$$

with $C_\rho = 2\|\rho\|_{L^\infty}$. Using the trace and Poincare inequalities we deduce that for any $\delta_1 > 0$

$$\int_\Gamma C_\rho(|\overline{u}\tilde{u}| + \tilde{u}^2) d\sigma_y \leq \int_\Gamma C_\rho \Big(\delta_1 \overline{u}^2 + \Big(\frac{1}{\delta_1} + 1\Big)\hat{u}^2\Big) d\sigma_y$$

$$\leq \int_\Gamma C_\rho \delta_1 u^2 d\sigma_y + C_1\Big(\frac{1}{\delta_1} + 1\Big) \int_\omega |\nabla u|^2 dy.$$

Combining the last two inequalities and choosing δ_1 in such a way that $C_\rho \delta_1 = \frac{1}{2}\overline{\rho}$ we obtain

$$\int_\Gamma \rho u^2 d\sigma_y \geq \int_\Gamma \frac{1}{2}\overline{\rho} u^2 d\sigma_y - C_1\Big(\frac{1}{\delta_1} + 1\Big) \int_\omega |\nabla u|^2 dy.$$

This yields

$$\mathcal{J}(u) \geq \int_\omega |\nabla u|^2 dy + \frac{\delta}{2} \int_\Gamma \overline{\rho} u^2 d\sigma_y - C_1 \delta \Big(\frac{1}{\delta_1} + 1\Big) \int_\omega |\nabla u|^2 dy.$$

Finally, taking δ such that $C_1\delta\big((1/\delta_1) + 1\big) \leq 1/2$, we get

$$\mathcal{J}(u) \geq \frac{1}{2} \int_\omega |\nabla u|^2 dy + \frac{\delta}{2} \int_\Gamma \overline{\rho} u^2 d\sigma_y \geq C(\delta)\|u\|^2_{H^1(\omega)}.$$

□

3 The case $\overline{\rho} > 0$

The aim of this section is to prove Theorem 1.4. We begin with an auxiliary statement.

Lemma 3.1. *Let $u_\varepsilon, u \in H_0^1(\Omega)$, $||u_\varepsilon||_{H_0^1} \leq c$, $u_\varepsilon \to u$ strongly in $L^2(\Omega)$ and $\overline{\rho} > 0$. Then*

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 \, d\sigma_x \to \overline{\rho}\sigma_y(\Gamma) \int_\Omega u^2 dx, \qquad \text{as } \varepsilon \to 0. \tag{3.73}$$

Moreover, for all $v \in H_0^1(\Omega)$

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v \, d\sigma_x \to \overline{\rho}\sigma_y(\Gamma) \int_\Omega uv dx, \qquad \text{as } \varepsilon \to 0. \tag{3.74}$$

Proof. Let us denote by $\hat{u}_\varepsilon$ the piece-wise constant function that takes the value of the average of u_ε in each ε-cell that is

$$\hat{u}_\varepsilon(x) = \hat{u_j^\varepsilon} \quad \text{if } x \in Y_\varepsilon^i,$$

$$\hat{u_j^\varepsilon} = \frac{1}{|\omega_j^\varepsilon|} \int_{\omega_j^\varepsilon} u_\varepsilon dx.$$

Note that, by our assumptions and Poincaré inequality, it follows that $\hat{u}_\varepsilon \to u$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$. In fact

$$\int_\Omega |u_\varepsilon - \hat{u}_\varepsilon|^2 dx = \sum_j \int_{Y_\varepsilon^i} |u_\varepsilon - \hat{u_j^\varepsilon}|^2 dx \leq c\varepsilon^2 \sum_j \int_{Y_\varepsilon^i} |\nabla u_\varepsilon|^2 dx \leq c\varepsilon^2. \tag{3.75}$$

In order to prove (3.73), we write

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 \, d\sigma_x = \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \hat{u}_\varepsilon^2 \, d\sigma_x + \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u_\varepsilon^2 - \hat{u}_\varepsilon^2) \, d\sigma_x. \tag{3.76}$$

The first term can be rearranged as follows

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \hat{u}_\varepsilon^2 \, d\sigma_x = \varepsilon \sum_j \int_{\Gamma_j^\varepsilon} \rho_\varepsilon (\hat{u_j^\varepsilon})^2 \delta\sigma_x =$$

$$= \varepsilon \sum_j (\hat{u_j^\varepsilon})^2 \varepsilon^{n-1} \int_\Gamma \rho(y) \, d\sigma_y = \overline{\rho}\sigma_y(\Gamma) \left(\int_\Omega (\hat{u}_\varepsilon)^2 dx + o(1) \right).$$

Hence, by (3.75), we can conclude that

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \hat{u}_\varepsilon^2 \, d\sigma_x \to \overline{\rho}\sigma_y(\Gamma) \int_\Omega u^2 \, dx$$

as $\varepsilon \to 0$. The second term in (3.76) is negligible, since

$$\left| \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u_\varepsilon^2 - \hat{u}_\varepsilon^2) \, d\sigma_x \right| \leq \varepsilon \int_{\Gamma_\varepsilon} |\rho_\varepsilon| \, |u_\varepsilon - \hat{u}_\varepsilon| \, |u_\varepsilon + \hat{u}_\varepsilon| \, d\sigma_x \leq$$

$$\leq \varepsilon \left(\int_{\Gamma_\varepsilon} |\rho_\varepsilon| \, |u_\varepsilon - \hat{u}_\varepsilon|^2 \, d\sigma_x \right)^{1/2} \left(\int_{\Gamma_\varepsilon} |\rho_\varepsilon| \, |u_\varepsilon + \hat{u}_\varepsilon|^2 \, d\sigma_x \right)^{1/2}.$$

The first term on the right hand side can be estimated by means of Poicaré inequality. We have

$$\varepsilon \left(\int_{\Gamma_\varepsilon} |\rho_\varepsilon| \, |u_\varepsilon - \hat{u}_\varepsilon|^2 \, d\sigma_x \right)^{1/2} \leq \varepsilon \|\rho\|_{L^\infty}^{1/2} \left(\varepsilon \int_{\Omega_\varepsilon} |\nabla u_e|^2 dx \right)^{1/2} \leq c\varepsilon^{3/2}.$$

The second term can be estimated by means of the trace inequality:

$$\left(\int_{\Gamma_\varepsilon} |\rho_\varepsilon| \, |u_\varepsilon + \hat{u}_\varepsilon|^2 \, d\sigma_x \right) \leq \left(2\|\rho\|_{L^\infty} \int_{\Gamma_\varepsilon} u_\varepsilon^2 \, d\sigma_x + 2\|\rho\|_{L^\infty} \sigma_y(\Gamma) \int_{\Omega_\varepsilon} u_\varepsilon^2 dx \right)^{1/2}$$

$$\leq c \left(\varepsilon^{-1} \int_{\Omega_\varepsilon} u_\varepsilon^2 dx + \varepsilon \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \right).$$

Hence, combining the last two inequalities, we finally have

$$\left| \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u_\varepsilon^2 - \hat{u}_\varepsilon^2) \, d\sigma_x \right| \leq c\varepsilon^{1/2},$$

and (3.73) follows.
To prove (3.74) it suffices to notice that

$$u_\varepsilon v = \frac{1}{2}(u_\varepsilon + v)^2 - \frac{1}{2}u_\varepsilon^2 - \frac{1}{2}v^2,$$

then (3.73) applies. □

Proof of Theorem 1.4.

We begin by obtaining the following estimates

$$c^- \leq \varepsilon^{-1} \lambda_j^\varepsilon \leq c_j \quad \text{for all } \varepsilon > 0, \qquad 0 < c^- < c_j < \infty. \tag{3.77}$$

Let us first justify the lower bound. Due to (2.61), (2.56) and the Poincaré inequality, one has

$$\frac{1}{\lambda_1^\varepsilon} = \mu_1^\varepsilon = \sup_{u \in H_\varepsilon} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx} \leq \|\rho\|_{L^\infty} \sup_{u \in H_\varepsilon} \frac{\int_{\Gamma_\varepsilon} u^2 \, d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx} \leq k\|\rho\|_{L^\infty} \varepsilon^{-1}$$

with a constant k that does not depend on ε. This yields the desired lower bound.

Let us now prove the upper bound in (3.77) for $j = 1$. From (2.61) we derive

$$\frac{1}{\lambda_1^\varepsilon} = \mu_1^\varepsilon = \sup_{u \in H_\varepsilon} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon \, d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 \, dx} \geq \frac{\int_{\Gamma_\varepsilon} \varphi^2 \rho_\varepsilon \, d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla \varphi|^2 \, dx}$$

for any $\varphi \in H_\varepsilon$. In particular, if we choose $\varphi \in \mathcal{C}_0^\infty(\Omega)$, $\varphi \neq 0$, then

$$\int_{\Omega_\varepsilon} |\nabla\varphi|^2\, dx \to |\omega| \int_\Omega |\nabla\varphi|^2\, dx > 0,$$

where $|\omega|$ denotes the Lebesgue measure of ω. By Lemma 3.1 with $u_\varepsilon = \varphi$, we get

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2\, d\sigma_x \to \overline{\rho}\sigma_y(\Gamma) \int_\Omega \varphi^2 dx > 0.$$

Therefore, there exist two constants $\varepsilon_0 > 0$ and $c > 0$ such that

$$\mu_1^\varepsilon \geq \frac{\int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2\, d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla\varphi|^2\, dx} \geq \frac{c}{\varepsilon} \quad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.78}$$

This implies the estimate

$$0 < \lambda_1^\varepsilon \leq \frac{\varepsilon}{c}.$$

It remains to denote $c_1 = 1/c$.

In order to justify the upper bound for $j > 1$, we consider a set of non-zero $C_0^\infty(\Omega)$ functions $\varphi_1, \ldots, \varphi_j$ with disjoint supports. Since these functions are orthogonal in H_ε, there is a non-trivial linear combination $\phi^\varepsilon = \gamma_1^\varepsilon \varphi_1 + \cdots + \gamma_j^\varepsilon \varphi_j$ such that

$$(\phi^\varepsilon, u_1^\varepsilon)_{H_\varepsilon} = \ldots = (\phi^\varepsilon, u_{j-1}^\varepsilon)_{H_\varepsilon} = 0.$$

Then, by (2.63),

$$\mu_j^\varepsilon \geq \frac{\int_{\Gamma_\varepsilon} (\phi^\varepsilon)^2 \rho_\varepsilon\, d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla\phi^\varepsilon|^2\, dx}$$

Using the fact that the functions φ_i have disjoint supports, it is easy to check that

$$\int_{\Gamma_\varepsilon} (\phi^\varepsilon)^2 \rho_\varepsilon\, d\sigma_x = \sum_{i=1}^{j} (\gamma_i^\varepsilon)^2 \int_{\Gamma_\varepsilon} (\varphi_i)^2 \rho_\varepsilon\, d\sigma_x, \quad \int_{\Omega_\varepsilon} |\nabla\phi^\varepsilon|^2\, dx = \sum_{i=1}^{j} (\gamma_i^\varepsilon)^2 \int_{\Omega_\varepsilon} |\nabla\varphi_i|^2\, dx.$$

By (3.78), there are $c > 0$ and $\varepsilon_0 > 0$ such that

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi_i^2\, d\sigma_x \geq \frac{c}{\varepsilon} \int_{\Omega_\varepsilon} |\nabla\varphi_i|^2\, dx \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \quad i = 1, \ldots, j.$$

Multiplying these inequalities by $(\gamma_i^\varepsilon)^2$ and summing up the resulting relations yields

$$\mu_j^\varepsilon \geq \frac{c}{\varepsilon}.$$

This implies the required upper bound in (3.77).

The proof of (1.40)-(1.52) relies on several technical statements.

Proposition 3.2. *Let $\{(\lambda^\varepsilon_{j(\varepsilon)}, u^\varepsilon_{j(\varepsilon)})\}$ be a family of normalized eigenpairs of problem (1.5) or, equivalently, (1.6), and assume that, perhaps for a subsequence,* $\dfrac{\lambda^\varepsilon_{j(\varepsilon)}}{\varepsilon} \to \overline{\lambda}$, *as* $\varepsilon \to 0$. *Then $\overline{\lambda}$ is an eigenvalue of the limit problem (1.16). If, in addition, $u^\varepsilon_{j(\varepsilon)}$ converges to $\overline{u}$ weakly in $H^1_0(\Omega)$ for the same subsequence of ε, then $\overline{u} \neq 0$, and $(\overline{\lambda}, \overline{u})$ is an eigenpair of (1.16).*

Proof. Since the family $\{u^\varepsilon_{j(\varepsilon)}\}$ is bounded in $H^1_0(\Omega)$, we may assume without loss of generality that $u^\varepsilon_{j(\varepsilon)} \rightharpoonup \overline{u}$ weakly in $H^1_0(\Omega)$. Then $u^\varepsilon_{j(\varepsilon)} \to \overline{u}$ in $L^2(\Omega)$, and by Lemma 3.1,

$$1 = \int_{\Omega_\varepsilon} |\nabla u_{j(\varepsilon)}|^2\, dx = \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u^2_{j(\varepsilon)}\, d\sigma_x \longrightarrow \overline{\lambda}\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u}^2 dx.$$

Therefore, $\int_\Omega \overline{u}^2 dx > 0$, and $\overline{u} \neq 0$.

Our goal is to show that

$$\int_\Omega a^{\text{eff}} \nabla \overline{u} \cdot \nabla \varphi dx = \overline{\lambda}\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u}\varphi dx \quad \forall \varphi \in H^1_0(\Omega) \tag{3.79}$$

with a^{eff} defined in (1.17). To this end, we consider the following auxiliary homogenization problem

$$\int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi dx = \frac{\overline{\lambda}\overline{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \overline{u}\varphi dx \quad \forall \varphi \in H_\varepsilon \tag{3.80}$$

stated in the perforated domain Ω_ε. It is well-known in homogenization theory (see, for instance, [?]) that, as $\varepsilon \to 0$, the (extended) solution v_ε tends weakly in $H^1_0(\Omega)$ and strongly in $L^2(\Omega)$ to a function $v \in H^1_0(\Omega)$ being a unique solution of the homogenized problem

$$\int_\Omega a^{\text{eff}} \nabla v \cdot \nabla \varphi dx = \overline{\lambda}\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u}\varphi dx \quad \forall \varphi \in H^1_0(\Omega). \tag{3.81}$$

By the lower-semicontinuity of the H^1-norm and the boundedness of the extension operators, we have

$$\int_\Omega |\nabla v - \nabla \overline{u}|^2 dx \le \liminf_{\varepsilon \to 0} \int_\Omega |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx \le c_0 \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx.$$

Using in equations (1.6) and (3.80) the test functions $\varphi = v_\varepsilon$ and $\varphi = u_\varepsilon$, yields

$$\int_{\Omega_\varepsilon} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 dx + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx - 2 \int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla u_\varepsilon dx =$$

$$= \frac{\overline{\lambda}\overline{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \overline{u} v_\varepsilon dx + \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u^2_\varepsilon\, d\sigma_x +$$

$$-\lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v_\varepsilon \, d\sigma_x - \frac{\overline{\lambda}\overline{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \overline{u} u_\varepsilon dx.$$

Since $u^\varepsilon \to \overline{u}$ and $v^\varepsilon \to v$ in $L^2(\Omega)$, the following limit relations hold, as $\varepsilon \to 0$:

$$\frac{\overline{\lambda}\overline{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \overline{u} v_\varepsilon dx \to \lambda\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u} v dx,$$

$$-\frac{\overline{\lambda}\overline{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \overline{u} u_\varepsilon dx \to -\overline{\lambda}\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u}^2 dx.$$

Furthermore, by Lemma 3.1,

$$\lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 \, d\sigma_x \to \overline{\lambda}\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u}^2 dx,$$

$$-\lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v_\varepsilon \, d\sigma_x \to -\lambda\overline{\rho}\sigma_y(\Gamma) \int_\Omega \overline{u} v dx.$$

Combining the above inequalities, we arrive at the estimate

$$\int_\Omega |\nabla v - \nabla \overline{u}|^2 dx \le c_0 \liminf_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx = 0,$$

which implies that $v = \overline{u}$. Thus, (3.79) holds true. □

The proof of the fact that any eigenpair of the limit operator is approached by the eigenpairs of ε-problems relies on the so-called Lemma on "eigenvalues and eigenvectors" (see [9]). For the reader's convenience we formulate it here.

Lemma 3.3. *Let $A : H \to H$ be a linear compact self-adjoint operator in a Hilbert space H. Suppose that there are a real number μ and a vector $u \in H$, such that $||u||_H = 1$ and*

$$||Au - \mu u||_H \le \alpha. \tag{3.82}$$

Then, there is an eigenvalue μ_i of the operator A such that

$$|\mu_i - \mu| \le \alpha. \tag{3.83}$$

Moreover, for any $d > \alpha$ there exists a vector $\overline{u}$ such that

$$||u - \overline{u}||_H \le 2\alpha d^{-1}, ||\overline{u}||_H = 1, \tag{3.84}$$

and $\overline{u}$ is a linear combination of eigenvectors of the operator A corresponding to eigenvalues of A in the closed segment $[\mu - d, \mu + d]$.

In the sequel we refer to μ and u in (3.82) as almost eigenvalue and eigenvector of A. We proceed with other technical statements.

Lemma 3.4. *Let $f \in L^\infty_{\mathrm{per}}(\omega)$ and $g \in L^\infty(\partial\omega)$ satisfy*

$$\int_\omega f(y)\,dy - \int_\Gamma g(y)\,d\sigma_y = 0. \tag{3.85}$$

Then there exists $c > 0$ such that

$$\left|\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv\,dx - \varepsilon\int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv\,d\sigma_x\right| \le c\varepsilon\|\nabla(uv)\|_{L^2(\Omega_\varepsilon)} \tag{3.86}$$

for all $u, v \in H_\varepsilon$ such that $\nabla(uv) \in L^2(\Omega_\varepsilon)$. Also, for any $u, v \in H_\varepsilon$ it holds

$$\left|\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv\,dx - \varepsilon\int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv\,d\sigma_x\right| \le c\varepsilon\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon}. \tag{3.87}$$

If for $f \in L^2_\#(\omega)$ and $g \in L^2(\partial\omega)$ condition (3.85) is fulfilled then there is $c > 0$ such that

$$\left|\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv\,dx - \varepsilon\int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv\,d\sigma_x\right| \le c\varepsilon\|\nabla(uv)\|_{L^2(\Omega_\varepsilon)} \tag{3.88}$$

for all $u \in W^{1,\infty}(\Omega)$ and $v \in H_\varepsilon$.

Proof. Let $\psi \in H^1(\omega)$ be a solution to problem

$$\begin{cases} \Delta\psi = f & \text{in } \omega, \\ \frac{\partial\psi}{\partial\nu} = g, & \text{on } \Gamma, \\ \psi & Y\text{-periodic.} \end{cases} \tag{3.89}$$

Then $\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$ is εY-periodic, it belongs to $H^1_{\mathrm{loc}}(\mathbb{R}^n)$ and satisfies

$$\nabla_x\psi_\varepsilon = \varepsilon^{-1}(\nabla_y\psi)\left(\frac{x}{\varepsilon}\right), \quad \Delta_x\psi_\varepsilon = \varepsilon^{-2}(\Delta_y\psi)\left(\frac{x}{\varepsilon}\right).$$

Writing down the integral identity

$$\int_{\Omega_\varepsilon} (\Delta_y\psi)\left(\frac{x}{\varepsilon}\right) uv\,dx = \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv\,dx,$$

after integration by parts one has

$$\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv\,dx - \varepsilon\int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv\,d\sigma_x = -\varepsilon\int_{\Omega_\varepsilon} (\nabla_y\psi)\left(\frac{x}{\varepsilon}\right)\nabla(uv)\,dx,$$

from which (3.86) and (3.88) follow immediately.
In order to justify (3.87) we consider the functions $\overline{u}_\varepsilon$ and $\overline{v}_\varepsilon$ introduced in (2.55). Notice that

$$\|\overline{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le \|u\|_{L^2(\Omega_\varepsilon)}, \qquad \varepsilon\|\overline{u}_\varepsilon\|^2_{L^2(\Gamma_\varepsilon)} \le c\|u\|^2_{L^2(\Omega_\varepsilon)}.$$

Denoting $f^\varepsilon = f(x/\varepsilon)$ and $g^\varepsilon = g(x/\varepsilon)$, and using (2.54) and Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left|\int_{\Omega_\varepsilon} f^\varepsilon uv\,dx - \varepsilon\int_{\Gamma_\varepsilon} g^\varepsilon uv\,d\sigma_x\right| \le \left|\int_{\Omega_\varepsilon} f^\varepsilon \overline{u}_\varepsilon v\,dx - \varepsilon\int_{\Gamma_\varepsilon} g^\varepsilon \overline{u}_\varepsilon v\,d\sigma_x\right| \\ + \left|\int_{\Omega_\varepsilon} f^\varepsilon (u-\overline{u}_\varepsilon) v\,dx - \varepsilon\int_{\Gamma_\varepsilon} g^\varepsilon (u-\overline{u}_\varepsilon) v\,d\sigma_x\right| \le \left|\int_{\Omega_\varepsilon} f^\varepsilon \overline{u}_\varepsilon v\,dx - \varepsilon\int_{\Gamma_\varepsilon} g^\varepsilon \overline{u}_\varepsilon v\,d\sigma_x\right| \\ + C\varepsilon\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon} \le \left|\int_{\Omega_\varepsilon} f^\varepsilon \overline{u}_\varepsilon \overline{v}_\varepsilon\,dx - \varepsilon\int_{\Gamma_\varepsilon} g^\varepsilon \overline{u}_\varepsilon \overline{v}_\varepsilon\,d\sigma_x\right| \\ + \left|\int_{\Omega_\varepsilon} f^\varepsilon \overline{u}_\varepsilon (v-\overline{v}_\varepsilon)\,dx - \varepsilon\int_{\Gamma_\varepsilon} g^\varepsilon \overline{u}_\varepsilon (v-\overline{v}_\varepsilon)\,d\sigma_x\right| + C\varepsilon\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon} \\ \le C\varepsilon\|u\|_{H_\varepsilon}\|v\|_{H_\varepsilon};\end{aligned}$$

here we have also used (3.85).

□

The proof of the next statement is quite similar to the proof of (3.87) and can be found, for instance, in [5, Ch.1, Lemma1.1].

Lemma 3.5. *Let $h \in L^\infty_\#(Y)$ be such that*

$$\int_Y h(y)\,dy = 0. \tag{3.90}$$

Then there exists $c > 0$ such that for all $u, v \in H^1_0(\Omega)$

$$\left|\int_\Omega h\left(\frac{x}{\varepsilon}\right) uv\,dx\right| \le c\varepsilon||u||_{H^1_0(\Omega)}||v||_{H^1_0(\Omega)}. \tag{3.91}$$

We will also need cut-off functions in the vicinity of the exterior boundary $\partial\Omega$. For $\gamma > 0$ denote $\Omega(\gamma) = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \gamma\}$.

Lemma 3.6. *Let $\psi \in L^2_{\mathrm{per}}(Y)$, and let $h > 0$ be a positive number. Then, there exists $c > 0$ such that*

$$\left|\int_{\Omega\setminus\overline{\Omega}(h\varepsilon)} \psi\left(\frac{x}{\varepsilon}\right) v\,dx\right| \le c\varepsilon^{3/2}\|\nabla v\|_{L^2(\Omega)}, \tag{3.92}$$

and, if $\Gamma_\varepsilon \cap (\Omega \setminus \overline{\Omega}(h\varepsilon)) \neq \emptyset$,

$$\left|\int_{\Gamma_\varepsilon\cap(\Omega\setminus\overline{\Omega}(h\varepsilon))} \psi\left(\frac{x}{\varepsilon}\right) v\,d\sigma_x\right| \le c\sqrt{\varepsilon}\|\nabla v\|_{L^2(\Omega)}. \tag{3.93}$$

for all $\varepsilon > 0$, and all $v \in H^1_0(\Omega)$.

Proof. By the Hardy inequality (see, for instance [10]), there exists a constant $c > 0$ such that

$$\|v\|_{L^2(\Omega\setminus\overline{\Omega}(\gamma))} \leq c\gamma\|\nabla v\|_{L^2(\Omega)} \tag{3.94}$$

for all constants $\gamma > 0$ and for all $v \in H_0^1(\Omega)$. Then, combining this estimate with the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left|\int_{\Omega\setminus\overline{\Omega}(h\varepsilon)} \psi\left(\tfrac{x}{\varepsilon}\right) v\,dx\right| &\leq \|\psi\left(\tfrac{x}{\varepsilon}\right)\|_{L^2(\Omega\setminus\overline{\Omega}(h\varepsilon))}\|v\|_{L^2(\Omega\setminus\overline{\Omega}(h\varepsilon))} \\ &\leq c\varepsilon\|\psi\left(\tfrac{x}{\varepsilon}\right)\|_{L^2(\Omega\setminus\overline{\Omega}(h\varepsilon))}\|\nabla v\|_{L^2(\Omega)}.\end{aligned} \tag{3.95}$$

Denote $J(h\varepsilon) = \{j \in \mathbb{Z}^n : \varepsilon(Y+j)\cap(\Omega\setminus\overline{\Omega}(h\varepsilon)) \neq \emptyset\}$, and let $\#J(h\varepsilon)$ be the cardinality (the number of elements) of $J(h\varepsilon)$. Clearly, $\#J(h\varepsilon) \leq C(h)\varepsilon^{1-n}$. Thus,

$$\|\psi\left(\frac{x}{\varepsilon}\right)\|^2_{L^2(\Omega\setminus\overline{\Omega}(h\varepsilon))} \leq \#J(h\varepsilon)\varepsilon^n\|\psi\|^2_{L^2(Y)} \leq C\varepsilon.$$

To obtain (3.92) it remains to combine this inequality with (3.95). The proof of (3.93) relies also on the fact that for all ε-cell Y_i^ε

$$\int_{\Gamma_\varepsilon\cap Y_i^\varepsilon} v^2\,d\sigma_x \leq c\left(\varepsilon^{-1}\int_{Y_i^\varepsilon} v^2dx + \varepsilon\int_{Y_i^\varepsilon}|\nabla v|^2dx\right).$$

Summing up these estimates over $i \in J(h\varepsilon)$ and using (3.94) yields

$$\int_{\Gamma_\varepsilon\cap\Omega\setminus\overline{\Omega}(h\varepsilon)} v^2\,d\sigma_x \leq c\left(\varepsilon^{-1}\int_{\Omega\setminus\overline{\Omega}(h\varepsilon)} v^2dx + \varepsilon\int_{\Omega\setminus\overline{\Omega}(h\varepsilon)}|\nabla v|^2dx\right).$$

Combining the preceding inequality with (3.94) and the fact that $|\Omega\setminus\overline{\Omega}(h\varepsilon))| \leq c\varepsilon$ we immediately obtain (3.93). □

Proposition 3.7. *Let λ_j, $j \in \mathbb{N}$, be an eigenvalue of problem (1.16). Then there exist a family $\{k(\varepsilon)\}_{\varepsilon>0}$, $k(\varepsilon) \in \mathbb{N}$, such that*

$$\frac{\lambda^\varepsilon_{k(\varepsilon)}}{\varepsilon} \to \lambda_j, \quad \text{as } \varepsilon \to 0, \tag{3.96}$$

where $\lambda^\varepsilon_{k(\varepsilon)}$ is an eigenvalue of problem (1.5).

Proof. Let Ψ^ε be a family of $C_0^\infty(\Omega)$ functions such that $\Psi^\varepsilon(x) = 1$ if the distance from x to $\partial\Omega$ is greater than 2ε, $0 \leq \Psi^\varepsilon \leq 1$, and $|\nabla\Psi^\varepsilon(x)| \leq 2/\varepsilon$ for all $x \in \Omega$.

Denote $\tilde{U}_j^\varepsilon(x) = u_j^0(x) + \varepsilon\Psi^\varepsilon(x)\chi(x/\varepsilon)\nabla u_j^0(x)$, and $U_j^\varepsilon(x) = u_j^0(x) + \varepsilon\chi(x/\varepsilon)\nabla u_j^0(x)$. It is straightforward to check that, under our assumptions on regularity of $\partial\Omega$, we have

$$\|\tilde{U}_j^\varepsilon - U_j^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{3/2}, \qquad \|\tilde{U}_j^\varepsilon - U_j^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}.$$

Let us compute the norm of $\tilde U_j^\varepsilon$ in H_ε. Denoting the unit $n\times n$ matrix by $\mathbf{I}$ we have

$$\int_{\Omega_\varepsilon}\nabla\tilde U_j^\varepsilon\cdot\nabla\tilde U_j^\varepsilon\,dx=\int_{\Omega_\varepsilon}|\tilde\nabla u_j^0+\Psi^\varepsilon\nabla_y\chi\Big(\frac{x}{\varepsilon}\Big)\nabla u_j^0+\varepsilon\Psi^\varepsilon\chi\Big(\frac{x}{\varepsilon}\Big)\nabla\nabla u_j^0+\varepsilon\nabla\Psi^\varepsilon\chi\Big(\frac{x}{\varepsilon}\Big)\nabla u_j^0|^2dx$$

$$=\int_{\Omega_\varepsilon}|(\mathbf{I}+\nabla_y\chi(x/\varepsilon))\nabla u_j^0|^2dx+O(\varepsilon);$$

here we have used the facts that $|\varepsilon\nabla\Psi^\varepsilon|\le C$, the support of $\nabla\Psi^\varepsilon$ is a subset of 2ε-neighbourhood of $\partial\Omega$, and u_j^0 is a $C^2(\overline\Omega)$ function. Recalling the formula for the effective matrix $a^{\rm eff}$, normalization condition (1.20), and using once again the C^2 smoothness of u_j^0 we conclude that

$$\int_{\Omega_\varepsilon}\nabla\tilde U_j^\varepsilon\cdot\nabla\tilde U_j^\varepsilon\,dx=\int_\Omega a^{\rm eff}\nabla u_j^0\cdot\nabla u_j^0\,dx+O(\varepsilon)=1+O(\varepsilon).\tag{3.97}$$

Similarly, one can show that

$$\Big|\int_{\Omega_\varepsilon}\nabla\tilde U_j^\varepsilon\cdot\nabla\varphi\,dx-\int_\Omega a^{\rm eff}\nabla u_j^0\cdot\nabla\varphi\,dx\Big|\le C\sqrt{\varepsilon}\|\varphi\|_{H_0^1(\Omega)}\tag{3.98}$$

for any $\varphi\in H_0^1(\Omega)$.

We proceed with estimating the norm $\|K^\varepsilon\tilde U^\varepsilon-(\varepsilon\lambda_j)^{-1}\tilde U^\varepsilon\|_{H_\varepsilon}$. After straightforward rearrangement we have

$$\big\|K^\varepsilon\tilde U^\varepsilon-\frac{1}{\varepsilon\lambda_j}\tilde U^\varepsilon\big\|_{H_\varepsilon}=\sup_{\varphi\in B^\varepsilon}\Big(K^\varepsilon\tilde U^\varepsilon-\frac{1}{\varepsilon\lambda_j}\tilde U^\varepsilon,\varphi\Big)_{H_\varepsilon}=\sup_{\varphi\in B^\varepsilon}\int_{\Omega_\varepsilon}\Big(\nabla(K^\varepsilon\tilde U^\varepsilon)\cdot\nabla\varphi-\frac{1}{\varepsilon\lambda_j}\nabla\tilde U^\varepsilon\cdot\nabla\varphi\Big)dx$$

$$=\sup_{\varphi\in B^\varepsilon}\Big(\int_{\Gamma_\varepsilon}\rho^\varepsilon\tilde U^\varepsilon\varphi\,d\sigma_x-\frac{1}{\varepsilon\lambda_j}\int_{\Omega_\varepsilon}\nabla\tilde U^\varepsilon\cdot\nabla\varphi\,dx\Big)$$

with $B^\varepsilon=\{\varphi\in H_0^1(\Omega)\,:\,\|\varphi\|_{H_\varepsilon}=1\}$. By Lemma 3.4,

$$\Big|\int_{\Gamma_\varepsilon}\rho^\varepsilon\tilde U^\varepsilon\varphi\,d\sigma_x-\int_{\Gamma_\varepsilon}\overline\rho\tilde U^\varepsilon\varphi\,d\sigma_x\Big|\le C\|\varphi\|_{H^1}.$$

Thus,

$$\big\|K^\varepsilon\tilde U^\varepsilon-\frac{1}{\varepsilon\lambda_j}\tilde U^\varepsilon\big\|_{H_\varepsilon}\le\sup_{\varphi\in B^\varepsilon}\Big(\int_{\Gamma_\varepsilon}\overline\rho\tilde U^\varepsilon\varphi\,d\sigma_x-\frac{1}{\varepsilon\lambda_j}\int_{\Omega_\varepsilon}\nabla\tilde U^\varepsilon\cdot\nabla\varphi\,dx\Big)+C.$$

It remains to use (3.98) and once again Lemma 3.4 to obtain

$$\big\|K^\varepsilon\tilde U^\varepsilon-\frac{1}{\varepsilon\lambda_j}\tilde U^\varepsilon\big\|_{H_\varepsilon}\le\sup_{\varphi\in B^\varepsilon}\Big(\frac{\overline\rho}{\varepsilon}\frac{\sigma_y(\Gamma)}{|\omega|}\int_{\Omega_\varepsilon}u_j^0\varphi\,dx-\frac{1}{\varepsilon\lambda_j}\int_\Omega a^{\rm eff}\nabla u_j^0\cdot\nabla\varphi\,dx\Big)+C+\frac{C_1}{\sqrt{\varepsilon}}$$

$$\leq \sup_{\varphi\in B^\varepsilon}\left(\frac{\overline{\rho}}{\varepsilon}\sigma_y(\Gamma)\int_\Omega u_j^0\varphi\,dx - \frac{1}{\varepsilon\lambda_j}\int_\Omega a^{\rm eff}\nabla u_j^0\cdot\nabla\varphi\,dx\right) + C + \frac{C_1}{\sqrt{\varepsilon}} = C + \frac{C_1}{\sqrt{\varepsilon}}.$$

This estimate combined with (3.97) and Lemma 3.3 yields

$$\left|\mu_k^\varepsilon - \frac{1}{\varepsilon\lambda_j}\right| \leq C + \frac{C_1}{\sqrt{\varepsilon}}$$

for some $k = k(\varepsilon)$ and for all sufficiently small ε. Therefore,

$$|\lambda^\varepsilon_{k(\varepsilon)} - \varepsilon\lambda_j| \leq C\varepsilon^{3/2}, \tag{3.99}$$

and (3.96) follows. □

We should also understand better the convergence of spectrum in the vicinity of multiple eigenvalues of the limit operator.

Lemma 3.8. *Let λ_j be an eigenvalue of (1.16) of multiplicity m, $\lambda_{j-1} < \lambda_j = \cdots = \lambda_{j+m-1} < \lambda_{j+m}$. Then there are at least m families $\{\lambda^\varepsilon_{k_1(\varepsilon)}\}$, …, $\{\lambda^\varepsilon_{k_m(\varepsilon)}\}$, $k_i(\varepsilon) \neq k_l(\varepsilon)$ if $i \neq l$, such that*

$$(\varepsilon)^{-1}\lambda^\varepsilon_{k_i(\varepsilon)} \longrightarrow \lambda_j, \qquad \text{as } \varepsilon \to 0.$$

Proof. For each $i \in \{0, 1, \ldots, m-1\}$ we construct $U^\varepsilon_{j+i} = u^0_{j+i} + \varepsilon\chi(x/\varepsilon)\phi^\varepsilon(x)\nabla u^0_{j+i}$ as in the proof of Proposition 3.7. Then

$$\left\|K^\varepsilon U^\varepsilon_{j+i} - \frac{1}{\varepsilon\lambda_i}U^\varepsilon_{j+i}\right\|_{H_\varepsilon} \leq \frac{C}{\sqrt{\varepsilon}}, \quad i = 0, \ldots, m-1. \tag{3.100}$$

In the same way as in the proof of Proposition 3.7 one can check that

$$|(U^\varepsilon_{j+i}, U^\varepsilon_{j+l})_{H_\varepsilon} - \delta_{il}| \leq C\sqrt{\varepsilon}, \quad 0 \leq i, l \leq m-1. \tag{3.101}$$

Denote by $\lambda^\varepsilon_{k_1(\varepsilon)}$,…, $\lambda^\varepsilon_{k_N(\varepsilon)}$ the eigenvalues that belong to the interval $\varepsilon(\lambda_j - \varepsilon^{1/4}, \lambda_j + \varepsilon^{1/4})$ with $N = N(\varepsilon)$. According to Lemma 3.3 there are linear combinations of the corresponding eigenfunctions $V_i^\varepsilon = \sum\limits_{s=1}^{N(\varepsilon)} \beta^\varepsilon_{is} u^\varepsilon_{k_s(\varepsilon)}$ such that $\|U^\varepsilon_{j+i} - V^\varepsilon_i\|_{H_\varepsilon} \leq C\varepsilon^{1/4}$. From (3.100) and (3.101) it follows that $N(\varepsilon) \geq m$ for all sufficiently small ε, this yields the desired statements. □

The opposite inequality is granted by

Lemma 3.9. *Assume that there are families $k_1(\varepsilon)$, …, $k_N(\varepsilon)$, $k_i \neq k_l$ if $i \neq l$, such that, for a subsequence,*

$$\frac{1}{\varepsilon}\lambda^\varepsilon_{k_i(\varepsilon)} \longrightarrow \lambda_j, \qquad \text{as } \varepsilon \to 0, \quad i = 1, \ldots, N.$$

Then the multiplicity of λ_j is at least N.

Proof. Consider the eigenpairs $(\lambda^\varepsilon_{k_i(\varepsilon)}, u^\varepsilon_{k_i(\varepsilon)})$ with the eigenfunctions satisfying (1.20). Then, for a subsequence,

$$u^\varepsilon_{k_i(\varepsilon)} \rightharpoonup v_i \quad \text{weakly in } H^1_0(\Omega), \quad i = 1, \dots, N.$$

It was shown in Proposition 3.2, that v_i are eigenfunctions of the homogenized problem with eigenvalue λ_j, and that

$$\delta_{il} = \lim_{\varepsilon \to 0} \left(u^\varepsilon_{k_i(\varepsilon)}, u^\varepsilon_{k_l(\varepsilon)}\right)_{H_\varepsilon} = \lambda_j \overline{\rho} \sigma_y(\Gamma)\, (v_i, v_l)_{L^2(\Omega)}.$$

Therefore, $\{v_i\}_{i=1}^N$ are nontrivial and orthogonal in $L^2(\Omega)$, and thus the multiplicity of λ_j is at least N. □

Now the statements *(i), (ii)* and *(iv)* of Theorem 1.4 are immediate consequence of Propositions 3.2 and 3.7, Lemmata 3.8 and 3.9 and estimate (3.99).

In order to justify the statement *(iii)* we consider an eigenvalue λ_j of (1.16) that has multiplicity m_j, $m_j \geq 1$, so that $\lambda_j = \dots = \lambda_{j+m_j-1}$. Choosing $d_j = \frac{1}{3}\min(1/\lambda_{j-1} - 1/\lambda_j, 1/\lambda_j - 1/\lambda_{j+m_j})$, with the help of item *(i)* we conclude that for all sufficiently small ε an eigenvalue $(\lambda^\varepsilon_i)^{-1}$ belongs to the interval $\varepsilon^{-1}\big((\lambda_j)^{-1} - d_j, (\lambda_j)^{-1} + d_j\big)$ if and only if $j \leq i \leq j + m_j - 1$. Using (3.100) and applying Lemma 3.3 with $d = \varepsilon^{-1} d_j$, we obtain that there exist β^ε_{il} such that

$$\|U^\varepsilon_{j+i} - \sum_{l=0}^{m_j-1} \beta^\varepsilon_{il} u^\varepsilon_{j+l}\|_{H_\varepsilon} \leq C\sqrt{\varepsilon}.$$

This estimate combined with (3.101) implies the desired statement *(iii)*. The proof can be found in []. We omit the details. This completes the proof of Theorem 1.4.

Remark 3.10. *If in the conditions of Theorem 1.4 we suppose that Ω and ω are just Lipschitz continuous domains then the statements on convergence of the spectrum remain valid, however, the estimates for the rate of convergence might fail to hold. More precisely, in the case of Lipschitz continuous $\partial\Omega$ and $\partial\omega$ the following statement holds:*

- *For any $j \in \mathbb{N}$ the limit relation (1.40) is valid.*
- *Let λ_j be an eigenvalue of (1.16) of multiplicity m_j with $m_j \geq 1$, that is $\lambda_j = \dots = \lambda_{j+m_j-1}$. Then there is a orthogonal matrix β^ε_{il}, $0 \leq i, l \leq m_j - 1$, such that*

$$\lim_{\varepsilon \to 0} \Big\| u^\varepsilon_{i+j} - \sum_{l=0}^{m_j-1} \beta^\varepsilon_{il} u_{l+j} \Big\|_{L^2(\Omega)} = 0. \tag{3.102}$$

The proof follows the same strategy as in the case of sooth domains Ω and ω. We leave the details to the reader.

Remark 3.11. *The convergence of eigenspaces related to multiple eigenvalues of the effective spectral problem can be expressed in terms of the so-called Mosco convergence, see [] for its definition. Namely, if λ_j is an eigenvalue of (1.16) of multiplicity m_j with $\lambda_j = \dots = \lambda_{j+m_j-1}$, $m_j \le 1$, then* $\mathrm{span}\{u_j^\varepsilon, \dots, u_{j+m_j-1}^\varepsilon\}$ *Mosco-converges to* $\mathrm{span}\{u_j, \dots, u_{j+m_j-1}\}$.

4 The case $\overline{\rho} = 0$

In this section we prove Theorem 1.7. Our first goal is to show that there exists $\kappa_1 > 0$ such that for all sufficiently small $\varepsilon > 0$ the estimate holds

$$\mu_1^\varepsilon \ge \kappa_1 \tag{4.103}$$

with μ_1^ε defined in (2.61). To justify this estimate we substitute in (2.61) a test function of the form $u_\varepsilon(x) = \varphi(x)(\overline{u} + \varepsilon\pi\left(\frac{x}{\varepsilon}\right))$ with $\overline{u} \in \mathbb{R}^+$, $\varphi \in C_0^\infty(\Omega)$, $\varphi \ne 0$, and $\pi \in C_\#^\infty(Y)$ such that

$$\gamma_\pi = \int_\Gamma \rho(y)\pi(y)\,d\sigma_y > 0.$$

It is easy to check that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \overline{u}^2|\omega|\int_\Omega |\nabla\varphi|^2\,dx + \int_\Omega \varphi^2 dx\int_Y |\nabla\pi(y)|^2 dy > 0.$$

The surface integral can be estimated as follows

$$\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2\,d\sigma_x = \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \rho_\varepsilon\varphi^2\overline{u}^2\,d\sigma_x$$

$$+\lim_{\varepsilon\to 0}\,2\varepsilon\int_{\Gamma_\varepsilon} \rho_\varepsilon\varphi^2\overline{u}\pi\left(\frac{x}{\varepsilon}\right)\,d\sigma_x + \lim_{\varepsilon\to 0}\,\varepsilon^2\int_{\Gamma_\varepsilon} \rho_\varepsilon\varphi^2\pi^2\left(\frac{x}{\varepsilon}\right)\,d\sigma_x =$$

$$= \overline{u}^2\int_\Gamma y\rho(y)\,d\sigma_y\,\int_\Omega \nabla(\varphi^2)dx + 2\gamma_\pi\overline{u}\int_\Omega(\varphi)^2\,dx = 2\gamma_\pi\overline{u}\int_\Omega(\varphi)^2\,dx > 0;$$

here we have also used Lemma 3.1. This implies (4.103) for all sufficiently small ε.

Similar lower bounds can be obtained for μ_j^ε with $j > 1$. However, since these bounds will follow from the asymptotics constructed later on in this section, we do not bother the reader with their proof here.

An upper bound for μ_1^ε easily follows from (3.87). Indeed, since $\overline{\rho} = 0$, for any $u \in H_0^1(\Omega)$ by (3.87) we have

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon u^2\,d\sigma_x \le c\|u\|^2_{H^1(\Omega_\varepsilon)} \le c\|\nabla u\|^2_{L^2(\Omega_\varepsilon)}.$$

In view of (2.61) this yields

$$\mu_1^\varepsilon \le \kappa_2. \tag{4.104}$$

Estimates (4.103) and (4.104) suggest that the asymptotic series for $\lambda_{\pm j}^\varepsilon$ and $u_{\pm j}^\varepsilon$ should be of the form

$$\lambda_{\pm j}^\varepsilon = \lambda + \varepsilon\lambda_1 + \dots, \qquad u_{\pm j}^\varepsilon = u(x) + \varepsilon u_1(x, x/\varepsilon) + \dots$$

with $u_1(x,y)$ being periodic in y. Substituting these series in (1.5) and collecting power-like terms in the resulting equation and boundary condition, we conclude that

$$u_1(x,y) = \varepsilon\chi(y)\nabla u(x) + \varepsilon\lambda_{\pm j}\theta(y)u(x),$$

where χ and θ are solutions of problems (1.18) and (1.44), respectively, and

$$-\mathrm{div}\big(a^{\mathrm{eff}}\nabla u\big) = \lambda^2 \Xi u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Remark 4.1. *Notice that the first order term in λ is not presented in the limit equation. Indeed, the formal derivation yields a first order term of the form*

$$\lambda Du(x) \cdot \Big(\int_\Gamma \theta(y)\nu(y)\, d\sigma_y - \int_\omega \nabla_y \theta(y) dy \Big),$$

this term is equal to zero.

Proof of Theorem 1.7.

The following statement can be proved in exactly the same way as Proposition 3.7 and Lemma 3.8 in the previous section. We leave its proof to the reader.

Lemma 4.2. *Let λ_j be an eigenvalue of (1.45) of multiplicity m, $m \ge 1$, $\lambda_{j-1} < \lambda_j = \dots = \lambda_{j+m-1} < \lambda_{j+m}$. Then there are at least m families $\{\lambda_{k_1(\varepsilon)}^\varepsilon\}$, …, $\{\lambda_{k_m(\varepsilon)}^\varepsilon\}$ such that $k_i(\varepsilon) \ne k_l(\varepsilon)$ if $i \ne l$, and*

$$\lambda_{k_i(\varepsilon)}^\varepsilon \longrightarrow \lambda_j, \qquad \text{as } \varepsilon \to 0.$$

The statements similar to those of Proposition 3.2 and Lemma 3.9 also remain valid.

Proposition 4.3. *Let $\{(\lambda_{j(\varepsilon)}^\varepsilon, u_{j(\varepsilon)}^\varepsilon)\}$ be a family of normalized eigenpairs of problem (1.5) or, equivalently, (1.6), and assume that, perhaps for a subsequence, $\lambda_{j(\varepsilon)}^\varepsilon \to \overline{\lambda}$, as $\varepsilon \to 0$. Then $\overline{\lambda}$ is an eigenvalue of the limit problem (1.45). If, in addition, $u_{j(\varepsilon)}^\varepsilon$ converges to $\overline{u}$ weakly in $H_0^1(\Omega)$ for the same subsequence of ε, then $\overline{u} \ne 0$, and $(\overline{\lambda}, \overline{u})$ is an eigenpair of (1.45).*

Lemma 4.4. *Assume that there are families $k_1(\varepsilon)$, …, $k_N(\varepsilon)$, $k_i \ne k_l$ if $i \ne l$, such that, for a subsequence,*

$$\lambda_{k_i(\varepsilon)}^\varepsilon \longrightarrow \lambda_j, \qquad \text{as } \varepsilon \to 0, \quad i = 1, \dots, N.$$

Then the multiplicity of λ_j is at least N.

Lemmata 4.2 and 4.4 and Proposition 4.3 imply the desired statements of Theorem 1.7. □

5 Proof of Theorem 1.5

The goal of this section is to prove Theorem 1.5. Thus it is assumed here that $\overline{\rho} > 0$. We begin by introducing a new unknown function and a new spectral parameter in (1.7). Namely, we set

$$u_\varepsilon(x) = p_{-1}\Big(\frac{x}{\varepsilon}\Big)v_\varepsilon(x), \qquad \lambda = \frac{\alpha_{-1}}{\varepsilon} + \tilde{\lambda}$$

with p_{-1} and α_{-1} defined in (1.22)–(1.26), respectively. Substituting these expressions in (1.7) we deduce after straightforward rearrangements that in terms of v_ε and $\tilde{\lambda}$ problem (1.7) reads

$$\begin{cases} -\mathrm{div}\big(\tilde{a}(x/\varepsilon)\nabla v_\varepsilon\big) = 0 & \text{in } \widetilde{\Omega}_\varepsilon, \\ \tilde{a}(x/\varepsilon)Dv_\varepsilon \cdot \nu_\varepsilon = \tilde{\lambda}(p_{-1}(x/\varepsilon))^2\rho(x/\varepsilon)v_\varepsilon & \text{on } \Gamma_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\widetilde{\Omega}_\varepsilon \setminus \Gamma_\varepsilon; \end{cases} \tag{5.105}$$

here we have denoted $\tilde{a}(y) = (p_{-1}(y))^2\,\mathbf{I}$. For the sake of brevity in this section we use the notation $\tilde{a}^\varepsilon(x) = \tilde{a}(x/\varepsilon)$ and $\tilde{\rho}^\varepsilon(x) = (p_{-1}(x/\varepsilon))^2\rho(x/\varepsilon)$. We remind that under our regularity assumptions, $p(\cdot)$ is a smooth positive function.

Since by construction (see (1.26))

$$\int_\Gamma (p_{-1}(y))^2\rho(y)\,d\sigma_y < 0,$$

Theorem 1.4 applies to the negative part of the spectrum of problem (5.105). Although, in contrast with (1.7), in (5.105) we do not deal with the Laplacian but with a more general divergence form elliptic operator with periodic coefficients, the results stated in Theorem 1.4 remain valid. Namely, using exactly the same arguments as in the proof of Theorem 1.4 one can show that the statements (*i*)–(*iv*) of Theorem 1.4 hold true for the negative part of the spectrum of problem (5.105).

In order to complete the proof of Theorem 1.5 it remains to prove that on the interval $\big(\frac{\alpha_{-1}}{\varepsilon}, 0\big)$ there are no eigenvalues of problem (1.7).

Proposition 5.1. *The interval* $\big(\frac{\alpha_{-1}}{\varepsilon}, 0\big)$ *belongs to the resolvent set of problem (1.7).*

Proof. The proof relies on Floquet-Bloch representation of u_ε and follows the line of the proof of Theorem 5 and Lemma 11 in [11].

□

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