Linear ODEs: an Algebraic Perspective

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Linear ODEs: an Algebraic Perspective

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Linear ODEs: an Algebraic Perspective

XXII ESCOLA DE ÁLGEBRA
XXII Brazilian Algebra Meeting
40 ANOS

Salvador – Bahia, Brazil – July 15 to 20, 2012

To Aron Simis, on the occasion of his seventieth birthday

Preface

This booklet was intended to provide a minimum of ready-to-use references for the minicourse given by the author during the XXII Éscola de Algebra (40 Anos), held in Salvador de Bahia (July 2012). The purpose of these lecture notes is twofold. On one hand they aim to introduce and advertise a natural, flexible and elegant purely combinatorial–algebraic approach to the well-known classical theory of linear ODEs withæ constant coefficients (Chapter 3), and to introduce generalised Wronskians associated to a fundamental system of solutions (Chapter 4). Elementary applications will be shown, e.g. to the computation of the exponential of a square matrix without reducing to the Jordan normal form (Chapter 5). On the other hand it wishes to bring to the fore a number of relationships with other branches of mathematics. Examples include the theory of symmetric functions (Example 2.1.3), the theory of universal decomposition algebras associated to a polynomial (Example 3.2.8 and Remark 6.1.8), derivations of the exterior algebra of a free module (Chapter 6), D-modules (Example 3.2.3), Schubert calculus for the complex Grassmannian (Section 6.2), boson–fermion correspondence in the representation theory of the Heisenberg or the Virasoro algebra, the latter seen as an infinite–dimensional analogue of Poincaré's duality for the complex Grassmannians. The present exposition is totally inspired by the paper [18] and must be considered an expanded version of it.

The level of the exposition is elementary, given that more than seventy percent of the material can be followed with a basic understanding of polynomial algebras and the Leibniz rule for the product of two differentiable functions. More advanced topics, like Schubert Calculus or the bosonic representation of the oscillator algebra have been only sketched in the last two chapters. A deeper knowledge of those subjects is not necessary for the purposes of the minicourse, as they have been treated just to provide further examples to certify the surprising ubiquity of the Jacobi-Trudy formula in mathematics.

The present lecture notes have been written on a short notice, so they will certainly contain misprints and omissions and possibly some mistakes. Corrections and/or integrations can be found in the author's web page at the url

`http://calvino.polito.it/~gatto/public/XXIIEA/bahia.htm`

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Very special thanks are due to *Inna Scherbak*, the ideal coauthor of these notes, who generously shared her insight and helped me with many advises. I also thank *Caterina Cumino* and *Taíse Santiago Mozzato* for many discussions and my special friend *Simon Chiossi* not only for enlightening discussions but especially for his constant encouragement. For the last sketchy chapter of these notes I am deeply indebted to *Maxim Kazarian*, from whom I first learned about the boson–fermion correspondence. For the friendly and careful reading of the notes and for pointing me misprints and some mistake, I want to thank *Peter Malcom Johnson.* I am also indebted with Professor *Louis Rowen* for questions and remarks during his patient and stimulating presence to my lectures in Salvador, although they were in portuguese.

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Estas notas de aula são dedicadas ao Aron Simis, por ocasião do septuagésimo aniversário dele, desejando-lhe mais outros setenta anos de feliz atividade matemática.

Sangano, 23 de Maio 2012

[1]Gruppo Nazionale Strutture Algebriche Geometriche e Applicazioni.
[2]Istituto Nazionale di Alta Matematica.
[3]Progetto di Rilevante Interesse Nazionale.
[4]Fundação de Amparo à Pesquisa do Estado São Paulo.

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Introduction

These lecture notes tell a story which begins with a rather simple observation: all the solutions of a linear ODE with constant complex coefficients are analytic, i.e. they can be expressed in terms of convergent power series. It is then natural to suspect that the corresponding theory can be carried out in a purely formal way, working with rings of formal power series with coefficients in an arbitrary $\mathbb{Q}$-algebra. This is indeed the case and, pursuing the task, one easily obtains a simple, economical and elegant theory which offers both practical advantages and a novel perspective for interpreting other mathematical phenomena.

One of the most relevant features of the theory is that it comes with a *universal basis of solutions* for linear homogeneous ODEs of order, say, $r+1$ (Chapter 3). The universal basis cosnists of formal power series with coefficients in the polynomial ring $E_r := \mathbb{Q}[e_1, \ldots, e_{r+1}]$, where the indeterminates $e_1, \ldots, e_{r+1}$ are the coefficients of the equation. For the reader convenience, basics on formal power series, exterior algebra and the philosophy of generating functions, through the well known example of those of binomial coefficients, are collected in Chapter 1 in order to keep the exposition as self contained as possible.

A linear ODE of order $r+1$ with coefficients $a_1, \ldots, a_{r+1}$ taken in any $\mathbb{Q}$-algebra A (for instance $A = \mathbb{R}$ or $A = \mathbb{C}$) induces on A a natural structure of E_r-algebra, and the module of solutions to the equation is nothing else than the module of universal solutions after extending the coefficients. In down–to–earth, yet suggestive, terms this amounts to solve all the linear ODEs at once, and once and for all.

The idea of solving linear ODEs using power series, of course, is not new, and is taught in any standard calculus textbook – see e.g. [2, pp. 169–172]. The subject of the present notes is also obviously related with linear recurrence sequences, see e.g [1, Section 212]. The cutting–edge aspect is the implementation of it, which is based on a purely algebraic language and some combinatorics inspired by the theory of symmetric functions. In the present context, in particular, the knowledge of the roots of the characteristic polynomial is no longer necessary for solving a linear ODE. As a matter of fact, standard bases of solutions constructed via the exponential of the roots of the characteristic polynomials are not as canonical as the aforementioned universal ones – see Example 3.2.6. The latter reveal themselves especially useful for computing the exponential of a square matrix without reducing it to Jordan normal form (Chapter 5), thus completing an observation made by Putzer [33] in 1966 (see also [2, p. 205]) and relatively more recently by Leonard [28] (1996) and Liz [27] (1998).

The motivations for investigating, jointly with I. Scherbak, the combinatorics behind the *universal ODE*, come from Schubert calculus for Grassmannians, which can be thought of as the generalization of the classical Bézout theorem[5], widely known for projective spaces, to more general Grassmann varieties $G(r, \mathbb{P}^d)$ which parameterize r-dimensional linear subvarieties of the d-dimensional projective space. In [13, 14, 17] Schubert calculus was dealt with in terms of *derivations on a Grassmann algebra*. The formalism indicates a kinship with *generalized Wronskians* (Chapter 4), associated to a basis of solutions of an ordinary ODE, and their derivatives (see also [15]). The main result of [18] is a kind of Giambelli-Jacobi-Trudy formula for *generalized Wronskians* (Section 4.4). It shows that, from a formal point of view, the celebrated Pieri's formula that governs Schubert Calculus is nothing but Leibniz's rule for suitable derivatives of a generalized Wronskian. The proof of such Jacobi-Trudy formula forces to look at the most general linear ODE, which eventually led us to find, or pos-

[5]Bézout's theorem is best known for the projective plane $\mathbb{P}^2$. It says that if C_1 and C_2 are two projective curves of degree d_1 and d_2 respectively, with no component in common, then they intersect at $d_1 d_2$ points, keeping intersection multiplicity into account.

sibly rediscover[6], the universal basis of solutions alluded above. The universal solution of the Cauchy problem for a (in general non homogeneous, like in Section 3.4) linear ODE with constant coefficients has a number of consequences, besides those already mentioned.

For example, it shows that many properties of the matrix exponential are purely formal and hold for square matrices with entries in any commutative ring. If, in addition, the latter is an integral domain one can easily prove that the determinant of the exponential of a square matrix is equal to the exponential of its trace. Using this property, we show in Example 5.4.5 an amusing generalization of the celebrated fundamental trigonometric identity $\cos^2 t + \sin^2 t = 1$.

In a second instance one (re)discovers in a natural way a formal Laplace transform defined on $A[[t]]$, which amounts to multiplying the coefficients of t^n of a formal power series by the factorial $n!$ (see Sections 1.4.2 and 3.3).

Combinatorial properties of generalized Wronskians associated to a universal fundamental system, following [18] and [19], as well as their relationships with Schubert calculus and derivations of a Grassmann algebra are also briefly discussed (Chapter 4). One shows that wedging altogether the elements of a universal fundamental system is the same as considering the Wronskian of it. The universal Cauchy formula (3.11), that gives the explicit expression of the unique solution to a linear ODE with given initial data, is a consequence of a purely combinatorial property. The latter exhibits an alternative basis of the ring $A[[t]]$ of formal power series in the indeterminate t (Chapter 2) which is related to universal solutions to linear ODE. Such combinatorial property leads in a very natural (and probably unavoidable) way to consider universal linear ODEs of infinite order. They possess a universal basis of solutions, whose elements are indexed by negative integers. In this case, the algebra E_r must be replaced by the polynomial algebra $E_\infty := \mathbb{Q}[e_1, e_2, \ldots]$ in infinitely many indeterminates. The latter will be interpreted in Chapter 7 as the *Fock space* of the theory of representations of infinite dimensional Lie algebras (Oscillator Algebra, Virasoro algebra), which in turn is isomorphic to each *fermion space* of total charge m: the latter can be identified with the $\mathbb{Q}$-algebra generated by certain infinite wedge

[6]We do not know any explicit reference for this.

products of solutions of the linear ODE of infinite order. The very natural isomorphism one obtains in this way, based on the universal Cauchy formula for infinite–order linear ODEs, is nothing but the so-called *boson-fermion* correspondence, as described for example, in the introductory book [22] – see also [3, 23, 30].

Chapter 1

Algebraic Preliminaries

This chapter collects some basic notions of commutative and exterior algebra which may be useful to follow the remaining part of these lecture notes.

1.1 Modules

1.1.1. Let $A := (A, +, \cdot)$ be a *commutative ring with unit*, i.e. $(A, +)$ is an Abelian group, the product "$\cdot$" is associative, satisfies the distributive laws over the sum "$+$", possesses a neutral element $1 \in A$ and $a \cdot b = b \cdot a$ for each $a, b \in A$. A *module over* A, briefly said an A*-module*, is an abelian group M together with a map

$$\begin{cases} A \times M & \longrightarrow & M, \\ (a, m) & \longmapsto & am, \end{cases}$$

such that $1 \cdot m = m$, $a(bm) = (ab)m$, $(a + b)m = am + bm$ and $a(m_1 + m_2) = am_1 + am_2$, for any arbitrary choice of $a, b \in A$ and $m, m_1, m_2 \in M$. A *vector space* is a module over a field. A *ring homomorphism* is a map $\psi : A \to B$ such that $\psi(a_1 + a_2) = \psi(a_1) + \psi(a_2)$ and $\psi(a_1 a_2) = \psi(a_1)\psi(a_2)$.

1.1.2. Given A-modules M, N and P, a map $\psi : M \times N \to P$ is *bilinear* if it is linear in both the first and second argument, i.e., if for each $a, b \in A$ and each $m, m_1, m_2 \in M$ and each $n, n_1, n_2 \in N$:

$$\psi(am_1 + bm_2, n) = a\psi(m_1, n) + b\psi(m_2, n),$$

$$\psi(m, an_1 + bn_2) = a\psi(m, n_1) + b\psi(m, n_2).$$

A *tensor product* of M and N is a pair (T, Ψ) where T is an A-module and $\Psi : M \times N \to T$ is a *universal bilinear map* in the sense that for each bilinear $\phi : M \times N \to P$ there is a unique A-homomorphism $f_\phi : M \otimes_A N \to P$ such that $\phi = f_\phi \circ \Psi$. The tensor product is unique up to a canonical isomorphism and is denoted by $M \otimes_A N$.

1.1.3. The multiplication of the elements of a module by the elements of a ring can be described through the A-linear map

$$\begin{cases} A \otimes M & \longrightarrow & M, \\ \\ a \otimes m & \longmapsto & am. \end{cases} \tag{1.1}$$

If M is an A-module and N a B-module, a morphism $M \to N$ is a pair (ϕ, ψ) such that $\phi : A \to B$ is a ring homomorphism, $\psi : M \to N$ a homomorphism of abelian groups and the diagram

$$\begin{array}{ccc} A \otimes M & \longrightarrow & M \\ \downarrow & & \downarrow \\ B \otimes N & \longrightarrow & N \end{array}$$

commutes. The horizontal maps are the defined by the module structures of M and N over A and B respectively, as in (1.1).

1.1.4. From now on and for all the rest of these notes, if $m_1, \ldots, m_h$ are elements of an A-module M, then

$$[m_1, \ldots, m_h]_A := \{\sum a_i m_i \mid a_i \in A\},$$

will denote their linear span. The A-module M is said to be free of rank n if there exist elements $m_1, \ldots, m_n$ such that each $m \in M$ can be uniquely written as:

$$m = a_1 m_1 + a_2 m_2 + \ldots + a_n m_n,$$

for unique $a_1, \ldots, a_n$. In particular $M := [m_1, \ldots, m_n]_A$.

1.1.5. If $(m_1, \dots, m_n)$ freely generate M, they are A-linearly independent, in the sense that $a_1m_1 + a_2m_2 + \dots + a_nm_n = 0$ implies $a_i = 0$ for all $i \in \{1, \dots, n\}$. The converse is not true. For instance

$$\begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are linearly independent in $\mathbb{Z}^2$ but they do not generate it. The A–module A^n is free as it is freely generated by the *coordinate columns* $\mathbf{e}_1, \dots, \mathbf{e}_n$, where the column $\mathbf{e}_i$ has all the entries zero but the ith, which is 1. A module is said to be *principal* if there is $m_0 \in M$ such that the map $A \to M$ given by $a \mapsto am_0$ is surjective. If such a map is also injective then M is said to be *invertible* .

1.1.6. Example. Let

$$M := \frac{\mathbb{Z}[x, y]}{(xy)}$$

Then M is obviously a $\mathbb{Z}[x, y]$-module, but it is not free. It is generated by $x + (xy)$ and $y + (xy)$. But, for instance

$$(xy) = y \cdot (x + (xy)) = x \cdot (y + (xy))$$

so we have found two distinct decompositions of $0 + (xy)$, the null element of M, as a linear combination of $x + (xy)$ and $y + (xy)$ with non zero coefficients in $\mathbb{Z}[x, y]$.

1.2 Algebras

1.2.1. Any ring homomorphism $\psi : A \to B$ turns B into an A-module by setting $a * b = \psi(a) \cdot b$, for each $(a, b) \in A \times B$. The ring B with respect to such an A-module structure is said to be an (associative, commutative, with unit) A-*algebra* (the homomorphism ψ being understood). An A-homomorphism of A-modules M and N is a map $\psi : M \to N$ such that the equality $\psi(am_1 + bm_2) = a\psi(m_1) + b\psi(m_2)$ holds for each choice of $a, b \in A$ and $m_1, m_2 \in M$.

1.2.2. Example. The set $\mathrm{End}_A(M)$ of all A-endomorphisms of an A-module M is an A-algebra with respect to the usual notion of linear combination

$$(a\psi_1 + b\psi_2)(m) = a \cdot \psi_1(m) + b \cdot \psi_2(m),$$

and the product "$\circ$" given by the composition of maps: $(\psi, \phi) \mapsto \phi \circ \psi$. In general it is not commutative. For example, the algebra $End_{\mathbb{Z}}(\mathbb{Z}^2)$ is isomorphic to the non–commutative algebra of 2×2 matrices with $\mathbb{Z}$-coefficients.

1.2.3. Example. The $\mathbb{R}$-vector space $\mathbb{R}^3$ of columns vectors with three (real) components, acquires a Euclidean structure via the inner product

$$< \mathbf{u}, \mathbf{v} >= \mathbf{u}^T \cdot \mathbf{v}, \tag{1.2}$$

where ^{T} denotes transposition. Equality (1.2) defines a positive definite symmetric bilinear form. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the *cross product* $\mathbf{u} \times \mathbf{v}$ is the unique vector of $\mathbb{R}^3$ such that

$$< \mathbf{u} \times \mathbf{v}, \mathbf{w} >= \det(\mathbf{u}, \mathbf{v}, \mathbf{w}),$$

where "det" denotes the determinant of the square matrix of the components of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$. It turns out that $(\mathbb{R}^3, \times)$ is a $\mathbb{R}$-algebra, but it is neither commutative ($\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$) nor associative. In fact if $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is the canonical basis of $\mathbb{R}^3$ one has $0 = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} \neq \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j})$. Indeed $(\mathbb{R}^3, \times)$ is the simplest example of (non commutative) *Lie algebra*, because it satisfies the *Jacobi identity*:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = 0.$$

1.2.4. Example. If A is any commutative ring and t an indeterminate, the polynomial ring $A[t]$ has an obvious structure of an associative commutative A-algebra, through the monomorphism $A \to A[t]$ mapping each $a \in A$ to the constant polynomial.

1.3 Exterior algebra of a free A-module

1.3.1. To each free A-module $M := [m_1, \ldots, m_n]_A := \sum_{i=1}^n Am_i$ of rank n, one may attach a sequence $\bigwedge^k M$ of A-modules, $k \geq 0$, as follows. By definition $\bigwedge^0 M = A$ and $\bigwedge^1 M = M$, and for $k > 1$ one sets $\bigwedge^k M$ to be the A-module generated by all elements of the form

$$m_{i_1} \wedge \ldots \wedge m_{i_k}$$

subject to the relation:

$$m_{i_{\tau(1)}} \wedge \ldots \wedge m_{i_{\tau(k)}} = \mathrm{sgn}(\tau) m_{i_1} \wedge \ldots \wedge m_{i_k}, \tag{1.3}$$

where $\tau \in S_n$ is a *permutation* on n-elements and $\mathrm{sgn}(\tau)$ is its sign, i.e. ± 1 according to the parity of τ (the number modulo 2 of pairs $i < j$ such that $\tau(i) > \tau(j)$). Keeping relation (1.3) into account, it turns out that $\bigwedge^k M$ is free over A, generated by all $m_{i_1} \wedge \ldots \wedge m_{i_k}$ with $i_1 < i_2 < \ldots < i_k$. Clearly $\bigwedge^k M = 0$ if $k > n$ and $\bigwedge^n M$ is free of rank 1 generated by $m_1 \wedge \ldots \wedge m_n$.

1.3.2. Definition. *The* exterior algebra *of M is the graded module*

$$\bigwedge M = \bigoplus_{k\geq 0} \bigwedge^k M = A \oplus M \oplus \bigwedge^2 M \oplus \ldots \oplus \bigwedge^n M,$$

with respect to the product $\wedge$ defined by juxtaposition:

$$(m_{i_1} \wedge \ldots \wedge m_{i_h}) \wedge (m_{i_{h+1}} \wedge \ldots \wedge m_{i_k}) = m_{i_1} \wedge \ldots \wedge m_{i_h} \wedge m_{i_{h+1}} \wedge \ldots \wedge m_{i_k}.$$

1.3.3. Each endomorphism $\psi \in \mathrm{End}_A(M)$ (Cf. Example 1.2.2) induces a distinguished A-endomorphism

$$\bigwedge^k \psi : \bigwedge^k M \longrightarrow \bigwedge^k M,$$

which is nothing but the A-linear extension of the map

$$\bigwedge^k \psi(m_{i_1} \wedge \ldots \bigwedge m_{i_k}) = \psi(m_{i_1}) \wedge \ldots \wedge \psi(m_{i_k}).$$

One says that the *rank* of ψ is k, and writes $\mathrm{rk}_A(\psi) = k$, if $\bigwedge^k \psi \neq 0$ and $\bigwedge^{k+1} \psi = 0$.

1.4 Formal Power (and Laurent) Series

1.4.1. A *formal power series* with A-coefficients is a formal infinite sum

$$\mathbf{a}(t) = \sum_{n\geq 0} a_n t^n, \qquad a_n \in A. \tag{1.4}$$

The set of all of formal power series is denoted by $A[[t]]$. If $\mathbf{a}(t) \in A[[t]]$, then $\mathbf{a} := (a_0, a_1, \ldots)$ is the sequence of its coefficients. Conversely if $\mathbf{a} := (a_0, a_1, \ldots)$ is any sequence one may construct a formal power series $\mathbf{a}(t)$ as in (1.4). If $\mathbf{b}(t) = \sum_{n\geq 0} b_n t^n$ and $\lambda, \mu \in A$ then

$$\lambda\mathbf{a}(t) + \mu\mathbf{b}(t) = \sum_{n\geq 0}(\lambda a_n + \mu b_n)t^n, \tag{1.5}$$

is the *linear combination* of $\mathbf{a}(t)$ and $\mathbf{b}(t)$ with coefficients λ, μ and $A[[t]]$ is clearly an A-module with respect to such a notion. The equality:

$$\mathbf{a}(t) \cdot \mathbf{b}(t) = \sum_{n\geq 0}\left(\sum_{h=0}^{n} a_b b_{n-h}\right) t^n, \tag{1.6}$$

defines the *product* of $\mathbf{a}(t), \mathbf{b}(t) \in A[[t]]$. Each ring homomorphism $\phi : A \to B$ induces a formal power series homomorphism $\widehat{\phi} : A[[t]] \to B[[t]]$:

$$\widehat{\phi}(\sum_{n\geq 0} a_n t^n) = \sum_{n\geq n} \phi(a_n)t^n.$$

The A-algebra $A[t]$ of polynomials in the indeterminate t will be seen as an A-sub-algebra of $A[[t]]$, by identifying a polynomial with a formal power series having all but finitely many zero coefficients.

1.4.2. Each commutative ring is naturally a $\mathbb{Z}$-module, by defining $n \cdot a = \underbrace{a + \ldots + a}_{n \text{ times}}$ if n is positive and $(-n) \cdot (-a)$ if n is negative. If $\mathbf{a}(t) \in A[[t]]$ is as in (1.4) we define:

$$L(\mathbf{a}(t)) = \sum_{n\geq 0} n! a_n t^n.$$

We also set $L(\mathbf{a}) := (a_0, a_1, 2!a_2, 3!a_3, \ldots)$, so that $L(\mathbf{a})(t) = L(\mathbf{a}(t))$. The map L will be called the *formal Laplace transform* and is not invertible, unless A is a $\mathbb{Q}$-algebra (i.e. elements of A can be multiplied by rational numbers).

1.4.3. The sequence $\mathbf{t} := (1, t, t^2, \ldots)$ is a multiplicative system in the A-algebra $A[[t]]$, i.e. the product of any two terms of the sequence belongs to the sequence itself ($t^m t^n = t^{m+n}$). Let

$$A[t^{-1}, t]] := A[[t]]_{(t)},$$

be the localization of $A[[t]]$ with respect to the multiplicative system $\mathbf{t}$. Any element of $A[t^{-1}, t]]$ is a ratio of the form

$$\frac{\mathbf{a}(t)}{t^k},$$

where $\mathbf{a}(t) \in A[[t]]$. The elements of the localization $A[t^{-1}, t]]$ are said to be a *formal Laurent series*, which can be written as a sum:

$$\sum_{n\in\mathbb{Z}} a_n t^n, \qquad a_n \in A,$$

where $a_i = 0$ for all but finitely many $i < 0$. The *order* of a formal Laurent series is $k \geq 0$ if and only if $a_{-k} \neq 0$ and $a_j = 0$ for all $j < -k$. The *residue* of $a(t) \in A[t^{-1}, t]]$ is a_{-1}. The set $A[t^{-1}, t]]$ is obviously an A-module and is an A-algebra with respect to the product

$$\mathbf{a}(t)\mathbf{b}(t) = \sum_{n\in\mathbb{Z}} \left(\sum_{h+k=n} a_h b_k \right) t^n. \tag{1.7}$$

The right–hand side of (1.7) is well defined, as the sum $\sum\limits_{h+k=n} a_k b_{n-k}$, by construction, is finite for all $n \in \mathbb{Z}$.

1.4.4. One similarly defines the A-algebra $A[[t^{-1}, t]$, which is the set of all formal series with at most finitely many non zero coefficients of positive powers of t. The set $A[t^{-1}, t]$ of the *Laurent polynomial* (all coefficients zero but finitely many) is obviously an A-subalgebra of both $A[t^{-1}, t]]$ and $A[[t^{-1}, t]$. Indeed:

$$A[t^{-1}, t] = A[t^{-1}, t]] \cap A[[t^{-1}, t].$$

Furthermore $A[t]$ is a sub-algebra of $A[t^{-1}, t]$ and of $A[[t]]$ and $A[[t]]$ is an A-sub algebra of $A[t^{-1}, t]]$.

1.4.5. Remark. One may wonder why one did not define formal Laurent series $A[[t^{-1}, t]]$, i.e. the set of all expressions

$$\sum_{n\in\mathbb{Z}} a_n t^n$$

with no restriction on the number of non zero coefficients of negative powers. The main reason is that while such a set is certainly meaningful as an A-module, it is not as an A-algebra if A is an arbitrary ring. This is because the sum occurring at the right hand side of (1.7) would be infinite and hence meaningless without any prescribed notion of convergence. However if A is a topological field like $\mathbb{R}$ (the reals) and $\mathbb{C}$ (the complex numbers), the product of formal Laurent series with infinitely many non zero positive and negative terms can be considered provided that the series occurring in (1.7) are convergent.

1.4.6. Invertible formal power series. Let $\mathbf{a}(t) := \sum_{n\geq 0} a_n t^n \in A[[t]]$. If a_0 is a unit in A, then $\mathbf{a}(t)$ is invertible in $A[[t]]$, i.e. there exists $\mathbf{b}(t) \in A[[t]]$ such that

$$\mathbf{a}(t)\cdot\mathbf{b}(t) = \mathbf{b}(t)\cdot\mathbf{a}(t) = 1_A. \tag{1.8}$$

To see this, it is sufficient to define $\mathbf{b}(t) = \sum_{n\in\mathbb{Z}} b_n t^n$ through the equality:

$$\begin{aligned}\sum_{n\in\mathbb{Z}} b_n t^n &= \frac{a_0^{-1}}{1 + a_0^{-1}\sum_{k\geq 1} a_k t^k}\\ &= a_0^{-1}(1 - (a_0^{-1}\sum_{k\geq 1} a_k t^k) + (a_0^{-1}\sum_{k\geq 1} a_k t^k)^2 - \ldots) \qquad (1.9)\\ &= a_0^{-1}\sum_{n\geq 0}(-1)^h(a_0^{-1}\sum_{k\geq 1} a_k t^k)^n.\end{aligned}$$

Because in (1.9) only positive powers of t occur , $b_j = 0$ for all $j < 0$, i.e. the inverse $\mathbf{b}(t)$ is a formal power series *strictu sensu* (no negative power of t involved). To explicitly determine the coefficients b_n of t^n, one may also observe that Equation (1.8) implies $a_0 b_0 = 1$ and, for each $k > 0$, the bilinear relations:

$$a_0 b_k + a_1 b_{k-1} + \ldots + a_k b_0 = 0,$$

showing that, inductively, all the b_i's are polynomial expressions in the a_j's:

$$b_0 = a_0^{-1}, \quad b_1 = -a_0^{-2}a_1, \quad b_2 = -a_0^{-3}a_1^2 + a_2 a_0^{-2}, \quad \ldots$$

1.5 The formal derivative

1.5.1. If B is any A-algebra, an A-derivation $d : B \to B$ is an A-linear map satisfying the Leibniz rule:

$$d(b_1 b_2) = b_1 d(b_2) + d(b_1) b_2.$$

The map $D : A[[t]] \to A[[t]]$ defined by

$$D \sum_{n \geq 0} a_n t^n = \sum_{n \geq 0} n a_n t^{n-1}, \tag{1.10}$$

is an A-derivation of $A[[t]]$. In fact it is easily seen that it is A-linear:

$$D(\lambda \mathbf{a}(t) + \mu \mathbf{b}(t)) = \lambda \cdot D(\mathbf{a}(t)) + \mu D(\mathbf{b}(t)),$$

$\ker D = A$ (constant formal power series are mapped to zero) and Leibniz's rule holds:

$$D(\mathbf{a}(t) \cdot \mathbf{b}(t)) = D(\mathbf{a}(t)) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot D(\mathbf{b}(t)).$$

for each $\mathbf{a}(t), \mathbf{b}(t)$. In fact, if $\mathbf{a}(t)$ and $\mathbf{b}(t)$ are as in 1.4.1, one has:

$$\begin{aligned}
D(\mathbf{a}(t) \cdot \mathbf{b}(t)) &= \sum_{n \geq 0} n \left(\sum_{h+k=n} a_h b_k \right) t^{n-1} \\
&= \sum_{n \geq 0} \left(\sum_{h+k=n} n a_h b_k \right) t^{n-1} \\
&= \sum_{n \geq 0} \left(\sum_{h+k=n} h a_h b_k + k a_h b_k \right) t^{n-1} \\
&= \sum_{n \geq 0} \sum_{h+k=n} h a_h b_k t^{n-1} + \sum_{n \geq 0} \sum_{h+k=n} k a_h b_k t^{n-1} \\
&= \sum_{h \geq 0} h a_h t^{h-1} \sum_{k \geq 0} b_k t^k + \sum_{h \geq 0} a_h t^h \sum_{k \geq 0} k b_k t^{k-1} \\
&= D(\mathbf{a}(t)) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot D(\mathbf{b}(t)).
\end{aligned}$$

If $\mathbf{a} := (a_0, a_1, \ldots)$, one sets $D\mathbf{a} = (0, a_1, 2a_2, 3a_3, \ldots)$ so that $(D\mathbf{a})(t) = D(\mathbf{a}(t))$.

1.5.2. The i-th iteration of the A-derivation D is a differential operator of order i: it is linear as well and satisfies a generalized Leibniz rule whose verification is based on an easy induction:

$$D^n(\mathbf{a}(t)\cdot\mathbf{b}(t)) = \sum_{k=0}^{n}\binom{n}{k}\mathbf{a}^{(k)}(t)\cdot\mathbf{b}^{(n-k)}(t), \tag{1.11}$$

where one has set $\mathbf{a}^{(i)}(t) := D^i(\mathbf{a}(t))$.

1.6 The generating function of binomial coefficients

1.6.1. To each pair (k,n) of integers one may attach a *binomial coefficient*

$$\binom{n}{k}.$$

By definition, it is the coefficient of t^k in the expansion of $(1+t)^n$. The latter is a polynomial for $n \geq 0$ and a formal power series for $n < 0$. This is one of the easiest examples where *generating functions* come into play in combinatorics. In fact one can say that $f(t) := (1+t)^n$ is the *generating function* of the binomial coefficients:

$$(1+t)^n = \sum_{n\in\mathbb{Z}}\binom{n}{k},$$

in the sense that the right hand side of the equality is defined through its left hand side. In particular

$$\binom{n}{k} = 0,$$

for all $k < 0$, as there is no negative power of t in the expansion of $(1+t)^n$. Similarly

$$\binom{0}{k} = \begin{cases} 0 & \text{if} \quad k \neq 0, \\ 1 & \text{if} \quad k = 0, \end{cases}$$

because $(1+t)^0 = 1 = 1 \cdot t^0$. As a further example, for each $k \geq 0$

$$\binom{-1}{k} = (-1)^k,$$

because of the equality[1]:

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots = \sum_{k \geq 0} (-1)^k t^k.$$

The binomial coefficients $\{\binom{-n}{k} \mid n \geq 0\}$ can be computed inductively. In fact:

$$\sum_{l \in \mathbb{Z}} \binom{-n}{l} t^l = \frac{1}{(1+t)^n} = \frac{1}{1+t} \cdot \frac{1}{(1+t)^{n-1}} = \frac{1}{1+t} \cdot \sum_{k \geq 0} \binom{1-n}{k} t^k.$$

Hence

$$\begin{aligned} \sum_{l \in \mathbb{Z}} \binom{-n}{l} t^l &= \sum_{h \geq 0} (-1)^h t^h \cdot \sum_{k \geq 0} \binom{1-n}{k} t^k \\ &= \sum_{l \geq 0} \left(\sum_{h,k \geq 0 \,\mid\, h+k=l} (-1)^h \binom{1-n}{k} \right) t^l, \end{aligned}$$

from which:

$$\binom{-n}{l} = \sum_{h,k \geq 0 \,\mid\, h+k=l} (-1)^h \binom{1-n}{k}. \tag{1.12}$$

1.6.2. Example. To compute

$$\binom{-2}{k} = (-1)^k (k+1) \tag{1.13}$$

one uses (1.12):

$$\begin{aligned} \binom{-2}{k} &= \sum_{h=0}^{k} (-1)^h \binom{-1}{k-h} \\ &= \binom{-1}{k} - \binom{-1}{k-1} + \ldots + (-1)^{k+1} \binom{-1}{0} \\ &= (-1)^k (k+1). \end{aligned}$$

[1] Holding in $A[[t]]$ for each commutative ring A (in particular $A = \mathbb{Z}$).

Formula (1.13) is equivalent to the equality:

$$\frac{1}{(1+t)^2} = 1 - 2t + 3t^2 - 4t^3 + \dots$$

1.6.3. Example. To compute the coefficient of t^3 in $(1+t)^{-4}$, one can use induction as follows:

$$\binom{-4}{3} = \binom{-3}{3} - \binom{-3}{2} + \binom{-3}{1} - \binom{-3}{0}$$

Now:

$$\begin{aligned}
\binom{-3}{3} &= \binom{-2}{3} - \binom{-2}{2} + \binom{-2}{1} - \binom{-2}{0} = -4 - 3 - 2 - 1 = -10;\\
\binom{-3}{2} &= \binom{-2}{2} - \binom{-2}{1} + \binom{-2}{0} = 3 + 2 + 1 = 6;\\
\binom{-3}{1} &= \binom{-2}{1} - \binom{-2}{0} = -2 - 1 = -3.
\end{aligned}$$

Therefore

$$\binom{-4}{3} = -10 - 6 - 3 - 1 = -20.$$

1.6.4. Similarly, for each $n > 0$, the right hand side of the equality

$$\sum_{k\in\mathbb{Z}} \binom{n}{k} t^k = (1+t)^n = (1+t)\cdot(1+t)^{n-1} = (1+t)\cdot \sum_{k=0}^{n-1} \binom{n-1}{k} t^k,$$

can be rewritten as

$$(1+t)\cdot \sum_{k=0}^{n-1} \binom{n-1}{k} t^k = \sum_{k=0}^{n-1} \binom{n-1}{k} t^k + \sum_{k=0}^{n-1} \binom{n-1}{k} t^{k+1}.$$

But:

$$\sum_{k=0}^{n-1} \binom{n-1}{k} t^k = \sum_{k=0}^{n} \binom{n-1}{k} t^k,$$

because $\binom{n-1}{n} = 0$ and

$$\sum_{k=0}^{n-1} \binom{n-1}{k} t^{k+1} = \sum_{k=0}^{n} \binom{n-1}{k-1} t^k,$$

obtained by substituting $k \to k+1$, changing the index of summation and using the fact that $\binom{n-1}{-1} = 0$. Thus:

$$\sum_{k=0}^{n} \binom{n}{k} t^k = \sum_{k=0}^{n} \left[\binom{n-1}{k} + \binom{n-1}{k-1} \right] t^k,$$

from which

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \tag{1.14}$$

for each $n \geq 0$ and each $k \in \mathbb{Z}$.

1.6.5. Exercise. Using (1.14) prove by induction that:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{1.15}$$

for $0 \leq k \leq n$.

1.6.6. Similarly, using (1.12) and (1.14) prove that for all $n \in \mathbb{Z}$ and each $k \geq 0$ [2]:

$$\binom{n}{k} = \frac{n(n-1) \cdot \ldots \cdot (n-k+1)}{k!}.$$

[2]Recall that the definition of the factorial $n!$ of a non negative integer n is inductive: $0! = 1$ and $n! = n \cdot (n-1)!$.

Chapter 2

Formal power series over $\mathbb{Q}$-algebras

2.1 Basics on $\mathbb{Q}$-algebras

2.1.1. An *associative commutative $\mathbb{Q}$-algebra with unit* (Cf. Section 1.2.1) is a $\mathbb{Q}$-vector space A equipped with a binary operation " $\cdot$ " , such that $(A, +, \cdot)$ is a commutative ring whose product has a neutral element. All overfields of $\mathbb{Q}$, in particular $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ (the rationals, the real and the complex numbers), are $\mathbb{Q}$-algebras. A ring of polynomials with coefficients in a $\mathbb{Q}$-algebra is a $\mathbb{Q}$-algebra. The expression $\mathbb{Q}$-algebra with no additional adjective will always mean an *associative commutative $\mathbb{Q}$-algebra with unit.*

2.1.2. A monic polynomial $P \in A[t]$ of degree $r+1$ with coefficients in any $\mathbb{Q}$-algebra A, will be written as:

$$P(t) := t^{r+1} - e_1(P)t^r + \ldots + (-1)^{r+1}e_{r+1}(P),$$

and the sequence $\mathbf{e}(P) = (e_1(P), e_2(P), \ldots, e_{r+1}(P))$ will be said, abusing terminology, the *sequence of the coefficients* of P.

2.1.3. Example. Suppose that $A := \mathbb{Q}[x_1, \ldots, x_{r+1}]$, where x_i is an indeterminate over $\mathbb{Q}$ $(1 \le i \le r+1)$. If

$$P := \prod_{i=1}^{r+1}(t - x_i) = (t - x_1) \cdot \ldots \cdot (t - x_r) \cdot (t - x_{r+1})$$

then $e_i(P)$ is the i-th elementary symmetric polynomial in the indeterminates $x_1, \ldots, x_r, x_{r+1}$:

$$\begin{cases} e_1(P) & = & x_1 + \ldots + x_r + x_{r+1}, \\ e_2(P) & = & \displaystyle\sum_{1 \le i < j \le r+1} x_i x_j, \\ \vdots & & \\ e_j(P) & = & \displaystyle\sum_{1 \le i_1 < i_2 < \ldots < i_j \le r+1} x_{i_1} \cdot \ldots \cdot x_{i_j} \\ \vdots & & \\ e_{r+1}(P) & = & x_1 \cdot \ldots \cdot x_r x_{r+1}. \end{cases}$$

This explains the notation, which follows that of [29, p. 12].

2.2 Formal power series in $\mathbb{Q}$-algebras

2.2.1. Since the elements of a $\mathbb{Q}$-algebra A can be multiplied by rational numbers, any formal power series $\mathbf{a}(t) \in A[[t]]$ can be alternatively written as an infinite linear combination of the monomials $\dfrac{t^n}{n!}$:

$$\mathbf{a}(t) = \sum_{n \ge 0} a_n \frac{t^n}{n!}, \qquad (a_n \in A)^1. \tag{2.1}$$

The *formal Laplace transform* (Cf. Section 1.4.2) of $\mathbf{a}(t) \in A[[t]]$ written like in (2.1) is

$$L(\mathbf{a}(t)) = \sum_{n \ge 0} a_n t^n,$$

[1]The coefficients a_n occurring in this expression have not the same meaning of those occurring in (1.4)

and so $L(\mathbf{a}) = (a_0, a_1, \ldots)$. Notice that L is now invertible:

$$L^{-1}(\sum_{n\geq 0} a_n t^n) = \sum_{n\geq 0} a_n \frac{t^n}{n!},$$

because it is possible to divide by $n!$. To each sequence $L(\mathbf{a}) := (a_0, a_1, \ldots)$ in the $\mathbb{Q}$-algebra A corresponds the formal power series $\mathbf{a}(t)$ as in (2.1). If $L(\mathbf{b}) := (b_0, b_1, \ldots)$, the *product* $\mathbf{a}(t)\mathbf{b}(t)$ can be now expressed as:

$$\mathbf{a}(t)\mathbf{b}(t) = \sum_{n\geq 0}\left[\sum_{h=0}^{n}\binom{n}{h} a_h b_{n-h}\right]\frac{t^n}{n!}.$$

In fact

$$\begin{aligned}
\mathbf{a}(t)\mathbf{b}(t) &= \sum_{n\geq 0} a_n \frac{t^n}{n!}\cdot\sum_{n\geq 0} b_n\frac{t^n}{n!} = \sum_{n\geq 0}\left[\sum_{k=0}^{n-k}\frac{a_k}{k!}\cdot\frac{b_{n-k}}{(n-k)!}\right]t^n \\
&= \sum_{n\geq 0}\left[\sum_{k=0}^{n} n!\cdot\frac{a_k}{k!}\cdot\frac{b_{n-k}}{(n-k)!}\right]\frac{t^n}{n!} \\
&= \sum_{n\geq 0}\left[\sum_{k=0}^{n}\binom{n}{k}\cdot a_k\cdot b_{n-k}\right]\frac{t^n}{n!}. \qquad (2.2)
\end{aligned}$$

The formulas above in particular show that $L(\mathbf{a}(t)\mathbf{b}(t)) \neq L(\mathbf{a}(t))\times L(\mathbf{b}(t))$.

2.2.2. Example. For each $a \in A$, the *exponential formal power series* is

$$\exp(at) = \sum_{n\geq 0} a^n \frac{t^n}{n!}.$$

For each choice $a, b \in A$, one has:

$$\exp(at)\exp(bt) = \sum_{n\geq 0}\left[\sum_{h=0}^{n}\binom{n}{k} a^k b^{n-k}\right]\frac{t^n}{n!} = \sum_{n\geq 0}(a+b)^n\frac{t^n}{n!} = \exp((a+b)t)$$

The formal Laplace transform of the exponential of at is

$$L(\exp(at)) = \sum_{n\geq 0} a^n t^n = \frac{1}{1-at}.$$

2.2.3. Example. Let $\mathbf{f}(t) \in \mathbb{C}[[t]]$ be a formal power series with complex coefficients and let

$$\frac{1}{R} := \lim_{n\to\infty} \sup \sqrt[n]{a_n}.$$

By Hadamard's Theorem (see e.g. [6, p. 20]), f defines a holomorphic function in the disc $D_R := \{z \in \mathbb{C} \mid |z| < R\}$ given by $f(z) = \sum_{n\geq 0} a_n z^n$. If $a \in \mathbb{C}$, $\exp(at)$ defines the entire (i.e. defined on the whole $\mathbb{C}$) exponential function $\exp(az)$.

2.2.4. A map $\psi : A \to B$ is a homomorphism of $\mathbb{Q}$-algebras if $\psi(\lambda\cdot a_1+\mu\cdot a_2) = \lambda\cdot\psi(a_1)+\mu\cdot\psi(a_2)$ and $\psi(a_1a_2) = \psi(a_1)\cdot\psi(a_2)$, for arbitrary choices of $\lambda, \mu \in \mathbb{Q}$ and $a_1, a_2 \in A$. If $\psi \in Hom_{\mathbb{Q}-alg}(A, B)$, then the induced $\mathbb{Q}$-algebra homomorphism $\widehat{\psi} : A[[t]] \to B[[t]]$ can be expressed as:

$$\widehat{\psi}\left(\sum_{n\geq 0} a_n \frac{t^n}{n!}\right) = \sum_{n\geq 0} \psi\left(\frac{a_n}{n!}\right) t^n = \sum_{n\geq 0} \psi(a_n)\frac{t^n}{n!}.$$

2.3 Polynomial $\mathbb{Q}$-algebras

2.3.1. Let us fix once and for all a sequence

$$\mathbf{e} := (e_1, e_2, \dots,) \tag{2.3}$$

of indeterminates over $\mathbb{Q}$. For each $r \geq 0$, denote by E_r the polynomial $\mathbb{Q}$-algebra

$$E_r := \mathbb{Q}[e_1, e_2, \dots, e_{r+1}]. \tag{2.4}$$

We set by convention $E_{-1} = \mathbb{Q}$. By giving degree i to the indeterminate e_i, the algebra E_r can be seen as a graded $\mathbb{Q}$-algebra:

$$E_r := \bigoplus_{w\geq 0} (E_r)_w,$$

where $(E_r)_w$ is the set of all weighted homogeneous polynomials of degree w. For example

$$(E_r)_1 = \mathbb{Q}\cdot e_1,\ (E_r)_2 = \mathbb{Q}\cdot e_1^2 \oplus \mathbb{Q}\cdot e_2,\ (E_r)_3 = \mathbb{Q}\cdot e_1^3 \oplus \mathbb{Q}\cdot e_1e_2 \oplus \mathbb{Q}\cdot e_3,\ \dots$$

2.3.2. If $-1 \leq s \leq r$, the module E_s may be viewed either as $\mathbb{Q}$-subalgebra of E_r or as a quotient of it under the map

$$p_{r,s} : E_r \to E_s \tag{2.5}$$

mapping $e_{s+1}, \ldots, e_{r+1}$ to zero. Notice that $p_{r_2,r_3} \circ p_{r_1,r_2} = p_{r_1,r_3}$ for each $r_1 \geq r_2 \geq r_3$ and hence the ring of infinitely many indeterminates

$$E_\infty := \mathbb{Q}[e_1, e_2, \ldots],$$

is in fact the inverse (or projective) limit of the algebras E_r in the category of graded $\mathbb{Q}$-algebras. This just means that if B is any $\mathbb{Q}$-algebra equipped with homomorphism $q_r : B \to E_r$, then there is a unique $\mathbb{Q}$-algebra homomorphism $\psi : B \to E_\infty$ such that $q_r = p_r \circ \psi$, where

$$p_r : E_\infty \to E_r \tag{2.6}$$

is the unique $\mathbb{Q}$-algebra epimorphism sending $e_i \mapsto e_i$ if $1 \leq i \leq r+1$ and $e_j \mapsto 0$ if $j > r+1$. An element $P \in E_\infty$ is a polynomial $P \in E_r$ for some $r \geq -1$. In other words $E_\infty := \bigcup_{r \geq -1} E_r$.

2.3.3. For each $0 \leq r \leq \infty$, the algebras E_r are the most economical ones, in the following sense. Given any other $\mathbb{Q}$-algebra A and an $(r+1)$-tuple $\mathbf{a} := (a_1, \ldots, a_r, a_{r+1}) \in A^{r+1}$, there is a natural evaluation morphism

$$\begin{cases} \mathrm{ev}_{\mathbf{a}} & : & E_r & \longrightarrow & A, \\ & & P & \mapsto & P(\mathbf{a}), \end{cases} \tag{2.7}$$

which is indeed the unique $\mathbb{Q}$-algebra homomorphism mapping $e_i \mapsto a_i$, for $1 \leq i \leq r+1$.

2.3.4. Definition. *The* universal $\mathbb{Q}$-polynomial $U_{r+1} \in E_r[t]$ *of degree* $r+1$ *is*

$$U_{r+1}(t) = t^{r+1} - e_1 t^r + \ldots + (-1)^{r+1} e_{r+1}. \tag{2.8}$$

In other words $e_i(U_{r+1}(t)) = e_i$. We also define, for each $r \geq 0$:

$$V_{r+1} = t^{r+1} U_{r+1}\left(\frac{1}{t}\right) = 1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}. \tag{2.9}$$

and (setting $e_0 = 1$):

$$V_\infty = \sum_{n\geq 0}(-1)^n e_n t^n = 1 - e_1 t + e_2 t^2 - \ldots \in E_\infty[[t]]. \qquad (2.10)$$

Notice that under the maps (2.5) $p_{r,s} : E_r \to E_s$ and (2.6) $p_r : E_\infty \to E_r$, one has $\widehat{p}_{r,s}(V_{r+1}) = V_{s+1}$ and $\widehat{p}_r(V_\infty) = V_{r+1}$.

2.3.5. To each sequence $L(\mathbf{a}) := (a_0, a_1, \ldots,)$ of elements of an arbitrary E_r-algebra A, one may attach the sequence $U_0(\mathbf{a})$, $U_1(\mathbf{a})$, $U_2(\mathbf{a}), \ldots$ by setting $U_0(\mathbf{a}) = a_0$ and

$$U_i(\mathbf{a}) := a_i - e_1 a_{i-1} + \ldots + (-1)^{r+1} e_{r+1} a_{i-r-1} \qquad (2.11)$$

for each $i \geq 1$, with the convention that $a_j = 0$ if $j < 0$. So, for instance,

$$U_1(\mathbf{a}) = a_1 - e_1 a_0, \quad U_2(\mathbf{a}) = a_2 - e_1 a_1 + e_2 a_0, \ldots$$

Any finite sequence in A will be thought of as an infinite sequence such that all but finitely many terms are zero.

2.3.6. Remark. For each $j \in \mathbb{Z}$, let $\mathrm{s}_j(\mathbf{a})$ be the sequence $\mathbf{a}$ shifted by j

$$\mathrm{s}_j(\mathbf{a}) = (a_j, a_{j+1}, \ldots).$$

In particular $\mathrm{s}_0(\mathbf{a}) = \mathbf{a}$. If $e_{r+1+j} = 0$ for each $j > 0$, then

$$U_{r+1+j}(\mathbf{a}) = U_{r+1}(\mathrm{s}_j(\mathbf{a})).$$

2.3.7. Combinatorial Lemma. *For each $0 \leq r \leq \infty$, the equality*

$$\sum_{j=0}(-1)^j e_j t^j \sum_{n\geq 0} a_n t^n = \sum_{i\geq 0} U_i(\mathbf{a}) t^i \qquad (2.12)$$

holds in $A[[t]]$.

Proof. By the rule (2.2) for multiplying formal power series, it turns out that the coefficient of t^i, for $i \geq 0$, is the sum of products of the form $(-1)^j e_j a_k$ such that $j + k = i$, i.e.:

$$a_i - e_1 a_{i-1} + \ldots + (-1)^i e_i a_0 = U_i(\mathbf{a}) \qquad (2.13)$$

and (2.12) is proven. ∎

2.3.8. Remark. Equations (2.13) for $0 \leq i \leq r$ can be summarized into the following equality:

$$\begin{pmatrix} U_0(\mathbf{a}) \\ U_1(\mathbf{a}) \\ U_2(\mathbf{a}) \\ \vdots \\ U_r(\mathbf{a}) \end{pmatrix} = \mathcal{E}_r \cdot \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_r \end{pmatrix} \tag{2.14}$$

where

$$\begin{aligned} \mathcal{E}_r &:= ((-1)^{i-j} e_{i-j}) \\ &= \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ -e_1 & 1 & 0 & \dots & 0 \\ e_2 & -e_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^r e_r & (-1)^{r-1} e_{r-1} & (-1)^{r-2} e_{r-2} & \dots & 1 \end{pmatrix}. \end{aligned}$$

2.3.9. Let $0 \leq s \leq \infty$ and let $\mathbf{h} = (h_j)_{j \in \mathbb{Z}}$ be the sequence of elements of E_s defined through the equality

$$\sum_{n \in \mathbb{Z}} h_n t^n = \frac{1}{V_{s+1}(t)}, \tag{2.15}$$

holding in $E_s[[t]]$. As the inverse of the polynomial $V_{s+1}(t) = \sum_{i=0}^{s+1} (-1)^i e_i$ is an A–linear combination of positive powers of t only, then $h_j = 0$ for all $j < 0$. In addition $h_0 = 1$. Equaltion (2.15) is equivalent to:

$$1 = \sum_{i=0}^{s+1} (-1)^i e_i t^i \cdot \sum_{n \geq 0} h_n t^n = \sum_{j \geq 0} U_j(\mathbf{h}) t^j, \tag{2.16}$$

which implies $U_0(\mathbf{h}) = 1$ and $U_j(\mathbf{h}) = 0$ for all $j \geq 1$. The conditions $U_0(\mathbf{h}) = 1$ and $U_j(\mathbf{h}) = 0$ for each $0 \leq r < s$ can be equivalently written as

$$\mathcal{E}_r \cdot \mathcal{H}_r = \mathcal{H}_r \cdot \mathcal{E}_r = \mathbf{1}, \tag{2.17}$$

where $\mathbf{1}$ denotes the $(r+1)\times(r+1)$ identity matrix and

$$\mathcal{H}_r := (h_{i-j}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ h_1 & 1 & 0 & \dots & 0 \\ h_2 & h_1 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_r & h_{r-1} & h_{r-2} & \dots & 1 \end{pmatrix}.$$

In particular, Equation (2.14) can be written as:

$$\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix} = \mathcal{H}_r \cdot \begin{pmatrix} U_0(\mathbf{a}) \\ U_1(\mathbf{a}) \\ \vdots \\ U_r(\mathbf{a}) \end{pmatrix}. \tag{2.18}$$

2.3.10. For each $j \in \mathbb{Z}$, let

$$\boxed{u^{(j)} = L^{-1}(\mathrm{s}_j(\mathbf{h})(t)) = \sum_{n\geq 0} h_{n+j}\frac{t^n}{n!}.} \tag{2.19}$$

Notice that $D^i u^{(j)} = u^{(i+j)}$, for all $i, j \in \mathbb{Z}$. Then:

2.3.11. Main Lemma. *If A is any E_r-algebra and $L(\mathbf{a}(t)) = \sum_{n\geq 0} a_n t^n \in A[[t]]$, then the formal power series $u^{(j)}$ are linearly independent in $A[[t]]$ and, in addition:*

$$\mathbf{a}(t) = \sum_{j\geq 0} a_n \frac{t^n}{n!} = \sum_{n\geq 0} U_j(\mathbf{a})u^{(-j)} = a_0u^{(0)} + U_1(\mathbf{a})u^{(-1)} + \dots \tag{2.20}$$

Lemma 2.3.11 says that each element of $A[[t]]$ is an infinite linear combination of the $u^{(-j)}$. Taking infinite linear combinations of the $u^{(-j)}$ is meaningful, as the sequence $(u^{(-j)})_{j\geq 0}$ is *summable* in the sense of [6, p. 11]: for each $k \geq 0$, all terms except a finite number have order greater that k (the order of a formal power series is the smallest power of t occurring in it).

Proof of Lemma 2.3.11. For each $j \geq 0$, the order of $u^{(-j)}$ is j:

$$u^{(-j)} = \frac{t^j}{j!} + \sum_{n\geq 1} h_n \frac{t^{j+n}}{(j+n)!}.$$

Hence all the $u^{(-j)}$ are linearly independent, because they have distinct orders. To prove (2.20) one considers the coefficient of t^n in the right hand side of it:

$$\begin{aligned}\left[\sum_{j\geq 0} U_j(\mathbf{a})u^{(-j)}\right]_n &= \sum_{j\geq 0} U_j(\mathbf{a})[u^{(-j)}(0)]_n = \sum_{j\geq 0} U_j(\mathbf{a})h_{n-j} \\ &= U_0(\mathbf{a})h_n + h_{n-1}U_1(\mathbf{a}) + \ldots + h_0U_n(\mathbf{a}) = a_n\end{aligned}$$

where $[\ ]_n$ denotes the coefficient of degree n and the last equality is due to (2.18). ∎

2.3.12. Example. For each $n \geq 0$ one has

$$t^n = u^{(-n)} - e_1u^{(-n-1)} + e_2u^{(-n-2)} + \ldots$$

Furthermore

$$e^{at} = u^{(0)} + U_1(a)u^{(-1)} + U_2(a)u^{(-2)} + \ldots$$

2.3.13. The map (2.7) induces a $\mathbb{Q}$-algebra homomorphism $\mathrm{ev}_{\mathbf{a}} : E_r[t] \to A[t]$, denoted in the same way abusing notation:

$$\begin{aligned}t^{r+1} - p_1(e_i)t^r + \ldots + (-1)^{r+1}p_{r+1}(e_i) \mapsto t^{r+1} - p_1(a_i)t^r \\ + \ldots + (-1)^{r+1}p_{r+1}(a_i),\end{aligned}$$

where $p_1, \ldots, p_{r+1}$ are arbitrary elements in E_r. The adjective *universal* used in Definition 2.3.4 is to emphasize the fact that for each monic $P \in A[t]$ of degree $r+1$, the unique evaluation morphism $\mathrm{ev}_{\mathbf{e}(P)}$ mapping $e_i \mapsto e_i(P)$, sends U_{r+1} to P:

$$\mathrm{ev}_{\mathbf{e}(P)}(U_{r+1}(t)) = P(t).$$

Chapter 3

Universal Solutions to Linear ODEs

In this chapter A will denote any E_r-algebra, fixed once and for all.

3.1 Universal Linear Homogeneous ODE

3.1.1. The E_r-algebra A is a $\mathbb{Q}$-algebra as well. Thus, formal power series in $A[[t]]$ will be written as in (2.1) and $D : A[[t]] \to A[[t]]$, the first–order *ordinary differential operator* (1.10), in the form

$$(D\mathbf{a})(t) := D\left(\sum_{n\geq 0} a_n \frac{t^n}{n!}\right) = \sum_{n\geq 0} a_{n+1}\frac{t^n}{n!}.$$

For each $i \in \mathbb{Z}$, let:

$$(D^i\mathbf{a})(t) := D^i\left(\sum_{n\geq 0} a_n \frac{t^n}{n!}\right) = \sum_{n\geq 0} a_{n+i}\frac{t^n}{n!}, \tag{3.1}$$

with the convention that $a_j = 0$ if $j < 0$. In particular $D^0 = \mathrm{id}_{A[[t]]}$ is the identity morphism of $A[[t]]$ and, if $i \geq 0$, the operator D^i is just the i-th iteration of the endomorphism D.

3.1.2. For $\mathbf{a}(t) \in A[[t]]$ and $n \geq 0$, denote by $(D^n\mathbf{a})(0)$ the class of $D^n\mathbf{a}(t)$ modulo the ideal (t^{n+1}), so that:

$$(D^n\mathbf{a})(0) = a_n.$$

Any formal power series $\mathbf{a}(t) \in A[[t]]$ can be expressed in *Taylor form*:

$$\mathbf{a}(t) = \sum_{n\geq 0}(D^n\mathbf{a})(0)\frac{t^n}{n!}.$$

One says that $\mathbf{a}(t)$ vanishes at 0 with multiplicity at least $n+1$ if $\mathbf{a}(t) \in (t^{n+1})$ or, equivalently, if $(D^j\mathbf{a})(0) = 0$ for all $0 \leq j \leq n$. For each integer $r \geq 0$, the map

$$A[[t]] \longrightarrow \frac{A[[t]]}{(t^{r+1})}$$

associates to each formal power series $\mathbf{a}(t)$ like (2.1) its truncation to the power t^r:

$$\mathbf{a}(t) \mapsto a_0 + a_1 t + \ldots + a_r\frac{t^r}{r!} \in A[t].$$

3.1.3. By *Universal differential operator* we shall mean the evaluation at D of the universal monic polynomial (2.8):

$$U_{r+1}(D) := D^{r+1} - e_1D^r + \ldots + (-1)^{r+1}e_{r+1} \in End_{\mathbb{Q}}(E_r[[t]]). \quad (3.2)$$

Let

$$\mathcal{U}_r := \ker U_{r+1}(D).$$

It is an E_r-submodule of $E_r[[t]]$. Then

$$\mathcal{U}_r \otimes_{E_r} A := \ker(U_{r+1}(D) \otimes_A 1_A)$$

is the A-submodule of $A[[t]]$ of solutions of the *universal ODE*:

$$U_{r+1}(D)y = y^{(r+1)} - e_1y^{(r)} + \ldots + (-1)^{r+1}e_{r+1}y = 0. \quad (3.3)$$

In general, if

$$\mathbf{f} := \sum_{n\geq 0} f_n\frac{t^n}{n!} \in A[[t]], \quad (3.4)$$

then a *solution of the linear Ordinary Differential Equation* (ODE)

$$U_{r+1}(D)y = \mathbf{f} \tag{3.5}$$

is an element of the set $U_{r+1}(D)^{-1}(\mathbf{f}) \subseteq A[[t]]$.

3.1.4. The linear ODE (3.5) is usually more explicitly written as:

$$D^{r+1}y - e_1 D^r y + \ldots + (-1)^{r+1} e_{r+1} y = \mathbf{f}. \tag{3.6}$$

If $\mathbf{f}$ is the zero formal power series, then $U_{r+1}(D)^{-1}(0) = \mathcal{U}_r \otimes A$ is a submodule of $A[[t]]$ called the *module of solutions* of the *homogeneous* linear ODE $P(D)y = 0$.

3.1.5. Proposition. Let $y_0(t) \in U_{r+1}(D)^{-1}(\mathbf{f})$. Then

$$\begin{aligned} U_{r+1}(D)^{-1}(\mathbf{f}) &= y_0(t) + \ker U_{r+1}(D) \\ &= \{y_0(t) + y(t) \mid y(t) \in \ker U_{r+1}(D)\}. \end{aligned}$$

Proof. It is standard linear algebra. The A-linearity of $U_{r+1}(D)$ implies the inclusion $y_0(t) + \ker U_{r+1}(D) \subseteq U_{r+1}(D)^{-1}(\mathbf{f})$. Conversely, if $y_1(t) \in U_{r+1}(D)^{-1}(f)$, then $y_1(t) - y_0(t) \in \ker U_{r+1}(D)$, and then $U_{r+1}(D)^{-1}(\mathbf{f}) \subseteq y_0(t) + \ker U_{r+1}(D)$. ■

3.1.6. Proposition. *Let* $\mathbf{a}(t) \in A[[t]]$ *as in (2.1). Then* $\mathbf{a}(t) \in \ker U_{r+1}(D)$ *if and only the bilinear relation:*

$$U_{n+r+1}(\mathbf{a}) := a_{n+r+1} - e_1 a_{n+r} + \ldots + (-1)^{r+1} e_{r+1} a_n = 0 \tag{3.7}$$

holds for all $n \geq 0$.

Proof. Substituting $y = \mathbf{a}(t)$ in the equation (3.6) with $f = 0$ and keeping into account the expression of D^i given by (3.1), one obtains:

$$\begin{aligned} &\sum_{n\geq 0} a_{n+r+1}\frac{t^n}{n!} - e_1 \sum_{n\geq 0} a_{n+r}\frac{t^n}{n!} + \ldots + (-1)^{r+1} e_{r+1} \sum_{n\geq 0} a_n \frac{t^n}{n!} = \\ = &\sum_{n\geq 0} \left(a_{n+r+1} - e_1 a_{n+r} + \ldots + (-1)^{r+1} e_{r+1} a_n\right) \frac{t^n}{n!}. \end{aligned} \tag{3.8}$$

Then $\mathbf{a}(t) \in \ker U_{r+1}(D)$ if the formal power series (3.8) vanishes, i.e. if and only if (3.7) holds for all $n \geq 0$. ■

3.1.7. Example. The formal power series $\exp(e_1 t)$ solves the differential equation $U_1(D)y = 0$. In fact

$$e_1^{n+1} - e_1 \cdot e_1^n = e_1^{n+1} - e_1 \cdot \lambda e_1^n = e_1^{n+1} - e_1^{n+1} = 0.$$

If A is any $\mathbb{Q}$-algebra and $a \in A$, then A has a structure of algebra over $E_0 := \mathbb{Q}[e_1]$ via the unique homomorphism mapping $e_1 \mapsto a$. The module structure is given by $e_1 \cdot 1_A = a$. Hence $\exp(at) \in A[[t]]$ can be seen as a solution of the equation $Dy - e_1 y = 0$ as well. In fact

$$a^{n+1} - e_1 \cdot a^n = a^{n+1} - a \cdot a^n = a^{n+1} - a^{n+1} = 0.$$

which is the same as saying that $\exp(at)$ is solution of the differential equation $Dy - ay = 0$. The equality

$$P(t) := t - a = U_1(t) \otimes 1_A,$$

implies

$$\ker P(D) = \ker U_1(D) \otimes_{E_1} A = \ker(D-a) = [\exp(at)] := \{\lambda \cdot \exp(at) \mid \lambda \in A\}.$$

3.1.8. Example. Suppose that $P \in A[t]$ is monic of degree $r+1$ and has a root in A, i.e. there exists $a \in A$ such that $P(a) = 0$. It turns out that $\exp(at)$ is a solution of $P(D)y = 0$. In fact, by Ruffini's theorem, $P(t)$ admits the decomposition

$$P(t) = Q(t)(t-a),$$

where $Q(t)$ is a monic polynomial of degree r. It follows that

$$P(D)\exp(at) = Q(D)(D-a)(\exp(at)) = 0$$

because $\exp(at) \in \ker(D-a)$ by Example 3.1.7.

3.1.9. When imposing a formal power series $\mathbf{a}(t)$ to belong to $\ker U_{r+1}(D)$, no condition is required on the first $r+1$ coefficients $a_0, a_1, \ldots, a_r$: these are the *initial conditions* of the solution: $a_j = (D^j \mathbf{a})(0)$, for $0 \leq j \leq r$.

3.1.10. Proposition. *If all the initial conditions $a_0, a_1, \ldots, a_r$ of $\mathbf{a}(t) \in \ker U_{r+1}(D)$ vanish, then $\mathbf{a}(t) = 0$, i.e. $a_n = 0$ for each $n \geq 0$.*

Proof. In fact if $\mathbf{a}(t)$ is a solution, equation (3.7) holds for all $n \geq 0$. In particular, for $n = 0$, it gives $a_{r+1} = 0$ and by induction, assuming that $a_0 = a_1 = \ldots = a_{r+k-1} = 0$, one proves that $a_{r+k} = 0$, by using (3.7) again. ∎

3.1.11. Corollary. *Two formal power series* $\mathbf{a}(t), \mathbf{b}(t) \in \ker U_{r+1}(D)$ *coincide if and only if they have the same initial conditions.*

Proof. If $\mathbf{a}(t) = \mathbf{b}(t)$, the initial conditions obviously coincide. Conversely, if they have the same initial conditions $\mathbf{a}(t) - \mathbf{b}(t) \in \ker U_{r+1}(D)$ and all initial conditions vanish, then $\mathbf{a}(t) - \mathbf{b}(t) = 0$, because of 3.1.10, i.e. $\mathbf{a}(t) = \mathbf{b}(t)$. ■

As in Section 2.3 let now $\mathbf{h} = (h_0, h_1, \dots)$ be given by

$$\sum_{n \in \mathbb{Z}} h_n t^n = \frac{1}{1 - e_1 t + \dots + (-1)^{r+1} e_{r+1} t^{r+1}}$$

and the $(u^{(j)})_{j \in \mathbb{Z}}$ as in (2.19).

3.2 Universal solutions

3.2.1. Theorem. *The A-module* $\mathcal{U}_r \otimes A := \ker U_{r+1}(D) \otimes A$ *is free of rank* $r+1$ *generated by*

$$(u^{(0)}, u^{(-1)}, \dots, u^{(-r)}). \tag{3.9}$$

Proof. We already know that $(u^{(-j)})_{j \geq 0}$ are linearly independent in $A[[t]]$. In addition, for all $0 \leq j \leq r$ one has:

$$U_{r+1}(D)u^{(-j)} = U_{r+1}(D)D^{r-j}u^{(-r)} = D^{r-j}U_{r+1}(D)u^{(-r)},$$

and then to show that $u^{(-j)} \in \mathcal{U}_r$ for $0 \leq j \leq r$, it suffices to show that $u^{(-r)} \in \mathcal{U}_r$. Now

$$L(u^{(-r)}) = \mathrm{s}_{-r}(\mathbf{h}) = (\underbrace{0 \dots, 0}_{r \text{ times}}, 1, h_1, \dots)$$

(recall that $h_j = 0$ if $j < 0$). The left–hand side of Equation (3.7) with $a_n = h_{n-r}$ is:

$$\begin{aligned} a_{n+r+1} - e_1 a_{n+r} &+ \dots + (-1)^{r+1} e_{r+1} a_n \\ &= h_{n+1} - e_1 h_n + \dots + (-1)^{r+1} e_{r+1} h_{n-r}, \end{aligned} \tag{3.10}$$

and the right–hand side of (3.10) is $U_{n+1}(\mathbf{h})$, the coefficient of t^{n+1} in expansion (2.16), which vanishes for all $n \geq 0$ by definition of the h_i.

By criterion 3.1.6, $u^{(-r)} \in \mathcal{U}_r$, and then $u^{(-j)} \in \mathcal{U}_r$ for all $0 \leq j \leq r$. To show that $(u^{(0)}, \ldots, u^{(-r)})$ generates $\mathcal{U}_r$ over A, recall that each

$$\mathbf{a}(t) = \sum_{n \geq 0} a_n \frac{t^n}{n!} \in A[[t]]$$

can be expressed as an infinite linear combination of the $(u^{(-j)})_{j \geq 0}$:

$$\begin{aligned}\mathbf{a}(t) = U_0(\mathbf{a})\mathbf{u}^{(0)} + U_1(\mathbf{a})u^{(-1)} + \ldots + U_r(\mathbf{a})u^{(-r)} \\ + \sum_{n \geq 0} U_{r+1+n}(\mathbf{a})t^{r+1+n},\end{aligned}$$

by (2.3.11). But $\mathbf{a}(t) \in \ker \mathcal{U}_r$ if and only if $U_{r+1+n}(\mathbf{a}) = 0$ for each $n \geq 0$, by Proposition 3.1.6, i.e. if and only if

$$\boxed{\mathbf{a}(t) = U_0(\mathbf{a})u^{(0)} + \ldots + U_r(\mathbf{a})u^{(-r)}}, \tag{3.11}$$

and the theorem is proven. ∎

Formula (3.11) will be called the *Universal Cauchy Formula*.

3.2.2. Remark. Notice that $(u^{(0)}, u^{(-1)}, \ldots, u^{(-r)})$ is a basis of $\ker U_{r+1}(D)$ such that each $u^{(-j)}$ is obtained by differentiating $r-j$ times the solution $u^{(-r)}$. Such kinds of bases are known in analysis. In particular $u^{(-r)}$ generates the *impulsive response kernel*. See e.g. the nice account [5] by Camporesi.

3.2.3. Example. The universal Cauchy formula says that the map $A[D] \to \mathcal{U}_r$ defined as the A-linear extension of $D^j \mapsto D^j u^{(-r)} = u^{(j-r)}$ is an epimorphism. In fact any element of $\ker U_{r+1}(D) \otimes A$ can be written uniquely as an A–linear combination of $u^{(-j)} = D^{r-j}u^{(-r)}$, for $0 \leq j \leq r$. The kernel is generated by the polynomial expression in D vanishing on $\mathcal{U}_r$, which is precisely $U_{r+1}(D)$. So, one obtains the A-algebra isomorphism:

$$\ker U_{r+1}(D) \cong \frac{A[D]}{(\ker U_{r+1}(D) \otimes A)}.$$

The latter is an A-algebra generated by the differential operator D. It is a *D-module*, i.e. a module generated by differential operators, see [9] for a nice and enlightening exposition. Much of the theory we have presented could be developed using solely this language, but we will not pursue this alternative approach just for reasons of space. Notice that

$$\frac{A[D]}{(\ker U_{r+1}(D)\otimes A)}=\frac{E_r[D]}{\ker U_{r+1}(D)}\otimes_{E_r} A$$

and so the right–hand side is the *universal* D-module associated to the differential operator D. It is a free E_r-module of rank $r+1$ generated by $1, D, \ldots, D^r$.

3.2.4. As suggested by Example 3.1.7, if A is any $\mathbb{Q}$-algebra and $P \in A[T]$ is any monic polynomial of degree $r+1$, the unique $\mathbb{Q}$-algebra homomorphism $\psi_P : E_r \to A$ mapping $e_i \mapsto e_i(P)$, for $1 \le i \le r+1$, makes A into an E_r-algebra. Then (3.11) holds as well in such a situation. In particular, by definition of the E_r-algebra structure of A, $e_j a = \psi_P(e_j)a = e_j(P)a$, for each $a \in A$, and then the E_r-module product $U_{r+1}(D)y$ is the same as $P(D)y \in A[[t]]$ in this case. It follows that $\mathbf{a}(t)$ is a solution of the equation $P(D)y = 0$ if and only if it is a solution of $U_{r+1}(D)y = 0$, thinking of y as an element of the E_r-algebra $A[[t]]$. We have then proven the following:

3.2.5. Theorem. *Given $P(D)y = 0$, a homogeneous linear ODE with coefficients in any $\mathbb{Q}$-algebra A, then*

$$\ker P(D) = \ker U_{r+1}(D) \otimes_{E_r} A,$$

where the tensor product is taken with respect to the E_r-algebra structure inherited by A via the unique homomorphism $\psi_P : E_r \to A$ mapping $e_i \mapsto e_i(P)$.

Theorem 3.2.5 can be rephrased by saying that $(u^{(0)}, u^{(-1)}, \ldots, u^{(-r)})$ is a universal basis of solutions, i.e. that each solution of $P(D)y = 0$ is an A-linear combination of the image of the universal basis of $\mathcal{U}_r$ under $\widehat{\psi}_P$.

3.2.6. Example. Let $A := \mathbb{Q}[x_1, x_2]$ where x_1, x_2 are indeterminates, and consider the E_1-algebra structure determined by the map $E_1 \to A$ given by $e_1 \mapsto x_1 + x_2$ and $e_2 \mapsto x_1x_2$. Then $(exp(x_1t)$ and $\exp(x_2t)$ are linearly independent solutions of $U_2(D)y = 0$. However, they are not a basis of $\ker U_2(D)$. In fact, by the universal Cauchy formula (3.11):

$$\begin{cases} \exp(x_1t) & = & u^{(0)} - x_2u^{(-1)}, \\ \\ \exp(x_2t) & = & u^{(0)} - x_1u^{(-1)} \end{cases}$$

from which, e.g.:

$$(x_1 - x_2)u^{(-1)} = \exp(x_1t) - \exp(x_2t).$$

It follows that $u^{(-1)}$ cannot be expressed as linear combination of $\exp(x_1t)$ and $\exp(x_2t)$, because $x_1 - x_2$ is not invertible in A. Clearly $(\exp(x_1t), \exp(x_2t))$ is a basis of $\ker(\mathcal{U}_2)$ thought of as a submodule of $A_{(x_1-x_2)}[[t]]$, where $A_{(x_1-x_2)}$ denotes localization at the multiplicative set $\{1, x_1 - x_2, (x_1 - x_2)^2, \ldots)$. This example shows that $u^{(0)}$ and $u^{(-1)}$ is a "proper" canonical basis of $\ker U_2(D)$.

3.2.7. Example. Consider the differential equation $y'' + y = 0$, thought of linear ODE with real (or even rational) coefficients. This is the same as $P(D)y = 0$ where $P(t) = t^2 + 1$. The above theorem says that all solutions are of the form

$$\lambda u^{(0)} + \mu u^{(1)} = \lambda \cdot \widehat{\psi}_P(u^{(0)}) + \mu \cdot \widehat{\psi}_P(u^{(-1)}), \quad (\lambda, \mu \in \mathbb{R})$$

where the linear combination is taken with respect to the unique E_1-module structure of $\mathbb{R}$ induced by the unique $\mathbb{Q}$-algebra homomorphism sending $e_1 \mapsto 0$ and $e_2 \mapsto 1$. Let $v^{(i)} = \widehat{\psi}_P(u^{(i)})$. First of all notice that

$$\frac{1}{\widehat{\psi}_P(V_2(t))} = \frac{1}{1+t^2} = 1 - t^2 + t^4 - \ldots$$

According to the recipe of theTheorem, then,

$$v^{(0)} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \ldots$$

and $v^{(-1)}$ is the unique primitive of $v^{(0)}$ vanishing at 0, i.e.:

$$v^{(-1)} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots,$$

Both series have convergence radius equal to ∞ and they define in fact analytic functions everywhere on $\mathbb{R}$, the well known *sine* and *cosine*.

3.2.8. Example: the universal Euler formula. Let $U_2(t) = t^2 - e_1t + e_2 \in E_1[t]$. The polynomial $U_2(t)$ has no roots in $E_1[t]$ but one may construct a larger $\mathbb{Q}$-algebra where U_2 splits as the product of two linear factors. The most economical one is the *universal splitting algebra* of $U_2(t)$ (see [10]) , constructed as follows. Let

$$E_1[x] := \frac{E_1[t]}{(U_2(t))}$$

where x denotes the class of t modulo the principal ideal generated by $U_1(t)$. Then

$$U_1(x) = 0,$$

by construction, and so x is a root of the polynomial $U_1(t) \in E_1[x][t]$. Indeed

$$U_1(t) = (t - x)(t - (e_1 - x)),$$

in $E_1[x][t]$. Clearly $E_1[x]$ is an E_1-algebra and $\exp(xt)$ is an element of $\ker U_1(D)$. By the universal Cauchy formula (3.11):

$$\boxed{\exp(xt) = u^{(0)} + (x - e_1)u^{(-1)}}\,, \tag{3.12}$$

because the initial conditions of $\exp(xt)$ are 1 and x. It is reasonable to call Equation (3.12) *universal Euler formula*. In fact, consider the differential equation

$$y'' + y = 0,$$

with coefficients in the $\mathbb{Q}$-algebra $\mathbb{C}$. Then $i := \sqrt{-1}$ is a root of the characteristic polynomial t^2+1 of the equation. Therefore $\exp(it) \in \ker(D^2+1)$, i.e.

$$\exp(it) = v^{(0)} + (i - e_1)v^{(-1)}$$

under the E_1-algebra structure of $\mathbb{C}$ induced by the unique homomorphism $\psi : E_1 \to \mathbb{C}$ sending $e_1 \mapsto 0$ and $e_2 \mapsto 1$, where we have defined $v_i = \widehat{\psi}(u_i)$. One has then:

$$\exp(it) = v^{(0)} + iv^{(-1)} \in \ker(D^2 + 1) \subseteq \mathbb{C}[[t]].$$

i.e.

$$\exp(it) = \cos t + i \sin t, \tag{3.13}$$

by Example 3.2.7.

3.2.9. Example. The Euler formula (3.13) generalizes as follows. If $v^{(0)}, v^{(-1)}, \ldots, v^{(-r)}$ are the canonical solutions of the equation $y^{r+1}+y = 0$ and if x is any $(r+1)$th root of unity, then

$$\exp(xt) = u^{(0)} + xu^{(-1)} + \ldots + x^ru^{(-r)}.$$

3.3 A few remarks on the formal Laplace transform

3.3.1. Recall that the formal Laplace transform $L : A[[t]] \to A[[t]]$ is defined by $L(t^n) = n!t^n$ and extends to infinite linear combinations of powers of t, i.e.:

$$L\left(\sum_{n\geq 0} a_n \frac{t^n}{n!}\right) = \sum_{n\geq 0} a_n \frac{L(t^n)}{n!} = \sum_{n\geq 0} a_n t^n.$$

3.3.2. Proposition. *For each $h \geq -1$ and each $\mathbf{a}(t) \in A[[t]]$ one has:*

$$t^{h+1} L(D^{(h+1)}\mathbf{a})(t) = L(\mathbf{a})(t) - a_0 - a_1 t - \ldots - a_h t^h. \qquad (3.14)$$

Proof. If $L(\mathbf{a}) = (a_0, a_1, a_2, \ldots)$, then

$$\begin{aligned} t^{h+1} L(D^{h+1}\mathbf{a})(t) &= t^{h+1} L\left(\sum_{n\geq 0} a_{n+h+1} \frac{t^n}{n!}\right) \\ &= t^{h+1} \sum_{n\geq 0} a_{n+h+1} t^n \\ &= \sum_{n\geq 0} a_{n+h+1} t^{n+h+1} \\ &= \sum_{n\geq 0} a_n t^n - a_0 - a_1 t - \ldots - a_h t^h \\ &= L(\mathbf{a})(t) - a_0 - a_1 t - \ldots - a_h t^h. \end{aligned}$$

■

3.3.3. Classically, the Laplace transform is defined for real functions $f : \mathbb{R} \to \mathbb{R}$ of one real variable. Even if the domain of f coincides with the entire real line, the Laplace transform

$$\mathcal{L}(f(t))(s) = \int_0^{\infty} \exp(-st) f(t) dt \qquad (3.15)$$

may be not, because of the restriction on the parameter s to guarantee the convergence of the integral (3.15). For intance, the Laplace transform of $f(t) = \exp(at)$ is, according to (3.15), $(s-a)^{-1}$, but this is defined only for $s > a$, otherwise the integral

$$\int_0^\infty \exp((a-s)t)dt$$

would not converge. The integral (3.15) is defined for analytic functions as well, and then it commutes with infinite sums. For analytic functions it is completely determined by the action on monomials t^n:

$$\mathcal{L}(t^n)(s) = \int_0^\infty \exp(-st)t^n dt = \frac{n!}{s^{n+1}}$$

Therefore on analytic functions we have the equation:

$$\mathcal{L}(f)(s) = \frac{1}{s} \cdot L(f)\left(\frac{1}{s}\right) \tag{3.16}$$

and, conversely

$$L(f) = \frac{1}{t}\mathcal{L}(f)\left(\frac{1}{t}\right) \tag{3.17}$$

If one considers the Laplace transform as acting on elements of $\mathbb{R}[[t]]$, equations (3.16) and (3.17) tell us that the pair $(L, \mathcal{L})$ is the local representation of a rational section of the tautological bundle $\mathcal{T}$ over the real projective line $\mathbb{RP}^1$, which is the *Möbius band*. The section $\mathbb{L}(t^n) : \mathbb{RP}^1 \to \mathcal{T}$ is given as follows

$$(x_0, x_1) \mapsto \mathbb{L}(t^n)(x_0 : x_1) = n!\frac{x_0^n}{x_1^{n+1}}$$

and extended by linearity to infinite sums. It is clearly not defined at the point $(1:0)$. Furthermore it is not a function, because it depends on the representative used to represent a point $P := (x_0 : x_1)$. On the open set $\mathcal{D}_0 := \{(x_0 : x_1) \mid x_0 \neq 0\} \subseteq \mathbb{RP}^1$ it is represented by

$$\mathbb{L}(t^n)(x_0 : x_1) = \frac{n!}{s^{n+1}}$$

where we set $s := x_0/x_1$, while on the open set $\mathcal{D}_1 := \{(x_1 : x_0) \,|\, x_1 \neq 0\}$ it takes the form

$$\mathbb{L}(t^n)(x_0 : x_1) = n!t^n$$

where $t = x_1/x_0$. The two "versions" of the Laplace transform may be viewed, at least for analytic functions, as two different local representations of a same rational section of the tautological bundle. Working over the complex numbers, one can say that the Laplace transform $\mathbb{L}(f)$ of a holomorphic function on the complex plane is a meromorphic section of the line bundle $O_{\mathbb{P}^1}(-1)$, which corresponds to a divisor of degree -1 on the projective line.

3.4 Linear ODEs with source

The best way to study non–homogeneous linear ODEs is via the formal Laplace transform $t^n \mapsto n!t^n$. See below.

3.4.1. Let $L(\mathbf{f}) = (f_0, f_1, \ldots)$ and $L(\mathbf{c}) = (c_0, c_1, \ldots, c_r)$ be two sequences of indeterminates over E_r. Let $F_r := E_r[\mathbf{c}, \mathbf{f}]$. From the infinite sequence $L(\mathbf{f})$ form

$$\mathbf{f}(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!} \in F_r[[t]].$$

Solving the *universal Cauchy problem*

$$\begin{cases} U_{r+1}(D)y & = & \mathbf{f}, \\ \\ (D^i y)(0) & = & c_i, \qquad 0 \leq i \leq r, \end{cases} \tag{3.18}$$

amounts to finding a formal power series $\mathbf{p} \in U_{r+1}(D)^{-1}(\mathbf{f})$ having $L(\mathbf{c})$ as a vector of initial conditions. The first of (3.18) implies the equality:

$$t^{r+1}L(U_{r+1}(D)y) = t^{r+1}L(\mathbf{f}), \qquad (r \geq 0)$$

from which

$$t^{r+1}(L(y^{(r+1)}) - e_1 L(y^{(r)}) + \ldots + (-1)^{r+1}e_{r+1}L(y)) = \sum_{n\geq 0} f_n t^{n+r+1},$$

i.e., applying Proposition 3.3.2:

$$L(y) - \sum_{i=0}^{r} c_i t^i - e_1 t \left(L(y) - \sum_{i=0}^{r-1} c_i t^i \right) + \ldots + (-1)^{r+1} t^{r+1} L(y) = \sum_{n \geq 0} f_n t^{n+r+1}.$$

Factorizing $L(y)$ and a few more easy computations yield:

$$L(y)(1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}) = U_0(\mathbf{c}) + U_1(\mathbf{c})t + \ldots + U_r(\mathbf{c})t^r + \sum_{n \geq 0} f_n t^{n+r+1}.$$

One eventually gets the expression:

$$L(y) = L(\mathbf{p}(t)),$$

where

$$L(\mathbf{p}(t)) = \frac{U_0(\mathbf{c}) + U_1(\mathbf{c})t + \ldots + U_r(\mathbf{c})t^r + \sum_{n \geq r+1} f_{n-r-1} t^n}{1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}}. \tag{3.19}$$

Writing $L(\mathbf{p}(t))$ as $\sum_{n \geq 0} p_n t^n$, with p_n determined by equality (3.19) holding in $F_r[[t]]$, then

$$\mathbf{p}(t) = \sum_{n \geq 0} p_n \frac{t^n}{n!}$$

is the unique solution of the Cauchy problem (3.18). ■

3.4.2. Corollary. *The formal Laplace transform of* $u^{(-j)}$ *for* $0 \leq j \leq r$ *is:*

$$L(u^{(-j)}) = \frac{t^j}{1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}} = t^j \sum_{n \geq 0} h_n t^n.$$

3.4.3. Remark. If $r < \infty$, a formal power series $\mathbf{a}(t)$ is a solution of $U_{r+1}(D)y = 0$ if and only if (Cf. (2.9)):

$$L(\mathbf{a}(t)) = \frac{1}{V_{r+1}(t)}, \tag{3.20}$$

where $V_{r+1}(t)$ is as in (2.9). If $r = \infty$, it is not possible to write $U_{r+1}(D)y = 0$. Equation (3.20) can however be displayed even if $r = \infty$ and it is then natural to think of $(u^{(0)}, u^{(-1)}, \ldots)$ as a basis of $\mathcal{U}_\infty$, regarded as the module of solutions of the linear ODE of *infinite order* with coefficients $\mathbf{e} := (e_1, e_2, \ldots)$. By Lemma 2.3.11 each formal power series is a linear combination of the $(u^{(-j)})_{j\geq 0}$ and hence $\mathcal{U}_\infty = A[[t]]$. Thus any formal power series with A-coefficients, where A is any E_r-algebra, can be thought of as a solution of an infinite order linear ODE, and its expression as an infinite linear combination of $u^{(0)}, u^{(-1)}, u^{(-2)}, \ldots$ is nothing but the Universal Cauchy formula (3.11)!

Chapter 4

Generalized Wronskians

4.1 Partitions

4.1.1. By a *partition* of length at most $r+1$ one will mean a non–increasing sequence $\lambda = (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r)$ of non negative integers. Each λ_i is said to be a *part* of the partition. The *length* $\ell(\lambda)$ is the number of its non zero parts and its *weight* is $|\lambda| = \sum_{i=0}^{r} \lambda_i$. Each partition is a partition of the integer $|\lambda|$. The set of all partitions of length $\leq r+1$ will be denoted by ℓ_r. If $\lambda, \mu \in \ell_r$, the partition $\lambda + \mu$ is the partition whose parts are the sum of the parts

$$\lambda + \mu = (\lambda_0 + \mu_0, \lambda_1 + \mu_1, \ldots, \lambda_r + \mu_r).$$

4.1.2. Example. $(3, 3, 1, 1, 0)$ is a partition of length 4 and of weight 8. It is a partition of the integer 8. Another partition of the integer 8 is e.g.:

$$(2, 2, 1, 1, 1, 1)$$

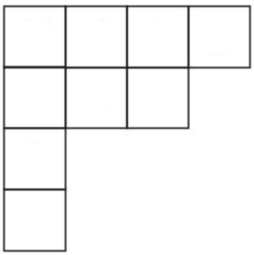
which has length 6.

4.1.3. The generating function of the integer $p(k)$ of all partitions of the non negative integer k is given by

$$\sum_{n \geq 0} p(n) t^n = \frac{1}{\prod_{i=1}^{\infty}(1 - t^i)} = \prod_{i=1}^{\infty} (\sum_{n \geq 0} t^{ni}),$$

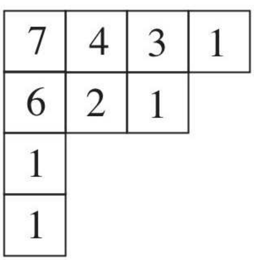
a result due to Euler (see e.g. [26, p. 211]).

4.1.4. A partition can be expressed in the form $(1^{m_1}2^{m_2}\dots k^{m_k})$ where m_j indicates how many times the integer j occurs in λ. For example, the partition $\lambda = (4,4,2,1,1,1)$ can be written as $(1^3 2^1 4^2)$. The partition $\underbrace{(1,\dots,1)}_{k\ times}$ can be written as 1^k. Each partition λ can be identified with its *Young diagram*, an array of left–justified rows, with λ_0 boxes in the first row, λ_1 boxes in the second row, ..., λ_r-boxes in the $(r+1)$th row.



The Young diagram of the partition $(4,3,1,1)$ of the integer 9.

Each box of a Young diagram determines a *hook*, consisting of that box and of all boxes lying below it or to its right. The *hook length* of a box is the number of boxes in its hook. So, for instance, filling the Young diagram of the partition $(3,2,1,1)$ with the hook length of each box gives the following *Young tableau*:



The hook length filling of $(3,2,1,1)$

We denote by $\mathcal{P}$ the set of all partitions and by $\mathcal{P}^{(r+1)\times(d-r)}$ the set of all partitions whose Young diagram is contained in a $(r+1)\times(d-r)$ rectangle, i.e. the set of all partitions λ such that:

$$d - r \geq \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_0 \geq 0.$$

One denotes by 0 the unique partition of weight 0. For more on partitions, see [12, 29].

4.2 Schur Polynomials

4.2.1. If $\mathbf{a} := (a_i)_{i\in\mathbb{Z}}$ is any sequence in A and $\lambda := (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r)$ a partition of length at most $r+1$, the *Schur polynomial* $\Delta_\lambda(\mathbf{a})$ associated to $\mathbf{a}$ and λ is by definition[1]:

$$\Delta_\lambda(\mathbf{a}) := \det(a_{\lambda_{j-1}-i+1}) = \begin{vmatrix} a_{\lambda_0} & a_{\lambda_1-1} & \cdots & a_{\lambda_r-r} \\ a_{\lambda_0+1} & a_{\lambda_1} & \cdots & a_{\lambda_r-r+1} \\ a_{\lambda_0+2} & a_{\lambda_1+1} & \cdots & a_{\lambda_r-(r-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\lambda_0+r} & a_{\lambda_1+r-1} & \cdots & a_{\lambda_r} \end{vmatrix} \in A.$$

The reader can easily check that adding a string of zeros to a partition λ does not change the Schur polynomial. In particular, if $\mathbf{e} = (e_1, e_2, \ldots)$ and $\mathbf{h} = (h_j)_{j\in\mathbb{Z}}$ are as in (2.15) one has:

$$h_k = \Delta_{(1^k)}(\mathbf{e})$$

i.e.

$$\begin{aligned} h_1 &= e_1; \\ h_2 &= \Delta_{(1^2)}(\mathbf{e}) = \begin{vmatrix} e_1 & e_2 \\ 1 & e_1 \end{vmatrix} = e_1^2 - e_2; \\ h_3 &= \Delta_{(1^3)}(\mathbf{e}) = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix} = e_1^3 - 2e_1e_2 + e_3. \end{aligned}$$

[1] There are other equivalent ways to express the Schur determinant, e.g. by transposing the matrix.

In order to keep a simple and neat notation, for each partition λ we shall write:

$$\boxed{h_\lambda := \Delta_\lambda(\mathbf{h})}\,. \tag{4.1}$$

So, e.g., $h_{(1,1)}$ stands for $h_1^2 - h_2$ and $h_{(2,1)} = h_1h_2 - h_3$.

4.3 Generalized Wronskians

4.3.1. Let

$$\mathbf{u}_r = (u^{(0)}, u^{(-1)}, \ldots, u^{(-r)})$$

be the row of the universal fundamental system. The *generalized wronskian* (determinant) of $\mathbf{u}_r$ corresponding to the partition λ : $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r$ will be denoted as:

$$W_\lambda(\mathbf{u}_r) := D^{\lambda_r}\mathbf{u}_r \wedge D^{1+\lambda_{r-1}}\mathbf{u}_r \wedge \ldots \wedge D^{r+\lambda_0}\mathbf{u}_r, \tag{4.2}$$

shorthand for:

$$W_\lambda(\mathbf{u}_r) := \begin{vmatrix} D^{\lambda_r}u^{(0)} & D^{\lambda_r}u^{(-1)} & \ldots & D^{\lambda_r}u^{(-r)} \\ D^{1+\lambda_{r-1}}u^{(0)} & D^{1+\lambda_{r-1}}u^{(-1)} & \ldots & D^{1+\lambda_{r-1}}u^{(-r)} \\ \vdots & \vdots & \ddots & \vdots \\ D^{r+\lambda_0}u^{(0)} & D^{r+\lambda_0}u^{(-1)} & \ldots & D^{r+\lambda_0}u^{(-r)} \end{vmatrix} \tag{4.3}$$

$$= \begin{vmatrix} u^{(\lambda_r)} & u^{(-1+\lambda_r)} & \ldots & u^{(-r+\lambda_r)} \\ u^{(1+\lambda_{r-1})} & u^{(\lambda_{r-1})} & \ldots & u^{(1-r+\lambda_{r-1})} \\ \vdots & \vdots & \ddots & \vdots \\ u^{(r+\lambda_0)} & u^{(-1+r+\lambda_0)} & \ldots & u^{(\lambda_0)} \end{vmatrix}.$$

It is nothing else than a formal power series generalizing the classical Wronskian determinant:

$$W_0(\mathbf{u}) = \begin{vmatrix} u^{(0)} & u^{(-1)} & \dots & u^{(-r)} \\ Du^{(0)} & Du^{(-1)} & \dots & Du^{(-r)} \\ \vdots & \vdots & \ddots & \vdots \\ D^r u^{(0)} & D^r u^{(-1)} & \dots & D^r u^{(-r)} \end{vmatrix}$$
$$= \begin{vmatrix} u^{(0)} & u^{(-1)} & \dots & u^{(-r)} \\ u^{(1)} & u^{(0)} & \dots & u^{(-r+1)} \\ \vdots & \vdots & \ddots & \vdots \\ u^{(r)} & u^{(-1+r)} & \dots & u^{(0)} \end{vmatrix}$$

which, accordingly, will be simply written as $\mathbf{u}_r \wedge D\mathbf{u}_r \wedge \ldots \wedge D^r\mathbf{u}_r$. Notice that $W_0(\mathbf{u}_r)(0) = 1$. We like expression (4.2) more than its determinantal expression (4.3), because the derivation operator D satisfies a Leibniz–like rule with respect to the wedge product $\wedge$, namely:

$$\begin{aligned} DW_\lambda &= D(D^{\lambda_r}\mathbf{u}_r \wedge D^{1+\lambda_{r-1}}\mathbf{u}_r \wedge \ldots \wedge D^{r+\lambda_0}\mathbf{u}_r) \\ &= \sum_{\substack{i_0+i_1+\ldots+i_r=1 \\ i_j\geq 0}} D^{i_0+\lambda_r}\mathbf{u}_r \wedge D^{1+i_1+\lambda_{r-1}}\mathbf{u}_r \\ &\qquad \wedge \ldots \wedge D^{r+i_r+\lambda_0}\mathbf{u}_r. \end{aligned} \tag{4.4}$$

4.4 The Jacobi-Trudy formula for Wronskians

The fact that each generalized Wronskian can be seen as a “deformation” of the usual Wronskian is shown in the following:

4.4.1. Proposition. *Let $\lambda \in \ell_r$ be any partition of length less than $r+1$. Then:*

$$W_\lambda(\mathbf{u}_r) = \sum_{n\geq 0}\sum_{\mu\in\ell_r, |\mu|=n} \binom{n}{\mu} h_{\lambda+\mu}\cdot\frac{t^n}{n!}, \tag{4.5}$$

where if $|\mu| = n$

$$\binom{n}{\mu} = \frac{n!}{\mu_0!\mu_1! \cdot \ldots \cdot \mu_r!}$$

is the coefficient of $x^\mu := x_0^{\mu_0} x_1^{\mu_1} \cdot \ldots \cdot x_r^{\mu_r}$ *in the expansion of* $(x_0+x_1+\ldots+x_r)^n$*. In particular, the constant term* $W_\lambda(\mathbf{u}_r)(0)$ *of* $W_\lambda(\mathbf{u}_r)$ *is* h_λ *(see (4.1)) which, for* $\lambda = 0$ *(the null partition), yields* $W_0(\mathbf{u})(0) = h_0 = 1$*.*

Proof. The transpose $\mathbf{u}_r^T$ of the row $\mathbf{u}_r$ can be written as:

$$\mathbf{u}_r^T = \sum_{n \geq 0} \begin{pmatrix} h_n \\ h_{n-1} \\ \vdots \\ h_{n-r} \end{pmatrix} \cdot \frac{t^n}{n!}.$$

By renaming n as μ_{r-i}, for each $0 \leq i \leq r$, one easily finds:

$$D^{i+\lambda_{r-i}} \mathbf{u}_r^T = \sum_{\mu_{r-i} \geq 0} \begin{pmatrix} h_{\lambda_{r-i}+\mu_{r-i}+i} \\ h_{\lambda_{r-i}+\mu_{r-i}+i-1} \\ \vdots \\ h_{\lambda_{r-i}+\mu_{r-i}+i-r} \end{pmatrix} \frac{t^{\mu_{r-i}}}{\mu_{r-i}!}.$$

Invoking the fact that the determinant of a matrix coincides with the determinant of the transpose, one can write:

$$W_\lambda(\mathbf{u}_r) = D^{\lambda_r} \mathbf{u}_r^T \wedge D^{1+\lambda_{r-1}} \mathbf{u}_r^T \wedge \ldots \wedge D^{r+\lambda_0} \mathbf{u}_r^T =$$

$$= \sum_{\mu \in \ell_r} \begin{pmatrix} h_{\lambda_r+\mu_r} \\ h_{\lambda_r+\mu_r-1} \\ \cdots \\ h_{\lambda_r+\mu_r-r} \end{pmatrix} \frac{t^{\mu_r}}{\mu_r!} \wedge \begin{pmatrix} h_{\lambda_{r-1}+\mu_{r-1}+1} \\ h_{\lambda_{r-1}+\mu_{r-1}} \\ \cdots \\ h_{\lambda_{r-1}+\mu_{r-1}-(r-1)} \end{pmatrix}$$

$$\frac{t^{\mu_{r-1}}}{\mu_{r-1}!} \wedge \ldots \wedge \begin{pmatrix} h_{\lambda_0+\mu_0+r} \\ h_{\lambda_0+\mu_0-1} \\ \cdots \\ h_{\lambda_0+\mu_0} \end{pmatrix} \frac{t^{\mu_0}}{\mu_0!} =$$

$$
= \sum_{\mu\in\ell_r} \begin{pmatrix} h_{\lambda_r+\mu_r} \\ h_{\lambda_r+\mu_r-1} \\ \dots \\ h_{\lambda_r+\mu_r-r} \end{pmatrix} \wedge \begin{pmatrix} h_{\lambda_{r-1}+\mu_{r-1}+1} \\ h_{\lambda_{r-1}+\mu_{r-1}} \\ \dots \\ h_{\lambda_{r-1}+\mu_{r-1}-(r-1)} \end{pmatrix}
$$

$$
\wedge \dots \wedge \begin{pmatrix} h_{\lambda_0+\mu_0+r} \\ h_{\lambda_0+\mu_0-1} \\ \dots \\ h_{\lambda_0+\mu_0} \end{pmatrix} \cdot \frac{t^{\mu_r}}{\mu_r!}\frac{t^{\mu_{r-1}}}{\mu_{r-1}!}\cdot \dots \cdot \frac{t^{\mu_0}}{\mu_0!} =
$$

$$
= \sum_{n\geq 0} \sum_{\{\mu\in\ell_r \big| |\mu|=n\}} \frac{n!}{\mu_0!\mu_1!\cdot\dots\cdot\mu_r!} \times
$$

$$
\times \begin{vmatrix} h_{\lambda_r+\mu_r} & h_{\lambda_{r-1}+\mu_{r-1}+1} & \dots & h_{\lambda_0+\mu_0+r} \\ h_{\lambda_r+\mu_r-1} & h_{\lambda_{r-1}+\mu_{r-1}} & \dots & h_{\lambda_0+\mu_0+r-1} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\lambda_r+\mu_r-r} & h_{\lambda_{r-1}+\mu_{r-1}-(r-1)} & \dots & h_{\lambda_0+\mu_0} \end{vmatrix} \frac{t^n}{n!} =
$$

$$
= \sum_{n\geq 0} \sum_{\{\mu\in\ell_r \big| |\mu|=n\}} \binom{n}{\mu} h_{\lambda+\mu} \cdot \frac{t^n}{n!}
$$

as desired. Notice that the constant term of $W_\lambda(\mathbf{u}_r)$ is obtained for $\mu = 0$, which has weight 0:

$$
W_\lambda(\mathbf{u}_r)(0) = h_\lambda.
$$

In particular, if $\lambda = 0$ one has $W_0(\mathbf{u}_r)(0) = h_0 = 1$. ∎

An immediate consequence of Proposition (4.4.1) is :

4.4.2. Theorem. (Cf. [18]) *The* Jacobi-Trudy *formula holds:*

$$
W_\lambda(\mathbf{u}_r) = h_\lambda W_0(\mathbf{u}_r) \tag{4.6}
$$

for each $\lambda \in \ell_r$.

Proof. One first notices that $W_\lambda(\mathbf{h})$ is an E_r-multiple of $W_0(\mathbf{h})$. In fact, each column of the form $D^{i_j}\mathbf{u}_r$ with $i_j \geq r+1$ occurring in the expression of $W_\lambda(\mathbf{u}_r)$ can be replaced by a linear combination of lower–order derivatives of $\mathbf{u}_r$, using (3.2) (and its consequences). One finally obtains the product of an element of E_r with a determinant involving derivatives of $\mathbf{u}_r$ of order $0 \leq h \leq r$ only, which is the Wronskian up to a permutation of the columns, i.e. $W_\lambda(\mathbf{u}_r) = \gamma_\lambda W_0(\mathbf{u}_r)$ for some $\gamma_\lambda \in E_r$. As two formal power series are proportional if and only if the coefficients of all powers of t are proportional, with the same coefficient of proportionality, it follows that:

$$\gamma_\lambda = \frac{W_\lambda(\mathbf{u}_r)(0)}{W(\mathbf{u}_r)(0)} = h_\lambda,$$

because of the last part of Proposition 4.4.1. ∎

4.4.3. Corollary (Liouville, Abel). *The derivative of the Wronskian is proportional to the Wronskian itself. Indeed:*

$$W_0(\mathbf{u}_r) = \exp(e_1 t).$$

Proof. By differentiating the Wronskian W_0

$$\begin{aligned} DW_0(\mathbf{u}_r) &= D(\mathbf{u}_r \wedge D\mathbf{u}_r \wedge \ldots \wedge D^{r-1}\mathbf{u}_r \wedge D^r\mathbf{u}_r) \\ &= \mathbf{u}_r \wedge D\mathbf{u}_r \wedge \ldots \wedge D^{r-1}\mathbf{u}_r \wedge D^{r+1}\mathbf{u}_r \\ &= W_{(1)}(\mathbf{u}_r) = h_1 W_0(\mathbf{u}_r) = e_1 W_0(\mathbf{u}_r), \end{aligned}$$

where the last row of equalities is based on both Theorem 4.4.2 and the equality $h_1 = e_1$. The equation $DW_0(\mathbf{u}_r) = e_1 W_0$ implies that $W_0(\mathbf{u}_r) = w_0 \exp(e_1 t)$, and since $w_0 = W_0(\mathbf{u}_r)(0) = 1$, the claim follows. ∎

4.4.4. Corollary. *Pieri's formula for generalized Wronskians holds:*

$$h_i W_\lambda(\mathbf{u}_r) = \textstyle\sum_\mu W_\mu(\mathbf{u}_r)$$

where the sum is over all partitions $\mu \in \ell_r$ such that $|\mu| = i + |\lambda|$ and

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r \tag{4.7}$$

Proof. By [11, Lemma A.9.4] one has the equality $h_i \Delta_\lambda(\mathbf{h}) = \sum_\mu \Delta_\mu(\mathbf{h})$, where the sum is over all partitions μ such that $|\mu| = |\lambda| + i$ satisfying inequalities (4.7). Thus, applying 4.4.2, one has

$h_iW_\lambda(\mathbf{u}_r) = (h_i\Delta_\lambda(\mathbf{h}))W_0(\mathbf{u}_r) = \sum_\mu \Delta_\mu(\mathbf{h}_r)W_0(\mathbf{u}_r) = \sum_\mu W_\mu(\mathbf{h}_r)$

as desired. ∎

4.4.5. It is well known that the multiplication in A of two Schur polynomials is an integral linear combination of Schur polynomials

$$\Delta_\lambda(\mathbf{a}) \cdot \Delta_\mu(\mathbf{a}) = \sum_{|\nu|=|\lambda|+|\mu|} L^\nu_{\lambda\mu} \cdot \Delta_\nu(\mathbf{a}).$$

The numbers $L^\nu_{\lambda\mu} \in \mathbb{Z}$ are known as *Littlewood-Richardson coefficients*. These are known to be non–negative (see e.g. [29]). They are very important at present as suggested by the following:

4.4.6. Theorem. *The product of two generalized Wronskians $W_\lambda(\mathbf{u}_r) \in E_r[[t]]$ obeys the following equality:*

$$W_\lambda(\mathbf{u}_r) \cdot W_\mu(\mathbf{u}_r) = \left(\sum_{|\nu|=|\lambda|+|\mu|} L^\nu_{\lambda\mu} \cdot W_\nu(\mathbf{u}_r) \right) \cdot W_0(\mathbf{u}_r). \tag{4.8}$$

Proof. In fact, by Theorem 4.4.2

$$\begin{aligned} W_\lambda(\mathbf{u}_r)W_\mu(\mathbf{u}_r) &= h_\lambda W_0(\mathbf{u}_r) \cdot h_\mu W_0(\mathbf{u}_r) \\ = (h_\lambda \cdot h_\mu W_0(\mathbf{u}_r))W_0(\mathbf{u}_r) &= \left(\sum_{|\nu|=|\lambda|+|\mu|} L^\nu_{\lambda\mu} h_\nu W_0(\mathbf{u}_r) \right) W_0(\mathbf{u}_r) \\ &= \left(\sum_{|\nu|=|\lambda|+|\mu|} L^\nu_{\lambda\mu} W_\nu(\mathbf{u}_r) \right) W_0(\mathbf{u}_r). \end{aligned}$$

where in the third equality the definition of the Littlewood-Richardson coefficients has been used and in the last the Jacobi-Trudy formula (4.6). ∎

4.5 The hook length formula

Generalized Wronskians are clearly related to derivatives of Wronskians. For instance one easily checks that

$$DW_0(\mathbf{u}_r) = W_{(1)}(\mathbf{u}_r).$$

Differentiating a determinant is the same as applying Leibniz's rule with respect to the wedge product $\wedge$. Using the fact that a determinant vanishes whenever two rows are equal, one gets:

$$\begin{aligned} DW_0(\mathbf{u}_r) &= D(\mathbf{u}_r \wedge D\mathbf{u}_r \wedge \ldots \wedge D^r\mathbf{u}_r) \\ &= \mathbf{u}_r \wedge D\mathbf{u}_r \wedge \ldots \wedge D^{r-1}\mathbf{u}_r \wedge D^{r+1}\mathbf{u}_r \\ &= W_{(1)}(\mathbf{u}_r). \end{aligned}$$

The reader can easily convince himself that, in general, the j-th derivative of a wronskian is a linear combination of generalized Wronskians:

$$D^jW_0(\mathbf{u}_r) = \sum_{|\lambda|=j} g_\lambda W_\lambda(\mathbf{u}_r). \tag{4.9}$$

This is a consequence of (4.4) and of an easy induction. It is well known that the Jacobi-Trudy formula, or its Pieri's consequence, implies that the coefficient of $W_\lambda(\mathbf{u}_r)$ in the expansion of $h_1^n W_0(\mathbf{u}_r)$ can be computed by means of the *hook length formula* (Cf. Section 4.5.1). In terms of generalized Wronskians, this property reads:

4.5.1. Theorem. (see [18]) *The coefficient g_λ in (4.9) can be computed by the* hook length *formula:*

$$g_\lambda = \frac{|\lambda|!}{k_1 \cdot \ldots \cdot k_j} = \frac{j!}{k_1 \cdot \ldots \cdot k_j} \tag{4.10}$$

where k_i is the hook length of each box of the Young diagram of the partition λ .

Proof. In fact by the Liouville formula $D^jW_\lambda(\mathbf{u}_r) = h^jW_0(\mathbf{u}_r)$, and then one may apply the hook length formula holding for the multiplication of h_1^j with a Schur polynomial h_λ, as in [12]. ∎

A consequence of Theorem 4.5.1 is the following remarkable fact, already observed in [14] (see also Chapter 6):

4.5.2. Corollary. *Let $(d-r)^{r+1}$ be the partition with $r+1$ parts equal to $d-r$. Then the coefficient $g_{(d-r)^{r+1}}$ multiplying $W_{(d-r)^{r+1}}(\mathbf{u}_r)$ in the expansion of $D^{(r+1)(d-r)}W_0(\mathbf{u}_r)$ is precisely the Plücker degree* *of the Grassmannian $G(r,\mathbb{P}^d)$, parameterizing r dimensional linear subvarieties in the complex projective space, i.e.:*

$$g_{(d-r)^{r+1}} = \frac{1!2!\cdot\ldots\cdot r!\cdot(r+1)(d-r)!}{(d-r)!(d-r+1)!\cdot\ldots\cdot d!}.$$

4.5.3. Example. Let $\mathbf{u}_1 = (u^{(0)}, u^{(-1)})$. Then:

$$\begin{aligned}
D^4W_0(\mathbf{u}_1) &:= D^4(\mathbf{u}_1\wedge D\mathbf{u}_1) = D^3\circ D(\mathbf{u}_1\wedge D\mathbf{u}_1)\\
&= D^3(\mathbf{u}_1\wedge D^2\mathbf{u}_1) = D^2(D\mathbf{u}_1\wedge D^2\mathbf{u}_1 + \mathbf{u}_1\wedge D^3\mathbf{u}_1)\\
&= D(2\cdot D\mathbf{u}_1\wedge D^3\mathbf{u}_1 + \mathbf{u}_1\wedge D^4\mathbf{u}_1)\\
&= 2\cdot D^2\mathbf{u}_1\wedge D^3\mathbf{u}_1 + 3\cdot D\mathbf{u}_1\wedge D^4\mathbf{u}_1 + \mathbf{u}_1\wedge D^5\mathbf{u}_1\\
&= 2\cdot D^{0+2}\mathbf{u}_1\wedge D^{1+2}\mathbf{u}_1 + 3\cdot D^{0+1}\mathbf{u}_1\wedge D^{1+3}\mathbf{f}\\
&\quad + \mathbf{u}_1\wedge D^{1+4}\mathbf{u}_1\\
&= 2\cdot W_{(2,2)}(\mathbf{u}_1) + 3\cdot W_{(3,1)}(\mathbf{u}_1) + W_{(4)}(\mathbf{u}_1).
\end{aligned}$$

The coefficient 2 multiplying $W_{(2,2)}(\mathbf{u}_1)$ is precisely the Plücker degree of the Grassmannian $G(1,\mathbb{P}^3)$ in its Plücker embedding, i.e the *number of lines meeting* 4 *others in general position* – see [13, 14] for details. All other coefficients have an enumerative interpretation in either the classical or the quantum cohomology of the Grassmannian (see e.g. [14]).

Chapter 5

Exponential of matrices

5.1 A brief historical account.

5.1.1. Any system of linear ODEs with constant coefficients:

$$\begin{cases} \dot{x}_1 & = & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ \dot{x}_2 & = & a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots & \vdots & \\ \dot{x}_n & = & a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n \end{cases}$$

in the unknown functions $x_i : D \subseteq \mathbb{C} \to \mathbb{C}$ can be written in matrix form as:

$$\dot{\mathbf{x}} = M \cdot \mathbf{x},$$

where $\mathbf{x}$ is the column of the unknown functions x_i and M is the (constant) matrix of the coefficients (a_{ij}). It is known that the solution of the system is of the form

$$\mathbf{x} = \exp(tM) \cdot \mathbf{c} \tag{5.1}$$

where $\mathbf{c} \in \mathbb{C}^n$ is a column of constants depending on the initial condition $\mathbf{x}(0)$. If M is a square matrix, its exponential $\exp(M)$ is an invertible square matrix defined by extending formally the definition of the exponential of a complex number:

$$\exp(M) := 1 + M + \frac{M^2}{2!} + \ldots = \sum_{n\geq 0} \frac{M^n}{n!}. \tag{5.2}$$

One is naturally led to wonder if the exponential of a matrix defined in this way is a matrix itself, i.e. if the series (5.2) is convergent. This can be easily done. Indeed

$$\exp(tM) = \sum_{n\geq 0} \frac{M^n t^n}{n!}$$

converges for each $t \in \mathbb{C}$ and then it defines a family of invertible square matrices. The second problem is to computing it. To this purpose one uses the fact that if P is any invertible matrix, then $\exp(P^{-1}MP) = P^{-1}\exp(M)P$. Over the complex numbers one knows that each matrix is conjugated to an essentially unique matrix in Jordan normal form J_M. Then

$$\exp(A) = \exp(P^{-1}J_A P) = P^{-1}\exp(J_A)P$$

and the problem is reduced to compute the exponential of a matrix which is already in Jordan normal form. This is easy, because any such matrix can be written as a block matrix, whose blocks are sums $D + N$ of a diagonal and a nilpotent matrix commuting with one another. The exponential of any such block is $\exp(N)\exp(D)$. Using such a procedure, all the technical issues due to the computation of the exponential of a matrix are reduced to the problem of computing a canonical Jordan form of it. Such a computation can be however avoided, as noted by E. J. Putzer, in a paper appeared on American Mathematical Monthly [33] in 1966. Before explaining Putzer's method let us make a comment. To show that $\exp(Mt)\cdot\mathbf{c}$ is a solution of (5.1) one has just to compute its first derivative with respect to t in a purely formal way:

$$\frac{d}{dt}\exp(tM) = \frac{d}{dt}\sum_{n\geq 0}\frac{M^n t^n}{n!} = \sum_{n\geq 1}\frac{M^n t^n}{n!} = M + M^2 t + M^3\frac{t^2}{2} + \ldots =$$

$$= M\left(\mathbf{1} + Mt + \frac{M^2t^2}{2!} + \ldots\right) = M\exp(tM).$$

Thus $\exp(tM)\mathbf{c}$ satisfies (5.1) for purely formal reasons, a remark suggesting that much important information can possibly be obtained by considering the exponential of a matrix in a purely formal way.

5.1.2. Putzer's idea. As well known, each square matrix $A \in \mathbb{C}^{n\times n}$ satisfies at least one polynomial identity (indeed many polynomial identities), e.g. it is a zero of its characteristic polynomial by the *Cayley–Hamilton theorem*. In other words, the exponential of M, if it exists, can be written as a complex linear combination of $\mathbf{1}, M, M^2, \ldots, M^{n-1}$,

$$\exp(M) = \alpha_0\mathbf{1} + \alpha_1 M + \alpha_2 M^2 + \ldots + \alpha_{n-1}M^{n-1},$$

and the task consists now in computing such coefficients. Putzer (1966, [33]) solves the problem by considering the exponential $\exp(Mt)$ which, because of the Cayley–Hamilton theorem, can be written in the form:

$$\exp(tM) = \alpha_0(t)\mathbf{1} + \alpha_1(t)M + \ldots + \alpha_{n-1}(t)M^{n-1}, \tag{5.3}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ should be suitable holomorphic functions. This is indeed the case and in [33] the author is able to describe precisely all coefficients α_i occurring into the expression (5.3). Let $P_M(t)$ be the characteristic polynomial of the matrix M. Then the coefficient $\alpha_{n-1}(t)$ is the unique solution of the Cauchy Problem $P_M(D)y = 0$ such that $y^{(i)} = 0$ for $0 \le i \le n-2$ and $y^{n-1}(0) = 1$ (which is a generator of the impulsive response kernel, see Remark 3.2.2). Once the coefficient α_{n-1} is known, Putzer gives a precise description of how to determine the remaining coefficients, based on the sole knowledge of α_{n-1}. To apply his method, however, the author needs to find the roots of the characteristic polynomial, in order to express α_{n-1} as a linear combination of a fundamental system of solutions.

5.1.3. Although Putzer's description of the computation of $\alpha_1, \ldots, \alpha_{n-2}$ occurring in (5.3) is very precise, the Author proposes no interpretation of them. The latter is offered in a couple of short papers by I. H. Leonard [28] (1996) and E. Liz [27] (1998), published in the SIAM journal. They show that, for each $0 \le i \le n-1$, the function α_i is the unique solution of the Cauchy problem $P_A(D)y = 0$

satisfying the initial condition $y_i^{(j)}(0) = \delta_{ij}$. This result, although very elementary, is beautiful and elegant. To compute the exponential of a matrix, however, even the aforementioned authors have to cope with the technical issue of finding the roots of the characteristic polynomial. The purpose of next section is to show that working formally the matter gets easier, basing on the results of the previous Chapters.

5.2 The matrix exponential

5.2.1. Let $M := (a_{ij}) \in A^{n\times n}$ be a square matrix with entries in the E_r-algebra A and let t be an indeterminate over A. Then $M \cdot t$ is a square matrix with entries in $A[t] \subseteq A[[t]]$. Define:

$$\exp(Mt) := \sum_{n\geq 0} \frac{M^n t^n}{n!}$$

tviewed as a matrix of formal power series, under the identification $A[[t]]^{n\times n} \cong A^{n\times n}[[t]]$. Notice that $\{P \in A[t] \mid P(M) = 0\}$ is non–empty. In fact $\mathbf{1}, M, M^2, \ldots, M^{n^2}$ are $n^2 + 1$ matrices living in the free module $A^{n\times n}$ of rank n^2, and thus they are linearly dependent. The Cayley-Hamilton theorem holds for A-valued matrices as well: each matrix $M \in A^{n\times n}$ is a zero of its characteristic polynomial, i.e.

$$P_M(t) = \det(M - t\mathbf{1}).$$

In other words M^{n+j} is an A-linear combination of $\mathbf{1}, M, \ldots, M^{n-1}$ and so $\exp(Mt)$ must be a linear combination of $(\mathbf{1}, M, \ldots, M^{n-1})$ as well, with coefficients in $A[[t]]$. To determine the coefficients we shall first analyze the case when M is the *companion matrix* associated to the universal ODE (3.3). See below.

5.3 The exponential of a companion matrix

5.3.1. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the canonical basis of A^n thought of as the A-module of the columns of n elements of A. The entries of the column $\mathbf{e}_j$ are 0 except the i–th which is equal to 1. One says that $\mathbf{a} := (a_i) \in A^n$ is the initial condition of a solution of the linear ODE: $P(D)y = 0$ if and only if $y^{(i)}(0) = a_i$.

5.3.2. The universal ODE: $U_{r+1}(D)y = 0$ can be equivalently written as a system of differential equations:

$$\begin{cases} y^{(1)} & = & y_1, \\ y_1^{(1)} & = & y_2, \\ \vdots & \vdots & \vdots \\ y_r^{(1)} & = & e_1 y_r - e_2 y_{r-1} + \ldots + (-1)^r e_{r+1} y, \end{cases}$$

or, more compactly:

$$Y^{(1)} = \mathcal{C}_r \cdot Y$$

where

$$Y := \begin{pmatrix} y \\ y_1 \\ \vdots \\ y_r \end{pmatrix}, \qquad \text{and}$$

$$\mathcal{C}_r := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^r e_{r+1} & (-1)^{r-1} e_r & (-1)^{r-2} e_{r-1} & \ldots & e_1 \end{pmatrix}. \tag{5.4}$$

The matrix $\mathcal{C}_r$ is the *companion matrix* associated to the polynomial $U_{r+1}(t) \in E_r[t]$ and $Y^{(1)}$ is the column of the derivatives of the entries of Y, with respect to the indeterminate t. For purely formal (and not substantial) reasons, we shall regard $\mathcal{C}_r$ as an endomorphism of the E_r-module $E_r \oplus E_r^r$ of columns of $(r+1)$ elements of E_r, with basis $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_r$:

$$\mathbf{e}_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 1 \oplus \mathbf{0}_{E_r^r}, \quad \mathbf{e}_1 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_r = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

So, in particular, the columns of $\mathcal{C}_r$ are numbered from 0 to r.

5.3.3. Proposition. *Let α_i, $0 \leq i \leq r$, be the unique element of* $\ker U_{r+1}(D)$ *such that* $(D^j\alpha_i)(0) = \delta_{ij}$. *The matrix* $\exp(\mathcal{C}_r t)$ *is the Wronskian matrix associated to the fundamental system of solutions* $(\alpha_0, \alpha_1, \ldots, \alpha_r)$.

Proof. For each $0 \leq j \leq r$

$$Y = \exp(\mathcal{C}_r t)\mathbf{e}_j$$

is a solution of the system such that $Y(0) = \mathbf{e}_j$, i.e. it corresponds to the unique solution $\alpha_j(t)$ of the linear ODE: $U_{r+1}(D)y = 0$, such that $y^{(j)}(0) = 1$ and $y^i(0) = 0$, for all $i \neq j$. On the other hand $\exp(\mathcal{C}_r t)\mathbf{e}_j$ is the $(j+1)$-th column of $\exp(\mathcal{C}_r t)$, i.e.:

$$\exp(\mathcal{C}_r t)\mathbf{e}_j = \begin{pmatrix} \alpha_j \\ \alpha_j' \\ \vdots \\ \alpha_j^{(r)} \end{pmatrix}.$$

Then

$$\exp(\mathcal{C}_r t) = \begin{pmatrix} \alpha_0 & \alpha_1 & \ldots & \alpha_r \\ \alpha_0^{(1)} & \alpha_1^{(1)} & \ldots & \alpha_r^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0^{(r)} & \alpha_1^{(r)} & \ldots & \alpha_r^{(r)} \end{pmatrix}$$

as claimed. ■

5.3.4. We are now in the position to give a precise expression of the exponential of $\mathcal{C}_r$. By the universal Cauchy formula (3.11) one has:

$$\alpha_j = \sum_{i=0}^{r} U_i(\mathbf{e}_j)u^{(-i)}$$

i.e.:

$$\begin{array}{rcl} \alpha_0 & = & u^{(0)} - e_1 u^{(-1)} + e_2 u^{(-2)} + \ldots + (-1)^r e_r u^{(-r)}. \\ \alpha_1 & = & u^{(-1)} - e_1 u^{(-2)} + \ldots + (-1)^r e_r u^{(-r)} \\ \vdots & = & \vdots \\ \alpha_r & = & u^{(-r)}. \end{array}$$

5.3.5. Example. To compute the exponential of

$$C_2 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ e_3 & -e_2 & e_1 \end{pmatrix}$$

we consider the solutions $\alpha_1, \alpha_2, \alpha_3$ of the linear ODE: $U_2(D)y = 0$, having $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ as initial conditions. Then:

$$\alpha_0 = u^{(0)} - e_1 u^{(-1)} + e_2 u^{(-2)}, \quad \alpha_1 = u^{(-1)} - e_1 u^{(-2)}, \quad \alpha_2 = u^{(-2)},$$

from which

$$\alpha_0^{(1)} = u^{(1)} - e_1 u^{(0)} + e_2 u^{(-1)}, \quad \alpha_1^{(1)} = u^{(0)} - e_1 u^{(-1)}, \quad \alpha_2 = u^{(-1)}.$$

One uses now the fact that

$$u^{(1)} = e_1 u^{(0)} - e_2 u^{(-1)} + e_3 u^{(-2)},$$

which substituted into the expression of $\alpha_0^{(1)}$ gives $\alpha_0^{(1)} = e_3 u^{(-2)}$. Hence:

$$\alpha_0^{(2)} = e_3 u^{(-1)}, \quad \alpha_1^{(2)} = -e_2 u^{(-1)} + e_3 u^{(-2)}, \quad \alpha_2 = u^{(0)}.$$

In conclusion, the exponential of $\mathcal{C}_2$ is:

$$\exp(\mathcal{C}_2 t) = \begin{pmatrix} u^{(0)} - e_1 u^{(-1)} + e_2 u^{(-2)} & u^{(-1)} - e_1 u^{(-2)} & u^{(-2)} \\ e_3 u^{(-2)} & u^{(0)} - e_1 u^{(-1)} & u^{(-1)} \\ e_3 u^{(-1)} & -e_2 u^{(-1)} + e_3 u^{(-2)} & u^{(0)} \end{pmatrix}.$$

5.3.6. Lemma. *For $1 \leq j \leq r$, all entries of the first row of $\mathcal{C}_r^j$ are 0 but the j–th, which is equal to 1.*

Proof. For $0 \leq i \leq r$, let $\epsilon^i = \mathbf{e}_i^T$ be the row which has all entries 0 but the i–th which is equal to 1. Then the first row of $\mathcal{C}_r^j$ is given by the matrix product $\epsilon^i \cdot \mathcal{C}_r^j$. Now

$$\mathcal{C}_r = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \\ \vdots \\ \epsilon^r \\ (-1)^r e_{r+1}\epsilon^0 + (-1)^{r-1} e_r \epsilon^1 + \ldots + e_1 \epsilon^r \end{pmatrix}.$$

We claim that for $j-1 \leq r-1$ one has:

$$\mathcal{C}_r^j = \begin{pmatrix} \epsilon^j \\ \epsilon^{j+1} \\ \vdots \\ \epsilon^r \\ \vdots \end{pmatrix}.$$

The property is true for $j=1$. Suppose it holds for $1 \leq j-1 \leq r-1$. Then

$$\mathcal{C}_r^j = \mathcal{C}_r \cdot \mathcal{C}_r^{j-1} = \begin{pmatrix} \epsilon^1 \cdot \mathcal{C}_r^{j-1} \\ \epsilon^2 \cdot \mathcal{C}_r^{j-1} \\ \vdots \\ \epsilon^{r-j-1} \cdot \mathcal{C}_r^{j-1} \\ \vdots \end{pmatrix} = \begin{pmatrix} \epsilon^j \\ \epsilon^{j+1} \\ \vdots \\ \epsilon^r \\ \vdots \end{pmatrix}$$

by inductive hypothesis. In particular the first row of $\mathcal{C}_r^j$ is $\epsilon^0 \cdot \mathcal{C}_r^j = \epsilon^j$ as claimed. ∎

5.3.7. Proposition. *The exponential of the companion matrix $\mathcal{C}_r$ is:*

$$\exp(\mathcal{C}_r t) = \alpha_1 \mathbf{1} + \alpha_2 \mathcal{C}_r + \ldots + \alpha_{r+1} \mathcal{C}_r^r. \tag{5.5}$$

Proof. The characteristic polynomial of $\mathcal{C}_r$ is precisely $U_{r+1}(t)$. Hence $\mathcal{C}_r^n$ is a linear combination of $\mathbf{1}$, $\mathcal{C}_r$, ..., $\mathcal{C}_r^r$ for each $n \geq r+1$, and thus the exponential of $\mathcal{C}_r t$ can be expressed as a linear combination

$$\exp(\mathcal{C}_r t) = \lambda_0 \mathbf{1} + \lambda_1 \mathcal{C}_r + \ldots + \lambda_r \mathcal{C}_r^r, \tag{5.6}$$

where $\lambda_0, \lambda_1, \ldots, \lambda_r \in E_r[[t]]$. Then (5.6) implies

$$\epsilon^0 \cdot \exp(\mathcal{C}_r t) = \lambda_0 \epsilon^0 \mathbf{1} + \lambda_1 \epsilon^0 \mathcal{C}_r + \ldots + \lambda_r \epsilon^0 \mathcal{C}_r^r.$$

By Lemma (5.3.6) the first row of $\mathcal{C}_r^j$ is precisely ϵ^j and so by Proposition 5.3.3 one has the equality:

$$(\alpha_0, \alpha_1, \ldots, \alpha_r) = \lambda_0 \epsilon^0 + \lambda_1 \epsilon^1 + \ldots + \lambda_r \epsilon^r = (\lambda_0, \lambda_1, \ldots, \lambda_r),$$

whence $\lambda_j = \alpha_j$, for all $0 \leq j \leq r$. ∎

5.3.8. If A is any $\mathbb{Q}$-algebra and $P \in A[t]$ is a monic polynomial of degree $r+1$, we shall denote by C_P the companion matrix associated to P, i.e.

$$C_P := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-1)^r e_{r+1}(P) & (-1)^{r-1}e_r(P) & (-1)^{r-2}e_{r-1}(P) & \dots & e_1(P) \end{pmatrix} \tag{5.7}$$

where $e_i(P) \in A$. Let $\beta_i := \widehat{\phi}_P(\alpha_i)$ be the image of the universal basis $(\alpha_0, \alpha_1, \dots, \alpha_r)$ of $\ker U_{r+1}(D)$ through the unique homomorphism $\phi : E_r \to A$ mapping $e_i \mapsto e_i(P)$. Then

$$\exp(C_P t) = \beta_0 \mathbf{1} + \beta_1 C_P + \dots + \beta_r C_P^r.$$

5.3.9. Given a $m \times m$ matrix, let $P \in A[t]$ be any monic polynomial of degree $r+1$ vanishing at M, i.e. $P(M) = 0$. If P were the characteristic polynomial of M, then $r+1 = m$, if P were the minimal polynomial of M then $r+1 \leq m$. The degree of P may also be bigger than m and, of course, any such polynomial would be a multiple of the minimal polynomial. Suppose now that $\beta_0, \beta_1, \dots, \beta_r \in A[[t]]$ are the unique solutions of the Cauchy problem $P(D)y = 0$ with initial conditions $\mathbf{e}_i$, $1 \leq i \leq n$.

5.3.10. Theorem.

$$\exp(M \cdot t) = \beta_0 \mathbf{1} + \beta_1 M + \dots + \beta_r M^r.$$

Proof. By definition of exponential of a matrix, one has:

$$\begin{aligned} \exp(Mt) &= \sum_{n \geq 0} M^n \frac{t^n}{n!} \\ &= \sum_{n=0}^{r} M^n \frac{t^n}{n!} + M^{r+1} \frac{t^{r+1}}{(r+1)!} + M^{r+2} \frac{t^{r+2}}{(r+2)!} + \dots \end{aligned}$$

Now $M^{r+1} = e_1(P)M^r + \dots + (-1)^r e_{r+1}(P)\mathbf{1}$ and, by induction, each M^n, for $n \geq r+1$, can be expressed as a linear combination of $\mathbf{1}, M, \dots, M^r$. Therefore the exponential of $M \cdot t$ is:

$$\exp(M \cdot t) = \lambda_0 \mathbf{1} + \lambda_1 M + \dots + \lambda_r M^r, \quad \lambda_i \in A[t]].$$

The coefficients λ_i are determined in a purely formal way using the relation $P(M) = 0$ and do not depend on the choice of the matrix. So, in particular, they are the same that one would find if M were the companion matrix C_P associated to P. In this case, by 5.3.8, $\lambda_0, \lambda_1, \ldots, \lambda_r$ are precisely the unique solutions $(\beta_0, \beta_1, \ldots, \beta_r)$ of the Cauchy problem $P(D)y = 0$ with initial conditions $(D^i\beta_j)(0) = \delta_{ij}$.■

5.4 Some properties of the exponential

The A-module $A^{n\times n}$ of A-valued $n \times n$ matrices is a Lie A-algebra, with respect to the commutator:

$$[M, N] = MN - NM.$$

5.4.1. Proposition. Let $M, N \in A^{n\times n}$. Then

$$\exp(M + N) = \exp(M)\exp(N) \tag{5.8}$$

if and only if $[M, N] = 0$.

Proof. It is enough to compare the summands of degree 2 on both sides of (5.8). The summand of degree 2 in the expansion of $\exp(M+N)$ is

$$\frac{(M+N)(M+N)}{2} = \frac{M^2 + MN + NM + N^2}{2},$$

while that in the expansion of $\exp(M)\exp(N)$ is

$$\frac{M^2 + 2MN + N^2}{2}.$$

Equality (5.8) implies

$$M^2 + MN + NM + N^2 = M^2 + 2MN + N^2,$$

i.e. $MN = NM$. Conversely if $[M, N] = 0$,

$$\frac{(M+N)^n}{n!} = \frac{\sum_{h=0}^{n}\binom{n}{h}M^hN^{n-h}}{n!}$$

which coincides precisely with the summand of degree n in the expansion of $\exp(A)\exp(B)$. ■

5.4.2. Let $M \in A^{m\times m}$ and $N \in A^{n\times n}$. Denote by $M \oplus N$ the block diagonal
$(m+n)\times(m+n)$ matrix

$$M \oplus N := \left(\begin{array}{c|c} \mathrm{M} & 0_{m\times n} \\ \hline 0_{n\times m} & \mathrm{N} \end{array}\right).$$

The notation is motivated by the fact that $M \oplus N$, when seen as an endomorphism of A^{m+n}, decomposes the latter into the direct sum of M and N-invariant subspaces:

$$A^{m+n} = (A^m \oplus \mathbf{0}_n) \oplus (\mathbf{0}_m \oplus A^n).$$

By induction one easily proves that:

$$\det(M_1 \oplus \ldots \oplus M_h) = \det(M_1)\cdot \ldots \cdot \det(M_h),$$

and

$$\mathrm{tr}(M_1 \oplus \ldots \oplus M_h) = \mathrm{tr}(M_1) + \ldots + \mathrm{tr}(M_h).$$

Finally, it is clear that

$$\exp((M_1 \oplus \ldots \oplus M_h)t) = \exp(M_1 t) \oplus \ldots \oplus \exp(M_h t).$$

5.4.3. Lemma. *Let C_P be the companion matrix associated to a monic polynomial $P \in A[t]$ of degree $r+1$ ($r \geq 0$). Then*

$$\det(\exp(C_P t)) = \exp(\mathrm{tr}(C_P)t).$$

Proof. By the expression (5.7) it is clear that $\mathrm{tr}(C_P) = e_1(P)$. Now $\exp(C_P t)$ is the Wronskian $W = W(\beta_0, \beta_1, \ldots, \beta_r)$ of the fundamental system of solutions β_i such that $D^j\beta_i(0) = \delta_{ij}$. By Proposition 4.4.3 one knows that

$$W = \exp(C_P t) = \exp(e_1(P)t) = \exp(\mathrm{tr}(C_P t)),$$

and the claim is proven. ∎

5.4.4. Theorem. *Let $M \in A^{(r+1)\times(r+1)}$. Then*

$$\det(\exp(Mt)) = \exp(\operatorname{tr}(Mt)). \tag{5.9}$$

Proof. We shall prove the property under the additional hypothesis that A is an integral domain. In this case we consider the matrix M as being an $(r+1)\times(r+1)$ matrix with entries in $\mathbb{Q}(A)$, the quotient field of A. Then M is conjugated to a block matrix such that each block is a companion matrix, i.e. there exists $g \in Gl_{r+1}(\mathbb{Q}(A))$ such that

$$g^{-1}Mg = C_{P_1} \oplus \ldots \oplus C_{P_h}$$

Therefore

$$\begin{aligned}\det(\exp(Mt)) &= \det(\exp(C_{P_1}t \oplus \ldots \oplus C_{P_h}t)) \\ &= \det(\exp(C_{P_1}t) \oplus \ldots \oplus \exp(C_{P_h}t))\end{aligned}$$

The last equality gives, again:

$$\begin{aligned}\det(\exp(Mt)) &= \det(\exp(C_{P_1}t)) \cdot \ldots \cdot \det(\exp(C_{P_h}t)) \\ &= \exp(\operatorname{tr}(C_{P_1}t))) \cdot \ldots \cdot \exp(\operatorname{tr}(C_{P_h}t))) \\ &= \exp(\operatorname{tr}(C_{P_1}t)) + \ldots + \operatorname{tr}(C_{P_h}t)) \\ &= \exp(\operatorname{tr}(C_{P_1} \oplus \ldots \oplus C_{P_h})t) = \exp(\operatorname{tr}(M)t).\end{aligned}$$

and the theorem is proven. ∎

5.4.5. Example. Let

$$M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in A^{2\times 2}$$

be a matrix with entries in the polynomial ring $A := \mathbb{Q}[a,b,c,d]$ which is certainly an integral domain. Let $P_M(t) := t^2 - e_1(M)t + e_2(M)$ be its characteristic polynomial, i.e. $e_1(M) := a + d$ and $e_2(M) = ad - bc$. Let v_0, v_1 be the canonical basis of $\ker P_M(D)$. Then

$$\begin{aligned}\exp(M\cdot t) &= v_0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + v_1\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} - e_1\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \\ &= \begin{pmatrix} v_0 - dv_1 & bv_1 \\ cv_1 & v_0 - av_1 \end{pmatrix}\end{aligned}$$

Equation (5.9) says that

$$\boxed{v_0^2 - e_1(M)v_0v_1 + e_2(M)v_2^2 = \exp(e_1t)}\,, \tag{5.10}$$

which is the generalization of the celebrated equation

$$\cos^2 t + \sin^2 t = 1.$$

In fact if $e_1(M) = 0$ and $e_2(M) = 1$, the solutions v_0, v_1 occurring in (5.10) are precisely $\cos t$ and $\sin t$ while $\exp(e_1(M)t) = \exp(0 \cdot t) = 1$.

Chapter 6

Derivations on an Exterior Algebra

This chapter aims to provide a quick overview of the theory developed in [13, 14, 17] which will be applied to the situation we are mostly interested in, regarding modules of solutions of linear ODEs. Those papers mainly motivated the research carried out in [18].

6.1 Schubert Calculus on a Grassmann Algebra

6.1.1. Given a graded integral $\mathbb{Q}$-algebra $A := \oplus_{i\geq 0} A_i$, finitely generated as an A_0-algebra, consider the quotient

$$M_{\mathtt{p}} := \frac{A[t]}{(\mathtt{p})}$$

where $\mathtt{p} := t^{d+1} - e_1(\mathtt{p})t^d + \ldots + (-1)^{d+1}e_{d+1}(\mathtt{p})$ is a monic polynomial of degree $d+1$, such that $e_i(\mathtt{p}) \in A_i$. For each $j \geq 0$ let

$$\epsilon^j := t^j + (\mathtt{p})$$

be the class of t^j modulo P. The A-module $M_{\mathtt{p}}$ is free of rank $d+1$ and generated by $\epsilon^0, \epsilon^1, \ldots, \epsilon^d$, because of the relation

$$\epsilon^{d+1+j} = e_1(\mathtt{p})\epsilon^{d+j} + \ldots + (-1)^{d+1+j} e_{d+1}(\mathtt{p})\epsilon^0, \quad (j \geq 0)$$

holding for each $j \geq 0$. There is a direct–sum decomposition

$$M_{\mathtt{p}} := \bigoplus_{w \geq 0} M_{\mathtt{p},w},$$

where $M_{\mathtt{p},w} := A_0\epsilon^w \oplus A_1\epsilon^{w-1} \oplus \ldots \oplus A_w$. Let

$$\bigwedge M_{\mathtt{p}} := A \oplus \bigoplus_{r \geq 0} \bigwedge^{r+1} M_{\mathtt{p}}$$

be the exterior algebra of $M_{\mathtt{p}}$ ($\bigwedge^0 M_{\mathtt{p}} = A$ and $\bigwedge^1 M_{\mathtt{p}} = M_{\mathtt{p}}$). The module $\bigwedge^{r+1} M_{\mathtt{p}}$, $0 \leq r \leq d$, is freely generated by

$$\{\epsilon^\lambda = \epsilon^{r+\lambda_0} \wedge \ldots \wedge \epsilon^{1+\lambda_{r-1}} \wedge \epsilon^{\lambda_r}\}, \tag{6.1}$$

where λ ranges over the set ℓ_r of all the partitions $\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r$ of length at most $r+1$. The *weight* of a homogeneous element ϵ^λ is by definition the weight $|\lambda| = \sum \lambda_i$ of the partition λ. There is a direct sum decomposition

$$\bigwedge^{r+1} M_{\mathtt{p}} = \bigoplus_{w \geq 0} (\bigwedge^{r+1} M_{\mathtt{p}})_w \tag{6.2}$$

where $(\bigwedge^{r+1} M_{\mathtt{p}})_w$ is the A-submodule of $\bigwedge^{r+1} M$ generated by

$$\Big\{ \sum_{\substack{\lambda \in \ell_r \\ 0 \leq |\lambda| \leq w}} a_\lambda \epsilon^\lambda \,|\, a_\lambda \in A_{w-|\lambda|} \Big\}.$$

Clearly $(\bigwedge^{r+1} M_{\mathtt{p}})_0$ is a free A_0-module of rank 1 generated by

$$\epsilon^{(0)} := \epsilon^r \wedge \epsilon^{r-1} \wedge \ldots \wedge \epsilon^0.$$

The decomposition (6.2) defines the *weight gradation* of $\bigwedge^{r+1} M_{\mathtt{p}}$.

6.1.2. Let us consider now the A-algebra $\bigwedge M_{\mathrm{p}}[[t]]$ of formal power series with $\bigwedge M_{\mathrm{p}}$-coefficients. The product is defined by

$$(\sum_{i\geq 0}\mathbf{p}_i t^i)\wedge(\sum_{j\geq 0}\mathbf{q}_j t^i)=\sum_{n\geq 0}(\sum_{i+j=n}\mathbf{p}_i\wedge\mathbf{q}_j)t^n.$$

Next, one considers $End_A(\bigwedge M_{\mathrm{p}})[[t]]$, the A-algebra of formal power series with coefficients in the (non commutative) A-algebra of endomorphisms of $\bigwedge M_{\mathrm{p}}$. Each element

$$D_t := D_0 + D_1 t + \ldots \in End_A(\bigwedge M_{\mathrm{p}})[[t]], \tag{6.3}$$

can be regarded as an A-module homomorphism $D_t : \bigwedge M_{\mathrm{p}} \to \bigwedge M_{\mathrm{p}}[[t]]$ via the obvious map $\mathbf{p} \mapsto D_t(\mathbf{p}) = \sum_{i\geq 0} D_i\mathbf{p}\cdot t^i$.

6.1.3. Definition. *A formal power series (6.3) is a* derivation on the exterior algebra *if*

$$D_t(\mathbf{p}\wedge\mathbf{q}) = D_t\mathbf{p}\wedge D_t\mathbf{q}, \quad (\mathbf{p},\mathbf{q}\in\bigwedge M), \tag{6.4}$$

i.e. if the map $D_t : \bigwedge M_{\mathrm{p}} \to \bigwedge M_{\mathrm{p}}[[t]]$ is an A-algebra homomorphism.

Condition (6.4) implies that the homomorphism D_t is uniquely defined once one knows how it operates on M – the degree 1 part of the exterior algebra.

6.1.4. Definition. *The* Schubert derivation *on $\bigwedge M_{\mathrm{p}}$ is the unique derivation such that that $D_j\epsilon^i = \epsilon^{i+j}$.*

The reason for such a terminology, adopted in [14], is that Equation (6.4), the *fundamental equation of Schubert Calculus*, presides the formalism of the intersection theory on Grassmann bundles – see below. Observe that each $D_i : \bigwedge M_{\mathrm{p}} \to \bigwedge M_{\mathrm{p}}$ is a homomorphism of degree 0, i.e. $D_i(\bigwedge^{1+r} M) \subseteq \bigwedge^{r+1} M_{\mathrm{p}}$. By its very definition, it is homogeneous of degree i on each $\bigwedge^{r+1} M_{\mathrm{p}}$ with respect to the *weight graduation*:

$$D_i(\bigwedge^{r+1} M_{\mathrm{p}})_w \subseteq (\bigwedge^{r+1} M_{\mathrm{p}})_{w+i}.$$

Moreover $D_iD_j = D_jD_i$, for each $i, j \geq 0$. Let

$$A[\mathbf{T}] := A[T_1, T_2, \ldots]$$

be the polynomial ring in infinitely many indeterminates with A-coefficients. Each indeterminate T_i is given degree i. For a polynomial $Q \in A[\mathbf{T}]$, let $Q(D)$ be its evaluation at $D := (D_0, D_1, \ldots,)$ through the map $T_i \mapsto D_i$. It turns out that $A[D] := \mathrm{ev}_D(A[\mathbf{T}])$ is a commutative subring of $End_A(\bigwedge M_{\mathtt{p}})$.

6.1.5. Proposition. *The natural evaluation map*

$$\begin{cases} \mathrm{ev}_{\epsilon^{(0)}} & : & A[D] & \longrightarrow & \bigwedge^{r+1} M_P \\ & & Q(D) & \longmapsto & Q(D)\epsilon^{(0)} \end{cases} \tag{6.5}$$

is onto.

Proof. To show the surjectivity of the map (6.5) it is sufficient to show that for each $\lambda \in \ell_r$ there exists $Q_\lambda \in A[\mathbf{T}]$ such that $\epsilon^\lambda = Q_\lambda(D) \cdot \epsilon^{(0)}$. Then one argues by *integration by parts* as in [14]. See the example below. ∎

6.1.6. Example. Consider $\epsilon^{(1,1)} = \epsilon^2 \wedge \epsilon^1$. Then

$$\begin{aligned} \epsilon^2 \wedge \epsilon^1 &= \epsilon^2 \wedge D_1\epsilon^0 \\ &= D_1(\epsilon^2 \wedge \epsilon^0) - (D_1\epsilon^2) \wedge \epsilon^0 \\ &= D_1(D_1\epsilon^1 \wedge \epsilon^0) - \epsilon^3 \wedge \epsilon^0 \\ &= D_1^2\epsilon^1 \wedge \epsilon^0 - D_2(\epsilon^1 \wedge \epsilon^0) = (D_1^2 - D_2)\epsilon^0. \end{aligned}$$

In other words:

$$\epsilon^{(1,1)} = \begin{vmatrix} D_1 & 1 \\ D_2 & D_1 \end{vmatrix} \epsilon^1 \wedge \epsilon^0 = \Delta_{(1,1)}(D).$$

In general, one can show that *Giambelli's formula* holds:

$$\epsilon^\lambda = \Delta_\lambda(D)\epsilon^0, \tag{6.6}$$

where $\Delta_\lambda(D)$ is the Schur determinant associated to the sequence $D := (D_0, D_1, \ldots)$ corresponding to the partition λ. Formula (6.6) is equivalent to *Pieri's formula*:

$$D_h\epsilon^\lambda = \sum_{\mu \in \ell_{r,h,\lambda}} \epsilon^\mu \qquad (h \geq 0),$$

where

$$\ell_{r,h,\lambda} = \left\{ \begin{array}{c} \text{partitions } \mu \in \ell_r \text{ such that } |\mu| = |\lambda| + h \text{ and} \\ \mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r \geq 0 \end{array} \right\}. \quad (6.7)$$

6.1.7. Let

$$\mathcal{A}^*(\bigwedge^{r+1} M_{\mathtt{p}}) := \frac{A[D]}{(\ker \mathrm{ev}_{\epsilon^0})}.$$

By construction, $\bigwedge^{r+1} M_{\mathtt{p}}$ is a free $\mathcal{A}^*(\bigwedge^{r+1} M_{\mathtt{p}})$-module of rank 1 generated by $\epsilon^{(0)} = \epsilon^r \wedge \ldots \wedge \epsilon^1 \wedge \epsilon^0$. The module multiplication is given by the A-bilinear map:

$$\mathcal{A}^*(\bigwedge^{r+1} M_{\mathtt{p}}) \otimes \bigwedge^{r+1} M_{\mathtt{p}} \longrightarrow \bigwedge^{r+1} M_{\mathtt{p}}$$

given by $(Q(D) + \ker \mathrm{ev}_{\epsilon^{(0)}}, \epsilon^{\lambda}) \mapsto Q(D)\epsilon^{\lambda}$.

6.1.8. Remark. It turns out that $\mathcal{A}^*(\bigwedge^{r+1} M_P)$ is the universal splitting algebra of the polynomial $\mathtt{p}$ as the product of two monic polynomials one of degree $r+1$. See e.g. [10, 24, 25].

6.1.9. Example. Let $\mathbf{p}, \mathbf{q} \in \bigwedge M_{\mathtt{p}}$. Then Newton's Binomial formula for D_1 holds, i.e.:

$$D_1^m(\mathbf{p} \wedge \mathbf{q}) = \sum_{k=0}^{m} \binom{m}{k} D_1^k \mathbf{p} \wedge D^{m-k}\mathbf{q}. \quad (6.8)$$

This is easily seen by induction. If $m = 1$ it is just Leibniz's rule:

$$D_1(\mathbf{p} \wedge \mathbf{q}) = D_1\mathbf{p} \wedge \mathbf{q} + \mathbf{p} \wedge D_1\mathbf{q}.$$

For $m = 2$:

$$\begin{aligned} D_1^2(\mathbf{p} \wedge \mathbf{q}) &= D_1[D_1\mathbf{p} \wedge \mathbf{q} + \mathbf{p} \wedge D_1\mathbf{q}] \\ &= D_1^1\mathbf{p} \wedge \mathbf{q} + D_1\mathbf{p} \wedge D_1\mathbf{q} + D_1\mathbf{p} \wedge D_1\mathbf{q} + \mathbf{p} \wedge D_1^2\mathbf{q} \\ &= D_1^1\mathbf{p} \wedge \mathbf{q} + 2 \cdot D_1\mathbf{p} \wedge D_1\mathbf{q} + \mathbf{p} \wedge D_1^2\mathbf{q}. \end{aligned}$$

If (6.8) holds for $h = m - 1 \geq 1$ it holds for m. In fact

$$\begin{aligned} D_1^m(\mathbf{p} \wedge \mathbf{q}) &= D_1(D_1^{(m-1}(\mathbf{p} \wedge \mathbf{q})) \\ &= D_1 \sum_{k=0}^{m-1} \binom{m-1}{k} D_1^k\mathbf{p} \wedge D_1^{m-1-k}\mathbf{q}. \end{aligned}$$

See [8] for generalizations and applications to the enumerative geometry of rational space curves.

6.2 Connection with Schubert Calculus

Schubert calculus rules the intersection theory on Grassmann bundles. We give here a quick account to illustrate its relationships with the derivations on a Grassmann algebra. Detailed references for this section are [13, 14, 17, 24, 25]. The original motivation was to study loci of Weierstrass points on families of possible singular Gorenstein curves (which exclude, therefore, the case studied in [34]). For a nice survey about some combinatorics related with ramification loci of linear systems which inspired the present work see e.g. [7].

6.2.1. Each smooth complex projective variety X can be equipped with an *intersection theory*, by attaching to X an Abelian group $A_*(X)$ and a $\mathbb{Z}$-algebra $A^*(X)$ which are isomorphic as $\mathbb{Z}$-modules (Poincaré duality). The group $A_*(X)$ is generated by *cycles modulo rational equivalence* and is a free module of rank 1 over $A^*(X)$ generated by the *fundamental class* $[X]$. The $A^*(X)$-module multiplication of $A_*(X)$ is said *cap product*:

$$A^*(X) \otimes_{\mathbb{Z}} A_*(X) \xrightarrow{\cap} A_*(X)$$

sending $(\alpha, c) \mapsto \alpha \cap c \in A_*(X)$. See [11].

6.2.2. Example. The intersection theory of the projective space may be described by saying that $A_*(\mathbb{P}^n)$ is a $\mathbb{Z}$-module freely generated by the class of linear subvarieties of codimension i, for $0 \leq i \leq n$:

$$A_*(\mathbb{P}^n) = \mathbb{Z}[\mathbb{P}^n] \oplus \mathbb{Z} \cdot H \oplus \mathbb{Z} \cdot H^2 \oplus \ldots \oplus \mathbb{Z} \cdot H^n.$$

The generator H^i is the class modulo rational equivalence of any linear subvariety of codimenion i. The generator $[\mathbb{P}^n]$ is the *fundamental class*, while H^n is the *class of a point*. The class of each subvariety of codimension i is of the form $d \cdot H^i$ and the integer d is called the *degree* of the subvariety. The Chow ring of $\mathbb{P}^n$ is

$$A^*(\mathbb{P}^n) = \frac{\mathbb{Z}[H]}{(H^{n+1})}$$

and says what is the dimension and the degree of the intersection of a hyperplane and a projective subvariety of $\mathbb{P}^n$ in general position. The relation $H^{n+1} = 0$ expresses the fact that the intersection of $(n+1)$-hyperplanes in general position is empty. The *cap product* $H \cap [S]$ is the operation of *cutting with a hyperplane*. If S_1 and S_2 are closed subvarieties of $\mathbb{P}^n$ in general position, of codimension i_1 and i_2 and degree d_1 and d_2 respectively, then $S_1 \sim d_1 H^{i_1} \cap [\mathbb{P}^n]$, $S_2 \sim d_2 H^{i_2} \cap [\mathbb{P}^n]$ and

$$[S_1 \cap S_2] = d_1 H^{i_1} \cap [S_2] = d_1 H^{i_1} \cap (d_2 H^{i_2} \cap [\mathbb{P}^n]) = d_1 d_2 H^{i_1+i_2} \cap [\mathbb{P}^n].$$

In other words two subvarieties of $\mathbb{P}^n$ of degree d_1 and d_2 in general position do intersect along a subvariety of degree $d_1 d_2$ having as codimension the sum of the codimensions. This is *Bezout's theorem* for $\mathbb{P}^n$.

6.2.3. The intersection theory of Grassmann bundles is only slightly more complicated. First of all recall that the set $G(r, \mathbb{P}^d)$, parameterizing all linear subvarieties of $\mathbb{P}^d$ of dimension r, is itself a smooth projective variety of dimension $(r+1)(d-r)$. It is called the *Grassmann variety* (of r-dimensional linear subavarieties of $\mathbb{P}^d$).

6.2.4. Example. Let $O_{\mathbb{P}^1}(d)$ be the line bundle over the projective line whose global sections are the homogeneous form of degree d in two indeterminates (say x_0 and x_1). A linear system of degree d and (projective) dimension r on $\mathbb{P}^1$, denoted by g^r_d, is a choice of an $(r+1)$-dimensional subspace of $H^0(O_{\mathbb{P}^1}(d))$. Hence the set of all g^r_ds on $\mathbb{P}^1$ are parameterized by the Grassmann variety $G(r, \mathbb{P}H^0(O_{\mathbb{P}^1}(d))$.

6.2.5. Let $\rho : E \to X$ be a vector bundle of rank $d+1$ over a smooth projective variety X and let $\rho_0 : \mathbb{P}E \to X$ be the associated projective bundle (the fiber over $x \in X$ is the projective space $\mathbb{P}E_x$). Let $\rho_r : G(r, \mathbb{P}E) \to X$ be the Grassmann bundle of all linear projective subvarieties of dimension r in the fibers of $\mathbb{P}E$:

$$G(r, \mathbb{P}E)_x = G(r, \mathbb{P}E_x).$$

Over $G(r, \mathbb{P}E)$ sits the universal exact sequence

$$0 \longrightarrow \mathcal{S}_r \longrightarrow \rho_r^* E \longrightarrow \mathcal{Q}_r \longrightarrow 0,$$

where $\mathcal{S}_r$ is the universal subbundle of $\rho_r^* E$: the fiber of $\mathcal{S}_r$ over $[\Lambda] \in G(r, \mathbb{P}E_x)$ is the r-linear subvariety $\Lambda \in G(r, \mathbb{P}E_x)$ parameterized by the point $[\Lambda]$ itself. Let

$$P_E(t) := t^{d+1} + c_1(E) t^d + \ldots + c_d(E) \in A^*(X)[t],$$

where $c_i(E)$ are the *Chern classes* of the vector bundle E. Then (Cf. [11]) the Chow ring of $\mathbb{P}(E)$ is given by

$$A^*(\mathbb{P}(E)) = \frac{A^*(X)[\xi]}{(P_E(\xi))},$$

where $P_E(\xi)$ is the evaluation of the polynomial P_E at $\xi = -c_1(\mathcal{S}_0)$, the class of any section of $\mathbb{P}E$ cutting fiberwise a hyperplane section. The Chow group $A_*(\mathbb{P}E)$ is an $A^*(X)$-module generated by $(\mathrm{m}^0, \mathrm{m}^1, \ldots)$ where $\mathrm{m}^i := \xi^i \cap [\mathbb{P}E]$. Indeed, $A_*(\mathbb{P}E)$ is a free $A^*(X)$–module of rank $d+1$, generated by $\mathrm{m}^0, \mathrm{m}^1, \ldots, \mathrm{m}^d$, because of the relation

$$\mathrm{m}^{d+i} + c_1(E)\mathrm{m}^{d-1+i} + \ldots + c_d(E)\mathrm{m}^i = 0,$$

holding for each $i \geq 0$. For each $j \geq 0$ define

$$D_j \mathrm{m}^i = \mathrm{m}^{i+j}.$$

In this case $D_j = D_1^j$. Then, trivially, $A_*(\mathbb{P}(E))$ is a free module of rank 1 over

$$\frac{A[D_1]}{(P_E(D_1))}$$

generated by m^0. The module $\bigwedge^{r+1} A_*(\mathbb{P}E)$ is a free $A^*(X)$-module generated by

$$\mathrm{m}^\lambda = \mathrm{m}^{r+\lambda_0} \wedge \ldots \wedge \mathrm{m}^{1+\lambda_{r-1}} \wedge \mathrm{m}^{\lambda_r}$$

and it turns out to be a free $\mathcal{A}^*(\bigwedge^{r+1} A_*(\mathbb{P}E))$-module of rank 1 generated by $\mathrm{m}^{(0)} := \mathrm{m}^r \wedge \ldots \wedge \mathrm{m}^1 \wedge \mathrm{m}^0$.

6.2.6. Let

$$\begin{aligned} c_t(\mathcal{Q}_r - \rho_r^* E) &= \frac{c_t(\mathcal{Q}_r)}{c_t(\rho_r^* E)} \\ &= (1 + c_1(\mathcal{Q}_r)t + c_2(\mathcal{Q}_r)t^2 + \ldots)(1 - c_1(\rho_r^* E)t \\ &\qquad + (c_1(\rho_r^* E)^2 - c_2(\rho_r^* E))t^2 + \ldots) \\ &= 1 + (c_1(\mathcal{Q}_r)t - c_1(\rho_r^* E))t \\ &\qquad + (c_2(\mathcal{Q}_r) - c_1(\mathcal{Q}_r)c_1(\rho_r^* E) - c_2(\rho_r^* E))t^2 + \ldots \end{aligned}$$

Then (Cf. [11, Ch. 14]) the Chow group $A_*(G(r, \mathbb{P}E)$ is a free $A^*(X)$-module generated by

$$\{\Delta_\lambda(c_t(\mathcal{Q}_r - \rho^* E)) \cap [G(r, \mathbb{P}E)] \mid \lambda \in \ell_r\}.$$

It follows that $A_*(G(r,\mathbb{P}E)$ is a free $A^*(G(r,\mathbb{P}E))$-module of rank 1, generated by the fundamental class $[G(r,\mathbb{P}E)]$. Furthermore *Pieri's formula* holds (see notation (6.7)):

$$c_h(\mathcal{Q}_r - \rho^*E) \cap \Delta_\lambda(c_t(\mathcal{Q}_r - \rho^*E)) \cap [G(r,\mathbb{P}E)] =$$

$$= \sum_{\mu \in \ell_{r,h,\lambda}} \Delta_\mu(c_t(\mathcal{Q}_r - \rho^*E)) \cap [G(r,\mathbb{P}E)].$$

Let :

$$\iota : A^*(G(r,\mathbb{P}E)) \to \mathcal{A}^*(\bigwedge^{r+1} M)$$

be defined by $c_h(\mathcal{Q}_r - \rho^*E) \mapsto D_h$ and

$$\jmath : A_*(G(r,\mathbb{P}E) \mapsto \bigwedge^{r+1} M$$

by $\Delta_\lambda(c_t(\mathcal{Q}_r - \rho^*E)) \cap [G(r,\mathbb{P}E)] \mapsto \mathrm{m}^\lambda$, where for the sake of brevity we set $M := A_*(\mathbb{P}E)$. The following result is a generalization of the theory developed in [13], obtained in [24] (see also [25]) in the language of symmetric functions and universal splitting algebras, and in [17] within the framework of derivations on an exterior algebra.

6.2.7. Theorem. *The diagram*

$$\begin{array}{ccc} H^*(G_{r,d}) \otimes_{\mathbb{Z}} H_*(G_{r,d}) & \xrightarrow{\cap} & H_*(G_{r,d}) \\ {\scriptstyle \iota \otimes \jmath}\big\downarrow & & \big\downarrow{\scriptstyle \jmath} \\ \mathcal{A}^*(\bigwedge^{r+1} M) \otimes_{\mathbb{Z}} \bigwedge^{r+1} M & \longrightarrow & \bigwedge^{r+1} M \end{array} \tag{6.9}$$

commutes and the vertical maps are isomorphisms.

Proof. The very easy argument is based on the construction of the Schubert derivation on the Grassmann algebra of a free A-module. In fact

$$\iota \otimes \jmath(c_h \cap \Delta_\lambda(c_t(\mathcal{Q}_r - \rho_r^*E)) \mapsto D_h \mathrm{m}^\lambda = \sum_{\mu \in \ell_{r,h,\lambda}} \mathrm{m}^\mu.$$

On the other hand

$$\jmath(c_h \cap \Delta_\lambda(c_t(\mathcal{Q}_r - \rho_r^*E) \cap [G(r, \mathbb{P}E)]) = \jmath(\sum_{\mu \in \ell_{r,h,\lambda}} c_\mu(\mathcal{Q} - \rho_r^*E))$$
$$= \sum_{\mu \in \ell_{r,h,\lambda}} \mathrm{m}^\mu.$$

and the commutativity of diagram (6.9) is proven. The fact that the vertical maps are isomorphisms is obvious. ∎

6.2.8. Remark. Theorem 6.2.7 says that making computations at the top level of diagram (6.9) is the same as performing them at the bottom level, where the basic computational recipe is Leibniz's rule for derivatives. Notice that the bottom line of diagram (6.2.7) is the algebraic translation of *Poincaré dualily* for Grassmann bundles, as it translates the fact that $A^*(G(r, \mathbb{P}E))$ is isomorphic as an $A^*(X)$-module to $A_*((G(r, \mathbb{P}E))$ via the *cap product*. This picture has been used in [16] to interpret the class of hyperelliptic locus in M_g in terms of Schubert calculus in a suitable vector bundle containing the universal curve as a section. However it is not yet clear how to extend such techniques in the situation studied, e.g, in [32].

6.2.9. Remark. If $X = \{pt\}$ is a point, then one obtains classical Schubert calculus on the Grassmannian $G(r, \mathbb{P}^d)$ parameterizing closed linear subvarieties of the d-dimensional projective space. In this case $P_E(t) = t^{d+1}$ and $A^*(\mathbb{P}(E)) = \mathbb{Z}[\xi]/(\xi^{n+1})$, where $\xi = c_1(O_{\mathbb{P}^d}(1))$. If $F^\bullet$ is a flag of linear subspaces of $\mathbb{P}^d$

$$F^\bullet : \mathbb{P}^d := F^0 \supset F^1 \supset \ldots \supset F^d \supset \emptyset$$

where F^i has codimension i, one sets

$$\Omega_\lambda(F^\bullet) := \{\Lambda \in G(r, \mathbb{P}^d) \mid \dim(F^{r-i+\lambda_i} \cap \Lambda) \geq i\}.$$

The class of $\Omega_\lambda(F^\bullet)$ in $A_*(G(r, \mathbb{P}^d)$ does not depend on the choice of the flag and is denoted by Ω_λ. Giambelli's formula says that

$$\Omega_\lambda = c_\lambda(\mathcal{Q}_r) \cap [G(r, \mathbb{P}^d)]$$

and is in fact equivalent to Pieri's formula:

$$\sigma_h \cap \Omega_\lambda = \sum_{\mu \in \ell_{r,h,\lambda}} \Omega_\mu.$$

where $\sigma_h = c_h(\mathcal{Q}_r)$. According to Theorem 6.2.7 each Schubert cycle Ω_λ corresponds to m^λ, and computing $\sigma_h \cap \Omega_\lambda$ amounts computing

$$D_h \mathrm{m}^\lambda = D_h(\mathrm{m}^{r+\lambda_0} \wedge \mathrm{m}^{r-1+\lambda_1} \wedge \ldots \wedge \mathrm{m}^{\lambda_r}).$$

6.2.10. Example. The Grassmannian $G(1, \mathbb{P}^d)$ of lines in $\mathbb{P}^d$ can be embedded à la Plücker in $\mathbb{P}^N$ where

$$N = \binom{d+1}{2} - 1.$$

The class of a hyperplane section in $A_*(G(1, \mathbb{P}^d))$ is $\sigma_1 \cap [G(1, \mathbb{P}^d)]$. Hence the Plücker degree $d_{1,d}$ of $G(1, \mathbb{P}^d)$ can be computed by calculating the degree of the intersection cycle of the Plücker embedding of $G(1, \mathbb{P}^d)$ with $2d-2$ (= the dimension of the Grassmannian) hyperplanes of $\mathbb{P}^N$ in general position, i.e.:[1]

$$d_{1,d} = \int \sigma_1^{2d-2} \cap [G(1, \mathbb{P}^d)].$$

According to the dictionary (6.9), this amounts to computing the coefficient $d_{1,d}$ of $\mathrm{m}^d \wedge \mathrm{m}^{d-1}$ in the expansion of $D_1^{2d-2} \mathrm{m}^1 \wedge \mathrm{m}^0$. Using Newton's binomial formula (6.8) one has:

$$D_1^{2d-2} \mathrm{m}^1 \wedge \mathrm{m}^0 = \sum_{k=0}^{2d-2} \binom{2d}{k} \mathrm{m}^{1+k} \wedge \mathrm{m}^{2d-2-k} \tag{6.10}$$

Since $D_1^{2d} \mathrm{m}^1 \wedge \mathrm{m}^0 \in (\bigwedge^2 A_*(\mathbb{P}^d))_{2d}$, which is free of rank 1 generated by $\mathrm{m}^d \wedge \mathrm{m}^{d-1}$, only the contributions corresponding to $k = d-1$ and to $k = d-2$ survive in the sum (6.10). Hence:

$$\begin{aligned} D_1^{2d-2} \mathrm{m}^1 \wedge \mathrm{m}^0 &= \left(\binom{2d-2}{d-1} - \binom{2d-2}{d-2} \right) \mathrm{m}^d \wedge \mathrm{m}^{d-1} \\ &= \frac{(2d-2)!}{d!(d-1)!} \mathrm{m}^d \wedge \mathrm{m}^{d-1} \end{aligned}$$

and

$$d_{1,d} = \frac{(2d-2)!}{d!(d-1)!}.$$

[1] The integral sign means just "taking the degree of the cycle".

6.3 Connection with Wronskians

6.3.1. Let $\mathbf{u}_r$ be the canonical basis of solutions of the linear universal ODE (3.2) and let

$$\mathcal{W}(\mathbf{u}_r) := \{\mathcal{W}_\lambda(\mathbf{u}) \,|\, |\lambda \in \ell_r\}$$

be the *Wronskian* $\mathbb{Q}$*-vector space*. It is an E_r–submodule of $E_r[[t]]$. The submodule of $E_r[[t]]$: $\mathcal{U}_r := \ker U_{r+1}(D)$ is a $\mathbb{Q}$-vector space as well, generated by $u^{(j)}$ with $j \geq -r$. It is an infinite dimensional vector space. Its $(r+1)$-th exterior power $\bigwedge^{r+1} \ker(U_{r+1}(D))$ is generated by

$$\{\mathbf{u}^\lambda := u^{(\lambda_0)} \wedge u^{(-1+\lambda_1)} \wedge \ldots \wedge u^{(-r+\lambda_r)} \,|\, \lambda \in \ell_r\}$$

The non negative integer $\mathrm{wt}(\mathbf{u}^\lambda) := |\lambda|$ is by definition the *weight* of $\mathbf{u}^\lambda$ and we have the direct sum decomposition:

$$\bigwedge^{r+1} \mathcal{U}_r = \bigoplus_{w \geq 0} (\bigwedge^{r+1} \mathcal{U}_r)_w.$$

Here is a useful

6.3.2. Lemma. *Let λ be a partition. Then*

$$\begin{vmatrix} U_0(\mathrm{s}_{m+\lambda_0}(\mathbf{h})) & U_0(\mathrm{s}_{m-1+\lambda_1}(\mathbf{h})) & \ldots & U_0(\mathrm{s}_{m-r+\lambda_r}(\mathbf{h})) \\ U_1(\mathrm{s}_{m+\lambda_0}(\mathbf{h})) & U_1(\mathrm{s}_{m-1+\lambda_1}(\mathbf{h})) & \ldots & U_1(\mathrm{s}_{m-r+\lambda_r}(\mathbf{h})) \\ \vdots & \vdots & \ddots & \vdots \\ U_r(\mathrm{s}_{m+\lambda_0}(\mathbf{h})) & U_r(\mathrm{s}_{m-1+\lambda_1}(\mathbf{h})) & \ldots & U_r(\mathrm{s}_{m-r+\lambda_r}(\mathbf{h})) \end{vmatrix} = \Delta_\lambda(s_m(\mathbf{h})). \tag{6.11}$$

Proof. Using the fact that the determinant of a matrix coincides with that of its transpose, one observes that the determinant occurring on the left hand side of (6.11) can be written as:

$$\begin{pmatrix} h_{m+\lambda_0} \\ h_{m-1+\lambda_1} \\ \vdots \\ h_{m-r+\lambda_r} \end{pmatrix} \wedge \left[\begin{pmatrix} h_{m+1+\lambda_0} \\ h_{m+\lambda_1} \\ \vdots \\ h_{m-r+1+\lambda_r} \end{pmatrix} - e_1 \begin{pmatrix} h_{m+\lambda_0} \\ h_{m-1+\lambda_1} \\ \vdots \\ h_{m-r+\lambda_r} \end{pmatrix} \right] \wedge \ldots \wedge$$

$$\wedge\left[\begin{pmatrix} h_{m+r+\lambda_0} \\ h_{m+r-1+\lambda_1} \\ \vdots \\ h_{m+\lambda_r} \end{pmatrix} + \sum_{i=1}^{r}(-1)^i e_i \begin{pmatrix} h_{m+r-i+\lambda_0} \\ h_{m+r-i-1+\lambda_1} \\ \vdots \\ h_{m-i+\lambda_r} \end{pmatrix}\right] =$$

$$= \begin{pmatrix} h_{m+\lambda_0} \\ h_{m-1+\lambda_1} \\ \vdots \\ h_{m-r+\lambda_r} \end{pmatrix} \wedge \begin{pmatrix} h_{m+1+\lambda_0} \\ h_{m+\lambda_1} \\ \vdots \\ h_{m-r+1+\lambda_r} \end{pmatrix} \wedge \ldots \wedge \begin{pmatrix} h_{m+r+\lambda_0} \\ h_{m+r-1+\lambda_1} \\ \vdots \\ h_{m+\lambda_r} \end{pmatrix}, \qquad (6.12)$$

where the last equality is because of cancellations due to the skew–symmetry of the determinant and (6.12) is precisely the explicit expression of $\Delta_\lambda(s_m(\mathbf{h}))$. ■

We can then prove the following:

6.3.3. Theorem. *The $\mathbb{Q}$-vector space $\bigwedge^{r+1}\mathcal{U}_r$ has a canonical structure of free E_r-module of rank 1 generated by $\mathbf{u}_r^{(0)}$.*

Proof. Before defining the module structure, one checks that the following Giambelli formula holds:

$$\mathbf{u}_r^\lambda = h_\lambda \cdot \mathbf{u}_r^{(0)}. \qquad (6.13)$$

Applying the universal Cauchy formula (3.11) to the solution $u^{(-i+\lambda_i)}$ one has:

$$\begin{aligned} u^{(-i+\lambda_i)} &= U_0(s_{-i+\lambda_i}(\mathbf{h}))u^{(0)} \\ &\quad + U_1(s_{-i+\lambda_i}(\mathbf{h}))u^{(-1)} + \ldots + U_r(s_{-i+\lambda_i}(\mathbf{h}))u^{(-r)}, \end{aligned}$$

from which

$$\mathbf{u}^\lambda = u^{(\lambda_0)} \wedge u^{(-1+\lambda_1)} \wedge \ldots \wedge u^{(-r+\lambda_r)} = \det[U_i(\mathrm{s}_{-j+\lambda_j}(\mathbf{h}))] \cdot \mathbf{u}^{(0)}.$$

According to Lemma 6.3.2, $\det[U_i(\mathrm{s}_{-j+\lambda_j}(\mathbf{h}))]=\Delta_\lambda(\mathrm{s}_0(\mathbf{h}))=\Delta_\lambda(\mathbf{h}) = h_\lambda$ and (6.13) follows. One can now define the module structure by means of Pieri's formula

$$h_j\mathbf{u}^\lambda = \sum_{\mu\in\ell_{r,j,\lambda}} \mathbf{u}^\mu.$$

which, as is well known, is equivalent to (6.13). ■

6.3.4. Theorem. *There is a canonical E_r-module isomorphism between the module $\mathcal{W}(\mathbf{u}_r)$ and $\bigwedge^{r+1}\mathcal{U}_r$.*

Proof. Let $\iota_r : \mathcal{W}(\mathbf{u}_r) \longrightarrow \bigwedge^{r+1}\mathcal{U}_r$ be the $\mathbb{Q}$-linear extension of the bijection

$$W_\lambda(\mathbf{u}_r) \mapsto \mathbf{u}_r^\lambda.$$

Then:

$$\begin{aligned} \iota_r(h_j W_\lambda(\mathbf{u}_r)) &= \iota_r\Big(\sum_{\mu\in\ell_{r,j,\lambda}} W_\mu(\mathbf{u}_r)\Big) \\ &= \sum_{\mu\in\ell_{r,j,\lambda}} \iota_r(W_\mu(\mathbf{u}_r)) \\ &= \sum_{\mu\in\ell_{r,j,\lambda}} \mathbf{u}_r^\mu = h_j\mathbf{u}_r^\lambda \end{aligned}$$

proving that ι_r is E_r-linear as well. ■

Chapter 7

The Boson-Fermion Correpondence

7.1 Introduction

7.1.1. The *oscillator algebra* (see [22]) is a complex Lie algebra with basis $\{a_n, \hbar\}$ obeying the commutation relations

$$[\hbar, a_m] = 0, \qquad [a_m, a_n] = m\delta_{m,-n}\hbar \qquad (m, n \in \mathbb{Z}).$$

The *Fock space* is by definition the ring $B := \mathbb{C}[x_1, x_2, \ldots]$ of polynomials in infinitely many indeterminates, where each x_i has degree i. To each pair μ and $\hbar$ of real numbers, one may attach a certain *bosonic* representation $\beta : \mathcal{A} \to End_{\mathbb{C}}(B)$ of $\mathcal{A}$ in the *Fock space* B, defined on generators by:

$$\begin{cases} \beta(a_n)(P) & = & \dfrac{\partial P}{\partial x_n}, & (n \in \mathbb{N}_{>0}) \\ \beta(a_{-n})(P) & = & \hbar n x_n P, & (n \in \mathbb{N}_{>0}) \\ \beta(a_0)(P) & = & \mu P, & \\ \beta(\hbar)(P) & = & \hbar P, & \end{cases}$$

where P denotes an arbitrary polynomial in B. For example

$$\beta(a_5)(3x_1 - x_2x_5) = 15\hbar x_1x_5 - 5\hbar x_5^2, \qquad \beta(a_5)(3x_1 - x_2x_5) = x_2.$$

The constant polynomial 1 is the *vacuum vector* of B. One obviously has:

$$\beta(a_n)(1) = 0, \quad \beta(a_0)(1) = \mu, \quad \beta(\hbar)(1) = \hbar, \quad \beta(a_{-n}) = n\hbar x_n.$$

From a mathematical point of view, such a representation is important because it allows to define certain Virasoro operators in the Fock representation of $\mathcal{A}$ for $a_0 = \mu$. The *Virasoro algebra* is a central extension of the Lie algebra of C^∞–vector fields on the unit circle S^1. If $\mathtt{d}_n = z^{n+1}\dfrac{d}{dz}$, the Virasoro algebra is generated by $\{\mathtt{d}_n; c\}_{n\in\mathbb{Z}}$ subject to the commutation relations

$$[\mathtt{d}_n, c] = 0 \qquad \text{and} \qquad [\mathtt{d}_m, \mathtt{d}_n] = (m-n)\mathtt{d}_{m+n}\alpha(m,n)c,$$

where α must be chosen in such a way that $[,]$ defines a Lie algebra structure on $\mathcal{V} := \mathbb{C}\cdot c \oplus_{n\in\mathbb{Z}} \mathbb{C}\mathtt{d}_n$. This forces α to be:

$$\alpha(m,n) = \delta_{m,-n}\frac{(m^3 - m)}{12}c.$$

7.1.2. According to [22], Dirac hole positron theory can be phrased in mathematical terms via a different kind of representation of the *oscillator algebra* $\mathcal{A}$, namely using a *semi-infinite* exterior power F of an infinite dimensional $\mathbb{C}$-vector space generated by a basis indexed by the integers. Such an infinite wedge space is called the *fermionic space*. Remarkably, there is an isomorphism between the bosonic and fermionic representations, known as *Boson-Fermion correspondence*. For more background on this exciting mathematical subject and its relationship with the theory of algebraically integrable dynamical systems, the reader can consult the excellent expositions [3, 23, 30] and references therein. See also [4] for a bi-hamiltonian approach.

7.1.3. In this last chapter we will show that to each representation of a given algebra in the Fock space B there corresponds, in a natural manner, an isomorphic fermionic one – see also the note by Neretin [31]. This is due to the fact that each semi-infinite exterior power of an infinite dimensional vector space $V := \oplus_{i\in\mathbb{Z}}\mathbb{C}\cdot v_i$ is naturally a free-module of rank 1 over the Fock space B generated by some reference *vacuum state vector*, and the fermionic representation of the oscillator algebra turns out to be the composition of its bosonic representation in B with a canonical $\mathbb{C}$-vector spaces isomorphism $B \to F$. It turns out that the isomorphism is the infinite-dimensional analogue of the Poincaré duality for Grassmannians phrased in terms of derivations of an exterior algebra. Such an isomorphism becomes especially natural when one identifies the vector space V with the space generated by the solutions of a *linear ODE of infinite order*. Indeed we shall prove that $\bigwedge^{\frac{\infty}{2}} E_\infty[[t]]$ is a free E_∞ module of rank 1 generated by a certain *perfect vacuum vector* Φ_0. Our main ingredient will be the universal Cauchy formula (3.11)! See below.

7.2 Linear ODEs of infinite order

7.2.1. As in Section 2.3 let $(e_1, e_2, \ldots)$ be an infinite sequence of indeterminates over $\mathbb{Q}$ and $\mathbf{h} := (h_0, h_1, \ldots)$ be defined through the equality

$$\sum_{n\in\mathbb{Z}} h_n t^n = \frac{1}{1 - e_t + e_2t^2 - \ldots} = \frac{1}{\sum_{i\geq 0}(-1)^i e_i t^i} \qquad (e_0 = 1).$$

Let $E_\infty := \mathbb{Q}[e_1, e_2, \ldots] = \mathbb{Q}[h_1, h_2, \ldots]$, and

$$u^{(j)} := \sum_{n\geq 0} h_{n+j}\frac{t^n}{n!} \in E_\infty.$$

Then $(u^{(-j)})_{j\geq 0}$ is an E_∞-basis of $A[[t]]$, where A denotes an arbitrary E_∞-algebra. In fact by Lemma 2.3.11:

$$\sum_{n\geq 0} a_n\frac{t^n}{n!} = \sum_{n=0}^{\infty} U_i(\mathbf{a})u^{(-j)},$$

where $L(\mathbf{a}(t)) = \sum_{n\geq 0} a_n t^n \in A[[t]]$ and $U_0(\mathbf{a}), U_1(\mathbf{a}), \ldots$ are defined as in (2.11). By Remark 3.4.3, the sequence

$$\mathbf{u} := (u^{(0)}, u^{(-1)}, \ldots)$$

can be regarded as a E_∞-basis of the space $\mathcal{U}_\infty$ of solutions of the linear ODE of infinite order with coefficients $(e_1, e_2, \ldots)$.

7.2.2. In particular, for each $m \geq 1$, the m-th derivative of $u^{(0)}$ is a(n infinite) linear combination of the $u^{(-j)}$ $(j \geq 0)$: it is the infinite–dimensional counterpart of the fact that the derivative of the solution of a linear ODE is a solution as well. More precisely one has:

$$u^{(m)} = \sum_{j\geq 0} U_j(\mathrm{s}_m(\mathbf{h}))u^{(-j)}, \tag{7.1}$$

which is nothing but the infinite-dimensional version of the Cauchy universal formula (3.11) applied to the solution $u^{(m)}$. Formula (7.1) can be generalized as follows:

$$u^{(i+m)} = \sum_{j\geq 0} U_j(\mathrm{s}_m(\mathbf{h}))u^{(i-j)}. \tag{7.2}$$

Notice that while the expression on the right of (7.1) is unique, the one on right–hand side of (7.2) is not, because all the $u^{(i-j)}$ such that $i - j \geq 0$ can be expressed as linear combination of $(u^{(0)}, u^{(-1)}, \ldots)$, which is a basis for $A[[t]]$, again by Lemma 2.3.11.

7.3 Fermionic spaces

7.3.1. According to [22], let F_m be the $\mathbb{Q}$-vector space generated by the semi-infinite monomials

$$\begin{aligned}\Phi^\lambda_m(\mathbf{u}) := u^{m+\lambda_0} &\wedge u^{(m-1+\lambda_1)}\\ &\wedge \ldots \wedge u^{(m-r+\lambda_r)} \wedge u^{(m-r-1)} \wedge u^{(m-r-2)} \wedge \ldots,\end{aligned} \tag{7.3}$$

where $\lambda \in \ell_r$ $(r \geq 0)$. Physicists would call F_m a *fermionic space of charge number* m.[1] The degree of a semi-infinite monomial like (7.3) is by definition the weight $|\lambda|$ of the partition λ. One defines the *Fermionic space* as being

$$F := \bigoplus_{m\in\mathbb{Z}} F_m.$$

[1] But of course the author has no idea of what this means, yet.

The *Bosonic Fock space* , instead, will be, by definition, the ring $E_\infty := \mathbb{Q}[h_1, h_2, \ldots]$. It turns out that each F^m is a free E_∞-module of rank 1 generated by

$$\Phi_m(\mathbf{u}) := \Phi_m^{(0)}(\mathbf{u}) = u^{(m)} \wedge u^{(m-1)} \wedge \ldots, \tag{7.4}$$

which is the *vacuum vector with charge number* m, where now (0) denotes the null partition $(0, 0, \ldots)$. The algebraic content of the boson-fermion correspondence as described in [22] and stated in our ODE language is described by:

7.3.2. Theorem (The Boson-Fermion Correspondence). *The equality:*

$$\Phi_m^\lambda(\mathbf{u}) = h_\lambda \cdot \Phi_m^{(0)}(\mathbf{u}) \tag{7.5}$$

holds for each partition λ*, where, as in (4.1),* $h_\lambda = \Delta_\lambda(\mathbf{h})$.

7.3.3. Remark. Formula (7.5) may be called *Jacobi–Trudy formula* for the fermionic vectors $\Phi_m^{(\lambda)}$.

Proof. Suppose $\lambda := (\lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r)$ for some $r \geq 0$, so that $\Phi_m^{(\lambda)}$ is as in (7.3). One uses formula (7.2):

$$\begin{aligned} u^{(m-i+\lambda_i)} = U_0(\mathrm{s}_{\lambda_i}(\mathbf{h}))u^{(m-i)} + U_1(\mathrm{s}_{\lambda_i}(\mathbf{h}))u^{(m-i-1)} \\ + \ldots + U_r(\mathrm{s}_{\lambda_i}(\mathbf{h})))u^{(m-i-r)} + \ldots \end{aligned}$$

Therefore

$$\begin{aligned} \Phi_m^\lambda(\mathbf{u}) &= \sum_{j_0 \geq 0} U_{j_0}(\mathrm{s}_{\lambda_0}(\mathbf{h}))u^{(m-j_0)} \wedge \sum_{j_1 \geq 0} U_{j_1}(\mathrm{s}_{\lambda_1}(\mathbf{h}))u^{(m-j_1)} \wedge \ldots \\ &\qquad \ldots \wedge \sum_{j_r \geq 0} U_{j_r}(\mathrm{s}_{\lambda_r}(\mathbf{h}))u^{(m-j_r)} \wedge \Phi_{(m-r-1)} \\ &= \sum_{j_0=0}^{r} U_{j_0}(\mathrm{s}_{\lambda_0}(\mathbf{h}))u^{(m-j_0)} \wedge \sum_{j_1=0}^{r} U_{j_1}(\mathrm{s}_{\lambda_1}(\mathbf{h}))u^{(m-j_1)} \\ &\qquad \wedge \ldots \wedge \sum_{j_r=0}^{r} U_{j_r}(\mathrm{s}_{\lambda_r}(\mathbf{h}))u^{(m-j_r)} \wedge \Phi_{(m-r-1)} \end{aligned} \tag{7.6}$$

In the last equality we restricted the summation range, because wedging with $\Phi_{(m-r-1)}(\mathbf{u})$ kills the summands containing $u^{(m-r-i)}$ with

$i \geq 1$. Now simple linear algebra (multilinearity of the wedge product) shows that (7.6) can be written as:

$$\begin{vmatrix} U_0(\mathrm{s}_{\lambda_0}(\mathbf{h})) & U_1(\mathrm{s}_{\lambda_0}(\mathbf{h})) & \dots & U_r(\mathrm{s}_{\lambda_0}(\mathbf{h})) \\ U_0(\mathrm{s}_{\lambda_1}(\mathbf{h})) & U_1(\mathrm{s}_{\lambda_1}(\mathbf{h})) & \dots & U_r(\mathrm{s}_{\lambda_1}(\mathbf{h})) \\ \vdots & \vdots & \ddots & \vdots \\ U_0(\mathrm{s}_{\lambda_r}(\mathbf{h})) & U_1(\mathrm{s}_{\lambda_r}(\mathbf{h})) & \dots & U_r(\mathrm{s}_{\lambda_r}(\mathbf{h})) \end{vmatrix} \times$$

$$\times\; u^{(m)} \wedge u^{(m-1)} \wedge \ldots \wedge u^{(m-r)} \wedge \Phi_{m-r+1},$$

Using (6.11), eventually:

$$\Phi_m^\lambda(\mathbf{u}) = \Delta_\lambda(\mathbf{h}) \cdot \Phi_m(\mathbf{u}) = h_\lambda \cdot \Phi_m(\mathbf{u})$$

a Jacobi-Trudy-like formula.

7.3.4. The *perfect vacuum* (see [22]) is the vector

$$\Phi_0(\mathbf{u}) := u^{(0)} \wedge u^{(-1)} \wedge u^{(-2)} \wedge \ldots .$$

The *excitation* of the vacuum by means of a Schur polynomial h_λ gives a vector Φ_0^λ which has as many *holes* (unoccupied energy states= number of missing $u^{(-j)}$ with $j \geq 0$) as positive energy states (number of $u^{(-j+\lambda_j)}$ with $-j + \lambda_j > -j$). In this case, the boson–fermion correspondence (7.5) reads as:

$$\Phi_0^\lambda(\mathbf{u}) = h_\lambda \Phi_0(\mathbf{u}). \tag{7.7}$$

which says, as anticipated, that $F_0 := \bigwedge^{\frac{\infty}{2}} \mathcal{U}_\infty = \bigwedge^{\frac{\infty}{2}} E_\infty$ is a free E_∞-module of rank 1 generated by Φ_0 given as in (7.7).

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