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# Steklov problems in perforated domains with a coefficient of indefinite sign

Valeria Chiado Piat, Sergey A. Nazarov, Andrey L. Piatnitski

## Abstract

We consider homogenization of Steklov spectral problem for a divergence form elliptic operator in periodically perforated domain under the assumption that the spectral weight function changes sign. We show that the limit behaviour of the spectrum depends essentially on whether the average of the weight function over the boundary of holes is positive, or negative or equal to zero. In all these cases we construct the asymptotics of the eigenpairs.

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## Introduction

The paper studies Steklov spectral problem in a periodically perforated domain for the Laplace operator or for more general divergence form elliptic operator with periodic coefficients, under the assumptions that the Steklov condition is imposed on the perforation boundary and that the corresponding periodic weight function changes sign.

Previously, periodic homogenization of a bulk spectral problem with sign-changing density for an elliptic operator or an elliptic system was carried out in recent works [12], [11]. It was shown that the asymptotic behaviour of spectrum depends crucially on whether the mean value of the weight function is positive, or negative, or equal to zero.

The idea of studying Steklov and other spectral problems with sign-changing weight function arose during the conference "Differential Equations and Related Topics" in Moscow in 2007. It occurred after the talk "Homogenization in perforated domains with Fourier boundary conditions" that focused on homogenization of elliptic problems with Fourier boundary condition on the perforation surface under the assumption that the coefficient of the boundary operator changes sign. It turned out that the limit behaviour of solutions depends crucially on whether the average of this coefficient over the perforation surface is positive, or negative, or equal to zero (see [4] for further details).

Steklov spectral problem, although has many common features with the bulk problem, differs essentially from the bulk problem due to the facts that the surface volume of the

perforation tends to infinity, as the period vanishes, and that the perforation geometry is asymptotically singular.

The detailed formulation of the studied Steklov problem is

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \quad (0.1)$$

here  $\Omega$  is a smooth bounded domain,  $\Omega_\varepsilon$  is the corresponding perforated domain,  $\Gamma_\varepsilon$  is the surface of a smooth periodic perforation consisting of disjoint inclusions,  $\nu_\varepsilon$  is the exterior unit normal on  $\Gamma_\varepsilon$ , and  $\varepsilon$  is a small positive parameter. We assume that the function  $\rho$  is periodic and changes sign (see Section 1 for further details).

We also study a slightly more general problem of the form

$$\begin{cases} -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon^a} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \quad (0.2)$$

with a periodic symmetric matrix  $a(y)$  that satisfies the uniform ellipticity conditions,  $\nu_\varepsilon^a = a(x/\varepsilon)\nu_\varepsilon$ .

We first prove that the spectrum of the considered Steklov problem is discrete and, since the weight function  $\rho$  defines an indefinite metric on the perforation border (see [?]), the spectrum consists of two infinite sequences, one converges to  $+\infty$  and another to  $-\infty$ .

We show that the asymptotic behaviour of spectrum in (0.1), as  $\varepsilon \rightarrow 0$ , depends essentially on whether the average of  $\rho$  over the surface of the hole is greater than zero, or less than zero, or equal to zero.

If the average of  $\rho$  is positive (negative), then the positive (negative) part of the spectrum behaves in a regular way and admits homogenization like in the classical case when  $\rho > 0$ . In particular, for any  $k \in \mathbb{N}$ , the  $k$ -th positive eigenvalue is of order  $\varepsilon$ , and the corresponding eigenfunction has a bounded  $H^1$  norm. The convergence result in this case is presented in Theorem 1.4.

If  $\rho$  has zero average then both positive and negative eigenvalues have finite limits and the limit behaviour of the corresponding eigenpairs can be described in terms of the effective quadratic operator pencil. This operator pencil has a very simple structure and can be reduced to a standard eigenvalue problem for an elliptic operator in  $\Omega$ . Notice that in this case the  $k$ -th negative and positive eigenfunctions are bounded in  $H^1$ -norm. The asymptotic behaviour of the spectrum in the case of zero average  $\rho$  is described in Theorem 1.7.

Finally, if the average of  $\rho$  is positive then the negative part of the spectrum of (0.1) (or (0.2)) shows a singular behaviour. Namely, for any  $k \in \mathbb{N}$  the  $k$ -th negative eigenvalue is of order  $1/\varepsilon$  and the corresponding eigenfunctions are rapidly oscillating.

We show that studying the negative part of the spectrum can be reduced to studying the negative part of the spectrum of an auxiliary problem that exhibits more regular behaviour. This reduction is done by means of factorization with the first negative eigenfunction of the corresponding cell periodic spectral problem. Further details can be found in Theorem 1.5 and its proof.

## 1 Setting of the problem and main results

In this section we provide a detailed set up of the studied Steklov spectral problem, introduce necessary notation and auxiliary problems, and then formulate the main results of the paper.

Let  $\Omega$  be a smooth bounded domain in  $\mathbb{R}^n$ . We denote by  $Y = (0, 1)^n$  the unit cube of  $\mathbb{R}^n$ , and by  $\omega = Y \setminus B$  the perforated reference cell, for a given closed set  $B \subset Y$  with sufficiently smooth boundary  $\partial B = \Gamma$ . Setting

$$J_\varepsilon = \{z \in \mathbb{Z}^n : \varepsilon(Y + z) \subset \Omega\}, \quad (1.3)$$

we define  $B_\varepsilon = \bigcup_{z \in J_\varepsilon} \varepsilon(z + B)$ ,  $\Gamma_\varepsilon = \bigcup_{z \in J_\varepsilon} \varepsilon(z + \Gamma)$ . Then a perforated domain is introduced as

$$\Omega_\varepsilon = \Omega \setminus B_\varepsilon.$$

Notice that, according to (1.3),  $B_\varepsilon$  does not intersect the external boundary  $\partial\Omega$ .

**Remark 1.1.** *Another possibility is not to remove the perforation in the vicinity of  $\partial\Omega$ . Instead, we can keep this part of perforation and impose the homogeneous Dirichlet boundary condition on it. We denote*

$$\tilde{\Omega}_\varepsilon = \Omega \setminus \bigcup_{z \in \mathbb{Z}^n} \varepsilon(z + B). \quad (1.4)$$

Throughout this paper we assume that the exterior boundary  $\partial\Omega$  has the regularity  $C^{2,\alpha}$ . In many our statements this regularity can be replaced with just Lipschitz continuity of the boundary. However, in this case we obtain only convergence results without estimating the rate of convergence.

In what follows the symbol  $\Gamma_\#$  stands for the periodic extension of  $\Gamma$  in  $\mathbb{R}^n$ . Also, the lower index  $\#$  in the functional space notation indicates that the corresponding functions are periodic.

Given a function  $\rho \in L^\infty_\#(\Gamma)$ , we study the asymptotic behaviour of the eigenvalue problems

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.5)$$

as  $\varepsilon \rightarrow 0$ . The corresponding weak formulation reads

$$\begin{cases} u_\varepsilon \in H_\varepsilon, \\ \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v d\sigma_x \quad \forall v \in H_\varepsilon, \end{cases} \quad (1.6)$$

where

$$H_\varepsilon = \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}$$

is a Hilbert space equipped with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx,$$

and  $\sigma_x$  denotes the  $(n-1)$ -dimensional surface measure.

We also consider a similar problem in  $\widetilde{\Omega}_\varepsilon$

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \widetilde{\Omega}_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = \lambda_\varepsilon \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\widetilde{\Omega}_\varepsilon \setminus \Gamma_\varepsilon. \end{cases} \quad (1.7)$$

Every solution  $u_\varepsilon$  of problem (1.5) or (1.7) can be extended to the whole domain  $\Omega$  as a function  $\tilde{u}_\varepsilon \in H_0^1(\Omega)$ , with uniform estimates

$$\int_{\Omega} |\nabla \tilde{u}_\varepsilon|^2 dx \leq c_0 \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx, \quad \int_{\Omega} |\tilde{u}_\varepsilon|^2 dx \leq c_0 \int_{\Omega_\varepsilon} |u_\varepsilon|^2 dx$$

for all  $\varepsilon > 0$  and for some  $c_0 > 0$  that does not depend on  $\varepsilon$  (see, for instance, [1]). In the sequel, abusing slightly the notation, we still denote this extension by  $u_\varepsilon$ . Let us notice that, thanks to the above inequality, the usual Friedrichs inequality in  $H_\varepsilon$  holds true with a constant  $c_1$  independent of  $\varepsilon$ , i.e.,

$$\int_{\Omega_\varepsilon} u^2 dx \leq c_1 \int_{\Omega_\varepsilon} |\nabla u|^2 dx \quad \forall u \in H_\varepsilon. \quad (1.8)$$

Throughout this paper we assume that the coefficient  $\rho$  satisfies the condition of **indefinite sign**

$$\sigma_y(\{y \in \Gamma : \rho(y) > 0\}) > 0 \quad \text{and} \quad \sigma_y(\{y \in \Gamma : \rho(y) < 0\}) > 0. \quad (1.9)$$

The limit behaviour of problems (1.5) appears to be different if the mean value  $\bar{\rho}$  of  $\rho$ ,

$$\bar{\rho} = \frac{1}{\sigma_y(\Gamma)} \int_{\Gamma} \rho(y) d\sigma_y(y), \quad (1.10)$$

is zero or non zero.

We begin by considering problem (1.5) for a fixed positive  $\varepsilon$ .

**Proposition 1.2.** *For each  $\varepsilon > 0$  the spectrum of problem (1.5) consists of two sequences of eigenvalues*

$$0 < \lambda_1^\varepsilon \leq \lambda_2^\varepsilon \leq \dots \leq \lambda_j^\varepsilon \rightarrow +\infty \quad (1.11)$$

$$0 > \lambda_{-1}^\varepsilon \geq \lambda_{-2}^\varepsilon \geq \dots \geq \lambda_{-j}^\varepsilon \rightarrow -\infty \quad \text{as } j \rightarrow +\infty \quad (1.12)$$

Moreover, for all  $\varepsilon > 0$  there exists an orthonormal basis in  $H_\varepsilon$  of eigenfunctions  $u_j^\varepsilon \in H_\varepsilon$  which are solutions to problem (1.5) corresponding to  $\lambda_\varepsilon = \lambda_j^\varepsilon$ , and for all  $i, j \in \mathbb{Z} \setminus \{0\}$

$$\int_{\Omega_\varepsilon} \nabla u_i^\varepsilon \cdot \nabla u_j^\varepsilon dx = \delta_{ij}. \quad (1.13)$$

Furthermore,

$$\lambda_1^\varepsilon \quad \text{and} \quad \lambda_{-1}^\varepsilon \quad \text{are simple.} \quad (1.14)$$

The proof of this proposition will be given in Section 2.

Similar statement holds true for problem (1.7). Orthogonality condition in this case reads

$$\int_{\tilde{\Omega}_\varepsilon} \nabla u_i^\varepsilon \cdot \nabla u_j^\varepsilon dx = \delta_{ij}. \quad (1.15)$$

If  $\bar{\rho} > 0$ , the asymptotic analysis of the positive eigenvalues (1.11) as  $\varepsilon \rightarrow 0$  involves the spectral properties of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(a^{\text{eff}} \nabla u) = \lambda \bar{\rho} \sigma_x(\Gamma) u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.16)$$

where  $a^{\text{eff}}$  is a symmetric positive definite constant  $(n \times n)$ -matrix whose associated quadratic form is defined by

$$a^{\text{eff}} \xi \cdot \xi = \inf \left\{ \int_{\omega} |\xi + \nabla w(y)|^2 dy : w \in H_{\#}^1(Y) \right\} \quad \forall \xi \in \mathbb{R}^n, \quad (1.17)$$

and  $H_{\#}^1(Y)$  denotes the space of  $Y$ -periodic functions  $\varphi(y)$  with finite norm

$$\|\varphi\|_{H_{\#}^1(Y)} = \left( \int_Y (|\varphi|^2 + |\nabla \varphi|^2) dy \right)^{1/2}.$$

The function  $w_\xi$  that provides a minimum in (1.17) has the form  $w_\xi = \xi \cdot \chi$  with the vector-function  $\chi$  being a periodic solution to the classical cell problem

$$\begin{cases} \Delta \chi = 0 & \text{in } \omega, \\ \nabla \chi \cdot \nu = -\nu(y) & \text{on } \Gamma. \end{cases} \quad (1.18)$$

From the classical theory of elliptic operators it follows that the spectrum of (1.16) is discrete and consists of a sequence  $\{\lambda_j\}_{j \in \mathbb{N}}$  of positive eigenvalues,

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_j \rightarrow +\infty \quad \text{as } j \rightarrow +\infty, \quad (1.19)$$

and that the corresponding eigenfunctions  $\{u_j\}_{j \in \mathbb{N}} \in H_0^1(\Omega)$  form, under proper normalization, an orthonormal basis in  $L^2(\Omega)$ . For our purposes it is convenient to normalize  $u_j$ ,  $j \in \mathbb{N}$ , as follows

$$\int_{\Omega} a^{\text{eff}} \nabla u_i \cdot \nabla u_j dx = \delta_{ij}. \quad (1.20)$$

Then

$$\int_{\Omega} u_i u_j dx = (\bar{\rho} \lambda_i \sigma_x(\Gamma))^{-1} \delta_{ij}. \quad (1.21)$$

In what follows we use the notation

$$\Lambda = \{\lambda_j : j \in \mathbb{N}\}.$$

The asymptotic analysis of negative eigenvalues in (1.12) as  $\varepsilon \rightarrow 0$  requires two more auxiliary spectral problems. The first one is stated in the periodicity cell with periodic boundary conditions:

$$\begin{cases} -\Delta p = 0 & \text{in } \omega, \\ \frac{\partial p}{\partial \nu} = \alpha \rho p, & \text{on } \Gamma, \\ p & \text{is } Y\text{-periodic.} \end{cases} \quad (1.22)$$

The corresponding weak formulation reads

$$\begin{cases} \int_{\omega} \nabla p \cdot \nabla w dy = \alpha \int_{\Gamma} \rho p w d\sigma_y & \forall w \in H_{\#}^1(Y), \\ p \in H_{\#}^1(Y). \end{cases} \quad (1.23)$$

Here,  $\alpha$  is the spectral parameter. The statement below describes the behaviour of spectrum of problem (1.22). This statement will be proved in Section 2. The proof is more involved than that of Proposition 1.2 because the quadratic form related to (1.23) is not coercive.

**Proposition 1.3.** *Let  $\bar{\rho} > 0$ . Then the spectrum of problem (1.22) is discrete and consists of two sequences of eigenvalues*

$$0 = \alpha_1 < \alpha_2 \leq \dots \leq \alpha_j \rightarrow +\infty \quad \text{as } j \rightarrow +\infty, \quad (1.24)$$

$$0 > \alpha_{-1} > \alpha_{-2} \geq \dots \geq \alpha_{-j} \rightarrow -\infty \quad \text{as } j \rightarrow +\infty. \quad (1.25)$$

Moreover  $\alpha_1, \alpha_{-1}$  are simple and the associated eigenfunctions  $p_1, p_{-1} \in H_{\#}^1(Y) \cap L^\infty(\omega)$  can be normalized as follows

$$p_{\pm 1} > 0 \quad \text{in } \omega, \quad \int_{\Gamma} \rho (p_{\pm 1})^2 d\sigma_y = \pm 1. \quad (1.26)$$

Finally, if  $\partial\omega \in \mathcal{C}^{2,\alpha}$  and  $\rho \in C^\alpha(\partial B)$ , then  $p_\pm \in \mathcal{C}^2(\bar{\omega})$ , and  $0 < C_- \leq p_\pm \leq C^+$  for some constants  $C_-$  and  $C^+$ .

Now, we introduce the second spectral problem, which is stated in the whole set  $\Omega$  and involves a new weight function  $\rho^* = \rho^*(y)$  and its mean value  $\bar{\rho}^*$ :

$$\rho^* = \rho p_{-1}^2, \quad (1.27)$$

$$\bar{\rho}^* = \frac{1}{\sigma_y(\Gamma)} \int_{\Gamma} \rho^*(y) d\sigma_y. \quad (1.28)$$

Due to Proposition 1.3,

$$\int_{\omega} |\nabla p_{-1}|^2 dy = \alpha_{-1} \int_{\Gamma} p_{-1}^2 \rho d\sigma_y > 0,$$

and hence

$$\bar{\rho}^* = \frac{1}{\sigma_y(\Gamma)} \int_{\Gamma} \rho^* d\sigma_y = \int_{\Gamma} p_{-1}^2 \rho d\sigma_y < 0. \quad (1.29)$$

Define by  $\tilde{a}^{\text{eff}}$  the constant positive definite  $(n \times n)$ -matrix whose associated quadratic form is defined by

$$\tilde{a}^{\text{eff}} \xi \cdot \xi = \inf \left\{ \int_{\omega} |\xi + \nabla w(y)|^2 (p_{-1}(y))^2 dy : w \in H_{\#}^1(Y) \right\} \quad \forall \xi \in \mathbb{R}^n, \quad (1.30)$$

Notice that a minimum in (1.30) is attained at the function  $\tilde{w}_{\xi} = \xi \cdot \tilde{\chi}$  with the vector-function  $\tilde{\chi}$  being a periodic solution to the following cell problem

$$\begin{cases} \operatorname{div}((p_{-1})^2(\mathbf{I} + \nabla \tilde{\chi})) = 0 & \text{in } \omega, \\ \nabla \tilde{\chi} \cdot \nu = -\nu(y) & \text{on } \Gamma, \end{cases} \quad (1.31)$$

here  $\mathbf{I}$  stands for the unit matrix.

We now introduce the effective spectral problem:

$$\begin{cases} -\operatorname{div}(\tilde{a}^{\text{eff}} \nabla v) = \varkappa \bar{\rho}^* \sigma_y(\Gamma) v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.32)$$

where  $\varkappa$  is a spectral parameter.

Problem (1.32) is classical. Since  $\bar{\rho}^* < 0$ , the spectrum of this problem consists of a sequence

$$0 > \varkappa_{-1} > \varkappa_{-2} \geq \varkappa_{-3} \geq \cdots \geq \varkappa_{-j} \longrightarrow -\infty, \quad \text{as } j \rightarrow \infty. \quad (1.33)$$

The corresponding eigenfunctions  $\{v_{-j}\}_{j \in \mathbb{N}}$ , under proper normalization, form an orthonormal basis in  $L^2(\Omega)$ . However, we normalize them in a different way. Namely, we assume that

$$\int_{\Omega} \tilde{a}^{\text{eff}} \nabla v_{-i} \cdot \nabla v_{-j} dx = \delta_{ij}. \quad (1.34)$$



The following results concern the case of  $\bar{\rho} > 0$ . It should be noted that, in this case, the positive and the negative parts of the spectrum show totally different behaviour. We first deal with the positive part of the spectrum.

**Theorem 1.4.** *Let  $\bar{\rho} > 0$ , and let  $(\lambda_j^\varepsilon, u_j^\varepsilon)$  be the  $j$ -th eigenpair of problem (1.5), (1.13), or problem (1.7) with  $j > 0$ . Then*

(i) *For all  $j \in \mathbb{N}$*

$$\frac{\lambda_j^\varepsilon}{\varepsilon} \rightarrow \lambda_j \quad \text{as } \varepsilon \rightarrow 0, \quad (1.35)$$

*where  $\lambda_j$  is the  $j$ -th eigenvalue of problem (1.16).*

(ii) *Under the additional assumption that  $\Omega$  is a  $\mathcal{C}^{2,\delta}$  domain with some  $\delta > 0$  the rate of convergence in (1.40) can be estimated as follows: for every  $j \in \mathbb{N}$  there exist constants  $\varepsilon_j, C_j > 0$  such that*

$$\left| \frac{\lambda_j^\varepsilon}{\varepsilon} - \lambda_j \right| \leq C_j \sqrt{\varepsilon} \quad \text{for all } \varepsilon \in (0, \varepsilon_j). \quad (1.36)$$

(iii) *If, for  $j \in \mathbb{N}$ ,  $\lambda_j$  is an eigenvalue of problem (1.16) of multiplicity  $m_j$ ,  $\lambda_{j-1} < \lambda_j = \lambda_{j+1} = \dots = \lambda_{j+m_j-1} < \lambda_{j+m_j}$ , then there exist orthogonal  $m_j \times m_j$  matrices  $\mathcal{U}^\varepsilon$  and constants  $\varepsilon_j > 0$  and  $C_j > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_j]$ ,*

$$\left\| u_{j+l-1}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon u_{j+k-1} \right\|_{L^2(\Omega)} \leq C_j \sqrt{\varepsilon}, \quad l = 1, \dots, m_j, \quad (1.37)$$

$$\left\| u_{j+l-1}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon U_{j+k-1}^\varepsilon \right\|_{H^\varepsilon(\Omega)} \leq C_j \sqrt{\varepsilon}, \quad l = 1, \dots, m_j \quad (1.38)$$

*with  $U_j^\varepsilon(x) = u_j(x) + \varepsilon \chi(x/\varepsilon) \nabla u_j(x)$ , here  $\chi$  is a solution of problem (1.18).*

(iv) *The function  $\{U_j^\varepsilon\}$  are almost orthogonal and normalized in  $H_\varepsilon$  that is*

$$\left| \langle U_k^\varepsilon, U_l^\varepsilon \rangle_{H_\varepsilon} - \delta_{k,l} \right| \leq C \sqrt{\varepsilon}. \quad (1.39)$$

The same results hold true for problem (1.7)

We turn to the negative part of the spectrum. Here, in addition to the above assumptions, we suppose that the boundary of  $B$  has regularity  $C^{2,\alpha}$  and that  $\rho$  is Hölder continuous,  $\rho \in C^\alpha(\partial B)$ . Here we only consider problem (1.7).

**Theorem 1.5.** *Let  $\bar{\rho} > 0$ , and let  $(\lambda_{-j}^\varepsilon, u_{-j}^\varepsilon)$  be the  $j$ -th negative eigenpair of problem (1.7), (1.15). Then*

(i) For all  $j \in \mathbb{N}$

$$\frac{1}{\varepsilon} \left( \lambda_{-j}^\varepsilon - \frac{\alpha_{-1}}{\varepsilon} \right) \rightarrow \kappa_{-j} \quad \text{as } \varepsilon \rightarrow 0, \quad (1.40)$$

where  $\alpha_{-1}$  is defined in (1.25), and  $\kappa_{-j}$  is the  $j$ -th (negative) eigenvalue of problem (1.32).

(ii) If  $\Omega$  is a  $\mathcal{C}^{2,\delta}$  domain for some  $\delta > 0$  then for every  $j \in \mathbb{N}$  there exist constants  $\varepsilon_j, C_j > 0$  such that

$$\left| \frac{1}{\varepsilon} \left( \lambda_{-j}^\varepsilon - \frac{\alpha_{-1}}{\varepsilon} \right) - \kappa_{-j} \right| \leq C_j \sqrt{\varepsilon} \quad \text{for all } \varepsilon \in (0, \varepsilon_j). \quad (1.41)$$

(iii) If, for  $j \in \mathbb{N}$ ,  $\kappa_{-j}$  is an eigenvalue of problem (1.32) of multiplicity  $m_{-j}$ ,  $\kappa_{-j} = \kappa_{-(j+1)} = \dots = \kappa_{-(j+m_j-1)}$ , then there exist orthogonal  $m_{-j} \times m_{-j}$  matrices  $\mathcal{U}^\varepsilon$  and constants  $\varepsilon_{-j} > 0$  and  $C_{-j} > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_{-j}]$ ,

$$\left\| \frac{u_{-(j+l-1)}^\varepsilon}{\|u_{-(j+l-1)}^\varepsilon\|_{L^2(\Omega)}} - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon v_{-(j+k-1)}^\varepsilon \right\|_{L^2(\Omega)} \leq C_{-j} \sqrt{\varepsilon}, \quad l = 1, \dots, m_j, \quad (1.42)$$

with  $v_{-j}^\varepsilon(x) = (\|v_{-j}\|_{L^2(\Omega)})^{-1} v_{-j}(x) \hat{p}_{-1}(x/\varepsilon)$ ; here  $\hat{p}_{-1}$  is the eigenfunction of problem (1.22) that corresponds to  $\alpha_{-1}$  and is normalized by

$$\int_{\omega} (\hat{p}_{-1}(y))^2 dy = 1.$$

(iv) The functions  $\{U_{-j}^\varepsilon\}$ ,  $U_{-j}^\varepsilon(x) = v_{-j}(x) + \varepsilon \tilde{\chi}(x/\varepsilon) \nabla v_{-j}(x)$ , are almost orthogonal and normalized in  $H_\varepsilon$  that is

$$\left| \langle U_{-k}^\varepsilon, U_{-l}^\varepsilon \rangle_{H_\varepsilon} - \delta_{k,l} \right| \leq C \sqrt{\varepsilon}. \quad (1.43)$$

**Remark 1.6.** In contrast with problem (1.7) we cannot assure that the interval  $(\alpha_{-1}/\varepsilon, 0)$  belongs to the resolvent set of spectral problem (1.5). If there are eigenvalues of problem (1.5) on this interval, then the corresponding eigenfunctions concentrate in the vicinity of  $\partial\Omega$  that is they are of boundary layer type.

In order to write down the limit problem in the case  $\bar{\rho} = 0$  we introduce one more cell problem:

$$\begin{cases} -\Delta\theta = 0 & \text{in } \omega, \\ \frac{\partial\theta}{\partial\nu} = \rho, & \text{on } \Gamma, \\ \theta & \text{is } Y\text{-periodic,} \end{cases} \quad (1.44)$$

Since  $\bar{\rho} = 0$ , this problem is solvable, its solution is unique up to an additive constant. Denote

$$\Xi = \int_{\Gamma} \rho(y) \theta(y) d\sigma_y = \int_{\omega} \nabla \theta(y) \cdot \nabla \theta(y) dy > 0,$$

and consider the following operator pencil

$$\begin{cases} -\operatorname{div}(a^{\text{eff}} \nabla u) = \lambda^2 \Xi u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.45)$$

and a spectral problem

$$\begin{cases} -\operatorname{div}(a^{\text{eff}} \nabla u) = \nu \Xi u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.46)$$

with  $a^{\text{eff}}$  defined in (1.17).

Since (1.46) has a discrete spectrum  $0 < \nu_1 < \nu_2 \leq \nu_3 \leq \dots \leq \nu_j \rightarrow \infty$ , and all the eigenvalues  $\nu_j$  are positive, the spectrum of (1.45) is discrete, real and consists of two series

$$\lambda_j^+ = \sqrt{\nu_j}, \quad \lambda_j^- = -\sqrt{\nu_j}, \quad j = 1, 2, \dots \quad (1.47)$$

Here, for the corresponding eigenfunctions, we impose the following normalization conditions

$$\int_{\Omega} a^{\text{eff}} \nabla u_i \cdot \nabla u_j dx + \Xi \sqrt{\nu_i \nu_j} \int_{\Omega} u_i u_j dx = \delta_{ij}. \quad (1.48)$$

**Theorem 1.7.** *Let  $\bar{\rho} = 0$ , and let  $(\lambda_j^\varepsilon, u_j^\varepsilon)$ ,  $j \in \mathbb{Z} \setminus \{0\}$ , be the  $j$ -th eigenpair of problem (??), (1.13). Then*

(i) *For all  $j \in \mathbb{N}$*

$$\lambda_{\pm j}^\varepsilon \rightarrow \lambda_j^\pm, \quad \text{as } \varepsilon \rightarrow 0, \quad (1.49)$$

*where  $\lambda_j^\pm$  are defined in (1.47).*

(ii) *Under the additional assumption that  $\Omega$  is a  $\mathcal{C}^{2,\delta}$  domain with some  $\delta > 0$ , for every  $j \in \mathbb{N}$  there exist constants  $\varepsilon_j, C_j > 0$  such that*

$$|\lambda_{\pm j}^\varepsilon - \lambda_j^\pm| \leq C_j \sqrt{\varepsilon} \quad \text{for all } \varepsilon \in (0, \varepsilon_j). \quad (1.50)$$

(iii) *If, for  $j \in \mathbb{N}$ ,  $\nu_j$  is an eigenvalue of problem (1.46) of multiplicity  $m_j$ ,  $\nu_{j-1} < \nu_j = \nu_{j+1} = \dots = \nu_{j+m_j-1} < \nu_{j+m_j}$ , then there exist orthogonal  $m_j \times m_j$  matrices  $\mathcal{U}^\varepsilon$  and constants  $\varepsilon_j > 0$  and  $C_j > 0$  such that, for all  $\varepsilon \in (0, \varepsilon_j]$ ,*

$$\left\| u_{\pm(j+l-1)}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon u_{j+k-1} \right\|_{L^2(\Omega)} \leq C_j \sqrt{\varepsilon}, \quad l = 1, \dots, m_j, \quad (1.51)$$

$$\left\| u_{\pm(j+l-1)}^\varepsilon - \sum_{k=1}^{m_j} \mathcal{U}_{lk}^\varepsilon U_{\pm(j+k-1)}^\varepsilon \right\|_{H^\varepsilon(\Omega)} \leq C_j \sqrt{\varepsilon}, \quad l = 1, \dots, m_j \quad (1.52)$$

with  $U_{\pm j}^\varepsilon(x) = u_j(x) + \varepsilon \chi(x/\varepsilon) \nabla u_j(x) + \lambda_j^\pm \theta(x/\varepsilon) u_j(x)$ , here  $\chi$  and  $\theta$  are solutions of problems (1.18) and (1.44), respectively.

(iv) The function  $\{U_j^\varepsilon\}$  are almost orthogonal and normalized in  $H_\varepsilon$  that is

$$\left| \langle U_k^\varepsilon, U_l^\varepsilon \rangle_{H_\varepsilon} - \delta_{k,l} \right| \leq C \sqrt{\varepsilon}, \quad k, j \in \mathbb{Z} \setminus \{0\}. \quad (1.53)$$

## 2 Preliminary statements

We begin this section by recalling some inequalities valid in  $H_\varepsilon$ . In what follows we denote

$$\omega_\varepsilon^i = \varepsilon(\omega + i), \quad \Gamma_\varepsilon^i = \varepsilon(\Gamma + i), \quad i \in \mathbb{Z}^n.$$

*Poincaré-Wirtinger inequality.* Under our assumptions on  $\Omega_\varepsilon$  and  $\Gamma_\varepsilon$ , there exist a positive constant  $k$  such that for each  $u \in H_\varepsilon$  the following inequality holds:

$$\int_{\Gamma_\varepsilon} |u - \bar{u}_\varepsilon|^2 d\sigma_x \leq k \varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2 dx, \quad (2.54)$$

where we denote by  $\bar{u}_\varepsilon(\cdot)$  the piece-wise constant function obtained by taking the mean value of  $u$  over each perforated cell  $\omega_\varepsilon^i$ , i.e.,

$$\bar{u}_\varepsilon(x) = \frac{1}{|\omega_\varepsilon^i|} \int_{\omega_\varepsilon^i} u(y) dy, \quad \text{if } x \in \omega_\varepsilon^i; \quad (2.55)$$

here  $|\omega_\varepsilon^i|$  stands for the Lebesgue measure of  $\omega_\varepsilon^i$ . The above inequality remains valid if  $\bar{u}_\varepsilon$  is replaced with the piece-wise constant function being equal in each  $\omega_\varepsilon^i$  to the surface average of  $u$  over  $\Gamma_\varepsilon^i$ .

*Trace inequality*

$$\int_{\Gamma_\varepsilon} |u|^2 d\sigma_x \leq k_t \left( \varepsilon^{-1} \int_{\Omega_\varepsilon} |u|^2 dx + \varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2 dx \right), \quad (2.56)$$

Both inequalities can be easily obtained from the standard Poincaré-Wirtinger and trace inequalities, (see [2], [14]) by means of scaling arguments.

Given  $g \in L^2(\Gamma_\varepsilon)$ , consider the following boundary value problem with non-homogeneous Neumann boundary conditions on  $\Gamma_\varepsilon$

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n_\varepsilon} = g, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.57)$$

The corresponding weak formulation reads

$$\begin{cases} u_\varepsilon \in H_\varepsilon, \\ \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla v \, dx = \int_{\Gamma_\varepsilon} g v \, d\sigma_x \quad \forall v \in H_\varepsilon, \end{cases} \quad (2.58)$$

where

$$H_\varepsilon = \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\}$$

is a Hilbert space equipped with the scalar product

$$(u, v)_{H_\varepsilon} = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx.$$

**Proposition 2.1.** *For every  $g \in L^2(\Gamma_\varepsilon)$  there exists a unique solution  $u_\varepsilon \in H_\varepsilon$  to problem (2.57). Moreover  $u_\varepsilon$  satisfies the following a-priori estimate*

$$\|u_\varepsilon\|_{H_\varepsilon} \leq c\varepsilon^{-1/2} \|g\|_{L^2(\Gamma_\varepsilon)}, \quad (2.59)$$

where the constant  $c > 0$  is independent of  $\varepsilon$ .

*Proof.* The existence and uniqueness of  $u_\varepsilon$  is a straightforward consequence of the Reisz representation theorem for the problem

$$a(u, v) = F(v) \quad \forall v \in H,$$

where

$$a(u, v) = \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx, \quad F(v) = \int_{\Gamma_\varepsilon} g v \, d\sigma_x, \quad H = H_\varepsilon.$$

Moreover, replacing  $v = u_\varepsilon$  in the weak formulation (2.58), and using Friedrichs and trace inequalities (1.8), (2.56), we obtain that

$$\begin{aligned} \|u_\varepsilon\|_{H_\varepsilon}^2 &= \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 \, dx = \int_{\Gamma_\varepsilon} g u_\varepsilon \, d\sigma_x \leq \|g\|_{L^2(\Gamma_\varepsilon)} \|u_\varepsilon\|_{L^2(\Gamma_\varepsilon)} \leq \\ &\leq \|g\|_{L^2(\Gamma_\varepsilon)} \left( k_t \left( \varepsilon^{-1} \int_{\Omega_\varepsilon} |u|^2 \, dx + \varepsilon \int_{\Omega_\varepsilon} |\nabla u|^2 \, dx \right) \right)^{1/2} \leq \\ &\leq c\varepsilon^{-1/2} \|g\|_{L^2(\Gamma_\varepsilon)} \|u_\varepsilon\|_{H_\varepsilon}. \end{aligned}$$

Dividing by  $\|u_\varepsilon\|_{H_\varepsilon}$  we obtain the desired inequality (2.59).  $\square$

We introduce the operator  $K_\varepsilon : H_\varepsilon \rightarrow H_\varepsilon$  in the following way. For every  $u \in H_\varepsilon$ , we define  $K_\varepsilon u$  as the unique solution to the problem

$$\int_{\Omega_\varepsilon} \nabla(K_\varepsilon u) \cdot \nabla v \, dx = \int_{\Gamma_\varepsilon} \rho_\varepsilon u v \, d\sigma_x, \quad \forall v \in H_\varepsilon. \quad (2.60)$$

The existence and uniqueness of  $K_\varepsilon u$  follows directly from Proposition 2.1.

**Proposition 2.2.** *The operator  $K_\varepsilon : H_\varepsilon \rightarrow H_\varepsilon$  is linear, compact and self-adjoint.*

*Proof.* The linearity and self-adjointness of  $K_\varepsilon$  follows directly from its definition (see (2.60)). In order to prove the compactness of  $K_\varepsilon$  notice that formula (2.60) defines a bounded linear operator  $\tilde{K}_\varepsilon$  that maps  $L^2(\Gamma_\varepsilon)$  in  $H_\varepsilon$ . Since  $K_\varepsilon$  is the composition of the trace operator  $H_\varepsilon \mapsto L^2(\Gamma_\varepsilon)$  and  $\tilde{K}_\varepsilon$ , the desired compactness follows from the compactness of the mentioned trace operator (see, for instance, [8]).  $\square$

Assume that  $\mu_\varepsilon \neq 0$  is an eigenvalue of the operator  $K_\varepsilon$  and  $u_\varepsilon$  is a corresponding eigenfunction. It means that

$$K_\varepsilon u_\varepsilon = \mu_\varepsilon u_\varepsilon$$

i.e.

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial n_\varepsilon} = \frac{1}{\mu_\varepsilon} \rho_\varepsilon u_\varepsilon, & \text{on } \Gamma_\varepsilon, \\ u_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases}$$

Thus,  $\lambda_\varepsilon = \frac{1}{\mu_\varepsilon}$  is an eigenvalue of problem (1.5). Now, we recall the spectral properties of  $K_\varepsilon$ .

From general spectral theory, the spectrum of the operator  $K_\varepsilon$  is at most countable, it consists of two sequences (possibly finite or empty) of positive and negative eigenvalues, and of zero. The latter implies the essential spectrum of  $K_\varepsilon$ . Every non-zero eigenvalue has finite multiplicity. We denote by  $\mu_j^\varepsilon$ ,  $\mu_{-j}^\varepsilon$  the positive and negative eigenvalues, for every  $j \in \mathbb{N} \setminus \{0\}$ , with the convention that the positive eigenvalues are enumerated in decreasing order, the negative ones in increasing order, and each eigenvalue is repeated a number of times equal to its multiplicity. Moreover, we denote by  $u_j^\varepsilon$ , and  $u_{-j}^\varepsilon$  a sequence of corresponding  $H_\varepsilon$ -normalized eigenfunctions. The following variational characterizations hold true

$$\mu_1^\varepsilon = \max_{\substack{u \in H_\varepsilon, \\ u \neq 0}} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx}, \quad (2.61)$$

$$\mu_{-1}^\varepsilon = \min_{\substack{u \in H_\varepsilon, \\ u \neq 0}} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx}. \quad (2.62)$$

For each  $j \in \mathbb{N}$ ,  $j \geq 2$  one has also

$$\mu_j^\varepsilon = \max_{\substack{(u, u_i^\varepsilon)_{H_\varepsilon} = 0, \\ i=1, \dots, j-1}} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx} = \min_{\dim V = j-1} \max_{u \in V^\perp} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx}, \quad (2.63)$$

$$\mu_{-j}^\varepsilon = \min_{\substack{(u, u_{-i}^\varepsilon)_{H_\varepsilon} = 0, \\ i=1, \dots, j-1}} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx} = \max_{\dim V = j-1} \min_{u \in V^\perp} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx}, \quad (2.64)$$

where  $V^\perp$  stands for the orthogonal complement of  $V$  in  $H_\varepsilon$ .

**Remark 2.3.** From (2.56) and the fact that  $\rho \in L^\infty(\Gamma)$ , it follows that there exists a positive constant  $k_0$  such that

$$\varepsilon \mu_j^\varepsilon \leq k_0 \left( \varepsilon^2 + \frac{1}{\beta_j^\varepsilon} \right) \quad \text{for all } \varepsilon > 0, j \in \mathbb{N}, \quad (2.65)$$

where  $\beta_j^\varepsilon$  is the  $j$ -th eigenvalue of the Laplacian with homogeneous Neumann boundary conditions at the boundary of the perforation. More precisely,  $\{\beta_j^\varepsilon\}_{j=1}^\infty$ ,  $0 < \beta_1^\varepsilon \leq \beta_2^\varepsilon \leq \dots$ , is the spectrum of the problem

$$\begin{cases} -\Delta v_j^\varepsilon = \beta_j^\varepsilon v_j^\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{\partial v_j^\varepsilon}{\partial \nu_\varepsilon} = 0, & \text{on } \Gamma_\varepsilon, \\ v_\varepsilon = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.66)$$

It is known (see, for instance, [13]) that for all  $j \in \mathbb{N}$

$$\beta_j^\varepsilon \rightarrow \beta_j \quad \text{as } \varepsilon \rightarrow 0, \quad (2.67)$$

with  $\beta_j$  eigenvalue of the corresponding homogenized problem

$$\begin{cases} -\operatorname{div}(a^{\text{eff}} \nabla v_j) = \beta_j |\omega| v_j & \text{in } \Omega, \\ v_j = 0, & \text{on } \partial\Omega, \end{cases} \quad (2.68)$$

and

$$\beta_j \rightarrow +\infty, \quad \text{as } j \rightarrow +\infty. \quad (2.69)$$

**Proposition 2.4.** *If  $\rho$  satisfies condition (1.9), then for each  $\varepsilon > 0$  the sets*

$$\{j \in \mathbb{N} : \mu_j^\varepsilon > 0\} \quad \text{and} \quad \{j \in \mathbb{N} : \mu_{-j}^\varepsilon < 0\}$$

*have infinitely many elements.*

*Proof.*

**Step 1.** We first prove that

$$\mu_{-1}^\varepsilon < 0 < \mu_1^\varepsilon.$$

Letting

$$\rho_\varepsilon^+ = \max\{\rho_\varepsilon, 0\} \quad \rho_\varepsilon^- = \min\{\rho_\varepsilon, 0\},$$

under our assumption (1.9) on  $\rho$  we have

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon \rho_\varepsilon^+ d\sigma_x > 0.$$

Denote by  $\{u_\eta\}_{\eta>0}$  a family of functions  $u_\eta \in H_\varepsilon$  such that  $\|\sqrt{\rho_\varepsilon^+} - u_\eta\|_{L^2(\Gamma_\varepsilon)} \rightarrow 0$ , as  $\eta \rightarrow 0$ . Such functions  $u_\eta$  can be easily constructed by means of smoothing  $\sqrt{\rho_\varepsilon^+}$  on  $\Gamma_\varepsilon$ . Since

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon u_\eta^2 d\sigma_x \rightarrow \int_{\Gamma_\varepsilon} \rho_\varepsilon \rho_\varepsilon^+ d\sigma_x,$$

as  $\eta \rightarrow 0$ , then for all sufficiently small  $\eta > 0$  it holds

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon u_\eta^2 d\sigma_x > 0. \quad (2.70)$$

It remains to combine the last inequality with (2.61) in order to conclude that  $\mu_1^\varepsilon > 0$ .

In a similar way, one can prove that  $\mu_{-1}^\varepsilon < 0$ .

**Step 2.** Our next goal is to show that for any  $j \in \mathbb{N}$  the inequalities  $\mu_{-j}^\varepsilon < 0$  and  $\mu_j^\varepsilon > 0$  hold.

Assume that  $\mu_1^\varepsilon > 0, \dots, \mu_{j-1}^\varepsilon > 0$ , and let  $u_1^\varepsilon, \dots, u_{j-1}^\varepsilon$  be the corresponding normalized eigenfunctions,  $\langle u_i^\varepsilon, u_k^\varepsilon \rangle_{H_\varepsilon} = \delta_{ik}$  with  $i, k = 1, 2, \dots, j-1$ .

Consider a collection of sets  $\{S_i^\varepsilon\}_{i=1}^j$  with  $S_i^\varepsilon \subset \{x \in \Gamma_\varepsilon : \rho(x) > 0\}$ ,  $\sigma_x(S_i^\varepsilon) > 0$ ,  $S_i^\varepsilon \cap S_k^\varepsilon = \emptyset$ ,  $i \neq k$ , and denote  $\chi_i^\varepsilon$  the characteristic functions of these sets.

Let  $\chi_1^{\delta, \varepsilon}, \dots, \chi_j^{\delta, \varepsilon}$  be elements of  $H_\varepsilon$  such that  $\|\chi_i^\varepsilon - \chi_i^{\delta, \varepsilon}\|_{L^2(\Gamma_\varepsilon)} \leq \delta$ ,  $i = 1, \dots, j$ . It is clear that for sufficiently small  $\delta > 0$  the functions  $\chi_1^{\delta, \varepsilon}, \dots, \chi_j^{\delta, \varepsilon}$  are linearly independent. Therefore, there is a non-trivial linear combination  $\Xi = \beta_1^{\delta, \varepsilon} \chi_1^{\delta, \varepsilon} + \dots + \beta_j^{\delta, \varepsilon} \chi_j^{\delta, \varepsilon}$  such that  $\langle \Xi, u_i^\varepsilon \rangle_{H_\varepsilon} = 0$ ,  $i = 1, \dots, j-1$ .

It is also clear that for sufficiently small  $\delta > 0$  we have

$$\int_{\Gamma_\varepsilon} \Xi^2 \rho^+ d\sigma_x > 0.$$

Using  $\Xi$  as a test function in (2.63) we conclude that  $\mu_j^\varepsilon > 0$ . In the same way one can show that  $\mu_{-j}^\varepsilon < 0$ .

It remains to use the induction.

*Proof of Proposition 1.2.* All the statements of Proposition 1.2 except for (1.14) follow from the spectral properties of the operator  $K_\varepsilon$ , the fact that  $\lambda_j^\varepsilon = (\mu_j^\varepsilon)^{-1}$ , and from Proposition 2.4.

It remains to prove (1.14): we will do it for  $\lambda_1^\varepsilon$ , the proof for  $\lambda_{-1}^\varepsilon$  being analogous. We first show that each eigenfunction  $u$  related to  $\lambda_1^\varepsilon$  does not change sign in  $\Omega_\varepsilon$ .



Assume the contrary. Then there is an eigenfunction  $u$  related to  $\lambda_1^\varepsilon$  such that  $u^+ = \max\{u, 0\}$  and  $u^- = \min\{u, 0\}$  are non-trivial functions. Clearly,

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon(u^+)^2 d\sigma_x > 0 \quad \text{and} \quad \int_{\Gamma_\varepsilon} \rho_\varepsilon(u^-)^2 d\sigma_x > 0.$$

Indeed, if  $\int_{\Gamma_\varepsilon} \rho_\varepsilon(u^+)^2 d\sigma_x \leq 0$ , then  $\int_{\Gamma_\varepsilon} \rho_\varepsilon(u^-)^2 d\sigma_x \geq 1$ . Since  $\int_{\Omega_\varepsilon} |\nabla u^-|^2 dx < \int_{\Omega_\varepsilon} |\nabla u|^2 dx$ , this contradicts the variational principle (2.63). Therefore,  $\int_{\Gamma_\varepsilon} \rho_\varepsilon(u^+)^2 d\sigma_x > 0$ . Similarly,  $\int_{\Gamma_\varepsilon} \rho_\varepsilon(u^-)^2 d\sigma_x > 0$ .

By (2.63) we have

$$\int_{\Omega_\varepsilon} |\nabla u^-|^2 dx \leq \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon(u^-)^2 d\sigma_x, \quad \int_{\Omega_\varepsilon} |\nabla u^+|^2 dx \leq \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon(u^+)^2 d\sigma_x.$$

Summing up these inequalities and considering the relation

$$\int_{\Omega_\varepsilon} |\nabla u|^2 dx = \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon(u)^2 d\sigma_x$$

we conclude that

$$\int_{\Omega_\varepsilon} |\nabla u^+|^2 dx = \lambda_1^\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon(u^+)^2 d\sigma_x.$$

Thus,  $u^+$  is an eigenfunction related to  $\lambda_1^\varepsilon$ . Then  $u^+$  is a non-negative solution of the equation  $\Delta u^+ = 0$  in  $\Omega_\varepsilon$ , and the fact that  $u^+$  is equal to zero at interior points of  $\Omega_\varepsilon$  contradicts the maximum principle.

If we assume that there are two linearly independent positive eigenfunctions  $u, v \in H_\varepsilon$  related to  $\lambda_1^\varepsilon$ , then

$$\int_{\Omega_\varepsilon} (u - cv) dx = 0, \quad \text{for } c = \left( \int_{\Omega_\varepsilon} v dx \right)^{-1} \int_{\Omega_\varepsilon} u dx.$$

Therefore,  $u - cv$  is an eigenfunction that changes sign. This contradiction shows that  $\lambda_1^\varepsilon$  is simple.  $\square$

*Proof of Proposition 1.3.* Our goal is to show that for sufficiently small  $\delta > 0$  the quadratic form

$$\mathcal{J}(u) = \int_{\omega} |\nabla u(y)|^2 dy + \delta \int_{\Gamma} \rho(y)(u(y))^2 d\sigma_y$$

is coercive that is

$$\mathcal{J}(u) \geq C(\delta) \|u\|_{H^1(\omega)}^2 \quad \text{for all } u \in H_{\#}^1(Y) \quad (2.71)$$

with  $C(\delta) > 0$ . The spectral problem for the operator associated with  $\mathcal{J}$  reads

$$\begin{cases} \int_{\omega} \nabla p \cdot \nabla w dy + \delta \int_{\Gamma} \rho p w d\sigma_y = \tilde{\alpha} \int_{\Gamma} \rho p w d\sigma_y & \forall w \in H_{\#}^1(Y), \\ p \in H_{\#}^1(Y). \end{cases} \quad (2.72)$$

The spectrum of this problem coincides with the spectrum of problem (1.23) shifted by  $\delta$ . Exploiting (2.71) by the same arguments as in the proof of Proposition 1.2 one can deduce that the spectrum of (2.72), and thus of (1.23), is discrete and consists of two infinite sequences of eigenvalues, one of these sequences tends to  $-\infty$ , another to  $+\infty$ .

Other statements of Proposition 1.3 can be justified in the same way as in the proof of Proposition 1.2.

To prove (2.71) we represent  $\rho$  as  $\rho = \bar{\rho} + \hat{\rho}$  with  $\bar{\rho} > 0$  defined in (1.10). For an arbitrary function  $u \in H_{\#}^1(Y)$  denote  $\bar{u} = (\sigma_y(\Gamma))^{-1} \int_{\Gamma} u(y) d\sigma_y$ ,  $\hat{u} = u - \bar{u}$ . Then

$$\begin{aligned} \int_{\Gamma} \rho u^2 d\sigma_y &= \int_{\Gamma} (\bar{\rho} u^2 + \hat{\rho}(\bar{u} + \hat{u})^2) d\sigma_y = \int_{\Gamma} (\bar{\rho} u^2 + 2\hat{\rho}\bar{u}\hat{u} + \hat{\rho}\hat{u}^2) d\sigma_y \\ &\geq \int_{\Gamma} (\bar{\rho} u^2 - C_{\rho}(|\bar{u}\hat{u}| + \hat{u}^2)) d\sigma_y \end{aligned}$$

with  $C_{\rho} = 2\|\rho\|_{L^{\infty}}$ . Using the trace and Poincaré inequalities we deduce that for any  $\delta_1 > 0$

$$\begin{aligned} \int_{\Gamma} C_{\rho}(|\bar{u}\hat{u}| + \hat{u}^2) d\sigma_y &\leq \int_{\Gamma} C_{\rho} \left( \delta_1 \bar{u}^2 + \left( \frac{1}{\delta_1} + 1 \right) \hat{u}^2 \right) d\sigma_y \\ &\leq \int_{\Gamma} C_{\rho} \delta_1 u^2 d\sigma_y + C_1 \left( \frac{1}{\delta_1} + 1 \right) \int_{\omega} |\nabla u|^2 dy. \end{aligned}$$

Combining the last two inequalities and choosing  $\delta_1$  in such a way that  $C_{\rho}\delta_1 = \frac{1}{2}\bar{\rho}$  we obtain

$$\int_{\Gamma} \rho u^2 d\sigma_y \geq \int_{\Gamma} \frac{1}{2} \bar{\rho} u^2 d\sigma_y - C_1 \left( \frac{1}{\delta_1} + 1 \right) \int_{\omega} |\nabla u|^2 dy.$$

This yields

$$\mathcal{J}(u) \geq \int_{\omega} |\nabla u|^2 dy + \frac{\delta}{2} \int_{\Gamma} \bar{\rho} u^2 d\sigma_y - C_1 \delta \left( \frac{1}{\delta_1} + 1 \right) \int_{\omega} |\nabla u|^2 dy.$$

Finally, taking  $\delta$  such that  $C_1\delta((1/\delta_1) + 1) \leq 1/2$ , we get

$$\mathcal{J}(u) \geq \frac{1}{2} \int_{\omega} |\nabla u|^2 dy + \frac{\delta}{2} \int_{\Gamma} \bar{\rho} u^2 d\sigma_y \geq C(\delta) \|u\|_{H^1(\omega)}^2.$$

□

### 3 The case $\bar{\rho} > 0$

The aim of this section is to prove Theorem 1.4. We begin with an auxiliary statement.

**Lemma 3.1.** *Let  $u_\varepsilon, u \in H_0^1(\Omega)$ ,  $\|u_\varepsilon\|_{H_0^1} \leq c$ ,  $u_\varepsilon \rightarrow u$  strongly in  $L^2(\Omega)$  and  $\bar{\rho} > 0$ . Then*

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 d\sigma_x \rightarrow \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} u^2 dx, \quad \text{as } \varepsilon \rightarrow 0. \quad (3.73)$$

Moreover, for all  $v \in H_0^1(\Omega)$

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v d\sigma_x \rightarrow \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} u v dx, \quad \text{as } \varepsilon \rightarrow 0. \quad (3.74)$$

*Proof.* Let us denote by  $\hat{u}_\varepsilon$  the piece-wise constant function that takes the value of the average of  $u_\varepsilon$  in each  $\varepsilon$ -cell that is

$$\hat{u}_\varepsilon(x) = \hat{u}_j^\varepsilon \quad \text{if } x \in Y_\varepsilon^i,$$

$$\hat{u}_j^\varepsilon = \frac{1}{|\omega_j^\varepsilon|} \int_{\omega_j^\varepsilon} u_\varepsilon dx.$$

Note that, by our assumptions and Poincaré inequality, it follows that  $\hat{u}_\varepsilon \rightarrow u$  strongly in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ . In fact

$$\int_{\Omega} |u_\varepsilon - \hat{u}_\varepsilon|^2 dx = \sum_j \int_{Y_\varepsilon^i} |u_\varepsilon - \hat{u}_j^\varepsilon|^2 dx \leq c\varepsilon^2 \sum_j \int_{Y_\varepsilon^i} |\nabla u_\varepsilon|^2 dx \leq c\varepsilon^2. \quad (3.75)$$

In order to prove (3.73), we write

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 d\sigma_x = \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \hat{u}_\varepsilon^2 d\sigma_x + \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u_\varepsilon^2 - \hat{u}_\varepsilon^2) d\sigma_x. \quad (3.76)$$

The first term can be rearranged as follows

$$\begin{aligned} \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \hat{u}_\varepsilon^2 d\sigma_x &= \varepsilon \sum_j \int_{\Gamma_j^\varepsilon} \rho_\varepsilon (\hat{u}_j^\varepsilon)^2 d\sigma_x = \\ &= \varepsilon \sum_j (\hat{u}_j^\varepsilon)^2 \varepsilon^{n-1} \int_{\Gamma} \rho(y) d\sigma_y = \bar{\rho} \sigma_y(\Gamma) \left( \int_{\Omega} (\hat{u}_\varepsilon)^2 dx + o(1) \right). \end{aligned}$$

Hence, by (3.75), we can conclude that

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \hat{u}_\varepsilon^2 d\sigma_x \rightarrow \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} u^2 dx$$

as  $\varepsilon \rightarrow 0$ . The second term in (3.76) is negligible, since

$$\left| \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u_\varepsilon^2 - \hat{u}_\varepsilon^2) d\sigma_x \right| \leq \varepsilon \int_{\Gamma_\varepsilon} |\rho_\varepsilon| |u_\varepsilon - \hat{u}_\varepsilon| |u_\varepsilon + \hat{u}_\varepsilon| d\sigma_x \leq$$

$$\leq \varepsilon \left( \int_{\Gamma_\varepsilon} |\rho_\varepsilon| |u_\varepsilon - \hat{u}_\varepsilon|^2 d\sigma_x \right)^{1/2} \left( \int_{\Gamma_\varepsilon} |\rho_\varepsilon| |u_\varepsilon + \hat{u}_\varepsilon|^2 d\sigma_x \right)^{1/2}.$$

The first term on the right hand side can be estimated by means of Poicaré inequality. We have

$$\varepsilon \left( \int_{\Gamma_\varepsilon} |\rho_\varepsilon| |u_\varepsilon - \hat{u}_\varepsilon|^2 d\sigma_x \right)^{1/2} \leq \varepsilon \|\rho\|_{L^\infty}^{1/2} \left( \varepsilon \int_{\Omega_\varepsilon} |\nabla u_e|^2 dx \right)^{1/2} \leq c\varepsilon^{3/2}.$$

The second term can be estimated by means of the trace inequality:

$$\begin{aligned} \left( \int_{\Gamma_\varepsilon} |\rho_\varepsilon| |u_\varepsilon + \hat{u}_\varepsilon|^2 d\sigma_x \right) &\leq \left( 2\|\rho\|_{L^\infty} \int_{\Gamma_\varepsilon} u_\varepsilon^2 d\sigma_x + 2\|\rho\|_{L^\infty} \sigma_y(\Gamma) \int_{\Omega_\varepsilon} u_\varepsilon^2 dx \right)^{1/2} \\ &\leq c \left( \varepsilon^{-1} \int_{\Omega_\varepsilon} u_\varepsilon^2 dx + \varepsilon \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx \right). \end{aligned}$$

Hence, combining the last two inequalities, we finally have

$$\left| \varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon (u_\varepsilon^2 - \hat{u}_\varepsilon^2) d\sigma_x \right| \leq c\varepsilon^{1/2},$$

and (3.73) follows.

To prove (3.74) it suffices to notice that

$$u_\varepsilon v = \frac{1}{2}(u_\varepsilon + v)^2 - \frac{1}{2}u_\varepsilon^2 - \frac{1}{2}v^2,$$

then (3.73) applies.  $\square$

#### Proof of Theorem 1.4.

We begin by obtaining the following estimates

$$c^- \leq \varepsilon^{-1} \lambda_j^\varepsilon \leq c_j \quad \text{for all } \varepsilon > 0, \quad 0 < c^- < c_j < \infty. \quad (3.77)$$

Let us first justify the lower bound. Due to (2.61), (2.56) and the Poincaré inequality, one has

$$\frac{1}{\lambda_1^\varepsilon} = \mu_1^\varepsilon = \sup_{u \in H_\varepsilon} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx} \leq \|\rho\|_{L^\infty} \sup_{u \in H_\varepsilon} \frac{\int_{\Gamma_\varepsilon} u^2 d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx} \leq k \|\rho\|_{L^\infty} \varepsilon^{-1}$$

with a constant  $k$  that does not depend on  $\varepsilon$ . This yields the desired lower bound.

Let us now prove the upper bound in (3.77) for  $j = 1$ . From (2.61) we derive

$$\frac{1}{\lambda_1^\varepsilon} = \mu_1^\varepsilon = \sup_{u \in H_\varepsilon} \frac{\int_{\Gamma_\varepsilon} u^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla u|^2 dx} \geq \frac{\int_{\Gamma_\varepsilon} \varphi^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla \varphi|^2 dx}$$

for any  $\varphi \in H_\varepsilon$ . In particular, if we choose  $\varphi \in \mathcal{C}_0^\infty(\Omega)$ ,  $\varphi \neq 0$ , then

$$\int_{\Omega_\varepsilon} |\nabla \varphi|^2 dx \rightarrow |\omega| \int_{\Omega} |\nabla \varphi|^2 dx > 0,$$

where  $|\omega|$  denotes the Lebesgue measure of  $\omega$ . By Lemma 3.1 with  $u_\varepsilon = \varphi$ , we get

$$\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2 d\sigma_x \rightarrow \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} \varphi^2 dx > 0.$$

Therefore, there exist two constants  $\varepsilon_0 > 0$  and  $c > 0$  such that

$$\mu_1^\varepsilon \geq \frac{\int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2 d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla \varphi|^2 dx} \geq \frac{c}{\varepsilon} \quad \forall \varepsilon \in (0, \varepsilon_0). \quad (3.78)$$

This implies the estimate

$$0 < \lambda_1^\varepsilon \leq \frac{\varepsilon}{c}.$$

It remains to denote  $c_1 = 1/c$ .

In order to justify the upper bound for  $j > 1$ , we consider a set of non-zero  $C_0^\infty(\Omega)$  functions  $\varphi_1, \dots, \varphi_j$  with disjoint supports. Since these functions are orthogonal in  $H_\varepsilon$ , there is a non-trivial linear combination  $\phi^\varepsilon = \gamma_1^\varepsilon \varphi_1 + \dots + \gamma_j^\varepsilon \varphi_j$  such that

$$(\phi^\varepsilon, u_1^\varepsilon)_{H_\varepsilon} = \dots = (\phi^\varepsilon, u_{j-1}^\varepsilon)_{H_\varepsilon} = 0.$$

Then, by (2.63),

$$\mu_j^\varepsilon \geq \frac{\int_{\Gamma_\varepsilon} (\phi^\varepsilon)^2 \rho_\varepsilon d\sigma_x}{\int_{\Omega_\varepsilon} |\nabla \phi^\varepsilon|^2 dx}$$

Using the fact that the functions  $\varphi_i$  have disjoint supports, it is easy to check that

$$\int_{\Gamma_\varepsilon} (\phi^\varepsilon)^2 \rho_\varepsilon d\sigma_x = \sum_{i=1}^j (\gamma_i^\varepsilon)^2 \int_{\Gamma_\varepsilon} (\varphi_i)^2 \rho_\varepsilon d\sigma_x, \quad \int_{\Omega_\varepsilon} |\nabla \phi^\varepsilon|^2 dx = \sum_{i=1}^j (\gamma_i^\varepsilon)^2 \int_{\Omega_\varepsilon} |\nabla \varphi_i|^2 dx.$$

By (3.78), there are  $c > 0$  and  $\varepsilon_0 > 0$  such that

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi_i^2 d\sigma_x \geq \frac{c}{\varepsilon} \int_{\Omega_\varepsilon} |\nabla \varphi_i|^2 dx \quad \text{for all } \varepsilon \in (0, \varepsilon_0), \quad i = 1, \dots, j.$$

Multiplying these inequalities by  $(\gamma_i^\varepsilon)^2$  and summing up the resulting relations yields

$$\mu_j^\varepsilon \geq \frac{c}{\varepsilon}.$$

This implies the required upper bound in (3.77).

The proof of (1.40)-(1.52) relies on several technical statements.

**Proposition 3.2.** *Let  $\{(\lambda_{j(\varepsilon)}^\varepsilon, u_{j(\varepsilon)}^\varepsilon)\}$  be a family of normalized eigenpairs of problem (1.5) or, equivalently, (1.6), and assume that, perhaps for a subsequence,  $\frac{\lambda_{j(\varepsilon)}^\varepsilon}{\varepsilon} \rightarrow \bar{\lambda}$ , as  $\varepsilon \rightarrow 0$ . Then  $\bar{\lambda}$  is an eigenvalue of the limit problem (1.16). If, in addition,  $u_{j(\varepsilon)}^\varepsilon$  converges to  $\bar{u}$  weakly in  $H_0^1(\Omega)$  for the same subsequence of  $\varepsilon$ , then  $\bar{u} \neq 0$ , and  $(\bar{\lambda}, \bar{u})$  is an eigenpair of (1.16).*

*Proof.* Since the family  $\{u_{j(\varepsilon)}^\varepsilon\}$  is bounded in  $H_0^1(\Omega)$ , we may assume without loss of generality that  $u_{j(\varepsilon)}^\varepsilon \rightharpoonup \bar{u}$  weakly in  $H_0^1(\Omega)$ . Then  $u_{j(\varepsilon)}^\varepsilon \rightarrow \bar{u}$  in  $L^2(\Omega)$ , and by Lemma 3.1,

$$1 = \int_{\Omega_\varepsilon} |\nabla u_{j(\varepsilon)}^\varepsilon|^2 dx = \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_{j(\varepsilon)}^2 d\sigma_x \longrightarrow \bar{\lambda} \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} \bar{u}^2 dx.$$

Therefore,  $\int_{\Omega} \bar{u}^2 dx > 0$ , and  $\bar{u} \neq 0$ .

Our goal is to show that

$$\int_{\Omega} a^{\text{eff}} \nabla \bar{u} \cdot \nabla \varphi dx = \bar{\lambda} \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} \bar{u} \varphi dx \quad \forall \varphi \in H_0^1(\Omega) \quad (3.79)$$

with  $a^{\text{eff}}$  defined in (1.17). To this end, we consider the following auxiliary homogenization problem

$$\int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla \varphi dx = \frac{\bar{\lambda} \bar{\rho} \sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \bar{u} \varphi dx \quad \forall \varphi \in H_\varepsilon \quad (3.80)$$

stated in the perforated domain  $\Omega_\varepsilon$ . It is well-known in homogenization theory (see, for instance, [?]) that, as  $\varepsilon \rightarrow 0$ , the (extended) solution  $v_\varepsilon$  tends weakly in  $H_0^1(\Omega)$  and strongly in  $L^2(\Omega)$  to a function  $v \in H_0^1(\Omega)$  being a unique solution of the homogenized problem

$$\int_{\Omega} a^{\text{eff}} \nabla v \cdot \nabla \varphi dx = \bar{\lambda} \bar{\rho} \sigma_y(\Gamma) \int_{\Omega} \bar{u} \varphi dx \quad \forall \varphi \in H_0^1(\Omega). \quad (3.81)$$

By the lower-semicontinuity of the  $H^1$ -norm and the boundedness of the extension operators, we have

$$\int_{\Omega} |\nabla v - \nabla \bar{u}|^2 dx \leq \liminf_{\varepsilon \rightarrow 0} \int_{\Omega} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx \leq c_0 \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx.$$

Using in equations (1.6) and (3.80) the test functions  $\varphi = v_\varepsilon$  and  $\varphi = u_\varepsilon$ , yields

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx &= \int_{\Omega_\varepsilon} |\nabla v_\varepsilon|^2 dx + \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx - 2 \int_{\Omega_\varepsilon} \nabla v_\varepsilon \cdot \nabla u_\varepsilon dx = \\ &= \frac{\bar{\lambda} \bar{\rho} \sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \bar{u} v_\varepsilon dx + \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 d\sigma_x + \end{aligned}$$

$$-\lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v_\varepsilon d\sigma_x - \frac{\bar{\lambda}\bar{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \bar{u} u_\varepsilon dx.$$

Since  $u^\varepsilon \rightarrow \bar{u}$  and  $v^\varepsilon \rightarrow v$  in  $L^2(\Omega)$ , the following limit relations hold, as  $\varepsilon \rightarrow 0$ :

$$\begin{aligned} \frac{\bar{\lambda}\bar{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \bar{u} v_\varepsilon dx &\rightarrow \lambda\bar{\rho}\sigma_y(\Gamma) \int_{\Omega} \bar{u} v dx, \\ -\frac{\bar{\lambda}\bar{\rho}\sigma_y(\Gamma)}{|\omega|} \int_{\Omega_\varepsilon} \bar{u} u_\varepsilon dx &\rightarrow -\bar{\lambda}\bar{\rho}\sigma_y(\Gamma) \int_{\Omega} \bar{u}^2 dx. \end{aligned}$$

Furthermore, by Lemma 3.1,

$$\begin{aligned} \lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 d\sigma_x &\rightarrow \bar{\lambda}\bar{\rho}\sigma_y(\Gamma) \int_{\Omega} \bar{u}^2 dx, \\ -\lambda_\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon v_\varepsilon d\sigma_x &\rightarrow -\lambda\bar{\rho}\sigma_y(\Gamma) \int_{\Omega} \bar{u} v dx. \end{aligned}$$

Combining the above inequalities, we arrive at the estimate

$$\int_{\Omega} |\nabla v - \nabla \bar{u}|^2 dx \leq c_0 \liminf_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla v_\varepsilon - \nabla u_\varepsilon|^2 dx = 0,$$

which implies that  $v = \bar{u}$ . Thus, (3.79) holds true.  $\square$

The proof of the fact that any eigenpair of the limit operator is approached by the eigenpairs of  $\varepsilon$ -problems relies on the so-called Lemma on "eigenvalues and eigenvectors" (see [9]). For the reader's convenience we formulate it here.

**Lemma 3.3.** *Let  $A : H \rightarrow H$  be a linear compact self-adjoint operator in a Hilbert space  $H$ . Suppose that there are a real number  $\mu$  and a vector  $u \in H$ , such that  $\|u\|_H = 1$  and*

$$\|Au - \mu u\|_H \leq \alpha. \quad (3.82)$$

*Then, there is an eigenvalue  $\mu_i$  of the operator  $A$  such that*

$$|\mu_i - \mu| \leq \alpha. \quad (3.83)$$

*Moreover, for any  $d > \alpha$  there exists a vector  $\bar{u}$  such that*

$$\|u - \bar{u}\|_H \leq 2\alpha d^{-1}, \|\bar{u}\|_H = 1, \quad (3.84)$$

*and  $\bar{u}$  is a linear combination of eigenvectors of the operator  $A$  corresponding to eigenvalues of  $A$  in the closed segment  $[\mu - d, \mu + d]$ .*

In the sequel we refer to  $\mu$  and  $u$  in (3.82) as almost eigenvalue and eigenvector of  $A$ . We proceed with other technical statements.

**Lemma 3.4.** *Let  $f \in L^\infty_{\text{per}}(\omega)$  and  $g \in L^\infty(\partial\omega)$  satisfy*

$$\int_{\omega} f(y) dy - \int_{\Gamma} g(y) d\sigma_y = 0. \quad (3.85)$$

*Then there exists  $c > 0$  such that*

$$\left| \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv dx - \varepsilon \int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv d\sigma_x \right| \leq c\varepsilon \|\nabla(uv)\|_{L^2(\Omega_\varepsilon)} \quad (3.86)$$

*for all  $u, v \in H_\varepsilon$  such that  $\nabla(uv) \in L^2(\Omega_\varepsilon)$ . Also, for any  $u, v \in H_\varepsilon$  it holds*

$$\left| \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv dx - \varepsilon \int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv d\sigma_x \right| \leq c\varepsilon \|u\|_{H_\varepsilon} \|v\|_{H_\varepsilon}. \quad (3.87)$$

*If for  $f \in L^2_{\#}(\omega)$  and  $g \in L^2(\partial\omega)$  condition (3.85) is fulfilled then there is  $c > 0$  such that*

$$\left| \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv dx - \varepsilon \int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv d\sigma_x \right| \leq c\varepsilon \|\nabla(uv)\|_{L^2(\Omega_\varepsilon)} \quad (3.88)$$

*for all  $u \in W^{1,\infty}(\Omega)$  and  $v \in H_\varepsilon$ .*

*Proof.* Let  $\psi \in H^1(\omega)$  be a solution to problem

$$\begin{cases} \Delta\psi = f & \text{in } \omega, \\ \frac{\partial\psi}{\partial\nu} = g, & \text{on } \Gamma, \\ \psi & Y\text{-periodic.} \end{cases} \quad (3.89)$$

Then  $\psi_\varepsilon(x) = \psi\left(\frac{x}{\varepsilon}\right)$  is  $\varepsilon Y$ -periodic, it belongs to  $H^1_{\text{loc}}(\mathbb{R}^n)$  and satisfies

$$\nabla_x \psi_\varepsilon = \varepsilon^{-1}(\nabla_y \psi)\left(\frac{x}{\varepsilon}\right), \quad \Delta_x \psi_\varepsilon = \varepsilon^{-2}(\Delta_y \psi)\left(\frac{x}{\varepsilon}\right).$$

Writing down the integral identity

$$\int_{\Omega_\varepsilon} (\Delta_y \psi)\left(\frac{x}{\varepsilon}\right) uv dx = \int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv dx,$$

after integration by parts one has

$$\int_{\Omega_\varepsilon} f\left(\frac{x}{\varepsilon}\right) uv dx - \varepsilon \int_{\Gamma_\varepsilon} g\left(\frac{x}{\varepsilon}\right) uv d\sigma_x = -\varepsilon \int_{\Omega_\varepsilon} (\nabla_y \psi)\left(\frac{x}{\varepsilon}\right) \nabla(uv) dx,$$



from which (3.86) and (3.88) follow immediately.

In order to justify (3.87) we consider the functions  $\bar{u}_\varepsilon$  and  $\bar{v}_\varepsilon$  introduced in (2.55). Notice that

$$\|\bar{u}_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq \|u\|_{L^2(\Omega_\varepsilon)}, \quad \varepsilon \|\bar{u}_\varepsilon\|_{L^2(\Gamma_\varepsilon)}^2 \leq c \|u\|_{L^2(\Omega_\varepsilon)}^2.$$

Denoting  $f^\varepsilon = f(x/\varepsilon)$  and  $g^\varepsilon = g(x/\varepsilon)$ , and using (2.54) and Cauchy-Schwartz inequality, we get

$$\begin{aligned} & \left| \int_{\Omega_\varepsilon} f^\varepsilon uv \, dx - \varepsilon \int_{\Gamma_\varepsilon} g^\varepsilon uv \, d\sigma_x \right| \leq \left| \int_{\Omega_\varepsilon} f^\varepsilon \bar{u}_\varepsilon v \, dx - \varepsilon \int_{\Gamma_\varepsilon} g^\varepsilon \bar{u}_\varepsilon v \, d\sigma_x \right| \\ & + \left| \int_{\Omega_\varepsilon} f^\varepsilon (u - \bar{u}_\varepsilon) v \, dx - \varepsilon \int_{\Gamma_\varepsilon} g^\varepsilon (u - \bar{u}_\varepsilon) v \, d\sigma_x \right| \leq \left| \int_{\Omega_\varepsilon} f^\varepsilon \bar{u}_\varepsilon v \, dx - \varepsilon \int_{\Gamma_\varepsilon} g^\varepsilon \bar{u}_\varepsilon v \, d\sigma_x \right| \\ & + C\varepsilon \|u\|_{H_\varepsilon} \|v\|_{H_\varepsilon} \leq \left| \int_{\Omega_\varepsilon} f^\varepsilon \bar{u}_\varepsilon \bar{v}_\varepsilon \, dx - \varepsilon \int_{\Gamma_\varepsilon} g^\varepsilon \bar{u}_\varepsilon \bar{v}_\varepsilon \, d\sigma_x \right| \\ & + \left| \int_{\Omega_\varepsilon} f^\varepsilon \bar{u}_\varepsilon (v - \bar{v}_\varepsilon) \, dx - \varepsilon \int_{\Gamma_\varepsilon} g^\varepsilon \bar{u}_\varepsilon (v - \bar{v}_\varepsilon) \, d\sigma_x \right| + C\varepsilon \|u\|_{H_\varepsilon} \|v\|_{H_\varepsilon} \\ & \leq C\varepsilon \|u\|_{H_\varepsilon} \|v\|_{H_\varepsilon}; \end{aligned}$$

here we have also used (3.85). □

The proof of the next statement is quite similar to the proof of (3.87) and can be found, for instance, in [5, Ch.1, Lemma1.1].

**Lemma 3.5.** *Let  $h \in L^\infty_\#(Y)$  be such that*

$$\int_Y h(y) \, dy = 0. \tag{3.90}$$

*Then there exists  $c > 0$  such that for all  $u, v \in H_0^1(\Omega)$*

$$\left| \int_\Omega h\left(\frac{x}{\varepsilon}\right) uv \, dx \right| \leq c\varepsilon \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}. \tag{3.91}$$

We will also need cut-off functions in the vicinity of the exterior boundary  $\partial\Omega$ . For  $\gamma > 0$  denote  $\Omega(\gamma) = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \gamma\}$ .

**Lemma 3.6.** *Let  $\psi \in L^2_{\text{per}}(Y)$ , and let  $h > 0$  be a positive number. Then, there exists  $c > 0$  such that*

$$\left| \int_{\Omega \setminus \bar{\Omega}(h\varepsilon)} \psi\left(\frac{x}{\varepsilon}\right) v \, dx \right| \leq c\varepsilon^{3/2} \|\nabla v\|_{L^2(\Omega)}, \tag{3.92}$$

*and, if  $\Gamma_\varepsilon \cap (\Omega \setminus \bar{\Omega}(h\varepsilon)) \neq \emptyset$ ,*

$$\left| \int_{\Gamma_\varepsilon \cap (\Omega \setminus \bar{\Omega}(h\varepsilon))} \psi\left(\frac{x}{\varepsilon}\right) v \, d\sigma_x \right| \leq c\sqrt{\varepsilon} \|\nabla v\|_{L^2(\Omega)}. \tag{3.93}$$

*for all  $\varepsilon > 0$ , and all  $v \in H_0^1(\Omega)$ .*

*Proof.* By the Hardy inequality (see, for instance [10]), there exists a constant  $c > 0$  such that

$$\|v\|_{L^2(\Omega \setminus \overline{\Omega}(\gamma))} \leq c\gamma \|\nabla v\|_{L^2(\Omega)} \quad (3.94)$$

for all constants  $\gamma > 0$  and for all  $v \in H_0^1(\Omega)$ . Then, combining this estimate with the Cauchy-Schwartz inequality, we get

$$\begin{aligned} \left| \int_{\Omega \setminus \overline{\Omega}(h\varepsilon)} \psi\left(\frac{x}{\varepsilon}\right) v \, dx \right| &\leq \left\| \psi\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega \setminus \overline{\Omega}(h\varepsilon))} \|v\|_{L^2(\Omega \setminus \overline{\Omega}(h\varepsilon))} \\ &\leq c\varepsilon \left\| \psi\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega \setminus \overline{\Omega}(h\varepsilon))} \|\nabla v\|_{L^2(\Omega)}. \end{aligned} \quad (3.95)$$

Denote  $J(h\varepsilon) = \{j \in \mathbb{Z}^n : \varepsilon(Y + j) \cap (\Omega \setminus \overline{\Omega}(h\varepsilon)) \neq \emptyset\}$ , and let  $\#J(h\varepsilon)$  be the cardinality (the number of elements) of  $J(h\varepsilon)$ . Clearly,  $\#J(h\varepsilon) \leq C(h)\varepsilon^{1-n}$ . Thus,

$$\left\| \psi\left(\frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega \setminus \overline{\Omega}(h\varepsilon))}^2 \leq \#J(h\varepsilon) \varepsilon^n \|\psi\|_{L^2(Y)}^2 \leq C\varepsilon.$$

To obtain (3.92) it remains to combine this inequality with (3.95). The proof of (3.93) relies also on the fact that for all  $\varepsilon$ -cell  $Y_i^\varepsilon$

$$\int_{\Gamma_\varepsilon \cap Y_i^\varepsilon} v^2 \, d\sigma_x \leq c \left( \varepsilon^{-1} \int_{Y_i^\varepsilon} v^2 \, dx + \varepsilon \int_{Y_i^\varepsilon} |\nabla v|^2 \, dx \right).$$

Summing up these estimates over  $i \in J(h\varepsilon)$  and using (3.94) yields

$$\int_{\Gamma_\varepsilon \cap \Omega \setminus \overline{\Omega}(h\varepsilon)} v^2 \, d\sigma_x \leq c \left( \varepsilon^{-1} \int_{\Omega \setminus \overline{\Omega}(h\varepsilon)} v^2 \, dx + \varepsilon \int_{\Omega \setminus \overline{\Omega}(h\varepsilon)} |\nabla v|^2 \, dx \right).$$

Combining the preceding inequality with (3.94) and the fact that  $|\Omega \setminus \overline{\Omega}(h\varepsilon)| \leq c\varepsilon$  we immediately obtain (3.93).  $\square$

**Proposition 3.7.** *Let  $\lambda_j$ ,  $j \in \mathbb{N}$ , be an eigenvalue of problem (1.16). Then there exist a family  $\{k(\varepsilon)\}_{\varepsilon>0}$ ,  $k(\varepsilon) \in \mathbb{N}$ , such that*

$$\frac{\lambda_{k(\varepsilon)}^\varepsilon}{\varepsilon} \rightarrow \lambda_j, \quad \text{as } \varepsilon \rightarrow 0, \quad (3.96)$$

where  $\lambda_{k(\varepsilon)}^\varepsilon$  is an eigenvalue of problem (1.5).

*Proof.* Let  $\Psi^\varepsilon$  be a family of  $C_0^\infty(\Omega)$  functions such that  $\Psi^\varepsilon(x) = 1$  if the distance from  $x$  to  $\partial\Omega$  is greater than  $2\varepsilon$ ,  $0 \leq \Psi^\varepsilon \leq 1$ , and  $|\nabla \Psi^\varepsilon(x)| \leq 2/\varepsilon$  for all  $x \in \Omega$ .

Denote  $\tilde{U}_j^\varepsilon(x) = u_j^0(x) + \varepsilon \Psi^\varepsilon(x) \chi(x/\varepsilon) \nabla u_j^0(x)$ , and  $U_j^\varepsilon(x) = u_j^0(x) + \varepsilon \chi(x/\varepsilon) \nabla u_j^0(x)$ . It is straightforward to check that, under our assumptions on regularity of  $\partial\Omega$ , we have

$$\|\tilde{U}_j^\varepsilon - U_j^\varepsilon\|_{L^2(\Omega)} \leq C\varepsilon^{3/2}, \quad \|\tilde{U}_j^\varepsilon - U_j^\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{1/2}.$$

Let us compute the norm of  $\tilde{U}_j^\varepsilon$  in  $H_\varepsilon$ . Denoting the unit  $n \times n$  matrix by  $\mathbf{I}$  we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla \tilde{U}_j^\varepsilon \cdot \nabla \tilde{U}_j^\varepsilon dx &= \int_{\Omega_\varepsilon} |\tilde{\nabla} u_j^0 + \Psi^\varepsilon \nabla_y \chi\left(\frac{x}{\varepsilon}\right) \nabla u_j^0 + \varepsilon \Psi^\varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla \nabla u_j^0 + \varepsilon \nabla \Psi^\varepsilon \chi\left(\frac{x}{\varepsilon}\right) \nabla u_j^0|^2 dx \\ &= \int_{\Omega_\varepsilon} |(\mathbf{I} + \nabla_y \chi(x/\varepsilon)) \nabla u_j^0|^2 dx + O(\varepsilon); \end{aligned}$$

here we have used the facts that  $|\varepsilon \nabla \Psi^\varepsilon| \leq C$ , the support of  $\nabla \Psi^\varepsilon$  is a subset of  $2\varepsilon$ -neighbourhood of  $\partial\Omega$ , and  $u_j^0$  is a  $C^2(\bar{\Omega})$  function. Recalling the formula for the effective matrix  $a^{\text{eff}}$ , normalization condition (1.20), and using once again the  $C^2$  smoothness of  $u_j^0$  we conclude that

$$\int_{\Omega_\varepsilon} \nabla \tilde{U}_j^\varepsilon \cdot \nabla \tilde{U}_j^\varepsilon dx = \int_{\Omega} a^{\text{eff}} \nabla u_j^0 \cdot \nabla u_j^0 dx + O(\varepsilon) = 1 + O(\varepsilon). \quad (3.97)$$

Similarly, one can show that

$$\left| \int_{\Omega_\varepsilon} \nabla \tilde{U}_j^\varepsilon \cdot \nabla \varphi dx - \int_{\Omega} a^{\text{eff}} \nabla u_j^0 \cdot \nabla \varphi dx \right| \leq C \sqrt{\varepsilon} \|\varphi\|_{H_0^1(\Omega)} \quad (3.98)$$

for any  $\varphi \in H_0^1(\Omega)$ .

We proceed with estimating the norm  $\|K^\varepsilon \tilde{U}^\varepsilon - (\varepsilon \lambda_j)^{-1} \tilde{U}^\varepsilon\|_{H_\varepsilon}$ . After straightforward rearrangement we have

$$\begin{aligned} \|K^\varepsilon \tilde{U}^\varepsilon - \frac{1}{\varepsilon \lambda_j} \tilde{U}^\varepsilon\|_{H_\varepsilon} &= \sup_{\varphi \in B^\varepsilon} \left( K^\varepsilon \tilde{U}^\varepsilon - \frac{1}{\varepsilon \lambda_j} \tilde{U}^\varepsilon, \varphi \right)_{H_\varepsilon} = \sup_{\varphi \in B^\varepsilon} \int_{\Omega_\varepsilon} \left( \nabla (K^\varepsilon \tilde{U}^\varepsilon) \cdot \nabla \varphi - \frac{1}{\varepsilon \lambda_j} \nabla \tilde{U}^\varepsilon \cdot \nabla \varphi \right) dx \\ &= \sup_{\varphi \in B^\varepsilon} \left( \int_{\Gamma_\varepsilon} \rho^\varepsilon \tilde{U}^\varepsilon \varphi d\sigma_x - \frac{1}{\varepsilon \lambda_j} \int_{\Omega_\varepsilon} \nabla \tilde{U}^\varepsilon \cdot \nabla \varphi dx \right) \end{aligned}$$

with  $B^\varepsilon = \{\varphi \in H_0^1(\Omega) : \|\varphi\|_{H_\varepsilon} = 1\}$ . By Lemma 3.4,

$$\left| \int_{\Gamma_\varepsilon} \rho^\varepsilon \tilde{U}^\varepsilon \varphi d\sigma_x - \int_{\Gamma_\varepsilon} \bar{\rho} \tilde{U}^\varepsilon \varphi d\sigma_x \right| \leq C \|\varphi\|_{H^1}.$$

Thus,

$$\|K^\varepsilon \tilde{U}^\varepsilon - \frac{1}{\varepsilon \lambda_j} \tilde{U}^\varepsilon\|_{H_\varepsilon} \leq \sup_{\varphi \in B^\varepsilon} \left( \int_{\Gamma_\varepsilon} \bar{\rho} \tilde{U}^\varepsilon \varphi d\sigma_x - \frac{1}{\varepsilon \lambda_j} \int_{\Omega_\varepsilon} \nabla \tilde{U}^\varepsilon \cdot \nabla \varphi dx \right) + C.$$

It remains to use (3.98) and once again Lemma 3.4 to obtain

$$\|K^\varepsilon \tilde{U}^\varepsilon - \frac{1}{\varepsilon \lambda_j} \tilde{U}^\varepsilon\|_{H_\varepsilon} \leq \sup_{\varphi \in B^\varepsilon} \left( \frac{\bar{\rho} \sigma_y(\Gamma)}{\varepsilon |\omega|} \int_{\Omega_\varepsilon} u_j^0 \varphi dx - \frac{1}{\varepsilon \lambda_j} \int_{\Omega} a^{\text{eff}} \nabla u_j^0 \cdot \nabla \varphi dx \right) + C + \frac{C_1}{\sqrt{\varepsilon}}$$

$$\leq \sup_{\varphi \in B^\varepsilon} \left( \frac{\bar{\rho}}{\varepsilon} \sigma_y(\Gamma) \int_{\Omega} u_j^0 \varphi \, dx - \frac{1}{\varepsilon \lambda_j} \int_{\Omega} a^{\text{eff}} \nabla u_j^0 \cdot \nabla \varphi \, dx \right) + C + \frac{C_1}{\sqrt{\varepsilon}} = C + \frac{C_1}{\sqrt{\varepsilon}}.$$

This estimate combined with (3.97) and Lemma 3.3 yields

$$\left| \mu_k^\varepsilon - \frac{1}{\varepsilon \lambda_j} \right| \leq C + \frac{C_1}{\sqrt{\varepsilon}}$$

for some  $k = k(\varepsilon)$  and for all sufficiently small  $\varepsilon$ . Therefore,

$$|\lambda_{k(\varepsilon)}^\varepsilon - \varepsilon \lambda_j| \leq C \varepsilon^{3/2}, \quad (3.99)$$

and (3.96) follows.  $\square$

We should also understand better the convergence of spectrum in the vicinity of multiple eigenvalues of the limit operator.

**Lemma 3.8.** *Let  $\lambda_j$  be an eigenvalue of (1.16) of multiplicity  $m$ ,  $\lambda_{j-1} < \lambda_j = \dots = \lambda_{j+m-1} < \lambda_{j+m}$ . Then there are at least  $m$  families  $\{\lambda_{k_1(\varepsilon)}^\varepsilon\}, \dots, \{\lambda_{k_m(\varepsilon)}^\varepsilon\}$ ,  $k_i(\varepsilon) \neq k_l(\varepsilon)$  if  $i \neq l$ , such that*

$$(\varepsilon)^{-1} \lambda_{k_i(\varepsilon)}^\varepsilon \longrightarrow \lambda_j, \quad \text{as } \varepsilon \rightarrow 0.$$

*Proof.* For each  $i \in \{0, 1, \dots, m-1\}$  we construct  $U_{j+i}^\varepsilon = u_{j+i}^0 + \varepsilon \chi(x/\varepsilon) \phi^\varepsilon(x) \nabla u_{j+i}^0$  as in the proof of Proposition 3.7. Then

$$\left\| K^\varepsilon U_{j+i}^\varepsilon - \frac{1}{\varepsilon \lambda_i} U_{j+i}^\varepsilon \right\|_{H_\varepsilon} \leq \frac{C}{\sqrt{\varepsilon}}, \quad i = 0, \dots, m-1. \quad (3.100)$$

In the same way as in the proof of Proposition 3.7 one can check that

$$|(U_{j+i}^\varepsilon, U_{j+l}^\varepsilon)_{H_\varepsilon} - \delta_{il}| \leq C \sqrt{\varepsilon}, \quad 0 \leq i, l \leq m-1. \quad (3.101)$$

Denote by  $\lambda_{k_1(\varepsilon)}^\varepsilon, \dots, \lambda_{k_N(\varepsilon)}^\varepsilon$  the eigenvalues that belong to the interval  $\varepsilon(\lambda_j - \varepsilon^{1/4}, \lambda_j + \varepsilon^{1/4})$  with  $N = N(\varepsilon)$ . According to Lemma 3.3 there are linear combinations of the corresponding eigenfunctions  $V_i^\varepsilon = \sum_{s=1}^{N(\varepsilon)} \beta_{is}^\varepsilon u_{k_s(\varepsilon)}^\varepsilon$  such that  $\|U_{j+i}^\varepsilon - V_i^\varepsilon\|_{H_\varepsilon} \leq C \varepsilon^{1/4}$ . From (3.100) and (3.101) it follows that  $N(\varepsilon) \geq m$  for all sufficiently small  $\varepsilon$ , this yields the desired statements.  $\square$

The opposite inequality is granted by

**Lemma 3.9.** *Assume that there are families  $k_1(\varepsilon), \dots, k_N(\varepsilon)$ ,  $k_i \neq k_l$  if  $i \neq l$ , such that, for a subsequence,*

$$\frac{1}{\varepsilon} \lambda_{k_i(\varepsilon)}^\varepsilon \longrightarrow \lambda_j, \quad \text{as } \varepsilon \rightarrow 0, \quad i = 1, \dots, N.$$

*Then the multiplicity of  $\lambda_j$  is at least  $N$ .*

*Proof.* Consider the eigenpairs  $(\lambda_{k_i(\varepsilon)}^\varepsilon, u_{k_i(\varepsilon)}^\varepsilon)$  with the eigenfunctions satisfying (1.20). Then, for a subsequence,

$$u_{k_i(\varepsilon)}^\varepsilon \rightharpoonup v_i \quad \text{weakly in } H_0^1(\Omega), \quad i = 1, \dots, N.$$

It was shown in Proposition 3.2, that  $v_i$  are eigenfunctions of the homogenized problem with eigenvalue  $\lambda_j$ , and that

$$\delta_{il} = \lim_{\varepsilon \rightarrow 0} (u_{k_i(\varepsilon)}^\varepsilon, u_{k_l(\varepsilon)}^\varepsilon)_{H_\varepsilon} = \lambda_j \bar{\rho} \sigma_y(\Gamma) (v_i, v_l)_{L^2(\Omega)}.$$

Therefore,  $\{v_i\}_{i=1}^N$  are nontrivial and orthogonal in  $L^2(\Omega)$ , and thus the multiplicity of  $\lambda_j$  is at least  $N$ .  $\square$

Now the statements (i), (ii) and (iv) of Theorem 1.4 are immediate consequence of Propositions 3.2 and 3.7, Lemmata 3.8 and 3.9 and estimate (3.99).

In order to justify the statement (iii) we consider an eigenvalue  $\lambda_j$  of (1.16) that has multiplicity  $m_j$ ,  $m_j \geq 1$ , so that  $\lambda_j = \dots = \lambda_{j+m_j-1}$ . Choosing  $d_j = \frac{1}{3} \min(1/\lambda_{j-1} - 1/\lambda_j, 1/\lambda_j - 1/\lambda_{j+m_j})$ , with the help of item (i) we conclude that for all sufficiently small  $\varepsilon$  an eigenvalue  $(\lambda_i^\varepsilon)^{-1}$  belongs to the interval  $\varepsilon^{-1}((\lambda_j)^{-1} - d_j, (\lambda_j)^{-1} + d_j)$  if and only if  $j \leq i \leq j + m_j - 1$ . Using (3.100) and applying Lemma 3.3 with  $d = \varepsilon^{-1}d_j$ , we obtain that there exist  $\beta_{il}^\varepsilon$  such that

$$\|U_{j+i}^\varepsilon - \sum_{l=0}^{m_j-1} \beta_{il}^\varepsilon u_{j+l}^\varepsilon\|_{H_\varepsilon} \leq C\sqrt{\varepsilon}.$$

This estimate combined with (3.101) implies the desired statement (iii). The proof can be found in [1]. We omit the details. This completes the proof of Theorem 1.4.

**Remark 3.10.** *If in the conditions of Theorem 1.4 we suppose that  $\Omega$  and  $\omega$  are just Lipschitz continuous domains then the statements on convergence of the spectrum remain valid, however, the estimates for the rate of convergence might fail to hold. More precisely, in the case of Lipschitz continuous  $\partial\Omega$  and  $\partial\omega$  the following statement holds:*

- For any  $j \in \mathbb{N}$  the limit relation (1.40) is valid.
- Let  $\lambda_j$  be an eigenvalue of (1.16) of multiplicity  $m_j$  with  $m_j \geq 1$ , that is  $\lambda_j = \dots = \lambda_{j+m_j-1}$ . Then there is a orthogonal matrix  $\beta_{il}^\varepsilon$ ,  $0 \leq i, l \leq m_j - 1$ , such that

$$\lim_{\varepsilon \rightarrow 0} \|u_{i+j}^\varepsilon - \sum_{l=0}^{m_j-1} \beta_{il}^\varepsilon u_{l+j}^\varepsilon\|_{L^2(\Omega)} = 0. \quad (3.102)$$

The proof follows the same strategy as in the case of smooth domains  $\Omega$  and  $\omega$ . We leave the details to the reader.

**Remark 3.11.** The convergence of eigenspaces related to multiple eigenvalues of the effective spectral problem can be expressed in terms of the so-called Mosco convergence, see [1] for its definition. Namely, if  $\lambda_j$  is an eigenvalue of (1.16) of multiplicity  $m_j$  with  $\lambda_j = \dots = \lambda_{j+m_j-1}$ ,  $m_j \leq 1$ , then  $\text{span}\{u_j^\varepsilon, \dots, u_{j+m_j-1}^\varepsilon\}$  Mosco-converges to  $\text{span}\{u_j, \dots, u_{j+m_j-1}\}$ .

## 4 The case $\bar{\rho} = 0$

In this section we prove Theorem 1.7. Our first goal is to show that there exists  $\kappa_1 > 0$  such that for all sufficiently small  $\varepsilon > 0$  the estimate holds

$$\mu_1^\varepsilon \geq \kappa_1 \quad (4.103)$$

with  $\mu_1^\varepsilon$  defined in (2.61). To justify this estimate we substitute in (2.61) a test function of the form  $u_\varepsilon(x) = \varphi(x)(\bar{u} + \varepsilon\pi(\frac{x}{\varepsilon}))$  with  $\bar{u} \in \mathbb{R}^+$ ,  $\varphi \in C_0^\infty(\Omega)$ ,  $\varphi \neq 0$ , and  $\pi \in C_\#^\infty(Y)$  such that

$$\gamma_\pi = \int_\Gamma \rho(y)\pi(y) d\sigma_y > 0.$$

It is easy to check that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \bar{u}^2 |\omega| \int_\Omega |\nabla \varphi|^2 dx + \int_\Omega \varphi^2 dx \int_Y |\nabla \pi(y)|^2 dy > 0.$$

The surface integral can be estimated as follows

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \rho_\varepsilon u_\varepsilon^2 d\sigma_x &= \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2 \bar{u}^2 d\sigma_x \\ &+ \lim_{\varepsilon \rightarrow 0} 2\varepsilon \int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2 \bar{u} \pi\left(\frac{x}{\varepsilon}\right) d\sigma_x + \lim_{\varepsilon \rightarrow 0} \varepsilon^2 \int_{\Gamma_\varepsilon} \rho_\varepsilon \varphi^2 \pi^2\left(\frac{x}{\varepsilon}\right) d\sigma_x = \\ &= \bar{u}^2 \int_\Gamma y \rho(y) d\sigma_y \int_\Omega \nabla(\varphi^2) dx + 2\gamma_\pi \bar{u} \int_\Omega (\varphi)^2 dx = 2\gamma_\pi \bar{u} \int_\Omega (\varphi)^2 dx > 0; \end{aligned}$$

here we have also used Lemma 3.1. This implies (4.103) for all sufficiently small  $\varepsilon$ .

Similar lower bounds can be obtained for  $\mu_j^\varepsilon$  with  $j > 1$ . However, since these bounds will follow from the asymptotics constructed later on in this section, we do not bother the reader with their proof here.

An upper bound for  $\mu_1^\varepsilon$  easily follows from (3.87). Indeed, since  $\bar{\rho} = 0$ , for any  $u \in H_0^1(\Omega)$  by (3.87) we have

$$\int_{\Gamma_\varepsilon} \rho_\varepsilon u^2 d\sigma_x \leq c \|u\|_{H^1(\Omega_\varepsilon)}^2 \leq c \|\nabla u\|_{L^2(\Omega_\varepsilon)}^2.$$

In view of (2.61) this yields

$$\mu_1^\varepsilon \leq \kappa_2. \quad (4.104)$$

Estimates (4.103) and (4.104) suggest that the asymptotic series for  $\lambda_{\pm j}^\varepsilon$  and  $u_{\pm j}^\varepsilon$  should be of the form

$$\lambda_{\pm j}^\varepsilon = \lambda + \varepsilon \lambda_1 + \dots, \quad u_{\pm j}^\varepsilon = u(x) + \varepsilon u_1(x, x/\varepsilon) + \dots$$

with  $u_1(x, y)$  being periodic in  $y$ . Substituting these series in (1.5) and collecting power-like terms in the resulting equation and boundary condition, we conclude that

$$u_1(x, y) = \varepsilon \chi(y) \nabla u(x) + \varepsilon \lambda_{\pm j} \theta(y) u(x),$$

where  $\chi$  and  $\theta$  are solutions of problems (1.18) and (1.44), respectively, and

$$-\operatorname{div}(a^{\text{eff}} \nabla u) = \lambda^2 \Xi u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

**Remark 4.1.** Notice that the first order term in  $\lambda$  is not presented in the limit equation. Indeed, the formal derivation yields a first order term of the form

$$\lambda Du(x) \cdot \left( \int_{\Gamma} \theta(y) \nu(y) d\sigma_y - \int_{\omega} \nabla_y \theta(y) dy \right),$$

this term is equal to zero.

*Proof of Theorem 1.7.*

The following statement can be proved in exactly the same way as Proposition 3.7 and Lemma 3.8 in the previous section. We leave its proof to the reader.

**Lemma 4.2.** Let  $\lambda_j$  be an eigenvalue of (1.45) of multiplicity  $m$ ,  $m \geq 1$ ,  $\lambda_{j-1} < \lambda_j = \dots = \lambda_{j+m-1} < \lambda_{j+m}$ . Then there are at least  $m$  families  $\{\lambda_{k_1(\varepsilon)}^\varepsilon\}, \dots, \{\lambda_{k_m(\varepsilon)}^\varepsilon\}$  such that  $k_i(\varepsilon) \neq k_l(\varepsilon)$  if  $i \neq l$ , and

$$\lambda_{k_i(\varepsilon)}^\varepsilon \longrightarrow \lambda_j, \quad \text{as } \varepsilon \rightarrow 0.$$

The statements similar to those of Proposition 3.2 and Lemma 3.9 also remain valid.

**Proposition 4.3.** Let  $\{(\lambda_{j(\varepsilon)}^\varepsilon, u_{j(\varepsilon)}^\varepsilon)\}$  be a family of normalized eigenpairs of problem (1.5) or, equivalently, (1.6), and assume that, perhaps for a subsequence,  $\lambda_{j(\varepsilon)}^\varepsilon \rightarrow \bar{\lambda}$ , as  $\varepsilon \rightarrow 0$ . Then  $\bar{\lambda}$  is an eigenvalue of the limit problem (1.45). If, in addition,  $u_{j(\varepsilon)}^\varepsilon$  converges to  $\bar{u}$  weakly in  $H_0^1(\Omega)$  for the same subsequence of  $\varepsilon$ , then  $\bar{u} \neq 0$ , and  $(\bar{\lambda}, \bar{u})$  is an eigenpair of (1.45).

**Lemma 4.4.** Assume that there are families  $k_1(\varepsilon), \dots, k_N(\varepsilon)$ ,  $k_i \neq k_l$  if  $i \neq l$ , such that, for a subsequence,

$$\lambda_{k_i(\varepsilon)}^\varepsilon \longrightarrow \lambda_j, \quad \text{as } \varepsilon \rightarrow 0, \quad i = 1, \dots, N.$$

Then the multiplicity of  $\lambda_j$  is at least  $N$ .

Lemmata 4.2 and 4.4 and Proposition 4.3 imply the desired statements of Theorem 1.7.  $\square$

## 5 Proof of Theorem 1.5

The goal of this section is to prove Theorem 1.5. Thus it is assumed here that  $\bar{\rho} > 0$ . We begin by introducing a new unknown function and a new spectral parameter in (1.7). Namely, we set

$$u_\varepsilon(x) = p_{-1}\left(\frac{x}{\varepsilon}\right)v_\varepsilon(x), \quad \lambda = \frac{\alpha_{-1}}{\varepsilon} + \tilde{\lambda}$$

with  $p_{-1}$  and  $\alpha_{-1}$  defined in (1.22)–(1.26), respectively. Substituting these expressions in (1.7) we deduce after straightforward rearrangements that in terms of  $v_\varepsilon$  and  $\tilde{\lambda}$  problem (1.7) reads

$$\begin{cases} -\operatorname{div}(\tilde{a}(x/\varepsilon)\nabla v_\varepsilon) = 0 & \text{in } \tilde{\Omega}_\varepsilon, \\ \tilde{a}(x/\varepsilon)Dv_\varepsilon \cdot \nu_\varepsilon = \tilde{\lambda}(p_{-1}(x/\varepsilon))^2\rho(x/\varepsilon)v_\varepsilon & \text{on } \Gamma_\varepsilon, \\ v_\varepsilon = 0 & \text{on } \partial\tilde{\Omega}_\varepsilon \setminus \Gamma_\varepsilon; \end{cases} \quad (5.105)$$

here we have denoted  $\tilde{a}(y) = (p_{-1}(y))^2 \mathbf{I}$ . For the sake of brevity in this section we use the notation  $\tilde{a}^\varepsilon(x) = \tilde{a}(x/\varepsilon)$  and  $\tilde{\rho}^\varepsilon(x) = (p_{-1}(x/\varepsilon))^2\rho(x/\varepsilon)$ . We remind that under our regularity assumptions,  $p(\cdot)$  is a smooth positive function.

Since by construction (see (1.26))

$$\int_{\Gamma} (p_{-1}(y))^2 \rho(y) d\sigma_y < 0,$$

Theorem 1.4 applies to the negative part of the spectrum of problem (5.105). Although, in contrast with (1.7), in (5.105) we do not deal with the Laplacian but with a more general divergence form elliptic operator with periodic coefficients, the results stated in Theorem 1.4 remain valid. Namely, using exactly the same arguments as in the proof of Theorem 1.4 one can show that the statements (i)–(iv) of Theorem 1.4 hold true for the negative part of the spectrum of problem (5.105).

In order to complete the proof of Theorem 1.5 it remains to prove that on the interval  $(\frac{\alpha_{-1}}{\varepsilon}, 0)$  there are no eigenvalues of problem (1.7).

**Proposition 5.1.** *The interval  $(\frac{\alpha_{-1}}{\varepsilon}, 0)$  belongs to the resolvent set of problem (1.7).*

*Proof.* The proof relies on Floquet-Bloch representation of  $u_\varepsilon$  and follows the line of the proof of Theorem 5 and Lemma 11 in [11]. □

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