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In this paper the essential spectrum of the linear problem on water-waves in a water layer and in a channel with gently corrugated bottom is studied. We show that under a certain geometric condition the essential spectrum has spectral gaps. In other words there exist intervals in the positive real semi-axis which are free of the spectrum but have their endpoints in it. The position and the length of the gaps are found out by applying an asymptotic analysis to the model problem in the periodicity cell.

Keywords: water waves, sufficient conditions for spectral gaps, spectral problem, self-adjoint trace operator

1. Introduction

(*a*) *The problem*

We study the essential spectrum of the linearised water wave equation in a liquid layer with the gently periodically corrugated bottom. Our aim is to show that under a certain geometric condition the essential spectrum of the problem has spectral gaps, which means that the propagation of waves with frequencies in a certain range cannot take place. For such frequencies, the water wave problem is uniquely solvable. The position and length of the gaps are found out by applying an asymptotic analysis to the model problem in the periodicity cell. We state the mathematical problem and we discuss in detail motivations and known results in the following subsections.

We consider a layer Ξ^ε of an incompressible inviscid liquid (water) above the slightly corrugated surface

$$\{(x,y,z) \in \mathbb{R}^3 : z = -d + \varepsilon h(y)\} \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter and h is a smooth periodic function. Without a loss of generality we may assume that the period is equal to 1 and that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} h(y)\,\mathrm{d}y = 0, \quad h\left(-\tfrac{1}{2}\right) = h\left(\tfrac{1}{2}\right) = 0. \tag{1.2}$$

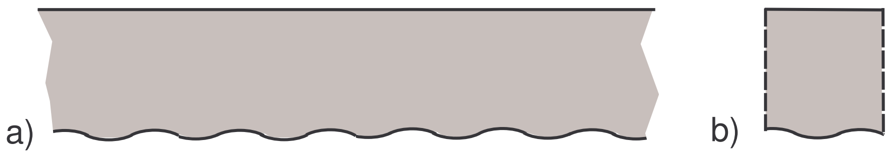

Figure 1. A channel with a corrugated bottom and the periodicity cell

This rescaling makes the Cartesian coordinates (x, y, z) and the positive parameters d, ε dimensionless. The reference position of the free surface is fixed at the xy-plane

$$\{(x, y, z) \in \mathbb{R}^3 : z = 0\}. \tag{1.3}$$

The small amplitude motion of the liquid is described (see, e.g., Kuznetsov *et al.* 2002) by a velocity potential $\varphi(x, y, z, t)$ satisfying the Laplace equation in Ξ^ε . The linearised kinematic/dynamic boundary condition

$$\partial_t^2 \varphi = -g \partial_z \varphi$$

is imposed at the free surface (1.3) and at the bottom (1.2) we set the Neumann boundary condition (no normal flow)

$$\partial_n \varphi = 0.$$

We search for solutions of the problem in the form

$$\varphi(x, y, z, t) = \mathrm{e}^{-i\omega^\varepsilon t} \mathrm{e}^{ikx} \Phi^\varepsilon(y, z) \tag{1.4}$$

with the frequency $\omega^\varepsilon > 0$ and the wave number $k \geq 0$ in the x-direction. Then the two-dimensional spectral boundary-value problem for Φ^ε reads as follows:

$$-\Delta\Phi^\varepsilon(y, z) + k^2\Phi^\varepsilon(y, z) = 0, \ (y, z) \in \Pi^\varepsilon, \tag{1.5}$$

$$\partial_z \Phi^\varepsilon(y, 0) = \lambda^\varepsilon \Phi^\varepsilon(y, 0), \ y \in \mathbb{R} = (-\infty, +\infty), \tag{1.6}$$

$$\partial_n \Phi^\varepsilon(y, -d + \varepsilon h(y)) = 0, \ y \in \mathbb{R}. \tag{1.7}$$

Here, $\lambda^\varepsilon = g^{-1}(\omega^\varepsilon)^2$ is the spectral parameter, $g > 0$ is the acceleration due to the gravity and Π^ε is a periodically curved strip (see Fig. 1,a) between the curves

$$\Gamma = \{(y, z) : z = 0\} \ \text{ and } \ \Gamma_d^\varepsilon = \{(y, z) : z = -d + \varepsilon h(y)\}. \tag{1.8}$$

The boundary value problem (1.5)-(1.7) requires, in principle, certain radiation conditions but since we are interested only on the examination of its spectrum, we do not need them for a while. The rigorous mathematical formulation of the spectral problem will be given in §4 *a* .

(*b*) *Spectra*

When $\varepsilon = 0$, the cross-section of the fluid layer is just the straight strip $\Pi^0 = \mathbb{R} \times (-d, 0)$. As well-known (see, e.g., Section 1.5.4 in Kuznetsov *et al.* 2002), in this

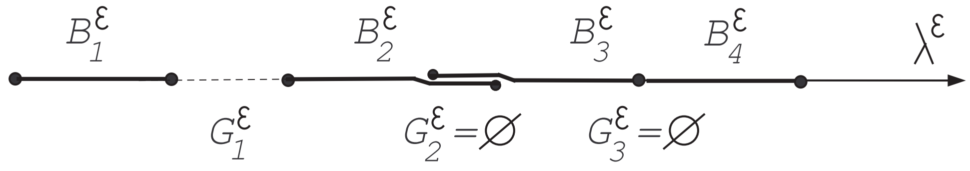

Figure 2. The band gap structure of the spectrum for a fixed k

case the continuous spectrum $\sigma_c^0(k)$ is the semi-infinite interval $[\lambda_\dagger^0(k), +\infty)$ for the corresponding reference ($\varepsilon = 0$) problem (1.5)-(1.7), where the cutoff point is

$$\lambda_\dagger^0(k) = k\frac{1 - \mathrm{e}^{-2kd}}{1 + \mathrm{e}^{-2kd}} = k \tanh kd =: D(k). \tag{1.9}$$

For any $\lambda^0 \geqslant \lambda_\dagger^0$ the function

$$w_\pm^0(y, z) = (\mathrm{e}^{Kz} + \mathrm{e}^{-K(z+2d)})\mathrm{e}^{\pm iy\sqrt{K^2-k^2}} \tag{1.10}$$

solves the problem (1.5)-(1.7) with $\varepsilon = 0$ and is called *a wave.* It is either a standing wave ($\lambda^0 = \lambda_\dagger^0(k)$ and $K = k$) or an oscillating wave propagating into infinity ($\lambda^0 > \lambda_\dagger^0(k)$ and $K > k$), the sign in the exponent informs the direction of the propagating wave. The number K is the only root of the transcendental equation

$$\lambda^0 = D(K).$$

The uniqueness of K follows from the strict monotonicity of the function $D(k)$ in (1.9), for $k \geq 0$. All calculations leading to this conclusions can be found out in Section 1.5.4 of Kuznetsov *et al.* (2002).

In the periodic strip Π^ε with $\varepsilon > 0$ the spectrum σ^ε has much more complicated structure. The Gel'fand transform, Gel'fand (1950), and the theory of Floquet-Bloch waves (see Skriganov 1985, Kuchment 1993, Nazarov & Plamenevsky 1994, and others) ensure that the essential spectrum $\sigma_e^\varepsilon(k)$ in problem (1.5)-(1.7) gets the band-gap structure and implies the union of closed segments (cf. Fig. 2)

$$\sigma_e^\varepsilon(k) = \bigcup_{m=1}^{\infty} B_m^\varepsilon \subset [0, +\infty). \tag{1.11}$$

These segments may touch or overlap each other (see on the right in Fig. 2) and, in principle, $\sigma_e^\varepsilon(k)$ can coincide with the interval $[\lambda_\dagger^\varepsilon(k), +\infty)$ with a non-negative cutoff point $\lambda_\dagger^\varepsilon(k)$ as well. However, the structure (1.11) permits spectral gaps (see on the left in Fig. 2), i.e., intervals in the positive real semi-axis $\mathbb{R}_+$ which have both endpoints in $\sigma_e^\varepsilon(k)$ but are free of the essential spectrum. In other words, a spectral gap forbids the propagation of waves with frequencies in a certain range. In this way the water-wave problem is uniquely solvable in the entire strip Π^ε, if λ^ε belongs to a spectral gap.

Since the profile function is 1-periodic it attains the Fourier representation

$$h(x) = a_0 + \sum_{p=1}^{\infty} a_p \cos(2\pi px) + b_p \sin(2\pi px). \tag{1.12}$$

We will show in this paper that the condition $|a_p|^2 + |b_p|^2 \neq 0$ for some $p \in \mathbb{N}$ together with $k^2 \neq \pi^2 p^2$ guarantee the opening of a spectral gap for a small ε. We will also provide its asymptotic position and length.

Our main result is stated in §3 *c*, while its proof provided is in §4.

(*c*) *A short review of known results*

Band gap engineering knows several approaches to discover a gap in the spectrum of an elliptic operator in a periodic media, which usually is assumed to be periodic in all spatial directions. The gap opening is then achieved by the manipulation of the coefficients of the differential operator, which are assumed to be contrasting, i.e. they depend on certain big (or small) parameters. Physically, this means a local periodic change in the material properties. For scalar problems, one can refer to the papers by Figotin & Kuchment (1996), Friedlander (2002) and Zhikov (2004), and for systems in the papers by Filonov 2003 and Nazarov 2010*c* and many others. Similar works can be found in the context of photonic structures (see, for example, Busch *et al.* 2006). The situation changes crucially, if the corresponding model problem has cylindrical (not periodic) outlets into infinity. In this case, the existing continuous spectrum cannot possess a gap in it (see Nazarov 2008*b*).

Our problem in water waves contains only one physical parameter - the wave number k. Therefore, the only possibility to create a spectral gap is to vary the geometry of the objects. Several results were obtained on the opening of a spectral gap for periodic wave guides by Yoshitomi (1998), Friedlander & Solomyak (2008), Nazarov (2010*a*) and Cardone *et al.* 2010, but only for the Dirichlet boundary condition.

The Steklov spectral boundary condition (1.6) requires new techniques, especially with respect to the asymptotic analysis. We emphasise that in the paper by Nazarov (2010*b*) an approach to detect gaps in the essential spectrum of an infinite periodic family of ponds connected by thin and short channels is based on an application of specific weighted Hardy inequalities, and the Max-min principle, but it does not employ the asymptotic behaviour of the eigenvalues, which is the key point in our study here.

In the literature there are numerous treatments on the propagation of water waves along periodic structures, e.g. along an array of identical obstacles, an underwater ridge or periodic coastlines, c.f. Evans & Porter (1999) and (2005). In the papers by O'Hare & Davies (1993), Porter & Porter (2003) the behaviour of water waves over periodic beds was considered in the two-dimensional context but with a different research objective so that spectral gaps were not examined and discovered. In McIver (2001), the propagation of water waves through a double periodic array, with the periods a in x-variable and b in y-variable, of vertical columns was studied mainly numerically. Roughly speaking these results concern the "passing" and "stopping" bands for the two-dimensional Bloch waves

$$\mathrm{e}^{i(xl+ym)}\Phi(x,y)\cosh(k(z+d)) \tag{1.13}$$

with the wave vector $(l,m) \in [0, 2\pi/a) \times [0, 2\pi/b)$ and a (a,b)-periodic function Φ which satisfies the Neumann problem for the Helmholtz equation $\Delta\Phi + k^2\Phi = 0$ in the perforated plane. If the wave vector (l,m) belongs to a passing band, the wave (1.13) exists, but when (l, m) belongs to the stopping band, it does not exist. However, the question of gap opening was not posed there. Notice that to discover a spectral gap is to point out a spectral parameter $\lambda = g^{-1}\omega^2$ for which no passing band occurs at all. This was done in a recent paper by Nazarov *et. al.* (2012). In Hu *et. al.* (2003) and Linton (2011) the band gaps for liquid surface waves propagating over periodic structures were found and investigated.

With arbitrary but fixed wave number $k \geqslant 0$ in the x-direction, we will prove rigorously the existence of gaps for the two-dimensional problem (1.5)-(1.7) which forbid waves (1.4) to propagate with the frequency in the corresponding intervals. The intrinsic interpretation of our result for the original three-dimensional problem in the water layer Ξ^ε is just the appearance of stopping bands for any $k \geqslant 0$. In this way the question of the existence of spectral gaps in three-dimensional water domain infinite in all horizontal directions remain open.

At the same time, in §5 we will show that the water-wave problem in the three-dimensional channel

$$\Xi^\varepsilon = \{(x, y, z) : -\tfrac{1}{2}l < x < \tfrac{1}{2}l,\ -\infty < y < \infty,\ -d + \varepsilon h(y) < z < 0\} \qquad (1.14)$$

with rectangular cross-sections surely gets spectral gaps under a certain restriction on the geometrical parameters l, d, and $h(y)$.

2. The model problem in the periodicity cell and Floquet waves

(*a*) *The model problem in the periodicity cell*

In this section, we will show how our original problem (1.5)-(1.7) is related to a family of spectral problems defined in a bounded domain. To this end, we need to recall the definition and main properties of the well-known Gel'fand transform (see Gel'fand 1950)

$$\Phi(y, z) \mapsto U(y, z; \eta) = \frac{1}{\sqrt{2\pi}} \sum_{p \in \mathbb{Z}} \mathrm{e}^{-i\eta p} \Phi(y + p, z) \qquad (2.1)$$

where η is called the Floquet parameter (see also, e.g., Kuchment 1993, Nazarov & Plamenevsky 1994) where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$. Notice that $(y, z) \in \Pi^\varepsilon$ on the left of (2.1) but on the right (y, z) belongs to the periodicity cell

$$\varpi^\varepsilon = \{(y, z) : |y| < 1/2, -d + \varepsilon h(y) < z < 0\}.$$

Moreover, the dual variable η may be assumed to stay in $[0, 2\pi)$ because the shift $\eta \mapsto \eta \pm 2\pi$ does not affect (2.1), since $\mathrm{e}^{\pm 2\pi i} = 1$.

For the reader's convenience, we recall here that the Gel'fand transform establishes the isometric isomorphism $L^2(\Pi^\varepsilon) \cong L^2(0, 2\pi; L^2(\varpi^\varepsilon))$, where $L^2(\Pi^\varepsilon)$ is the usual Lebesgue space and $L^2(0, 2\pi; \mathcal{B})$ is the Banach-valued Lebesgue space with the norm

$$||u; L^2(0, 2\pi; \mathcal{B})|| = \left(\int_0^{2\pi} \|u(\eta); \mathcal{B}\|^2 \, \mathrm{d}\eta \right)^{\frac{1}{2}}.$$

Furthermore, if Φ belongs to the Sobolev space $H^1(\Pi^\varepsilon)$, then $U \in L^2(0,2\pi;H^1(\varpi^\varepsilon))$ and we have the norm equivalence

$$c||\Phi;H^1(\Pi^\varepsilon)|| \leqslant ||U;L^2(0,2\pi;H^1(\varpi^\varepsilon))|| \leq C||\Phi;H^1(\Pi^\varepsilon)||,\ 0<c\leqslant C$$

and, moreover, U satisfies the quasi-periodicity condition

$$U\big(-\tfrac{1}{2},z;\eta\big) = \mathrm{e}^{-i\eta}U(\tfrac{1}{2},z;\eta),\ \text{for a.e. } z \in \big(-d+\varepsilon h(\tfrac{1}{2}),0\big) = (-d,0),\ \eta \in [0,2\pi). \tag{2.2}$$

The inverse transform is given by

$$U(y,z;\eta) \mapsto \Phi(y,z) = \frac{1}{\sqrt{2\pi}}\int_0^{2\pi} \mathrm{e}^{i\eta[y]}U(y-[y],z;\eta)\,\mathrm{d}\eta, \tag{2.3}$$

where $[y]$ denotes the integer part of the real number y.

Applying the Gel'fand transform to the variational formulation of the spectral problem (1.5)-(1.7) in Π^ε,

$$(\nabla\Phi^\varepsilon,\nabla\Psi^\varepsilon)_{\Pi^\varepsilon} + k^2(\Phi^\varepsilon,\Psi^\varepsilon)_{\Pi^\varepsilon} = \lambda^\varepsilon(\Phi^\varepsilon,\Psi^\varepsilon)_\Gamma, \quad \Psi^\varepsilon \in H^1(\Pi^\varepsilon), \tag{2.4}$$

and making use of the Parseval identity

$$(\Phi^\varepsilon,\Psi^\varepsilon)_{\Pi^\varepsilon} = \int_0^{2\pi}(U^\varepsilon(\cdot;\eta),V^\varepsilon(\cdot;\eta))_{\varpi^\varepsilon}\,\mathrm{d}\eta$$

we arrive at the spectral variational problem

$$(\nabla U^\varepsilon,\nabla V^\varepsilon)_{\varpi^\varepsilon} + k^2(U^\varepsilon,V^\varepsilon)_{\varpi^\varepsilon} = \Lambda^\varepsilon(U^\varepsilon,V^\varepsilon)_\gamma,\ V^\varepsilon \in H^1_{(\eta)}(\varpi^\varepsilon) \tag{2.5}$$

in the periodicity cell ϖ^ε. Here, $(\cdot,\cdot)_\Xi$ is the natural scalar product in $L^2(\Xi)$, U^ε and V^ε are the Gel'fand images of Φ^ε and Ψ^ε. Λ^ε is the new notation for the spectral parameter λ^ε. The free surface of the periodicity cell is denoted by $\gamma = \{(y,z) : |y| < \frac{1}{2}, z = 0\}$. The space $H^1_{(\eta)}(\varpi^\varepsilon)$ is the subspace of the Sobolev space $H^1(\varpi^\varepsilon)$ satisfying the quasi-periodicity conditions (2.2) with $\eta \in [0,2\pi)$. The corresponding differential formulation of the problem (2.5) reads now as

$$-\Delta U^\varepsilon(y,z;\eta) + k^2U^\varepsilon(y,z;\eta) = 0,\ (y,z) \in \varpi^\varepsilon, \tag{2.6}$$

$$\partial_z U^\varepsilon(y,z;\eta) = \Lambda^\varepsilon U^\varepsilon(y,z;\eta),\ (y,z) \in \gamma, \tag{2.7}$$

$$\partial_n U^\varepsilon(y,z;\eta) = 0,\ (y,z) \in \gamma^\varepsilon_d, \tag{2.8}$$

$$U^\varepsilon(-\tfrac{1}{2},z;\eta) = \mathrm{e}^{-i\eta}U^\varepsilon(\tfrac{1}{2},z;\eta) \quad , \quad \partial_y U^\varepsilon(-\tfrac{1}{2},z;\eta) = \mathrm{e}^{-i\eta}\partial_y U^\varepsilon(\tfrac{1}{2},z;\eta). \tag{2.9}$$

Here we have denoted by $\gamma^\varepsilon_d = \{(y,z) : |y| < \frac{1}{2}, z = -d + \varepsilon h(y)\}$ the bottom part (1.1) of the periodicity cell. The quasi-periodicity conditions (2.9) are imposed on the lateral sides $\tau_\pm = \{\pm\frac{1}{2}\} \times (-d,0)$ of the cell.

Since the sesquilinear forms on the left and the right-hand side of (2.5) are positive for every $\eta \in [0,2\pi)$ and the embedding $H^1(\varpi^\varepsilon) \subset L^2(\gamma)$ is compact, the problem (2.5) (or (2.6)-(2.9) in the differential form) with $\eta \in [0,2\pi)$ admits the eigenvalue sequence

$$0 \leq \Lambda^\varepsilon_1(\eta) \leq \Lambda^\varepsilon_2(\eta) \leq \ldots \leq \Lambda^\varepsilon_m(\eta) \leq \ldots \to +\infty. \tag{2.10}$$

Clearly the functions $\eta \mapsto \Lambda^\varepsilon_m(\eta)$ are continuous and 2π-periodic (see for example Kato 1966, Ch. 7). Hence the sets

$$B^\varepsilon_m = \{\Lambda^\varepsilon_m(\eta) : \eta \in [0,2\pi)\} \tag{2.11}$$

are bounded, closed, connected segments.

(*b*) *Floquet waves*

Let $f^\varepsilon \in H^1(\Pi^\varepsilon)^*$ be a linear bounded functional in the space $H^1(\Pi^\varepsilon)$. We can associate with the variational form of the inhomogeneous problem (1.5)-(1.7), cf. (2.4),

$$(\nabla\Phi^\varepsilon, \nabla\Psi^\varepsilon)_{\Pi^\varepsilon} + k^2(\Phi^\varepsilon, \Psi^\varepsilon)_{\Pi^\varepsilon} - \lambda^\varepsilon(\Phi^\varepsilon, \Psi^\varepsilon)_\Gamma = f^\varepsilon(\Psi^\varepsilon), \quad \Psi^\varepsilon \in H^1(\Pi^\varepsilon), \quad (2.12)$$

the continuous operator $A^\varepsilon(\lambda^\varepsilon)$

$$H^1(\Pi^\varepsilon) \ni \Phi^\varepsilon \mapsto A^\varepsilon(\lambda^\varepsilon)\Phi^\varepsilon = f^\varepsilon \in H^1(\Pi^\varepsilon)^*. \quad (2.13)$$

Let us note that, by definition, the essential spectrum σ_e^ε consists of those values λ^ε for which a continuous inverse of the operator $A^\varepsilon(\lambda^\varepsilon)$ cannot be defined on the range $\operatorname{Im} A^\varepsilon(\lambda^\varepsilon) = A^\varepsilon(\lambda^\varepsilon)H^1(\Pi^\varepsilon) \subset H^1(\Pi^\varepsilon)^*$. In this way the formula (1.11) with segments (2.11) is assured by the following assertion, which is due to general results on elliptic boundary-value problems in domains with periodic outlets to infinity, in Nazarov (1981) (see also Thm 3.4.6 in Nazarov & Plamenevsky 1994).

Proposition 2.1. *The operator $A^\varepsilon(\lambda^\varepsilon)$ of problem (2.12) is an isomorphism if and only if λ^ε does not coincide with $\Lambda_m^\varepsilon(\eta)$ for any $m \in \mathbb{N}$ and $\eta \in [0, 2\pi)$. In the case when $\lambda^\varepsilon = \Lambda_m^\varepsilon(\eta)$, the range $\operatorname{Im} A^\varepsilon(\lambda^\varepsilon)$ is not a closed subspace in $H^1(\Pi^\varepsilon)^*$.*

For the time being let $U_m^\varepsilon(y, z; \eta_m)$ be an eigenfunction of problem (2.6)-(2.9) with the parameters $\eta_m \in [0, 2\pi)$ and $\Lambda^\varepsilon = \Lambda_m^\varepsilon(\eta_m)$. Then the Floquet wave W_m^ε defined by

$$W_m^\varepsilon(y, z) = \mathrm{e}^{i\eta_m j} U_m^\varepsilon(y - j, z; \eta_m), \ (y - j, z) \in \varpi^\varepsilon, \ j \in \mathbb{Z}, \quad (2.14)$$

is a (non decaying) solution of the problem (1.5)-(1.7) in Π^ε with $\lambda^\varepsilon = \Lambda_m^\varepsilon(\eta_m)$. We emphasise here that the function (2.14) is smooth due to the quasi-periodicity conditions (2.9). Changing from one periodicity cell $\varpi_j^\epsilon = \{(y, z) \in \Pi^\varepsilon : |y - j| < \frac{1}{2}\}$ to the next ϖ_{j+1}^ϵ the only change in the Floquet wave is the factor $\mathrm{e}^{i\eta_m}$. Hence the energy of the wave (2.14) in any piece of the waveguide with length one is constant. The same property is attributed to waves (1.10) in the straight strip. This is just a different way to characterise the points in the essential spectrum (cf. Kuznetsov *et al.* 1998).

On the other hand, if λ^ε falls into the spectral gap, the inhomogeneous problem (2.5) is uniquely solvable for any $\eta \in [0, 2\pi)$. Then the inverse transform (2.3) of the solution delivers a unique solution of the original problem in Π^ε. This means that the operator $T^\varepsilon(\lambda^\varepsilon)$ is an isomorphism.

(*c*) *Floquet waves in* Π^0

Since the straight strip Π^0 can be also regarded as a periodic domain with the rectangular periodicity cell $\varpi^0 = (-\frac{1}{2}, \frac{1}{2}) \times (-d, 0)$, the waves (1.10) can be represented in the form (2.14). To do so we introduce the variable

$$\zeta = \pm\sqrt{K^2 - k^2} \ \text{ for } K \geqslant k$$

and rewrite (1.10) as follows

$$w^0(y, z; \zeta) = \left(\mathrm{e}^{z\sqrt{\zeta^2 + k^2}} + \mathrm{e}^{-(z+2d)\sqrt{\zeta^2+k^2}}\right)\mathrm{e}^{i\zeta y}. \quad (2.15)$$

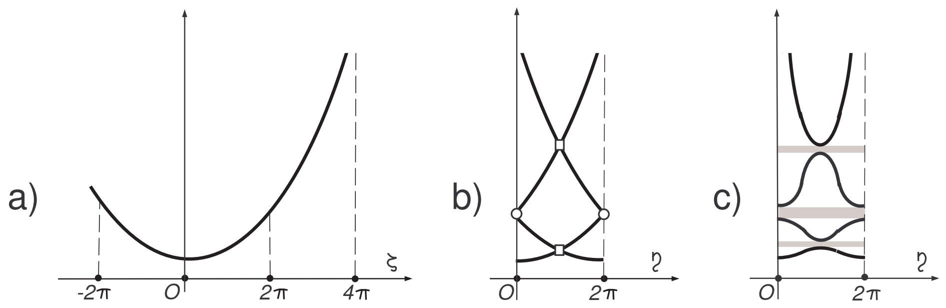

Figure 3. The dispersion curves ($k > 0$) for the reference problem (a) and (b) and for the perturbed problem (c)

Using the notation $D(k)$ in (1.9) we then set

$$\eta = \zeta - 2\pi q \in [0, 2\pi),\ q = \left[(2\pi)^{-1}\zeta\right] \in \mathbb{Z}, \tag{2.16}$$

$$\Lambda^{0,q}(\eta) = D(\sqrt{(\eta + 2\pi q)^2 + k^2}), \tag{2.17}$$

$$U^{0,q}(y, z; \eta) = \left(e^{z\sqrt{(\eta+2\pi q)^2+k^2}} + e^{-(z+2d)\sqrt{(\eta+2\pi q)^2+k^2}}\right) e^{i(\eta+2\pi q)y}. \tag{2.18}$$

Obviously $U^{0,q}(\cdot;\eta)$ is an eigenfunction and $\Lambda^{0,q}(\eta)$ the corresponding eigenvalue of the reference problem (2.6)-(2.9) with $\varepsilon = 0$. Moreover, the Floquet wave (2.14) constructed from $U^{0,q}(y, z; \zeta - 2\pi q)$ coincides with the wave (2.15). Note that the eigenpairs $\{\Lambda^{0,q}(\eta), U^{0,q}(\cdot;\eta)\}$ are supplied with the index $q \in \mathbb{Z}$. To get the eigenvalue sequence (2.10) at $\varepsilon = 0$, one needs to enumerate them in the increasing order. This can be done by setting

$$\begin{aligned} \Lambda^0_{2j-1}(\eta) &= \min\{\Lambda^{0,j-1}(\eta), \Lambda^{0,-j}(\eta)\}, \\ \Lambda^0_{2j}(\eta) &= \max\{\Lambda^{0,j-1}(\eta), \Lambda^{0,-j}(\eta)\},\ j \in \mathbb{N} \setminus \{0\}. \end{aligned} \tag{2.19}$$

The dispersion curve given by the equation $\Lambda(\zeta) = D(\sqrt{\zeta^2 + k^2})$ is drawn in Fig. 3a, for $k > 0$. The transition from ζ-variable to η-variable according to the formulae (2.16) and (2.17) cuts the curve in fig. 3a into a set of finite arcs and joins them into the curved truss as in Fig. 3b. The intersection of $\Lambda^0_{2j-1}(\eta)$ and $\Lambda^0_{2j}(\eta)$ at $\eta = \pi$ corresponds to an eigenvalue of multiplicity two. The same happens at $\eta = 0$.

In the next section we will derive a condition for the profile function h in (1.1) to ensure that for $\varepsilon > 0$ the truss frame will be disconnected into curves as shown in Fig. 3c. This disconnection of the dispersion curves provides spectral gaps indicated by the shadowed area.

3. Asymptotic analysis

(a) *Asymptotic ansätze for $\eta = \pi$.*

Let us consider two intersecting dispersion curves for the straight strip redrawn in fig. 4a. The perturbation ϖ^ε of the rectangular cell ϖ^0 may lead to two different

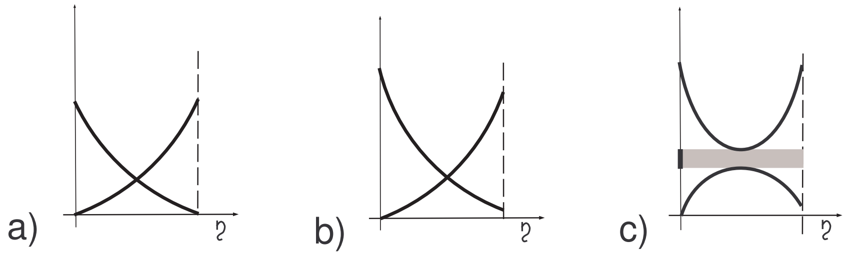

Figure 4. Perturbation of two intersecting dispersion curves (a): no gap (b) and the opening of the gap (c)

situations. In the first one, Fig. 4b, after the perturbation the curves still intersect each other preventing the existence of band gap. Quite on the contrary, the disjoint curves in Fig. 4c represent a gap in the essential spectrum $\sigma_e^\varepsilon(k)$.

From now on we fix $q \in \mathbb{N}$ and consider the graphs of the functions

$$[0, 2\pi) \ni \eta \mapsto \begin{cases} \Lambda^0_{q+}(\eta) = D(\sqrt{(\eta + 2\pi(q-1))^2 + k^2}), \\ \Lambda^0_{q-}(\eta) = D(\sqrt{(\eta - 2\pi q)^2 + k^2}), \end{cases} \tag{3.1}$$

which intersect each other at the point $\eta = \pi$. Notice that $\Lambda^0_{q+}(\eta) = \Lambda^{0,q-1}(\eta)$ and $\Lambda^0_{q-}(\eta) = \Lambda^{0,-q}(\eta)$ in the notation (2.17).

To investigate the behaviour of the functions $\eta \mapsto \Lambda^\varepsilon_{q\pm}(\eta)$ of the perturbed problem (2.6)-(2.8) in the vicinity of the point $\eta = \pi$, following the approach adopted from Nazarov (2010*a*), we introduce the deviation parameter δ, which will be varied in the proof of Theorem 3.1, and set

$$\eta = \pi + \varepsilon\delta. \tag{3.2}$$

To see whether a gap is opened near the point

$$\Lambda^0_{q\pm}(\pi) = D(\sqrt{\pi^2(2q-1)^2 + k^2}), \tag{3.3}$$

we need to construct the asymptotics of $\Lambda^\varepsilon_{q\pm}(\pi+\varepsilon\delta)$. For this we accept the following asymptotic ansätze in a domain with regularly perturbed boundary (see Ch. VII in Kato 1966, Ch. 5, in Mazya *et al.* 2000, and others)

$$\Lambda^\varepsilon_{q\pm}(\pi + \varepsilon\delta) = \Lambda^0_{q\pm}(\pi) + \varepsilon\Lambda'_{q\pm}(\delta) + \widetilde{\Lambda}^\varepsilon_{q\pm}(\pi + \varepsilon\delta), \tag{3.4}$$

$$U^\varepsilon_{q\pm}(y, z; \pi + \varepsilon\delta) = \mathcal{U}^0_{q\pm}(y, z; \pi) + \varepsilon\mathcal{U}'_{q\pm}(y, z; \delta) + \widetilde{U}^\varepsilon_{q\pm}(y, z; \pi + \varepsilon\delta), \tag{3.5}$$

where, according to (2.17) and (2.18), the main terms are given by (3.3) and

$$\begin{aligned} \mathcal{U}^0_{q\pm}(y, z; \pi) &= a^\pm_+ U^0_{q+}(y, z; \pi) + a^\pm_- U^0_{q-}(y, z; \pi), \qquad (3.6) \\ U^0_{q\pm}(y, z; \pi) &= (\mathrm{e}^{z\sqrt{\pi^2(2q-1)^2+k^2}} + \mathrm{e}^{-(z+2d)\sqrt{\pi^2(2q-1)^2+k^2}})\mathrm{e}^{\pm i\pi(2q-1)y}. \end{aligned}$$

Our task is now to find the correction terms $\Lambda'_{q\pm}(\delta), \mathcal{U}'_{q\pm}(\cdot; \delta)$ and the coefficients $a^\pm_\alpha$. The asymptotic remainders $\widetilde{\Lambda}^\varepsilon_{q\pm}(\pi + \varepsilon\delta)$ and $\widetilde{U}^\varepsilon_{q\pm}(y, z; \pi + \varepsilon\delta)$ in (3.4) and (3.5) will be estimated in §4.

(b) The asymptotic construction for $\eta = \pi$

We deal in details with the case $\eta = \pi$. Let us insert the formulae (3.4), (3.5) into the equations (2.6)-(2.9) with parameter (3.2) and extract multipliers of ε to compose a problem for $\Lambda'_{q\pm}(\delta)$ and $\mathcal{U}'_{q\pm}(\cdot;\delta)$. From (2.6) and (2.7) we immediately derive that

$$-\Delta\mathcal{U}'_{q\pm}(y,z;\delta) + k^2\mathcal{U}'_{q\pm}(y,z;\delta) = 0,\ (y,z)\in\varpi^0, \tag{3.7}$$

$$\partial_z\mathcal{U}'_{q\pm}(y,0;\delta) = \Lambda^0_{q\pm}(\pi)\mathcal{U}'_{q\pm}(y,0;\delta) + \Lambda'_{q\pm}(\delta)\mathcal{U}^0_{q\pm}(y,0;\delta),\ |y|<\tfrac{1}{2}. \tag{3.8}$$

Based on our assumption on the bottom profile (1.1) the outward normal derivative at γ^ε is given by

$$\partial_n = (1+\varepsilon^2|\partial_y h(y)|^2)^{-\frac{1}{2}}(-\partial_z + \varepsilon\partial_y h(y)\partial_y). \tag{3.9}$$

Then the Taylor formula at the point $\varepsilon = 0$ for $\partial_n\mathcal{U}^0_{q\pm}(y,-d+\varepsilon h(y);\pi)$ leads to the relation

$$\begin{aligned}\partial_n\mathcal{U}^0_{q\pm}(y,z;\pi)\big|_{\gamma^\varepsilon} = {}& (1+\varepsilon^2|h'(y)|^2)^{-\frac{1}{2}}\big(\varepsilon h'(y,-d;\pi)\partial_y\mathcal{U}^0_{q\pm}(y,-d;\pi)\\ & - \partial_z\mathcal{U}^0_{q\pm}(y,-d;\pi) - \varepsilon h(y,-d;\pi)\partial_z^2\mathcal{U}^0_{q\pm}(y,-d;\pi) + O(\varepsilon^2)\big).\end{aligned}$$

Using the boundary condition (2.8) at the bottom of the channel when $\varepsilon = 0$ and the formula (3.6) we obtain that $\partial_z\mathcal{U}^0_{q\pm}(y,-d;\pi) = 0$. Then applying the relation (2.6) at $\varepsilon = 0$ we have $-\partial_z^2\mathcal{U}^0_{q\pm} = \left(\partial_y^2 - k^2\right)\mathcal{U}^0_{q\pm}$. Hence the boundary condition at the rectified bottom γ^0 reads now as

$$-\partial_z\mathcal{U}'_{q\pm}(y,-d;\delta) = -\partial_y(h(y)\partial_y\mathcal{U}^0_{q\pm}(y,-d;\pi)) + k^2h(y)\mathcal{U}^0_{q\pm}(y,-d;\pi),\ \ |y|<\tfrac{1}{2}. \tag{3.10}$$

Finally, by (3.2) and $\mathrm{e}^{-i(\pi+\varepsilon\delta)} = \mathrm{e}^{-i\pi}(1-i\varepsilon\delta+O(\varepsilon^2))$, we derive the following quasi-periodicity conditions, or, more precisely, *inhomogeneous anti-periodicity* conditions on the lateral boundary

$$\mathcal{U}'_{q\pm}(-\tfrac{1}{2},z;\delta) = \mathrm{e}^{-i\pi}\left(\mathcal{U}'_{q\pm}(\tfrac{1}{2},z;\delta) - i\delta\mathcal{U}^0_{q\pm}(\tfrac{1}{2},z;\pi)\right), \tag{3.11}$$

$$\partial_y\mathcal{U}'_{q\pm}(-\tfrac{1}{2},z;\delta) = \mathrm{e}^{-i\pi}\left(\partial_y\mathcal{U}'_{q\pm}(\tfrac{1}{2},z;\delta) - i\delta\partial_y\mathcal{U}^0_{q\pm}(\tfrac{1}{2},z;\pi)\right). \tag{3.12}$$

Since (3.3) is an eigenvalue with multiplicity 2, we conclude by the Fredholm alternative that the problem (3.7)-(3.12) admits a solution if and only if two ($\alpha = \pm$) compatibility conditions are satisfied. To derive these conditions, one may directly insert the eigenfunction $U^0_{q\alpha}$ and the solution $\mathcal{U}'_{q\pm}$, which is assumed to exist, into the Green formula on ϖ^0. In this way we take into account all the boundary conditions (3.8)-(3.12) to obtain

$$\begin{aligned}&\Lambda'_{q\pm}(\delta)\int_{-\frac{1}{2}}^{\frac{1}{2}}\mathcal{U}^0_{q\pm}(y,0;\delta)\overline{U^0_{q\alpha}(y,0;\pi)}\,\mathrm{d}y\\ &= -\int_{-\frac{1}{2}}^{\frac{1}{2}} h(y)\big(\partial_y\mathcal{U}^0_{q\pm}(y,-d;\delta)\overline{\partial_y U^0_{q\alpha}(y,-d;\pi)} + k^2\mathcal{U}^0_{q\pm}(y,-d;\delta)\overline{U^0_{q\alpha}(y,-d;\pi)}\big)\,\mathrm{d}y\\ &-i\delta\int_{-d}^{0}\big(\partial_y\mathcal{U}^0_{q\pm}(\tfrac{1}{2},z;\delta)\overline{U^0_{q\alpha}(\tfrac{1}{2},z;\pi)} - \mathcal{U}^0_{q\pm}(\tfrac{1}{2},z;\delta)\overline{\partial_y U^0_{q\alpha}(\tfrac{1}{2},z;\pi)}\big)\,\mathrm{d}z.\end{aligned}$$

By a direct calculation together with the formula (1.2) these equations turn into the following system of equations for the column vector $a^{\pm} = (a_+^{\pm}, a_-^{\pm})^\top$:

$$M^{2q-1}(\delta)a^{\pm} = \Lambda'_{q\pm}(\delta)a^{\pm}, \quad M^{2q-1}(\delta) = \begin{pmatrix} K_{2q-1}\delta & F_{2q-1}H_{2q-1} \\ F_{2q-1}\overline{H_{2q-1}} & -K_{2q-1}\delta \end{pmatrix} \tag{3.13}$$

where, for $p = 2q-1$, we have

$$\begin{aligned} K_p &= \frac{2\pi p}{(1+\mathrm{e}^{-2dQ_p})^2}\Big(\frac{1-\mathrm{e}^{-4dQ_p}}{2Q_p} + 2d\mathrm{e}^{-2dQ_p}\Big) > 0, \\ H_p &= \int_{-\frac{1}{2}}^{\frac{1}{2}} h(y)\mathrm{e}^{-2\pi i p y}\mathrm{d}y, \quad F_p = \frac{4(\pi^2p^2-k^2)\mathrm{e}^{-2dQ_p}}{(1+\mathrm{e}^{-2dQ_p})^2} > 0, \\ Q_p &= \sqrt{\pi^2p^2+k^2}, \qquad p = 1,2,\ldots \end{aligned} \tag{3.14}$$

In other words the main asymptotic correction $\Lambda'_{q\pm}(\delta)$ is the eigenvalue of 2×2-matrix $M^{2q-1}(\delta)$ in (3.13), which is given by

$$\Lambda'_{q\pm}(\delta) = \pm\sqrt{F_{2q-1}^2|H_{2q-1}|^2 + K_{2q-1}^2\delta^2}. \tag{3.15}$$

(*c*) *The case* $\eta = 0$

A similar analysis ought to be performed for $\eta = 0 + \varepsilon\delta$, in order to detect a gap near the point

$$\Lambda^0_{q\pm}(0) = D(\sqrt{4\pi^2q^2+k^2}). \tag{3.16}$$

In this case we set

$$\Lambda^\varepsilon_{q\pm}(\varepsilon\delta) = \Lambda^0_{q\pm}(0) + \varepsilon\Lambda'_{q\pm}(\delta) + \widetilde{\Lambda}^\varepsilon_{q\pm}(\varepsilon\delta), \tag{3.17}$$

$$U^\varepsilon_{q\pm}(y,z;\varepsilon\delta) = \mathcal{U}^0_{q\pm}(y,z;0) + \varepsilon\mathcal{U}'_{q\pm}(y,z;\delta) + \widetilde{U}^\varepsilon_{q\pm}(y,z;\varepsilon\delta), \tag{3.18}$$

where, according to (2.17) and (2.18), the main terms are given by

$$\begin{aligned} \mathcal{U}^0_{q\pm}(y,z;0) &= a_+^{\pm}U^0_{q+}(y,z;0) + a_-^{\pm}U^0_{q-}(y,z;0), \\ U^0_{q\pm}(y,z;0) &= (\mathrm{e}^{z\sqrt{4\pi^2q^2+k^2}} + \mathrm{e}^{-(z+2d)\sqrt{4\pi^2q^2+k^2}})\mathrm{e}^{\pm i\pi 2q}. \end{aligned} \tag{3.19}$$

The correction terms $\Lambda'_{q\pm}(\delta), \mathcal{U}'_{q\pm}(\cdot;\delta)$ and the coefficients $a_\alpha^{\pm}$ can be determined as in the previous subsection. Concerning the characterization of $\Lambda'_{q\pm}(\delta), \mathcal{U}'_{q\pm}(\cdot;\delta)$ and the coefficients $a_\alpha^{\pm}$, equations (3.7), (3.8) stay unchanged. In condition (3.10) $\eta = \pi$ must be replaced by $\eta = 0$, so that it now reads

$$\partial_z\mathcal{U}'_{q\pm}(y,-d;\delta) = -\partial_y(h(y)\partial_y\mathcal{U}^0_{q\pm}(y,-d;0)) + k^2h(y)\mathcal{U}^0_{q\pm}(y,-d;0), \ |y| < \tfrac{1}{2}. \tag{3.20}$$

The quasi-periodicity conditions (3.11), (3.12) transform into the *inhomogeneous periodicity* conditions

$$\mathcal{U}'_{q\pm}(-\tfrac{1}{2},z;\delta) = \mathcal{U}'_{q\pm}(\tfrac{1}{2},z;\delta) - i\delta\mathcal{U}^0_{q\pm}(\tfrac{1}{2},z;0), \tag{3.21}$$

$$\partial_y\mathcal{U}'_{q\pm}(-\tfrac{1}{2},z;\delta) = \partial_y\mathcal{U}'_{q\pm}(\tfrac{1}{2},z;\delta) - i\delta\partial_y\mathcal{U}^0_{q\pm}(\tfrac{1}{2},z;0). \tag{3.22}$$

A direct calculation, similar to that of the case $\eta = \pi$, shows that the compatibility conditions turn into the algebraic system

$$M^{2q}(\delta)a^{\pm} = \Lambda'_{q\pm}(\delta)a^{\pm}, \text{ where } M^{2q}(\delta) = \begin{pmatrix} -K_{2q}\delta & F_{2q}H_{2q} \\ F_{2q}\overline{H_{2q}} & K_{2q}\delta \end{pmatrix} \tag{3.23}$$

and the constants F_p, H_p and K_p are still given by (3.14), with $p = 2q$. Again, as for $\eta = \pi$, the main asymptotic correction $\Lambda'_{q\pm}(\delta)$ is the eigenvalue of 2×2-matrix $M^q(\delta)$, which is given by

$$\Lambda'_{q\pm}(\delta) = \pm\sqrt{F_{2q}^2|H_{2q}|^2 + K_{2q}^2\delta^2}. \tag{3.24}$$

(*d*) *Formulation of the main theorem and some comments to it*

Under the specification (3.15), (3.24), the eigenvalues (3.4), (3.17) of the problem (2.6)-(2.9) has the dispersion curves as in Fig. 4c, provided

$$|H_p|^2 = \Big|\int_{-\frac{1}{2}}^{\frac{1}{2}} h(y)\cos(2\pi py)\mathrm{d}y\Big|^2 + \Big|\int_{-\frac{1}{2}}^{\frac{1}{2}} h(y)\sin(2\pi py)\mathrm{d}y\Big|^2 > 0 \tag{3.25}$$

and

$$\pi^2 p^2 \neq k^2. \tag{3.26}$$

We emphasise that the two coefficients in (1.12) are the same as in (3.25). Moreover, when both (3.25) and (3.26) are fulfilled, then the constant F_pH_p in (3.14) is nonzero; notice that (3.26) is always met if $k = 0$. Estimating the asymptotic remainders $\widetilde{\Lambda}^{\varepsilon}_{q\pm}(\pi + \varepsilon\delta)$, $\widetilde{\Lambda}^{\varepsilon}_{q\pm}(\varepsilon\delta)$ in §4 will prove the following assertion.

Theorem 3.1. *Let $p = 1, 2, \dots$ be fixed and let condition the (3.25), (3.26) be met. There exist $\varepsilon_p > 0$ and $c_p > 0$ such that for $\varepsilon \in (0, \varepsilon_p)$ the essential spectrum $\sigma_e^{\varepsilon}(k)$ of problem (1.5)-(1.7) has the gap*

$$G_p^{\varepsilon} = \Big(D(\sqrt{\pi^2 p + k^2}) - g^{\varepsilon}_{p-}, D(\sqrt{\pi^2 p + k^2}) + g^{\varepsilon}_{p+}\Big) \tag{3.27}$$

where

$$|g^{\varepsilon}_{p\pm} - \varepsilon F_p|H_p|| \leq c_p\varepsilon^{5/4}.$$

If the condition (3.25) is satisfied for several $q_1, \dots, q_N \in \mathbb{N}$, we may find common constants $\varepsilon_* > 0$ and $c_* > 0$ such that the above theorem holds true for all $q_1, \dots, q_N$ simultaneously. It means that if all the Fourier coefficients of the profile function h are non-zero, then the number of opened gaps grows indefinitely when $\varepsilon \to 0$. On the contrary, if we take h as in O'Hare & Davies 1993 and in Porter & Porter 2003, it will have only one non-zero Fourier coefficient H_1 while other coefficients vanish. Then we only can conclude the existence of at least one gap in the spectrum by Theorem 3.1.

If $F_pH_p = 0$, the formula (3.15), (3.24) turn into $\Lambda'_{q\pm}(\delta) = \pm K_{2q-1}\delta$, $\Lambda'_{q\pm}(\delta) = \pm K_{2q}\delta$, respectively. At the first sight then the situation in Fig. 4b occurs. However, we cannot deduce from this the absence of the gap (3.27), because of the higher order asymptotic terms which are not calculated here (cf. a discussion in Nazarov

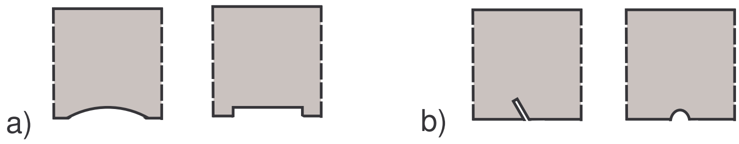

Figure 5. Regular piecewise smooth (a) and singular (b) perturbations of the periodicity cell

2010a). What we only can infer here is: $\sigma_e^\varepsilon(k)$ cannot have a gap of length $O(\varepsilon)$. The proof for the absence of a gap requires other ideas and techniques.

Our assumption on the smoothness of the bottom perturbation (1.1) is made to simplify the demonstration. Clearly the proof in §4 *a* works also in the case $h \in C^2_{per}([0,1])$, while for $h \in C^1_{per}([0,1])$ we need to deal only with the variational formulation of the problem. A slight modification of our method is needed if the profile function h is piecewise smooth, see Fig. 5a. Moreover, in this case the asymptotic formulae for the gaps remain literally the same. At the same time a singular perturbation of the bottom, see Fig. 5b, requires for new techniques (see Mazya *et al.* 2000) and it will lead to different asymptotic formulae, depending on the singularity: this case will be considered in our future work.

4. Justification of the asymptotics

(*a*) *Operator formulation of the model problem*

Here we follow the approach of the paper of Nazarov 2008*a* and introduce a specific scalar product together with the trace operator $T^\varepsilon(\eta)$ in $H^1_\eta(\varpi^\varepsilon)$,

$$\begin{aligned} \langle U^\varepsilon, V^\varepsilon\rangle_\eta &= (\nabla U^\varepsilon, \nabla V^\varepsilon)_{\varpi^\varepsilon} + k^2(U^\varepsilon, V^\varepsilon)_{\varpi^\varepsilon} + (U^\varepsilon, V^\varepsilon)_\gamma, \\ \langle T^\varepsilon(\eta) U^\varepsilon, V^\varepsilon\rangle_\eta &= (U^\varepsilon, V^\varepsilon)_\gamma \, \forall U^\varepsilon, V^\varepsilon \in H^1_{(\eta)}(\varpi^\varepsilon). \end{aligned} \tag{4.1}$$

The problem (2.5) becomes then equivalent to the abstract equation

$$T^\varepsilon(\eta)U^\varepsilon = \tau^\varepsilon(\eta)U^\varepsilon \quad \text{in } H^1_{(\eta)}(\varpi^\varepsilon)$$

with the spectral parameter

$$\tau^\varepsilon(\eta) = (1 + \Lambda^\varepsilon(\eta))^{-1}. \tag{4.2}$$

Formula (4.2) demonstrates, how the eigenvalues in (2.10) transform into eigenvalues of the operator $T^\varepsilon(\eta)$ which is positive, compact and self-adjoint, due to the compact embedding of $L^2(\gamma)$ into $H^1(\varpi^\varepsilon)^*$. The latter properties provide the following lemma on "almost eigenvalues" (see Vishik & Lyusternik 1957 and Ch. 6 in Birman & Solomjak 1987) which will be applied in §4 *c*.

Lemma 4.1. *Let* $u^\varepsilon \in H^1_{(\eta)}(\varpi^\varepsilon)$ *and* $t^\varepsilon \in \mathbb{R}_+$ *be such that*

$$||u^\varepsilon; H^1_{(\eta)}(\varpi^\varepsilon)|| = 1 \;\text{ and }\; \|T^\varepsilon(\eta)u^\varepsilon - t^\varepsilon u^\varepsilon; H^1_{(\eta)}(\varpi^\varepsilon)\|^{1/2} = \kappa^\varepsilon \in (0, t^\varepsilon), \tag{4.3}$$

Then there exists an eigenvalue $\tau^\varepsilon_m(\eta)$ *of the operator* $T^\varepsilon(\eta)$ *subject to the inequality*

$$|\tau^\varepsilon_m(\eta) - t^\varepsilon| \leqslant \kappa^\varepsilon.$$

(*b*) *Comparing eigenvalues outside a neighbourhood of the intersection point*

The coordinate change

$$(y, z) \mapsto (Y, Z) = \big(y, (1 - \varepsilon d^{-1} h(y))^{-1} z\big), \tag{4.4}$$

transforms the cell ϖ^ε into the cell ϖ^0. With the help of this transform the variational problem (2.5) turns into the problem

$$\langle J^\varepsilon Q^\varepsilon \nabla \mathbf{U}^\varepsilon, Q^\varepsilon \nabla \mathbf{V}^\varepsilon \rangle_{\varpi^0} + k^2 (J^\varepsilon \mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon)_{\varpi^0} = \Lambda^\varepsilon (\mathbf{U}^\varepsilon, \mathbf{V}^\varepsilon) \, \forall\, \mathbf{V}^\varepsilon \in H^1_\eta(\varpi^0) \tag{4.5}$$

posed in ϖ^0. Note that in the coordinate change (4.4) the free surface γ and the lateral sides $\tau_\pm = \{(y, z) : y = \pm\frac{1}{2}, z \in [-d, 0]\}$ remain unaltered. In (4.5) $\mathbf{U}^\varepsilon$ and $\mathbf{V}^\varepsilon$ are the functions U^ε and V^ε written in the (Y, Z)-coordinates. The Jacobian $Q^\varepsilon(y, z)$ and the inverse of its determinant are given by

$$Q^\varepsilon(y, z) = \begin{pmatrix} 1 & -\varepsilon d^{-1} z (1 - \varepsilon d^{-1} h(y, z))^{-2} \partial_y h(y) \\ 0 & (1 - \varepsilon d h(y))^{-1} \end{pmatrix}, \; J^\varepsilon(y, z) = (\det Q^\varepsilon(y, z))^{-1}.$$

Due to the evident estimates

$$||Q^\varepsilon(y, z) - \mathbb{I}; \mathbb{R}^{2\times 2}|| \leqslant c\varepsilon, \quad |J^\varepsilon(y, z) - 1| \leqslant c\varepsilon,$$

where $\mathbb{I}$ is the 2×2 identity matrix, the operator $T^\varepsilon(\eta)$ of the problem (4.1) is a perturbation of order ε of the analogous operator $T^0(\eta)$ related to the reference problem (2.5) in the rectangular cell ϖ^0. Thus, according to Ch.VII.6.2 in Kato (1966), we conclude that the eigenvalues of the model problems in ϖ^ε and ϖ^0 have the relationship

$$|\Lambda^\varepsilon_m(\eta) - \Lambda^0_m(\eta)| \le c_m \varepsilon \quad \text{for } \varepsilon \in (0, \varepsilon_m), \tag{4.6}$$

where the positive numbers c_m and ε_m depend on the eigenvalue number m (2.10) but are independent of $\varepsilon \in (0, \varepsilon_m)$ and $\eta \in [0, 2\pi)$.

Owing to formulae (2.17), (2.19) and (3.1) we observe that

$$\pm(\Lambda^0_{q\pm}(\eta) - D(\sqrt{\pi^2(2q-1)^2 + k^2})) \ge C_q |\eta - \pi|, \; C_q > 0,$$

and, therefore, in view of (4.6) we have

$$\pm\Lambda^\varepsilon_{q\pm}(\eta) \ge c_q C_q \varepsilon^{3/4} \pm D(\sqrt{\pi^2(2q-1)^2 + k^2}) \text{ for } \eta \in [0, 2\pi), |\eta - \pi| \ge c_q \varepsilon^{3/4}. \tag{4.7}$$

This information is crucial in §4 *c* where an argument providing the existence of a the gap (3.27) will be proved. Only evident changes in all formulas are needed to treat the case $\eta = 0$. The same holds true for the next subsection.

(*c*) *Comparing eigenvalues in the vicinity of the intersection point*

To apply the above Lemma 4.1 we choose the approximating eigenpair $(t^\varepsilon, u^\varepsilon)$ as follows:

$$t^\varepsilon_{q\pm} = (1 + \Lambda^0_{q\pm}(\pi) + \varepsilon \Lambda'_{q\pm}(\delta))^{-1}, \; u^\varepsilon_{q\pm} = \langle \mathcal{U}^\varepsilon_{q\pm}, \mathcal{U}^\varepsilon_{q\pm} \rangle^{-1/2}_{\pi + \varepsilon\delta} \mathcal{U}^\varepsilon_{q\pm}, \tag{4.8}$$

where

$$\mathcal{U}^\varepsilon_{q\pm}(y, z) = \mathcal{U}^0_{q\pm}(y, z; \pi) + \varepsilon \mathcal{U}'_{q\pm}(y, z; \delta) + \varepsilon^2 \widetilde{\mathcal{U}}^\varepsilon_{q\pm}(y, z). \tag{4.9}$$

Let us explain the notation used here. Firstly $\mathcal{U}^0_{q\pm}$ is the linear combination as in (3.6), where $a^\pm(\delta) = (a^\pm_+(\delta), a^\pm_-(\delta))$ is an eigenvector of the matrix $M^q(\delta)$ in (3.13) and $\|a^\pm(\delta)\| = 1$. Secondly $\mathcal{U}'_{q\pm}$ is a solution of the problem (3.7)-(3.12) (recall that the compatibility conditions of this problem have been satisfied by solving the algebraic system in (3.13)). Moreover, $\mathcal{U}'_{q\pm}$ is smooth in $\overline{\varpi^0}$ and can be extended below the line $\{(y,z) : z = -d\}$ so that the smooth extension, still denoted by $\mathcal{U}'_{q\pm}$, belongs to $H^3(\varpi^\varepsilon)$ satisfying the estimate

$$||\mathcal{U}'_{q\pm}; H^3(\varpi^\varepsilon)|| \le c_q(1 + |\delta|).$$

Notice that the harmonics $\mathcal{U}^0_{q\pm}$ are defined by formulae (3.6) for any $(y,z) \in \mathbb{R}^2$, thus in ϖ^ε.

The bound is a result of a simple analysis of the right-hand sides of problem (3.7)-(3.12). The last term $\widetilde{\mathcal{U}}^\varepsilon_{q\pm}$ in (4.9) is fixed to compensate the discrepancies of the sum $\mathcal{U}^0_{q\pm} + \varepsilon\mathcal{U}'_{q\pm}$ in the quasi-periodicity conditions (2.9) with $\eta = \pi + \varepsilon\delta$. Namely, in the first condition we have

$$\begin{aligned}
&\mathcal{U}^0_{q\pm}(-\tfrac{1}{2}, z; \pi) + \varepsilon\mathcal{U}'_{q\pm}(-\tfrac{1}{2}, z; \delta) - \mathrm{e}^{-i(\pi+\varepsilon\delta)}\left(\mathcal{U}^0_{q\pm}(\tfrac{1}{2}, z; \pi) + \varepsilon\mathcal{U}'_{q\pm}(\tfrac{1}{2}, z; \delta)\right) \\
&\quad = \varepsilon\left(\mathcal{U}'_{q\pm}(-\tfrac{1}{2}, z; \delta) - \mathrm{e}^{-i\pi}(\mathcal{U}'_{q\pm}(\tfrac{1}{2}, z; \delta) - i\delta\mathcal{U}^0_{q\pm}(\tfrac{1}{2}, z; \pi))\right) \\
&\quad\quad - i\mathrm{e}^{-i\pi}\frac{\varepsilon^2\delta^2}{2}\mathcal{U}^0_{q\pm}(\tfrac{1}{2}, z; \pi) - i\mathrm{e}^{-i\pi}\varepsilon^2\delta\mathcal{U}'_{q\pm}(\tfrac{1}{2}, z; \delta) + O(\varepsilon^2\delta^2). \quad (4.10)
\end{aligned}$$

The multiplier of ε in (4.10) on the right-hand side vanishes according to (3.11), so that the smooth discrepancy is of order $O(\varepsilon^2\delta(1+\delta))$. A similar discrepancy appears in the second periodicity condition. Thus we may find a function $\widetilde{\mathcal{U}}^\varepsilon_{q\pm} \in H^3(\varpi^\varepsilon)$ which compensates for both the discrepancies and satisfies the estimate

$$||\widetilde{\mathcal{U}}^\varepsilon_{q\pm}; H^3(\varpi^\varepsilon)|| \le c_q\delta(1 + \delta).$$

Furthermore, by virtue of (3.9) and (3.10)-(3.12), we have

$$\begin{aligned}
g^\varepsilon_0(y) &:= \partial_z\mathcal{U}^\varepsilon_{q\pm}(y,0) - (\Lambda^0_{q\pm}(\pi) + \varepsilon\Lambda'_{q\pm}(\delta))\mathcal{U}^\varepsilon_{q\pm}(y,0) \\
&= \varepsilon^2(\partial_z\widetilde{\mathcal{U}}^\varepsilon_{q\pm}(y,0) - (\Lambda^0_{q\pm}(\pi) + \varepsilon\Lambda'_{q\pm}(\delta)))\widetilde{\mathcal{U}}^\varepsilon_{q\pm}(y,0) \\
&- \varepsilon^2\Lambda'_{q\pm}(\delta)\mathcal{U}'_{q\pm}(y,0;\delta)
\end{aligned}$$

in the Steklov boundary condition (2.7) and

$$\begin{aligned}
g^\varepsilon_d(y) &:= \partial_n\mathcal{U}^\varepsilon_{q\pm}(y, -d + \varepsilon h(y)) \\
&= ((1 + \varepsilon^2|\partial_y h(y)|^2)^{-1/2} - 1)\partial_n\mathcal{U}^\varepsilon_{q\pm}(y, -d + \varepsilon h(y)) \\
&+ \varepsilon^2(-\partial_z\widetilde{\mathcal{U}}^\varepsilon_{q\pm}(y, -d + \varepsilon h(y)) + \varepsilon\partial_y h(y)\partial_y\widetilde{\mathcal{U}}^\varepsilon_{q\pm}(y, -d + \varepsilon h(y))) \\
&- \left(\partial_z\mathcal{U}^0_{q\pm}(y, -d + \varepsilon h(y); \pi) - \partial_z\mathcal{U}^0_{q\pm}(y, -d; \pi) + \varepsilon h(y)\partial_z^2\mathcal{U}^0_{q\pm}(y, -d; \pi)\right) \\
&+ \varepsilon\partial_y h(y)\left(\partial_y\mathcal{U}^0_{q\pm}(y, -d + \varepsilon h(y); \pi) - \partial_y\mathcal{U}^0_{q\pm}(y, -d; \pi)\right) \\
&- \varepsilon(\partial_z\mathcal{U}'_{q\pm}(y, -d + \varepsilon h(y); \delta) - \partial_z\mathcal{U}'_{q\pm}(y, -d; \delta)) \\
&+ \varepsilon^2\partial_y h(y)\partial_y\mathcal{U}'_{q\pm}(y, -d + \varepsilon h(y); \delta)
\end{aligned}$$

in the Neumann condition (2.8).

These formulae imply the estimate

$$||g_0^\varepsilon; L^2(\gamma)|| + ||g_d^\varepsilon; L^2(\gamma_d^\varepsilon)|| \le c_q \varepsilon^2 (1+\delta^2)(1+\varepsilon|\delta|).$$

We finally mention that $\mathcal{U}^0_{q\pm}$ satisfies the equation (2.6) in ϖ^ε but $\mathcal{U}'_{q\pm}$ does it only in ϖ^0. Therefore, recalling the smooth extension of $\mathcal{U}'_{q\pm}$, we obtain

$$\begin{aligned} \varepsilon\|\Delta\mathcal{U}'_{q\pm} - k^2\mathcal{U}'_{q\pm}; L^2(\varpi^\varepsilon)\| &= \varepsilon\|\Delta\mathcal{U}'_{q\pm} - k^2\mathcal{U}'_{q\pm}; L^2(\varpi^\varepsilon \setminus \varpi^0)\| \\ &\leqslant c\varepsilon^{3/2}\|\mathcal{U}'_{q\pm}; H^3(\varpi^\varepsilon)\| \\ &\le c_q \varepsilon^{3/2}(1+|\delta|). \end{aligned}$$

Here we have taken into account that $\varpi^\varepsilon \setminus \varpi^0$ is a thin curved strip of width $O(\varepsilon)$.

For the computation of $\kappa^\varepsilon = \kappa^\varepsilon_{q\pm}$ in (4.3) we use the definitions of $t^\varepsilon_{q\pm}$ and $u^\varepsilon_{q\pm}$ in (4.8) to obtain

$$\begin{aligned} \kappa^\varepsilon_{q\pm} &= \langle \mathcal{U}^\varepsilon_{q\pm}, \mathcal{U}^\varepsilon_{q\pm}\rangle^{-1/2}_{\pi+\varepsilon\delta} t^\varepsilon_{q\pm} \sup|(1+\Lambda^0_{q\pm}(\pi) + \varepsilon\Lambda'_{q\pm}(\delta)(\mathcal{U}^\varepsilon_{q\pm}, v^\varepsilon)_\gamma \\ &- (\nabla\mathcal{U}^\varepsilon_{q\pm}, \nabla v^\varepsilon)_{\varpi^\varepsilon} - k^2(\mathcal{U}^\varepsilon_{q\pm}, v^\varepsilon)_{\varpi^\varepsilon} - (\mathcal{U}^\varepsilon_{q\pm}, v^\varepsilon)_\gamma| \\ &= \langle \mathcal{U}^\varepsilon_{q\pm}, \mathcal{U}^\varepsilon_{q\pm}\rangle^{-1/2}_{\pi+\varepsilon\delta} t^\varepsilon_{q\pm} \sup|(-\Delta\mathcal{U}^\varepsilon_{q\pm} + k^2\mathcal{U}^\varepsilon_{q\pm}, v^\varepsilon)_{\varpi^\varepsilon} \\ &+ (g^\varepsilon_0, v^\varepsilon)_\gamma + (g^\varepsilon_d, v^\varepsilon)_{\gamma^\varepsilon_d}|. \end{aligned} \tag{4.11}$$

Here the supremum is calculated over all functions $v^\varepsilon \in H^1_{\pi+\varepsilon\delta}(\varpi^\varepsilon)$ such that $\langle v^\varepsilon, v^\varepsilon\rangle_{\pi+\varepsilon\delta} = 1$. Clearly,

$$||v^\varepsilon; L^2(\varpi^\varepsilon)|| + ||v^\varepsilon; L^2(\gamma)|| + ||v^\varepsilon; L^2(\gamma^\varepsilon_d)|| \le c.$$

In the sequel, we assume that $|\delta| \le c_q\varepsilon^{5/4}$. We then observe that

$$\begin{aligned} &|t^\varepsilon_{q\pm}| \le c_q(1+\varepsilon|\delta|) \le C_q, \\ &\langle \mathcal{U}^\varepsilon_{q\pm}, \mathcal{U}^\varepsilon_{q\pm}\rangle \ge c_q(1 - \varepsilon(1+|\delta|) - \varepsilon^2\delta(1+\delta)) \geqslant \tfrac{1}{2}c_q > 0. \end{aligned}$$

where c_q, C_q stand for different positive constants which may depend on q but are independent of $\varepsilon \in (0, \varepsilon_q)$. Collecting the above estimates we convert the relation (4.11) into

$$\kappa^\varepsilon_{q\pm} \le (\varepsilon^{3/2}(1+|\delta|) + \varepsilon^2(1+\delta^2)(1+\varepsilon|\delta|)) \le c_q\varepsilon^{5/4}.$$

Hence by Lemma 4.1 there exist eigenvalues $\tau^\varepsilon_{q\pm}(\pi+\varepsilon\delta)$ of the operator $T^\varepsilon(\pi+\varepsilon\delta)$ such that

$$|\tau^\varepsilon_{q\pm}(\pi+\varepsilon\delta) - (1+\Lambda^0_{q\pm}(\pi) + \varepsilon\Lambda'_{q\pm}(\delta))^{-1}| \le c_q\varepsilon^{5/4},$$

or, in view of (4.2),

$$|\Lambda^\varepsilon_{q\pm}(\pi+\varepsilon\delta) - \Lambda^0_{q\pm}(\pi) - \varepsilon\Lambda'_{q\pm}(\delta)| \le c_q\varepsilon^{5/4}. \tag{4.12}$$

Due to the formula (3.15) and the assumption (3.25) the eigenvalues $\Lambda^\varepsilon_{q+}(\pi+\varepsilon\delta)$ and $\Lambda^\varepsilon_{q-}(\pi+\varepsilon\delta)$ are different from each other. Moreover they stay in a $c\varepsilon$-neighbourhood of the point $\Lambda^0_{q\pm}(\pi)$ which, according to §4 *b*, contains the eigenvalues $\Lambda^\varepsilon_{2q-1}(\pi+\varepsilon\delta)$ and $\Lambda^\varepsilon_{2q}(\pi+\varepsilon\delta)$ only. Thus, these eigenvalues are distinct and satisfy the relation (4.12). In view of (2.17), (3.15) and (3.25) this observation together with formula (4.7) proves Theorem 3.1.

5. Conclusions and some open problems

We have verified that under the condition (3.25) any appropriate small perturbation (1.8) of the straight bottom $\{(y,z) : z = -d\}$ of the two-dimensional water domain $\Pi^0 = \{(y,z) : -d < z < 0\}$ produces gaps in the essential spectrum of the water-wave problem (1.5)-(1.7). Moreover, the number of opened gaps is proved to increase indefinitely when the small perturbation parameter ε tends to zero and infinitely many Fourier coefficients of the profile function h differ from zero. At the same time the asymptotic analysis applied in the paper does not allow us to conclude whether the number of spectral gaps is infinite or not. This question remains open as well as in the geometrical situation investigated in Nazarov (2010*b*). We also do not know what happens with the gap G_q^ε in the case when the Fourier coefficients in (3.25) vanish.

By the definition† the essential spectrum $\sigma_e^\varepsilon(k)$ is the union of the continuous spectrum σ_c^ε and the set $\sigma_\infty^\varepsilon \subset \sigma_p^\varepsilon$ of eigenvalues with infinite multiplicity. Such eigenvalues appear only in the case when a spectral band B_m^ε in (1.11) collapses into a single point. A simple consequence of our asymptotic analysis above shows that for any $J \in \mathbb{N} = \{1,2,3,\ldots\}$ there exists $\varepsilon_J > 0$ such that $B_1^\varepsilon, \ldots, B_J^\varepsilon$ have positive lengths for $\varepsilon \in (0, \varepsilon_J)$. However, the formulae $\sigma_\infty^\varepsilon = \varnothing$ and $\sigma_e^\varepsilon(k) = \sigma_c^\varepsilon$ have not yet been proved.

Introducing an obstacle θ into the periodic strip Π^ε, that is considering the cylinder $\Theta = \mathbb{R} \times \theta$ in the water layer between the surfaces (1.3) and (1.1), brings a compact perturbation which does not affect the essential spectrum of the problem but may create a point spectrum (eigenvalues). The classical results of Ursell (1951), Ursell (1987), and Garipov (1967) show that, for $k > 0$, any submerged obstacle θ in the straight strip Π^0 (see problem (1.5)-(1.7) at $\varepsilon = 0$) creates at least one eigenvalue in the discrete spectrum $\sigma_d^0 \subset (0, \lambda_\dagger^0)(k))$. In the papers by Nazarov (2009*a*) and (Nazarov 2009*b*) a simple approach was proposed which provides an elementary proof of this fact establishing also a simple sufficient condition for the existence of a trapped mode supported by surface-piercing obstacles, and showing that the discrete spectrum σ_d^ε for the problem (1.5)-(1.7) in the channel with periodic bottom $\Omega^\varepsilon = \Pi^\varepsilon \setminus \theta$ is not empty for any positive k and ε. The eigenvalue detected in the paper of Nazarov (2009*a*) is always below the lower bound of the essential spectrum $\sigma_{\text{ess}}^\varepsilon$. At the moment no example is known where an eigenvalue appears inside either the essential spectrum (embedded eigenvalue) or a spectral gap (an eigenvalue from the discrete spectrum).

Let us consider the water-wave problem in the channel (1.14):

$$
\begin{aligned}
-\Delta\varphi^\varepsilon(x,y,z) &= 0, \text{ in } \Xi^\varepsilon, && (5.1)\\
\partial_z\varphi^\varepsilon(x,y,0) &= \lambda^\varepsilon\varphi^\varepsilon(x,y,0),\ (x,y) \in (-\tfrac{1}{2}l, \tfrac{1}{2}l) \times \mathbb{R}, && (5.2)\\
\partial_n\varphi^\varepsilon(x,y,z) &= 0,\ (x,y) \in (-\tfrac{1}{2}l, \tfrac{1}{2}l) \times \mathbb{R},\ z = -d + \varepsilon h(y), && (5.3)\\
\pm\partial_x\varphi^\varepsilon(\pm\tfrac{l}{,}y,z) &= 0,\ y \in \mathbb{R},\ -d + \varepsilon h(y) < z < 0. && (5.4)
\end{aligned}
$$

† To specify components of the spectra, we use standard terminology in the theory of self-adjoint operators in Hilbert space (cf. Birman, M. Sh. & Solomjak, M.Z. 1987). An approach by Nazarov 2008*a* to treat problem (1.5)-(1.7) in this framework has been explained in §4*a*.

Separating the variables we may search for solutions in the form

$$\varphi(x,y,z) = \cos\Big(\frac{\pi j}{l}(x+\frac{l}{2})\Big)\Phi_j^\varepsilon(y,z),\ j\in\mathbb{N}_0.$$

Thus we arrive at the family of problems (1.5)-(1.7) in Π^ε with the parameters $k = k_j := l^{-1}\pi j$. The spectrum σ^ε of the three-dimensional problem (5.1)-(5.4) becomes the union of the spectra $\sigma_e^\varepsilon(k_j)$ of the two-dimensional problems (1.5)-(1.7) with $k = k_j$, $j\in\mathbb{N}_0$.

Owing to the formula (1.9) and the regular perturbation analysis in §4 *b* for any $\delta > 0$ we can find $\epsilon_\delta > 0$ such that the following inequality is valid:

$$\lambda_\dagger^\varepsilon(k_n) \geqslant \lambda_\dagger^\varepsilon(k_1) \geqslant (1-\delta)D(l^{-1}\pi),\ \text{for}\ \epsilon\in(0,\epsilon_\delta)\ \text{and}\ n\in\mathbb{N}.. \tag{5.5}$$

It is difficult to detect a spectral gap above the first threshold $\lambda_\dagger^\varepsilon(k_1)$ because this range of the spectrum is composed from the spectral bands which are generated by different numbers $k_1, k_2, \dots$ and they may overlap each other. However, within the interval $(0,(1-\delta)D(l^{-1}\pi))$ (see (5.5)) the spectrum σ^ε coincides with the unique spectrum $\sigma_e^\varepsilon(0)$ of the problem (1.5)-(1.7). Hence Theorem 3.1 ensures the gap opening provided

$$|F_pH_p| \neq 0\ \text{and}\ D(\pi p) < (1-\delta)D(l^{-1}\pi)$$

for some $p\in\mathbb{N}$. Since $D(l^{-1}\pi)\to\infty$ as $l\to\infty$, we can open as many gaps in σ^ε as we wish by diminishing l and ε while choosing the bottom profile h properly. Notice that in view of the initial rescaling the small width parameter l requires that the channel is deep and the corrugation period is large.

The methods used in this paper and in Nazarov (2010*b*) does not allow us to study the band-gap structure of the spectrum if the perturbation of the bottom is not small. On the other hand there is still a lack of efficient numerical methods for determining and evaluating spectral gaps and bands.

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