Simulating the tail of the interference in a Poisson network model

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Abstract

Interference among simultaneous transmissions represents the main limitation factor for the capacity and connectivity of dense wireless networks. In this paper we provide efficient simulation laws for the tail of the interference in a simple wireless ad hoc network model. Particularly, we consider node locations distributed according to a Poisson point process and various classes of light-tailed fading distributions.

Keywords: Fading channels; Importance sampling; Large deviations; Poisson process; Poisson shot noise.

I. Introduction

Mutual interference among simultaneous transmissions constitutes the main limitation factor to the performance of dense wireless networks, severely reducing the capacity of the whole system (see [19], [21], [23], [30] and [31].)

The availability of efficient analytical/numerical techniques to tightly characterize the interference produced by transmitting nodes operating over the same channel is a key ingredient to better predict performance of such complex systems as well as to design new Medium Access Control (MAC) protocols and more advanced transmission schemes that better use the system bandwidth. Just as matter of example, we shall explain in Section II, how the tail of the interference is directly related to the probability that the communication does not succeed, in the case when a single input/single output transmission scheme is adopted.

In this paper we consider a simple wireless network setting in which nodes are placed according to a Poisson process on the plane and employ a simple ALOHA MAC protocol (see [2], [4], [5], [6], [7], [12], [13], [15] and [20]). We propose a provably efficient numerical methodology to estimate the tail

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of the interference, under natural assumptions on fading and attenuation. If the tail of the interference is not too small, one may exploit a crude Monte Carlo approach to evaluate the complementary of the cumulative distribution function of the interference. However, when the tail of the interference is small the crude Monte Carlo method becomes inefficient, and different numerical techniques are needed. The methodology used in this paper is based on (state-dependent) importance sampling (see e.g. [3] and [8].)

Despite the fact that a significant body of work has attempted a characterization of the interference in large wireless networks (see [2], [4], [5], [6], [7], [12], [13], [15], [16] and [20]), we are not aware of previous work proposing provably efficient numerical algorithms to estimate the tail of the interference, assuming that the fading has a light-tail distribution and the attenuation decays sub-exponentially with the distance. Actually, most of the existing literature on the subject focuses on analytical characterizations of either the interference distribution or the outage probability, under specific assumptions on fading and attenuation. For instance, if the attenuation is of the form $\|x\|^{-\alpha}$, $x \in \mathbb{R}^2$, $\alpha > 2$, where the symbol $\|\cdot\|$ denotes the Euclidean norm, and the fading is constant (i.e. there is a purely geometric attenuation) or distributed according to a Rayleigh law, closed form expressions for the Laplace transform of the interference are derived e.g. in [2], [5] and [20]. However, only in exceptional cases the Laplace transform may be inverted to obtain the law of the interference. This is possible, for instance, if $\alpha = 4$ and the system is subjected to a purely geometric attenuation [16]. When the analytical inversion of the Laplace transform is not feasible, estimates of the tail of the interference may be obtained by inverting numerically its Laplace transform. However, numerical inversion techniques typically provide results with large accuracy only at a large computational cost, see e.g. [1], [9], [29]. Furthermore, to the best of our knowledge, to estimate the approximation error is usually a hard task. For these reasons, alternative efficient numerical techniques are highly desirable.

Under more general assumptions on fading and attenuation, explicit bounds on the tail of the interference may be found in [16]. In [15] a large deviations approach is employed to study the asymptotic behavior of the logarithm of the tail of the interference, for a quite general fading (possibly heavy-tail) and ideal Hertzian propagation, i.e. of the form $\max(R, \|x\|)^{-\alpha}$, $R > 0$, $\alpha > 2$. The results in [15] constitute the starting point to build provably efficient numerical algorithms to estimate the tail of the interference.

Under general assumptions on the node distribution, the fading distribution and the attenuation function, asymptotic estimates for the outage probability, as the intensity of the nodes goes to zero, are derived in [17] and [18]. Finally, a Monte Carlo algorithm to estimate the density of the interference for a quite general wireless network model has been proposed in [25].

The methodology proposed in this paper complements the previously mentioned results, providing an efficient and accurate Monte Carlo algorithm to compute the tail of the interference in cases where the analytical approach is not feasible. We believe that the proposed methodology may yield hints for a successive development of Monte Carlo procedures that allow fast and accurate evaluations of the tail of the interference when the transmitting nodes are distributed according to more general point processes models.

II. The system model and organization of the paper

We consider the following simple model of wireless network, which accounts for interference effects that arise when several nodes transmit at the same time.

Suppose that transmitting nodes (antennas) are located according to a Poisson process $\{X_k\}_{k\geq 1}$ on the plane with a locally integrable intensity function $\lambda(x)$, $x\in\mathbb{R}^2$, i.e. X_n is the location of node n. Denote by $P_n\in(0,\infty)$ the transmission power of node n. Assume that a new receiver is added at the origin and that a new transmitter is added at $x\in\mathbb{R}^2$. Let w be a positive constant which describes the thermal noise average power at the receiver, and suppose that the physical propagation of the signal is described by a measurable positive function $L:\mathbb{R}^2\to(0,\infty)$, which gives the attenuation or path-loss of the signal power. In addition, the signal undergoes random fading (due to occluding objects, reflections, multi-path interference, etc.). We denote by H_n the random power fading gain between node n and the receiver, and define $Y_n := P_nH_n$. Thus $Y_nL(X_n)$ is the received power at the origin due to node n. Similarly, we denote by $YL(x)$, the received power at the origin due to the transmitter at x. We assume that $\{Y,\{Y_k\}_{k\geq 1}\}$ is a sequence of independent and identically distributed (i.i.d.) random variables (r.v.'s), independent of locations, and we suppose that the marked Poisson process $\{(X_k,Y_k)\}_{k\geq 1}$ is defined on a probability space $(\Omega,\mathcal{F},P)$. In the following (with an abuse of terminology) we shall call the r.v.'s Y_k signals.

This paper provides a computationally efficient (state-dependent) importance sampling algorithm for the characterization of the total interference power at the origin, which is given by the Poisson shot noise r.v. $V := \sum_{k\geq 1} Y_k L(X_k)$. We emphasize that a tight characterization of the tail of the interference $\psi(\beta) := P(V > \beta)$ is needed to predict the performance of large scale wireless networks. In particular, the tail of the interference is related to the probability of successfully decoding the signal from the transmitter at x. Indeed, given the adopted modulation and encoding scheme, we assume that the receiver at the origin can successfully decode the signal from the transmitter at x if the Signal to Interference plus Noise Ratio (SINR) is greater than a given threshold, say $\tau > 0$ (which depends on the adopted scheme), i.e.

$$\frac{YL(x)}{w+V} \geq \tau.$$

So, conditional to the event $\{Y = y\}$, the probability that the communication succeeds is given by

$$P\left(\frac{YL(x)}{w+V} \geq \tau \,\Big|\, Y = y\right) = P\left(\frac{yL(x)}{w+V} \geq \tau\right) = P(V \leq yL(x)\tau^{-1} - w). \tag{1}$$

The attenuation function is often taken to be of the form $L(x) = \ell(\|x\|) = \|x\|^{-\alpha}$ or $(1+\|x\|)^{-\alpha}$ or $\max(R, \|x\|)^{-\alpha}$, where $\alpha > 2$ and $R > 0$ are positive constants. Setting $\tau = \theta\tau'$ in (1), where $\theta > 0$ and $\tau' > 0$ are two positive constants, we have

$$P\left(\frac{YL(x)}{\tau'(w+V)} \geq \theta \,\Big|\, Y = y\right) = P\left(\frac{yL(x)}{\tau'(w+V)} \geq \theta\right) = P\left(V \leq \frac{yL(x)}{\theta}\tau'^{-1} - w\right). \tag{2}$$

The high-reliability regime corresponds to the high-SINR regime, i.e. the regime where $\tau' \to 0$ (see [17] and [18] for the analysis of the high-SINR regime as the intensity of the nodes goes to zero.) Thus, for large values of β, the probability $\psi(\beta)$ is also related to the outage probability in the high-SINR regime.

Note that whenever $V < \infty$ almost surely, a.s. for short, (a sufficient condition for this is e.g. $\mathrm{E}[V] < \infty$, i.e. $\mathrm{E}[Y_1] < \infty$ and $\int_{\mathbb{R}^2} L(x)\lambda(x)\,\mathrm{d}x < \infty$) $\psi(\beta) \to 0$, as $\beta \to +\infty$, so the event $\{V > \beta\}$ is rare as β increases, and this rises questions about the numerical estimation of the small probabilities $\psi(\beta)$ via a Monte Carlo algorithm. The importance sampling technique proposed in this paper can be successfully used to obtain accurate estimates of $\psi(\beta)$ for values of β that correspond to small $\psi(\beta)$ (note that such values of β may be moderately large, see Section VI.) This permits to unveil how different system's parameters, such as the intensity of the nodes, the path-loss exponent and the fading distribution, impact on the system performance. For these reasons, we believe that our approach is complementary with respect to the previously proposed analytical approaches that capture either the asymptotic behavior, as $\beta \to \infty$, of the tail of the interference ([15]) or the asymptotic behavior, as the intensity of the nodes goes to zero, of the outage probability ([17], [18].)

The paper is organized as follows. In Section III we describe the importance sampling methodology in our context. In Sections IV and V we analyze networks with nodes distributed according to a stationary Poisson process on $\mathbb{R}^2$ with intensity $\lambda > 0$ and attenuation function of the form $L(x) = \ell(\|x\|) = \max(R, \|x\|)^{-\alpha}$, $\alpha > 2$, $R > 0$. More particularly, in Section IV we provide asymptotically admissible simulation laws for $\psi(\beta)$, as $\beta \to +\infty$, under a quite general light tail assumption on the distribution of the signals. In Section V we give asymptotically efficient simulation laws for $\psi(\beta)$, as $\beta \to +\infty$, when the signals are bounded, Weibull super-exponential or Exponential. In Section VI we provide numerical illustrations. Finally Section VII concludes the paper.

III. Importance sampling

Suppose $V < \infty$ a.s. and let $b(O, r)$ be the ball centered at the origin with radius $r > 0$. Define the r.v.'s

$$V_r := \sum_{k \geq 1} Y_k L(X_k) \mathbf{1}_{b(O,r)}(X_k)$$

and, for $\beta > 0$,

$$r_\beta := \inf\{n \in \mathbb{N} : \; V_n > \beta\} \quad \text{if } \{n \in \mathbb{N} : \; V_n > \beta\} \neq \emptyset, \quad r_\beta = +\infty \quad \text{otherwise.}$$

Let $M \geq 1$ be an integer and consider the crude Monte Carlo estimator

$$\widehat{\psi}_{CMC}(\beta, M) := \frac{1}{N} \sum_{i=1}^{N} (\mathbf{1}\{r_\beta \leq M\})^{(i)},$$

where $(\mathbf{1}\{r_\beta \leq M\})^{(1)}, \ldots, (\mathbf{1}\{r_\beta \leq M\})^{(N)}$ are N i.i.d. replica of the r.v. $\mathbf{1}\{r_\beta \leq M\}$. For any $\beta > 0$, the crude Monte Carlo estimator is an unbiased estimator of $\psi_M(\beta) := P(V_M > \beta)$ and an asymptotically unbiased estimator of $\psi(\beta)$. Indeed,

$$\mathrm{E}[\widehat{\psi}_{CMC}(\beta, M)] = P(r_\beta \leq M) = \psi_M(\beta), \quad \forall \; \beta > 0$$

and

$$\lim_{M\to+\infty} \mathrm{E}[\widehat{\psi}_{CMC}(\beta,M)] = \lim_{M\to+\infty} \psi_M(\beta) = \psi(\beta), \quad \forall\ \beta > 0.$$

In particular, for $\beta > 0$ such that $\psi_M(\beta)$ is not too small we may resort to a classical crude Monte Carlo method to evaluate the probability $\psi_M(\beta)$ and consequently $\psi(\beta)$. However, if $\beta > 0$ is such that $\psi_M(\beta)$ is very small the classical crude Monte Carlo method becomes inefficient to estimate $\psi_M(\beta)$ and consequently $\psi(\beta)$, as the following argument shows. Suppose that we wish to have at most a 5% error on $\psi_M(\beta)$ with 95% confidence. This means that we must have

$$P(|\psi_M(\beta) - \widehat{\psi}_{CMC}(\beta,M)| \le 0.05\psi_M(\beta)) = 0.95.$$

Note that, by the expression of the variance for a r.v. with a Bernoulli distribution,

$$\mathrm{Var}(\widehat{\psi}_{CMC}(\beta,M)) = \frac{\psi_M(\beta)(1-\psi_M(\beta))}{N}.$$

Since $\psi_M(\beta)$ is very small the following approximation is allowed:

$$\mathrm{Var}(\widehat{\psi}_{CMC}(\beta,M)) \simeq \psi_M(\beta)/N,$$

and by the Central Limit Theorem, for N large, we deduce

$$\begin{aligned}
&P(|\psi_M(\beta) - \widehat{\psi}_{CMC}(\beta,M)| \le 0.05\psi_M(\beta)) \\
&= P\left(\Big|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(\frac{(\mathbf{1}\{r_\beta \le M\})^{(i)} - \psi_M(\beta)}{\sqrt{\psi_M(\beta)}}\right)\Big| \le 0.05\sqrt{N\psi_M(\beta)}\right) \\
&\simeq P(|Z| \le 0.05\sqrt{N\psi_M(\beta)}),
\end{aligned}$$

where Z is a standard Gaussian r.v.. Now, using the tables, we have that the equality $P(|Z| \le z) = 0.95$ implies $z \simeq 2$. So, to have at most a 5% error on $\psi_M(\beta)$ with 95% confidence, we must have

$$N \simeq 1600/\psi_M(\beta). \tag{3}$$

Since $\psi_M(\beta)$ is very small, this means that we need a huge number of replica to reach a desired precision of $\widehat{\psi}_{CMC}(\beta,M)$.

Now, we start describing an alternative Monte Carlo estimator which allows to overcome these problems (see also the discussion at the beginning of Sections IV and V.) The idea is to use a suitable change of law. Note that by the well-known formula for Laplace functionals of (independently marked) Poisson processes (see e.g. [10]), for any measurable function $f : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ for which the integral in the right-hand side of (4) is well-defined, we have

$$\mathrm{E}[\mathrm{e}^{\sum_{k\geq 1} f(X_k,Y_k)}] = \exp\left(\int_{\mathbb{R}^2} (\mathrm{E}[\mathrm{e}^{f(x,Y_1)}]-1)\lambda(x)\,\mathrm{d}x\right). \tag{4}$$

In particular,

$$\mathrm{E}[\mathrm{e}^{tV_r}] = \exp\left(\int_{b(O,r)} (\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]-1)\lambda(x)\,\mathrm{d}x\right) \quad \text{for any } r,t>0. \tag{5}$$

Denote by $\mathcal{F}_r$ the σ-field generated by the points of the Poisson process on $b(O,r)$ and the corresponding marks, and by $\mathcal{F}_\infty$ the smallest σ-field containing $\bigcup_{r>0}\mathcal{F}_r$. Let $t>0$ be such that

$$\int_{b(O,r)} (\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]-1)\lambda(x)\,\mathrm{d}x < \infty, \quad \text{for any } r>0. \tag{6}$$

We shall check later on (see Lemma 3.1 below) that the stochastic process $\{\mathrm{e}^{tV_r}/\mathrm{E}[\mathrm{e}^{tV_r}]\}_{r\geq 0}$ is an $\mathcal{F}_r$-martingale (we refer the reader to e.g. [26] for the definition). Then, by e.g. Corollary 10.2.1 and Lemma 10.2.2 in [26], letting $P_t^{(r)}$ denote the probability measure on $(\Omega,\mathcal{F}_r)$ defined by

$$P_t^{(r)}(A) := \mathrm{E}\left[\frac{\mathrm{e}^{tV_r}}{\mathrm{E}[\mathrm{e}^{tV_r}]}\mathbf{1}_A\right], \tag{7}$$

we have that there exists a unique probability measure P_t on $(\Omega,\mathcal{F}_\infty)$ such that $P_t(A) = P_t^{(r)}(A)$, for all $A \in \mathcal{F}_r$. Moreover, if τ is an $\mathcal{F}_r$-stopping time and $A \subseteq \{\tau < \infty\}$ is such that $A \in \mathcal{F}_\tau$, being $\mathcal{F}_\tau$ the stopping σ-field (see e.g. [26] for the formal definition), then

$$P_t(A) = \mathrm{E}\left[\frac{\mathrm{e}^{tV_\tau}}{[\mathrm{E}[\mathrm{e}^{tV_r}]]_{r=\tau}}\mathbf{1}_A\right], \tag{8}$$

where the symbol $[\mathrm{E}[\mathrm{e}^{tV_r}]]_{r=\tau}$ denotes the quantity $\mathrm{E}[\mathrm{e}^{tV_r}]$ computed at $r=\tau$.

Lemma 3.1: Let $t>0$ be such that (6) holds, then $\{\mathrm{e}^{tV_r}/\mathrm{E}[\mathrm{e}^{tV_r}]\}_{r\geq 0}$ is an $\mathcal{F}_r$-martingale.

Proof By the properties of the Poisson process and the definition of the σ-field $\mathcal{F}_r$, for any $r' > r > 0$, we have that the r.v. e^{tV_r} is $\mathcal{F}_r$-measurable, the r.v. $\mathrm{e}^{t(V_{r'}-V_r)}$ is independent of the σ-field $\mathcal{F}_r$, and the r.v.'s e^{tV_r} and $\mathrm{e}^{t(V_{r'}-V_r)}$ are independent. Therefore

$$\mathrm{E}\left[\frac{\mathrm{e}^{tV_{r'}}}{\mathrm{E}[\mathrm{e}^{tV_{r'}}]}\,\Big|\,\mathcal{F}_r\right] = \mathrm{E}\left[\frac{\mathrm{e}^{tV_r}}{\mathrm{E}[\mathrm{e}^{tV_r}]}\frac{\mathrm{e}^{t(V_{r'}-V_r)}}{\mathrm{E}[\mathrm{e}^{t(V_{r'}-V_r)}]}\,\Big|\,\mathcal{F}_r\right]$$

$$= \frac{\mathrm{e}^{tV_r}}{\mathrm{E}[\mathrm{e}^{tV_r}]}\mathrm{E}\left[\frac{\mathrm{e}^{t(V_{r'}-V_r)}}{\mathrm{E}[\mathrm{e}^{t(V_{r'}-V_r)}]}\,\Big|\,\mathcal{F}_r\right] = \frac{\mathrm{e}^{tV_r}}{\mathrm{E}[\mathrm{e}^{tV_r}]},$$

and the claim follows.

□

Next theorem provides the probabilistic structure of the marked point process $\{(X_k, Y_k)\}_{k\geq 1}$, under P_t. In the following, we denote by $P_t^{(Y_k\,|\,X_k=x)}$ the conditional law of Y_k, given $\{X_k = x\}$, under P_t, and by $P^{(Y_1)}$ the common law of the Y's under P.

Theorem 3.2: Let $t > 0$ be such that (6) holds. Then, under P_t, the marked point process $\{(X_k, Y_k)\}_{k\geq 1}$ is distributed as follows: $\{X_k\}_{k\geq 1}$ is a Poisson process on $\mathbb{R}^2$ with intensity function $\Lambda_t(x) := \lambda(x)\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]$; given the ground process $\{X_k\}_{k\geq 1}$, the marks $\{Y_k\}_{k\geq 1}$ are mutually independent, with conditional distribution (or mark kernel)

$$\mathrm{d}P_t^{(Y_k\,|\,X_k=x)}(y) = \mathrm{M}_t(\mathrm{d}y\,|\,x) := \frac{\mathrm{e}^{tL(x)y}}{\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]}\,\mathrm{d}P^{(Y_1)}(y).$$

Proof Recall that the law of a point process $\{(X'_k, Y'_k)\}_{k\geq 1}$ on $\mathbb{R}^2 \times [0,\infty)$ is characterized by the Laplace functionals of the form $\mathrm{E}\left[\exp\left(-\sum_{k\geq 1} f(X'_k, Y'_k)\right)\right]$, where $f : \mathbb{R}^2 \times [0,\infty) \to [0,\infty)$ is a non-negative measurable function such that $f(x', y') = 0$ for all $(x', y') \in (\mathbb{R}^2 \setminus K) \times [0,\infty)$ and some compact $K \subset \mathbb{R}^2$ (see e.g. [10]). Take f as above and let $r > 0$ be such that $K \subset b(O, r)$. By the exponential change of measure (7) and the expression of the Laplace functional of a Poisson process (4), we have

$$\begin{aligned}
\mathrm{E}_{P_t}\left[\mathrm{e}^{-\sum_{k\geq 1} f(X_k,Y_k)}\right] &= \frac{\mathrm{E}[\mathrm{e}^{tV_r-\sum_{k\geq 1} f(X_k,Y_k)}]}{\mathrm{E}[\mathrm{e}^{tV_r}]}\\
&= \exp\left(-\int_{b(O,r)}(\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]-1)\lambda(x)\,\mathrm{d}x\right)\\
&\qquad \times \exp\left(\int_{b(O,r)}(\mathrm{E}[\mathrm{e}^{tL(x)Y_1-f(x,Y_1)}]-1)\lambda(x)\,\mathrm{d}x\right)\\
&= \exp\left\{\int_{b(O,r)}\left(\frac{\mathrm{E}[\mathrm{e}^{tL(x)Y_1-f(x,Y_1)}]}{\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]}-1\right)\lambda(x)\mathrm{E}[\mathrm{e}^{tL(x)Y_1}]\,\mathrm{d}x\right\}\\
&= \exp\left\{\int_{b(O,r)\times[0,\infty)}(\mathrm{e}^{-f(x,y)}-1)\mathrm{M}_t(\mathrm{d}y\,|\,x)\Lambda_t(x)\,\mathrm{d}x\right\}\\
&= \exp\left\{\int_{\mathbb{R}^2\times[0,\infty)}(\mathrm{e}^{-f(x,y)}-1)\mathrm{M}_t(\mathrm{d}y\,|\,x)\Lambda_t(x)\,\mathrm{d}x\right\}
\end{aligned}$$

which is exactly the Laplace functional of a point process $\{(X_k,Y_k)\}_{k\geq 1}$ such that $\{X_k\}_{k\geq 1}$ is a Poisson process on $\mathbb{R}^2$ with intensity function $\Lambda_t(x)$ and the marks $\{Y_k\}_{k\geq 1}$ are distributed as in the statement (see e.g. Proposition 6.4.IV in [10]).

□

For $t>0$ such that (6) holds and $M\in\mathbb{N}$, define the r.v.

$$L_{t,M}^{(\beta)} := \mathbf{1}\{r_\beta \leq M\}\mathrm{e}^{-tV_{r_\beta}}[\mathrm{E}[\mathrm{e}^{tV_n}]]_{n=r_\beta},$$

where the Laplace transform $\mathrm{E}[\mathrm{e}^{tV_n}]$ is given by (5) and the symbol $[\mathrm{E}[\mathrm{e}^{tV_n}]]_{n=r_\beta}$ denotes the quantity $\mathrm{E}[\mathrm{e}^{tV_n}]$ computed at $n=r_\beta$. Clearly r_β is an $\{\mathcal{F}_n\}_{n\geq 1}$-stopping time. So by (8) we have

$$\psi_M(\beta)=\mathrm{E}[\mathbf{1}\{r_\beta\leq M\}]=\mathrm{E}_{P_t}[L_{t,M}^{(\beta)}]. \tag{9}$$

We define the importance sampling estimator by

$$\widehat{\psi}_{IS}(\beta,t,M) := \frac{1}{N}\sum_{i=1}^{N}(L_{t,M}^{(\beta)})^{(i)}, \tag{10}$$

where $(L_{t,M}^{(\beta)})^{(1)}, \ldots, (L_{t,M}^{(\beta)})^{(N)}$ are N i.i.d. replica of the r.v. $L_{t,M}^{(\beta)}$, under the importance sampling law P_t. Note that by (9) it follows that, under P_t, for any $\beta > 0$, the importance sampling estimator is an unbiased estimator of $\psi_M(\beta)$ and an asymptotically unbiased estimator for $\psi(\beta)$, indeed

$$\lim_{M \to +\infty} \mathrm{E}_{P_t}[\widehat{\psi}_{IS}(\beta, t, M)] = \lim_{M \to \infty} \psi_M(\beta) = \psi(\beta), \quad \forall\ \beta > 0.$$

IV. Stationary Poisson networks with ideal Hertzian propagation: asymptotically admissible laws

To simulate the importance sampling estimator $\widehat{\psi}_{IS}(\beta, t, M)$ under P_t, where $t > 0$ is such that (6) holds, we need to generate the r.v. $\mathbf{1}\{r_\beta \leq M\}$ under the importance sampling law P_t. However, under P_t, as β increases, the probability of the event $\{r_\beta \leq M\}$ may be very small. In such a case the estimate provided by the importance sampling estimator is clearly very poor. So we need to introduce importance sampling laws under which the probability of the event $\{r_\beta \leq M\}$ is high, as $\beta \to +\infty$.

In this section we address this problem in the case when the nodes are distributed according to a stationary Poisson process with intensity $\lambda > 0$ and the attenuation function is given by $L(x) = \ell(\|x\|) = \max(R, \|x\|)^{-\alpha}$, $\alpha > 2$.

The stationarity assumption on the Poisson process is done only for convenience and the generalization of our result to the non-stationary case is possible with minor modifications. We shall assume the following light tail condition on the signals:

$$\mathrm{E}[\mathrm{e}^{tY_1}] < \infty \text{ for all } t \text{ in a right neighborhood of zero with supremum } \mathrm{S} \in (0, +\infty]. \tag{11}$$

For later purposes, we note that if this light-tail condition holds then

$$\text{For any function } t(\beta) \text{ with domain } (\bar{\beta}, +\infty),\ \bar{\beta} > 0, \text{ and codomain contained in } (0, R^\alpha \mathrm{S}) \tag{12}$$

$$\text{we have } \mathrm{E}[\mathrm{e}^{t(\beta) R^{-\alpha} Y_1}] < \infty \text{ for all } \beta > \bar{\beta}.$$

In particular, assuming the light tail condition (11) and choosing $t(\beta)$ as in (12) we have that condition (6) holds with $t = t(\beta)$, $L(x) = \max(R, \|x\|)^{-\alpha}$ and $\lambda(x) \equiv \lambda$. Consequently, there exist the probability measures $P_{t(\beta)}$, $\beta > \bar{\beta}$ (see the related discussion in the previous section). We say that the laws $P_{t(\beta)}$ are asymptotically admissible if

$$\lim_{\beta \to +\infty} P_{t(\beta)}(r_\beta \leq M) = 1, \quad \forall\ M \geq R.$$

Throughout this paper, using a standard notation, we denote by $o(x)$ a positive function such that $o(x)/x \to 0$, as $x \to +\infty$.

Theorem 4.1: Assume (11) and consider a function $t(\beta)$ as in (12). If moreover there exist a positive function φ and $K \in (1, +\infty]$ such that

$$\varphi(\beta) = o(\beta) \quad \text{as } \beta \to +\infty \text{ and } \quad \lim_{\beta\to+\infty} \frac{\lambda\pi\varphi(\beta)\mathrm{E}[\mathbf{1}\{Y_1 > \varphi(\beta)\}\mathrm{e}^{t(\beta)R^{-\alpha}Y_1}]}{\beta R^{\alpha-2}} = K, \tag{13}$$

then the laws $P_{t(\beta)}$ are asymptotically admissible.

Proof By (11) and (13) easily follows that $P(Y_1 > \varphi(\beta)) > 0$, for any β large enough. Recall that the thinning with retention probability $p(x)$, $x \in \mathbb{R}^2$, of a Poisson process on the plane with intensity function $f(x)$ is a Poisson process on the plane with intensity function $p(x)f(x)$ (see e.g. [10], see also [5].) By Theorem 3.2, for any $x \in \mathbb{R}^2$ and β large, we have

$$p(x) := P_{t(\beta)}(Y_k > \varphi(\beta) \,|\, X_k = x)$$

$$= \int_{[0,\infty)} \mathbf{1}\{y > \varphi(\beta)\} \frac{\mathrm{e}^{t(\beta)\ell(\|x\|)y}}{\mathrm{E}[\mathrm{e}^{t(\beta)\ell(\|x\|)Y_1}]} \,\mathrm{d}P^{(Y_1)}(y) > 0.$$

Using again Theorem 3.2, for β large, under $P_{t(\beta)}$, $\{X_k\}_{k\geq1}$ is a non-homogeneous Poisson process on $\mathbb{R}^2$ with intensity function $\Lambda_{t(\beta)}(x) = \lambda\mathrm{E}[\mathrm{e}^{t(\beta)\ell(\|x\|)Y_1}]$. So, for any $M \geq R$ and β large,

$$P_{t(\beta)}(r_\beta \leq M) = P_{t(\beta)}(V_M > \beta) \geq P_{t(\beta)}\left(\sum_{k\geq1} Y_k \mathbf{1}_{b(O,R)}(X_k)\mathbf{1}\{Y_k > \varphi(\beta)\} > \beta R^\alpha\right) \tag{14}$$

$$\geq P_{t(\beta)}\left(\sum_{k\geq1} \mathbf{1}_{b(O,R)}(X_k)\mathbf{1}\{Y_k > \varphi(\beta)\} > \frac{\beta R^\alpha}{\varphi(\beta)}\right)$$

$$\geq P_{t(\beta)}\left(\mathrm{N} > \left[\frac{\beta R^\alpha}{\varphi(\beta)}\right] + 1\right) \tag{15}$$

where, under $P_{t(\beta)}$, the r.v. N has a Poisson distribution with parameter

$$\lambda_{P_{t(\beta)}} := \int_{b(O,R)} p(x)\Lambda_{t(\beta)}(x)\,\mathrm{d}x = \lambda\pi R^2 \int_{[0,\infty)} \mathbf{1}\{y > \varphi(\beta)\}\mathrm{e}^{t(\beta)R^{-\alpha}y}\,\mathrm{d}P^{(Y_1)}(y)$$

and the symbol $[x]$ denotes the integer part of x. Here the inequality in (14) follows by the definition of V_M and the fact that $M \geq R$; the inequality (15) is consequence of the thinning property of the Poisson process, which guarantees that, under $P_{t(\beta)}$, the r.v. $\sum_{k\geq 1} \mathbf{1}_{b(O,R)}(X_k)\mathbf{1}\{Y_k > \varphi(\beta)\}$ has the same law of N. By the usual bounds on the Poisson distribution we have (see e.g. Lemma 1.2 in [24]), for any β such that

$$\left[\frac{\beta R^\alpha}{\varphi(\beta)}\right] + 1 \leq \lambda_{P_{t(\beta)}}$$

(note that this inequality is satisfied, for all β large enough, due to assumption (13)),

$$P_{t(\beta)}\left(\mathrm{N} > \left[\frac{\beta R^\alpha}{\varphi(\beta)}\right] + 1\right) \geq 1 - \exp\left(-\lambda_{P_{t(\beta)}} H\left(\frac{\left[\frac{\beta R^\alpha}{\varphi(\beta)}\right] + 1}{\lambda_{P_{t(\beta)}}}\right)\right), \tag{16}$$

where the function $H : [0,\infty) \to [0,\infty)$ is defined by $H(0) := 1$, $H(x) := 1 - x + x\log x$, $x > 0$. The claim follows combining the inequalities (15), (16) and letting β tend to $+\infty$ (note that due to the assumption (13) we have that $\lambda_{P_{t(\beta)}} \to +\infty$, as $\beta \to +\infty$, and that $([\beta R^\alpha/\varphi(\beta)] + 1)/\lambda_{P_{t(\beta)}}$ converges to a positive constant, as $\beta \to +\infty$).

□

We conclude this section with some examples of asymptotically admissible laws. In Section V we shall see that the laws described in the following examples are indeed asymptotically efficient.

Constant signals

Suppose that the signals Y_k have a bounded support with supremum $b > 0$. Then condition (11) clearly holds with $\mathrm{S} = +\infty$. Assumption (13) is satisfied if, in particular, the positive function $t(\beta)$, with domain on some interval $(\bar{\beta}, +\infty)$, $\bar{\beta} > 0$, is chosen in such a way that

$$\lim_{\beta\to+\infty} \frac{\lambda\pi\overline{b}\mathrm{E}[\mathbf{1}\{Y_1 > \overline{b}\}\mathrm{e}^{t(\beta)R^{-\alpha}Y_1}]}{\beta R^{\alpha-2}} = K, \quad \text{for some } \overline{b} \in (0,b) \text{ and } K \in (1,\infty]. \tag{17}$$

For instance, for constant signals all equal to $b > 0$, if we set $t_1(\beta) := (R^\alpha/b)\log\beta$, $\beta > 1$, then the laws $P_{t_1(\beta)}$ are asymptotically admissible if the parameters satisfy the condition $\lambda\pi R^{2-\alpha}b > 1$. Now, for a fixed $c \in (0,1)$, define $t_2(\beta) := (R^\alpha/b)\log(\beta(\log\beta)^c)$, $\beta > 3$. In such a case a straightforward computation shows that the laws $P_{t_2(\beta)}$ are asymptotically admissible for any choice of the parameters (the limit (17) holds with $K=+\infty$.)

Weibull superexponential signals

Suppose that the signals Y_k, $k\geq 1$, are Weibull super-exponential distributed with parameters $\gamma_1>0$ and $\gamma_2>1$, i.e.

$$P(Y_1>y)=\mathrm{e}^{-\gamma_1 y^{\gamma_2}},\quad y>0$$

(this case is particularly appealing in the context of wireless networks, see [27] for motivations.) It is well-known that the Laplace transform of a Weibull super-exponential law is always finite, and so condition (11) holds with $\mathrm{S}=+\infty$. We are going to propose a couple of choices for a positive function $t(\beta)$, with domain on some interval $(\bar{\beta},+\infty)$, $\bar{\beta}>0$, in such a way that condition (13) is satisfied. Note that $\eta:=1-(1/\gamma_2)\in(0,1)$. Define $t_1(\beta):=\gamma_2(\gamma_2-1)^{-\eta}\gamma_1^{1/\gamma_2}R^{\alpha}\log^{\eta}\beta$, $\beta>1$. For any positive function φ, we have

$$\frac{\lambda\pi\varphi(\beta)\mathrm{E}[\mathbf{1}\{Y_1>\varphi(\beta)\}\mathrm{e}^{t_1(\beta)R^{-\alpha}Y_1}]}{\beta R^{\alpha-2}}\geq\frac{\lambda\pi R^{2-\alpha}\varphi(\beta)\mathrm{e}^{t_1(\beta)\varphi(\beta)R^{-\alpha}}P(Y_1>\varphi(\beta))}{\beta}$$

$$=\frac{\lambda\pi R^{2-\alpha}\varphi(\beta)\mathrm{e}^{\varphi(\beta)^{\gamma_2}[t_1(\beta)\varphi(\beta)^{1-\gamma_2}R^{-\alpha}-\gamma_1]}}{\beta}.$$

Taking $\varphi(\beta):=\log^{1/\gamma_2}\beta$, $\beta>1$, we deduce

$$\frac{\lambda\pi R^{2-\alpha}\varphi(\beta)\mathrm{e}^{\varphi(\beta)^{\gamma_2}[t_1(\beta)\varphi(\beta)^{1-\gamma_2}R^{-\alpha}-\gamma_1]}}{\beta}=\lambda\pi R^{2-\alpha}\beta^{\gamma_2(\gamma_2-1)^{-\eta}\gamma_1^{1/\gamma_2}-(\gamma_1+1)}\log^{1/\gamma_2}\beta.$$

This latter term tends to infinity if

$$\gamma_2(\gamma_2-1)^{-\eta}\gamma_1^{1/\gamma_2}\geq\gamma_1+1. \tag{18}$$

Therefore, under the above condition on the parameters, assumption (13) holds, and by Theorem 4.1 we have that the laws $P_{t_1(\beta)}$ are asymptotically admissible. Now, define $t_2(\beta):=c\log^{\eta}\beta$, $\beta>1$, where

$$c\geq\max\{R^{\alpha}(\gamma_1+1),\gamma_2(\gamma_2-1)^{-\eta}\gamma_1^{1/\gamma_2}R^{\alpha}\}. \tag{19}$$

Taking again $\varphi(\beta) := \log^{1/\gamma_2} \beta$, $\beta > 1$, and arguing as in the first part of this example it can be checked that the laws $P_{t_2(\beta)}$ are asymptotically admissible for any choice of the parameters.

Exponential signals

Suppose that the signals are exponentially distributed with mean γ_3^{-1}. In this case we clearly have that condition (11) is satisfied with $\mathrm{S} = \gamma_3$. We are going to propose a couple of choices for a function $t(\beta)$, with domain on some $(\bar{\beta}, +\infty)$, $\bar{\beta} > 0$, and codomain in $(0, \gamma_3 R^\alpha)$, in such a way that the assumption (13) is satisfied. Let $\widetilde{\varphi}$ be a positive function such that

$$\widetilde{\varphi}(\beta) \uparrow +\infty \quad \text{and} \quad \lim_{\beta \to +\infty} \frac{\lambda \pi R^2 \gamma_3 \varphi(\beta) \widetilde{\varphi}(\beta) \mathrm{e}^{-R^{-\alpha} \varphi(\beta)/\widetilde{\varphi}(\beta)}}{\beta} \in (1, +\infty] \tag{20}$$

for some positive function φ such that $\varphi(\beta) = o(\beta)$, as $\beta \to +\infty$. Define $t(\beta) := \gamma_3 R^\alpha - \widetilde{\varphi}(\beta)^{-1}$, $\beta > \widetilde{\varphi}^{-1}(\gamma_3^{-1} R^{-\alpha})$. Since

$$\begin{aligned}
\mathrm{E}[\mathbf{1}\{Y_1 > \varphi(\beta)\} \mathrm{e}^{t(\beta) R^{-\alpha} Y_1}] &= \frac{\gamma_3}{\gamma_3 - t(\beta) R^{-\alpha}} \mathrm{e}^{-(\gamma_3 - t(\beta) R^{-\alpha}) \varphi(\beta)} \\
&= \gamma_3 R^\alpha \widetilde{\varphi}(\beta) \mathrm{e}^{-R^{-\alpha} \varphi(\beta)/\widetilde{\varphi}(\beta)},
\end{aligned}$$

assumption (13) holds and by Theorem 4.1 we have that the laws $P_{t(\beta)}$ are asymptotically admissible. In particular, if $\widetilde{\varphi}(\beta) := \sqrt{\beta}$, $\beta > 0$, and the parameters satisfy

$$\lambda \pi R^2 \gamma_3 \mathrm{e}^{-R^{-\alpha}} > 1, \tag{21}$$

then the laws $P_{t(\beta)}$ are asymptotically admissible. Indeed, in such a case (20) holds with $\varphi(\beta) := \sqrt{\beta}$, $\beta > 0$. If we take $\widetilde{\varphi}(\beta) := \beta^{2/3}$, $\beta > 0$, then the laws $P_{t(\beta)}$ are asymptotically admissible for any choice of the parameters. Indeed, in such a case (20) holds with $\varphi(\beta) := \beta^{2/3}$, $\beta > 0$.

V. STATIONARY POISSON NETWORKS WITH IDEAL HERTZIAN PROPAGATION: ASYMPTOTICALLY EFFICIENT LAWS

Assume (11) and let the function $t(\beta)$ be as in (12). The laws $P_{t(\beta)}$ are called asymptotically efficient if they are asymptotically admissible and

$$\liminf_{\beta\to+\infty} \frac{\log\sqrt{\mathrm{E}_{P_{t(\beta)}}[(L^{(\beta)}_{t(\beta),M})^2]}}{\log\psi_M(\beta)} \geq 1, \quad \forall\ M\geq R. \tag{22}$$

Note that this inequality implies, $\forall\ M\geq R$,

$$\lim_{\beta\to+\infty} \frac{\mathrm{Var}_{P_{t(\beta)}}(L^{(\beta)}_{t(\beta),M})}{\psi_M(\beta)^{2-\varepsilon}} = 0, \qquad \forall\ \varepsilon>0 \tag{23}$$

(see e.g. [3].) This guarantees a gain in terms of asymptotic efficiency, as the following argument show. Suppose that we wish to have at most a 5% error on $\psi_M(\beta)$ with 95% confidence. This means that we must have

$$P_{t(\beta)}(|\psi_M(\beta)-\widehat{\psi}_{IS}(\beta,t,M)|\leq 0.05\psi_M(\beta))=0.95.$$

By the Central Limit Theorem, for N large, we deduce

$$\begin{aligned}
&P_{t(\beta)}(|\psi_M(\beta)-\widehat{\psi}_{IS}(\beta,t(\beta),M)|\leq 0.05\psi_M(\beta))\\
&=P_{t(\beta)}\left(\frac{1}{N}\Big|\sum_{i=1}^N\Big((L^{(\beta)}_{t(\beta),M})^{(i)}-\psi_M(\beta)\Big)\Big|\leq 0.05\psi_M(\beta)\right)\\
&=P_{t(\beta)}\left(\Big|\frac{\sum_{i=1}^N(L^{(\beta)}_{t(\beta),M})^{(i)}-N\psi_M(\beta)}{\sqrt{N\mathrm{Var}_{P_{t(\beta)}}(L^{(\beta)}_{t(\beta),M})}}\Big|\leq 0.05\psi_M(\beta)\sqrt{\frac{N}{\mathrm{Var}_{P_{t(\beta)}}(L^{(\beta)}_{t(\beta),M})}}\right)\\
&\simeq P\left(|Z|\leq 0.05\psi_M(\beta)\sqrt{\frac{N}{\mathrm{Var}_{P_{t(\beta)}}(L^{(\beta)}_{t(\beta),M})}}\right),
\end{aligned}$$

where Z is a standard Gaussian r.v.. Now, using the tables, we have that the equality $P(|Z|\leq z)=0.95$ implies $z\simeq 2$. So, to have at most a 5% error on $\psi_M(\beta)$ with 95% confidence, we must have

$$N\simeq 1600\frac{\mathrm{Var}_{P_{t(\beta)}}(L^{(\beta)}_{t(\beta),M})}{\psi_M(\beta)^2}$$

Comparing this approximation with (3), thanks to (23), we immediately realize that using the importance sampling estimator one may reach a desired precision with a smaller number of replica (choose $\varepsilon \in (0,1)$).

In this section we provide asymptotically efficient simulation laws in the case when the nodes are distributed according to a stationary Poisson process with intensity $\lambda > 0$, the attenuation function is given by $L(x) = \ell(\|x\|) = \max(R, \|x\|)^{-\alpha}$, $\alpha > 2$, $R > 0$, and the signals are distributed according to three different light tail laws.

Next Propositions 5.1, 5.2 and 5.3 give, $\forall\, M \geq R$, the asymptotic behavior of $\log \psi_M(\beta)$, as $\beta \to +\infty$, in the case of bounded, Weibull super-exponential and Exponential signals, respectively. The proofs are based on the large deviation results proved in [15] (the reader is referred to [11] for an introduction on large deviations theory.) In the following we write $f(x) \sim g(x)$ if $f(\cdot)$ and $g(\cdot)$ are two functions such that $f(x)/g(x) \to 1$, as $x \to +\infty$.

Proposition 5.1: Assume that

$$Y_1 \text{ has a bounded support with supremum } b > 0. \tag{24}$$

Then, for any $M \geq R$,

$$\log \psi_M(\beta) \sim -(R^\alpha/b)\beta \log \beta, \quad \text{as } \beta \to +\infty.$$

Proposition 5.2: Assume that

$$\text{There exist constants } \gamma_1 > 0 \text{ and } \gamma_2 > 1\text{: } P(Y_1 > y) \sim \mathrm{e}^{-\gamma_1 y^{\gamma_2}}, \text{ as } y \to +\infty \tag{25}$$

and define $\eta := 1 - (1/\gamma_2)$. Then, for any $M \geq R$,

$$\log \psi_M(\beta) \sim -\gamma_2(\gamma_2 - 1)^{-\eta} \gamma_1^{1/\gamma_2} R^\alpha \beta \log^\eta \beta, \quad \text{as } \beta \to +\infty.$$

Proposition 5.3: Assume that

$$\text{There exists a constant } \gamma_3 > 0\text{: } P(Y_1 > y) \sim \mathrm{e}^{-\gamma_3 y}, \text{ as } y \to +\infty. \tag{26}$$

Then, for any $M \geq R$,

$$\log \psi_M(\beta) \sim -\gamma_3 R^\alpha \beta, \quad \text{as } \beta \to +\infty.$$

Proof of Proposition 5.1 Note that by the large deviation principles in [15] we know that, under the foregoing assumptions, the family of random variables $\{\varepsilon V\}_{\varepsilon > 0}$ and $\{\varepsilon V_R\}_{\varepsilon > 0}$ obey a large deviation principle on $[0, \infty)$ with speed $\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}$ and rate function $I(x) = R^{\alpha} x/b$. Therefore,

$$\lim_{\varepsilon \to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(V > x/\varepsilon) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{\log(1/\varepsilon)} \log P(V_R > x/\varepsilon) = -R^{\alpha} x/b.$$

The claim follows noticing that, as $M \geq R$, we have

$$P(V_R > \beta) \leq P(V_M > \beta) = \psi_M(\beta) \leq P(V > \beta) \quad \forall\ \beta > 0.$$

$\square$

Proof of Proposition 5.2 The proof is similar to the case of bounded signal powers. The main difference is that in the super-exponential Weibull case we have to use the following large deviation principles, again proved in [15]: the family of random variables $\{\varepsilon V\}_{\varepsilon>0}$ and $\{\varepsilon V_R\}_{\varepsilon>0}$ obey a large deviation principle on $[0, \infty)$ with speed $\frac{1}{\varepsilon} \log^{\eta}(\frac{1}{\varepsilon})$ and rate function $I(x) = \gamma_2(\gamma_2 - 1)^{-\eta} \gamma_1^{1/\gamma_2} R^{\alpha} x$.

$\square$

Proof of Proposition 5.3 Here again, the proof is similar to the case of bounded signal powers, but we have to use the following large deviation principles, proved in [15]: the family of random variables $\{\varepsilon V\}_{\varepsilon>0}$ and $\{\varepsilon V_R\}_{\varepsilon>0}$ obey a large deviation principle on $[0, \infty)$ with speed $\frac{1}{\varepsilon}$ and rate function $I(x) = \gamma_3 R^{\alpha} x$.

$\square$

Before providing the asymptotically efficient simulation laws, we compute the Laplace transform of

V_n. By a polar change of coordinates and (5) we have, for any $t>0$ and $n\in\mathbb{N}$,

$$
\begin{aligned}
\mathrm{E}[\mathrm{e}^{tV_n}] &= \exp\left(\lambda\int_{b(O,n)}(\mathrm{E}[\mathrm{e}^{t\ell(\|x\|)Y_1}]-1)\,\mathrm{d}x\right)=\exp\left(2\lambda\pi\int_0^n(\mathrm{E}[\mathrm{e}^{t\ell(\rho)Y_1}]-1)\rho\,\mathrm{d}\rho\right)\\
&= \mathbf{1}\{n\le R\}\exp\left(2\lambda\pi\int_0^n(\mathrm{E}[\mathrm{e}^{t\ell(\rho)Y_1}]-1)\rho\,\mathrm{d}\rho\right)\\
&\quad+\mathbf{1}\{n>R\}\exp\left(2\lambda\pi\int_0^n(\mathrm{E}[\mathrm{e}^{t\ell(\rho)Y_1}]-1)\rho\,\mathrm{d}\rho\right)\\
&= \mathbf{1}\{n\le R\}\exp\left(\lambda\pi n^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}]-1)\right)\\
&\quad+\mathbf{1}\{n>R\}\exp\left(\lambda\pi R^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}]-1)\right)\exp\left(2\lambda\pi\int_R^n(\mathrm{E}[\mathrm{e}^{t\rho^{-\alpha}Y_1}]-1)\rho\,\mathrm{d}\rho\right)\\
&= \mathbf{1}\{n\le R\}\exp\left(\lambda\pi n^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}]-1)\right)\\
&\quad+\mathbf{1}\{n>R\}\exp\left(\lambda\pi R^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}]-1)\right)\exp\left(2\lambda\pi\,\mathrm{E}\left[\int_R^n(\mathrm{e}^{t\rho^{-\alpha}Y_1}-1)\rho\,\mathrm{d}\rho\right]\right) \qquad (27)\\
&= \mathbf{1}\{n\le R\}\exp\left(\lambda\pi n^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}]-1)\right)\\
&\quad+\mathbf{1}\{n>R\}\exp\left(\lambda\pi R^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}]-1)\right)\exp\left(2\lambda\pi\sum_{k\ge1}\frac{t^k(R^{2-\alpha k}-n^{2-\alpha k})}{k!(\alpha k-2)}\mathrm{E}[Y_1^k]\right),
\end{aligned}
$$

(28)

where (27) follows by Fubini's theorem and (28) by the following computation

$$
\begin{aligned}
\int_R^n(\mathrm{e}^{t\rho^{-\alpha}Y_1}-1)\rho\,\mathrm{d}\rho &= \int_R^n\left(\sum_{k\ge1}\frac{(t\rho^{-\alpha}Y_1)^k}{k!}\right)\rho\,\mathrm{d}\rho\\
&= \sum_{k\ge1}\frac{(tY_1)^k}{k!}\int_R^n\rho^{1-\alpha k}\,\mathrm{d}\rho\\
&= \sum_{k\ge1}\frac{(tY_1)^k(R^{2-\alpha k}-n^{2-\alpha k})}{k!(\alpha k-2)}.
\end{aligned}
$$

In particular, as $\alpha > 2$,

$$\mathrm{E}[\mathrm{e}^{tV}] = \exp\left(\lambda\pi R^2(\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}] - 1)\right)\exp\left(2\lambda\pi R^2\sum_{k\geq 1}\frac{(tR^{-\alpha})^k}{k!(\alpha k-2)}\mathrm{E}[Y_1^k]\right), \tag{29}$$

and so $\mathrm{E}[\mathrm{e}^{tV}] < \infty$ for any $t > 0$ such that $\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}] < \infty$.

Now, we give the asymptotically efficient simulation laws. The following theorems hold.

Theorem 5.4: Assume (24) and let $t(\beta)$ be as in (12) with $\mathrm{S} = +\infty$. In addition suppose: (13),

$$\liminf_{\beta\to+\infty}\frac{2t(\beta)}{(R^\alpha/b)\log\beta} = K_1 \in [2, +\infty] \tag{30}$$

and

$$\liminf_{\beta\to+\infty}\frac{\mathrm{e}^{t(\beta)R^{-\alpha}b}}{-(R^\alpha/b)\beta\log\beta} = K_2 \in (-\infty, 0] \tag{31}$$

where the constants K_1 and K_2 are such that

$$K_1 + \frac{2\lambda\alpha\pi R^2}{\alpha-2}K_2 \geq 2. \tag{32}$$

Then the laws $P_{t(\beta)}$ are asymptotically efficient.

Theorem 5.5: Assume (25), let $t(\beta)$ be as in (12) with $\mathrm{S} = +\infty$ and set $\eta := 1 - (1/\gamma_2)$. In addition suppose: (13),

$$\liminf_{\beta\to+\infty}\frac{2t(\beta)}{\gamma_2(\gamma_2-1)^{-\eta}\gamma_1^{1/\gamma_2}R^\alpha\log^\eta\beta} = K_1 \in [2, +\infty], \tag{33}$$

there exist a positive function G such that

$$\lim_{\beta\to+\infty}\frac{G(R^{-\alpha}t(\beta))}{\mathrm{e}^{K_2 t(\beta)}} = +\infty, \quad \forall\ K_2 \in (0, \infty) \tag{34}$$

and a positive constant $B > 0$ such that

$$\sup_{\beta\geq B}\frac{G(R^{-\alpha}t(\beta))}{\beta\log^\eta\beta} \leq K_3, \quad \text{for some } K_3 \in (0, \infty). \tag{35}$$

Then the laws $P_{t(\beta)}$ are asymptotically efficient.

Theorem 5.6: Assume (26) and let $t(\beta)$ be as in (12) with $\mathrm{S} = \gamma_3 R^\alpha$. In addition suppose: (13),

$$t(\beta) \uparrow \gamma_3 R^\alpha, \quad \text{as } \beta \to +\infty \tag{36}$$

and

$$\lim_{\beta \to +\infty} \frac{t(\beta)}{(\gamma_3 R^\alpha - t(\beta))\beta} = 0. \tag{37}$$

Then the laws $P_{t(\beta)}$ are asymptotically efficient.

Proof of Theorem 5.4 By Theorem 4.1 the laws $P_{t(\beta)}$ are asymptotically admissible. It remains to prove (22). We start bounding the second moment of $L_{t,M}^{(\beta)}$, for any fixed $t, \beta > 0$ and $M \geq R$, under P_t. Using the equality (29), we deduce

$$\begin{aligned}
\mathrm{E}_{P_t}[(L_{t,M}^{(\beta)})^2] &\leq \mathrm{E}_{P_t}[\mathbf{1}\{r_\beta < +\infty\} \mathrm{e}^{-2tV_{r_\beta}} ([\mathrm{E}[\mathrm{e}^{tV_n}]]_{n=r_\beta})^2] \\
&\leq \mathrm{e}^{-2t\beta} \exp\left(2\lambda\pi R^2 (\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}] - 1)\right) \exp\left(4\lambda\pi R^2 \sum_{k\geq 1} \frac{(tR^{-\alpha})^k}{k!(\alpha k - 2)} \mathrm{E}[Y_1^k]\right) \\
&\leq \mathrm{e}^{-2t\beta} \exp\left(2\lambda\pi R^2 (\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}] - 1)\right) \exp\left(\frac{4\lambda\pi R^2}{\alpha - 2} (\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}] - 1)\right) \\
&= \mathrm{e}^{-2t\beta} \exp\left(\frac{2\lambda\alpha\pi R^2}{\alpha - 2} (\mathrm{E}[\mathrm{e}^{tR^{-\alpha}Y_1}] - 1)\right) && (38) \\
&\leq \mathrm{e}^{-2t\beta} \exp\left(\frac{2\lambda\alpha\pi R^2}{\alpha - 2} (\mathrm{e}^{tR^{-\alpha}b} - 1)\right), && (39)
\end{aligned}$$

where (39) is consequence of (24). Set $t = t(\beta)$. Then, taking the logarithm in the above inequality and dividing by $\log \psi_M(\beta)$ we have, for all β large enough,

$$\frac{\log \mathrm{E}_{P_{t(\beta)}}[(L_{t(\beta),M}^{(\beta)})^2]}{\log \psi_M(\beta)} \geq -\frac{2\beta t(\beta)}{\log \psi_M(\beta)} + \frac{2\lambda\alpha\pi R^2 (\mathrm{e}^{t(\beta)R^{-\alpha}b} - 1)}{(\alpha - 2)\log \psi_M(\beta)}.$$

Passing to the $\liminf$ as $\beta \to +\infty$ in the above inequality, by Proposition 5.1, (30), (31) and (32) we have (22).

□

Proof of Theorem 5.5 Here again, by Theorem 4.1 the laws $P_{t(\beta)}$ are asymptotically admissible. So we only need to prove (22). We first show that

$$\lim_{\beta\to+\infty}\frac{\mathrm{E}[\mathrm{e}^{R^{-\alpha}t(\beta)Y_1}]-1}{G(R^{-\alpha}t(\beta))}=0. \tag{40}$$

Since Y_1 is a non-negative r.v., we have

$$\mathrm{E}[\mathrm{e}^{\theta Y_1}]-1=\theta\int_0^\infty \mathrm{e}^{\theta y}P(Y_1>y)\,\mathrm{d}y,\quad \theta\in\mathbb{R}. \tag{41}$$

Thus, by (25) we deduce that $\forall\ \varepsilon>0$ there exists y_ε such that $\forall\ y\geq y_\varepsilon$ we have

$$\begin{aligned}\mathrm{E}[\mathrm{e}^{\theta Y_1}]-1&=\theta\int_0^{y_\varepsilon}\mathrm{e}^{\theta y}P(Y_1>y)\,\mathrm{d}y+\theta\int_{y_\varepsilon}^{+\infty}\mathrm{e}^{\theta y}P(Y_1>y)\,\mathrm{d}y\\&\leq \mathrm{e}^{\theta y_\varepsilon}-1+\theta(1+\varepsilon)\int_{y_\varepsilon}^{+\infty}\mathrm{e}^{\theta y-\gamma_1 y^{\gamma_2}}\,\mathrm{d}y.\end{aligned} \tag{42}$$

Using the substitution $y=\mathrm{e}^{-x}$, we have

$$\int_{y_\varepsilon}^{+\infty}\mathrm{e}^{\theta y-\gamma_1 y^{\gamma_2}}\,\mathrm{d}y=\int_0^{\mathrm{e}^{-y_\varepsilon}}\mathrm{e}^{-\gamma_1\mathrm{e}^{-\gamma_2 x}-x}\,\mathrm{e}^{\theta\mathrm{e}^{-x}}\,\mathrm{d}x. \tag{43}$$

The Laplace method for integrals (see e.g. formula (2.38) p.35 in [22]) yields

$$\int_0^{\mathrm{e}^{-y_\varepsilon}}\mathrm{e}^{-\gamma_1\mathrm{e}^{-\gamma_2 x}-x}\,\mathrm{e}^{\theta\mathrm{e}^{-x}}\,\mathrm{d}x\sim\frac{\mathrm{e}^\theta}{\theta}\mathrm{e}^{-\gamma_1}\quad\text{as }\theta\to+\infty. \tag{44}$$

Taking $\theta=R^{-\alpha}t(\beta)$ in (42), noticing that $t(\beta)\to+\infty$, as $\beta\to+\infty$, and using (43), (44) and (34) we have

$$\limsup_{\beta\to+\infty}\frac{\mathrm{E}[\mathrm{e}^{R^{-\alpha}t(\beta)Y_1}]-1}{G(R^{-\alpha}t(\beta))}\leq\lim_{\beta\to+\infty}\frac{\mathrm{e}^{y_\varepsilon R^{-\alpha}t(\beta)}-1+(1+\varepsilon)\mathrm{e}^{-\gamma_1}\mathrm{e}^{R^{-\alpha}t(\beta)}}{G(R^{-\alpha}t(\beta))}=0,$$

and (40) follows. Now, set $t=t(\beta)$ in the inequality (38) and take the logarithm. Dividing by $\log\psi_M(\beta)$ we have, for any $M\geq R$ and all β large enough,

$$\frac{\log\mathrm{E}_{P_{t(\beta)}}[(L^{(\beta)}_{t(\beta),M})^2]}{\log\psi_M(\beta)}\geq-\frac{2\beta t(\beta)}{\log\psi_M(\beta)}+\frac{2\lambda\alpha\pi R^2}{(\alpha-2)}\frac{\mathrm{E}[\mathrm{e}^{R^{-\alpha}t(\beta)Y_1}]-1}{G(R^{-\alpha}t(\beta))}\times\frac{G(R^{-\alpha}t(\beta))}{\log\psi_M(\beta)}.$$

The claim follows taking the $\liminf$ as $\beta \to +\infty$ on this inequality and using Proposition 5.2, (33), (40) and (35).

□

Proof of Theorem 5.6 As usual, we prove (22). Take $\theta < \gamma_3$. By (26) and (41), we deduce that $\forall\ \varepsilon > 0$ there exists y_ε such that $\forall\ y \geq y_\varepsilon$ we have

$$\mathrm{E}[\mathrm{e}^{\theta Y_1}] - 1 \leq \mathrm{e}^{\theta y_\varepsilon} - 1 + \theta(1+\varepsilon)\int_{y_\varepsilon}^{+\infty} \mathrm{e}^{-(\gamma_3-\theta)y}\,\mathrm{d}y$$

$$= \mathrm{e}^{\theta y_\varepsilon} - 1 + (1+\varepsilon)\frac{\theta}{\gamma_3-\theta}\mathrm{e}^{-(\gamma_3-\theta)y_\varepsilon}.$$

Take $\theta = R^{-\alpha}t(\beta)$ in the above inequality to have

$$\mathrm{E}[\mathrm{e}^{R^{-\alpha}t(\beta)Y_1}] - 1 \leq \mathrm{e}^{R^{-\alpha}t(\beta)y_\varepsilon} - 1 + (1+\varepsilon)\frac{t(\beta)}{\gamma_3 R^\alpha - t(\beta)}\mathrm{e}^{-(\gamma_3 - R^{-\alpha}t(\beta))y_\varepsilon}.$$

Dividing the above relation by β and letting β tend to infinity, by the assumptions (36) and (37) it follows

$$\lim_{\beta\to+\infty}\frac{\mathrm{E}[\mathrm{e}^{R^{-\alpha}t(\beta)Y_1}]-1}{\beta} = 0. \tag{45}$$

Note that in this case (where the Laplace transform of the signals is finite on $(-\infty,\gamma_3)$) the inequality (38) yields a non-trivial upper bound on the second moment of $L_{t,M}^{(\beta)}$ for all $0 < t < R^\alpha\gamma_3$ and $\beta > 0$. Set $t = t(\beta)$ in (38) and take the logarithm. Dividing by $\log\psi_M(\beta)$ we have, for all $M \geq R$ and β large enough,

$$\frac{\log \mathrm{E}_{P_{t(\beta)}}[(L_{t(\beta),M}^{(\beta)})^2]}{\log\psi_M(\beta)} \geq -\frac{2\beta t(\beta)}{\log\psi_M(\beta)} + \frac{2\lambda\alpha\pi R^2}{(\alpha-2)\log\psi_M(\beta)}(\mathrm{E}[\mathrm{e}^{R^{-\alpha}t(\beta)Y_1}]-1).$$

The claim follows taking the limit as β tends to infinity in this inequality and using (45), (36) and Proposition 5.3.

□

We conclude this section with some examples.

Constant signals (**Continued**)

Suppose that the signals are all equal to a positive constant $b>0$. We have already checked that for $t_1(\beta):=(R^\alpha/b)\log\beta$, $\beta>1$, the laws $P_{t_1(\beta)}$ are asymptotically admissible if the parameters satisfy the condition $\lambda\pi R^{2-\alpha}b>1$ (indeed, we checked condition (13).) Such laws are indeed asymptotically efficient because the assumptions (30), (31) and (32) of Theorem 5.4 are satisfied with $K_1=2$ and $K_2=0$. Now, consider $t_2(\beta):=(R^\alpha/b)\log(\beta(\log\beta)^c)$, where $\beta>3$ and $c\in(0,1)$. We have already noticed that the laws $P_{t_2(\beta)}$ are asymptotically admissible for any choice of the parameters (also in this case we checked condition (13).) Such laws are indeed asymptotically efficient because the assumptions (30), (31) and (32) of Theorem 5.4 are again satisfied with $K_1=2$ and $K_2=0$.

Weibull superexponential signals (Continued)

Suppose that the signals Y_k, $k\geq 1$, are Weibull distributed with parameters $\gamma_1>0$ and $\gamma_2>1$, i.e. $P(Y_1>y)=\mathrm{e}^{-\gamma_1 y^{\gamma_2}}$, $y>0$. Define $t_1(\beta):=\gamma_2(\gamma_2-1)^{-\eta}\gamma_1^{1/\gamma_2}R^\alpha\log^\eta\beta$, $\beta>1$, where $\eta:=1-(1/\gamma_2)$, and assume that the parameters satisfy condition (18). We have already checked that in such a case the laws $P_{t_1(\beta)}$ are asymptotically admissible (indeed, condition (13) is satisfied.) Such laws are indeed asymptotically efficient because condition (33) of Theorem 5.5 is satisfied with $K_1=2$ and assumptions (34) and (35) of Theorem 5.5 can be easily checked setting

$$G(\beta):=\mathrm{e}^{(\gamma_2^{-1}(\gamma_2-1)^\eta\gamma_1^{-1/\gamma_2}\beta)^{1/\eta}},\quad \beta>0$$

(note that $G(R^{-\alpha}t_1(\beta))=\beta$ and $\eta\in(0,1)$.) Now, define $t_2(\beta):=c\log^\eta\beta$, $\beta>1$, where the constant c satisfies (19). We have already noticed that the laws $P_{t_2(\beta)}$ are asymptotically admissible for any choice of the parameters (also in this case condition (13) is satisfied.) Such laws are indeed asymptotically efficient because condition (33) of Theorem 5.5 follows by (19) and assumptions (34) and (35) of Theorem 5.5 can be easily checked defining

$$G(\beta):=\mathrm{e}^{(c^{-1}R^\alpha\beta)^{1/\eta}},\quad \beta>0$$

(here again note that $G(R^{-\alpha}t_2(\beta))=\beta$ and $\eta\in(0,1)$.)

Exponential signals (Continued)

Suppose that the signals are exponentially distributed with mean γ_3^{-1}. Let $\widetilde{\varphi}$ be a positive function satisfying (20) (for some positive function φ such that $\varphi(\beta) = o(\beta)$, as $\beta \to +\infty$) and define $t(\beta) := \gamma_3 R^\alpha - \widetilde{\varphi}(\beta)^{-1}$, $\beta > \widetilde{\varphi}^{-1}(\gamma_3^{-1} R^{-\alpha})$. We have already checked that the laws $P_{t(\beta)}$ are asymptotically admissible (indeed, we checked condition (13).) Such laws are asymptotically efficient if moreover the function $\widetilde{\varphi}$ is such that $\widetilde{\varphi}(\beta) = o(\beta)$, as $\beta \to +\infty$. Indeed, in such a case condition (37) of Theorem 5.6 is satisfied. As we already checked, if $\widetilde{\varphi}(\beta) := \sqrt{\beta}$, $\beta > 0$, and the parameters satisfy condition (21), then (20) holds and therefore the laws $P_{t(\beta)}$ are asymptotically efficient; we also verified that (20) holds for any choice of the parameters if $\widetilde{\varphi}(\beta) := \beta^{2/3}$, $\beta > 0$, and so in such a case the laws $P_{t(\beta)}$ are asymptotically efficient for any choice of the parameters.

VI. Numerical illustrations

In this section we report an extensive set of numerical results for the three examples previously considered. We shall use the importance sampling estimator defined by (10). So, for fixed M and β, we simulate independent replica of the r.v. $L^{(\beta)}_{t(\beta),M}$, under a suitable chosen importance sampling law $P_{t(\beta)}$, and then we average. More in detail, following the approach described in Section III, the importance sampling estimator is defined as in (10) with $t(\beta)$ in place of t, i.e.

$$\widehat{\psi}_{IS}(\beta, t(\beta), M) := \frac{1}{N}\sum_{i=1}^{N}(L^{(\beta)}_{t(\beta),M})^{(i)},$$

where

$$L^{(\beta)}_{t(\beta),M} := \mathbf{1}\{r_\beta \leq M\}\mathrm{e}^{-t(\beta)V_{r_\beta}}[\mathrm{E}[\mathrm{e}^{t(\beta)V_n}]]_{n=r_\beta}.$$

We simulate the independent r.v.'s $(L^{(\beta)}_{t(\beta),M})^{(i)}$, under the law $P_{t(\beta)}$, according to the following algorithm. The truncated interference V_n, $n \geq 1$, caused by nodes in $b(O,n)$, is generated for an increasing sequence of radii $n = 1, 2, 3\cdots$, exploiting the recursion $V_n = V_{n-1} + \bar{V}_n$, where $V_0 = 0$ and $\bar{V}_n$ is the contribution provided by nodes in the annulus $b(O,n) \setminus b(O,n-1)$, where we set $b(0,0) := \emptyset$. The algorithm stops as soon as we find $n' \leq M$ such that $V_{n'} > \beta$ or for all $n \leq M$ we have $V_n \leq \beta$. In the first case we set $(L^{(\beta)}_{t(\beta),M})^{(i)} := \mathrm{e}^{-t(\beta)V_{n'}}\mathrm{E}[\mathrm{e}^{t(\beta)V_{n'}}]$, in the second case we set $(L^{(\beta)}_{t(\beta),M})^{(i)} := 0$. Note that, for any $n \leq M$, the quantity $\mathrm{E}[\mathrm{e}^{t(\beta)V_n}]$ can be numerically evaluated from (28). In Table I we report the detailed pseudo-code to generate the importance sampling estimator.

TABLE I

PSEUDO-CODE TO SIMULATE THE IMPORTANCE SAMPLING ESTIMATOR

Algorithm VI.1: IMPORTANCE SAMPLING(β, M)

procedure CONTRIBUTION_INTEFERENCE_BY_NODES_ANNULUS (n)

comment: Points are Poisson with intensity $\Lambda_{t(\beta)}(\cdot)$ on the annulus $b(O,n) \setminus b(O,n-1)$

$I \leftarrow 0$ **comment:** initialization

$N_{\text{points}} \leftarrow$ POISSON($\int_{b(O,n)\setminus b(O,n-1)} \Lambda_{t(\beta)}(x)\,\mathrm{d}x$)

for $i \leftarrow 1$ **to** N_{points}

do $\begin{cases} d \leftarrow \text{NODE\_DISTANCE\_FROM\_0}(n) & \textbf{comment: } \text{a acceptance-rejection method is employed} \\ Y \leftarrow \text{EXTRACT\_THE\_SIGNAL\_STRENGTH}(d) & \textbf{comment: } \text{signal strength depends on } d \text{ under } P_{t(\beta)} \\ I \leftarrow I + YL(d) \end{cases}$

return (I) **comment:** interference contribution by nodes in the annulus

procedure COMPUTE_AVE_FIELD(n, β)

$Z \leftarrow \lambda \int_{b(O,n)} (\mathrm{E}[\mathrm{e}^{t(\beta)L(x)Y_1}] - 1)\,\mathrm{d}x$

comment: Z is evaluated using standard numerical integration techniques

return $(\exp(Z))$

procedure $L^{(\beta)}_{t(\beta),M}$_SAMPLES_GENERATION(β)

$V \leftarrow 0$ **comment:** initialization

$r_\beta \leftarrow \infty$

Flag $\leftarrow FALSE$

$n \leftarrow 0$

repeat

comment: loop on n

$\begin{cases} n \leftarrow n+1 \\ \bar{V} \leftarrow \text{EVALUATE\_CONTRIBUTION\_INTEFERENCE\_BY\_NODES\_ANULUS}(n) \\ V \leftarrow V + \bar{V} \\ \textbf{if } V > \beta \\ \quad \textbf{then } \begin{cases} r_\beta \leftarrow n \\ \text{Flag} \leftarrow TRUE \end{cases} \end{cases}$

until Flag $= FALSE$ **and** $n \leq M$

if Flag $= TRUE$

then $W \leftarrow$ COMPUTE_AVE_FIELD(n, β)

return $(\exp(-t(\beta)V)W)$

else if return (0)

main

$\widehat{\psi}_{IS}(\beta, t(\beta), M) \leftarrow 0$ **comment:** Initialization

for $i \leftarrow 1$ **to** S **comment:** Main Loop; S = Number of samples

do $\widehat{\psi}_{IS}(\beta, t(\beta), M) \leftarrow \widehat{\psi}_{IS}(\beta, t(\beta), M) +$ SAMPLES_GENERATION(β)

$\widehat{\psi}_{IS}(\beta, t(\beta), M) \leftarrow \widehat{\psi}_{IS}(\beta, t(\beta), M)/S$

return $(\widehat{\psi}_{IS}(\beta, t(\beta), M))$

Constant signals (Continued)

Suppose that the signals Y_k are all equal to a constant b. Typically, this choice corresponds to the case in which transmitters and receivers are in line of sight (open space environment) and fading/shadowing effects on transmissions are negligible.

Applying Theorem 3.2 we easily have that, under P_t, $t>0$, $\{X_k\}_{k\geq 1}$ is a non-homogeneous Poisson process with intensity function $\Lambda_t(x)=\lambda \mathrm{e}^{tb\max(R,\|x\|)^{-\alpha}}$ and the signals $\{Y_k\}_{k\geq 1}$ are again all equal to b. In Figure 1 we compare the numerical estimates of $\psi(\beta,M)$ given by the crude Monte Carlo estimator $\widehat{\psi}_{CMC}$ and the importance sampling estimator $\widehat{\psi}_{IS}$. More precisely, we compare such estimates, as $\beta>3$ varies, setting $M=80$, $\lambda=1/\pi$, $R=1$, $\alpha=3$, $b=1$ and considering the asymptotically efficient law defined by $t(\beta)=\log(\beta(\log\beta)^{0.2})$. For both estimators $N=10^5$ samples were simulated (divided in 50 batches of 2000 samples.) Confidence intervals are represented on the plots. Note that while $\widehat{\psi}_{CMC}$ is able to estimate, with a sufficient degree of accuracy, only those probabilities that are one order larger than $1/N$, $\widehat{\psi}_{IS}$ allows to estimate accurately even probabilities that are several order of magnitude smaller than $1/N$. For $\beta>10$ the crude Monte Carlo estimator is unable to provide even a rough estimate of $\psi(\beta,M)$, since no samples of the interference above the threshold have been observed. Figure 2 refers to the case in which $\alpha=5$ and the other parameters are as in Figure 1. Similar considerations hold also in this case.

	$N=10^3$	$N=10^4$	$N=10^5$
$\widehat{\psi}_{CMC}$	9.00e-3	9.70e-3	1.03e-02
$\widehat{\psi}_{IS}$	1.01e-2	1.05e-2	1.06e-2
hw_{CMC}	7.99e-3	2.61e-3	9.01e-4
hw_{IS}	1.71e-3	6.39e-4	3.82e-4

TABLE II

COMPARISON BETWEEN THE CMC ESTIMATOR AND THE IS ESTIMATOR FOR DIFFERENT VALUES OF N AND $M=80$, $\lambda=1/\pi$, $R=1$, $\alpha=5$, $b=1$, $t(\beta)=\log(\beta(\log\beta)^{0.2})$, $\beta=5$.

To better appreciate the different degree of accuracy provided by the two numerical methods, in Tables II and III we directly compare the estimates $\widehat{\psi}_{CMC}$ and $\widehat{\psi}_{IS}$ for different values of N and $\beta=5$ and $\beta=7$, respectively. The system parameters and the importance sampling law are chosen as in Figure

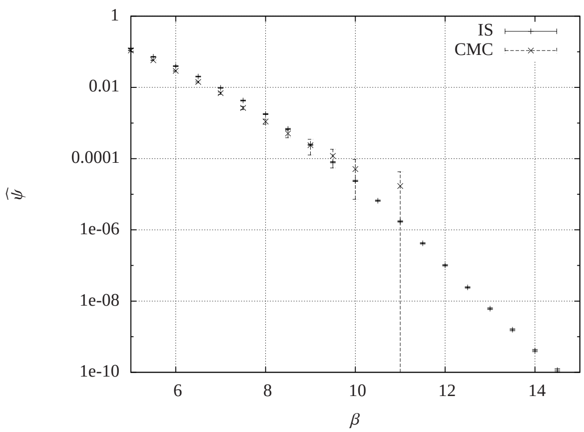

Fig. 1. Constant signals: comparison between the CMC estimator and the IS estimator (as β varies) for the following choice of the parameters: $M = 80$, $\lambda = 1/\pi$, $R = 1$, $\alpha = 3$, $b = 1$, $t(\beta) = \log(\beta(\log\beta)^{0.2})$.

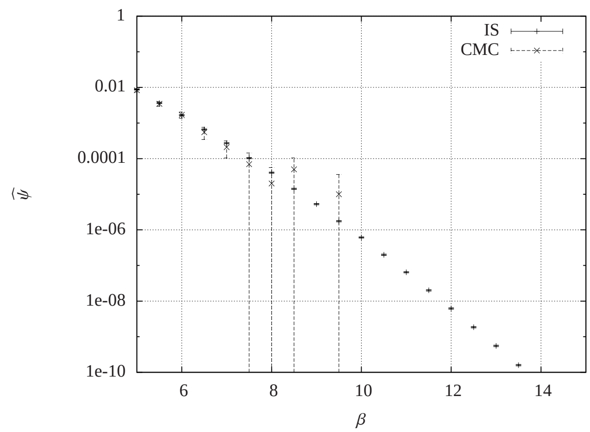

Fig. 2. Constant signals: comparison between the CMC estimator and the IS estimator (as β varies) for the following choice of the parameters: $M = 80$, $\lambda = 1/\pi$, $R = 1$, $\alpha = 5$, $b = 1$ $t(\beta) = \log(\beta(\log\beta)^{0.2})$.

	$N = 10^3$	$N = 10^4$	$N = 10^5$
$\widehat{\psi}_{CMC}$	-	2.00e-4	3.80e-04
$\widehat{\psi}_{IS}$	4.31e-4	4.42e-4	4.48e-4
hw_{CMC}	-	3.61e-4	1.05e-4
hw_{IS}	7.84e-5	3.29e-5	1.72e-5

TABLE III

COMPARISON BETWEEN THE CMC ESTIMATOR AND THE IS ESTIMATOR FOR DIFFERENT VALUES OF N AND $M = 80$, $\lambda = 1/\pi$, $R = 1$, $\alpha = 5$, $b = 1$, $t(\beta) = \log(\beta(\log\beta)^{0.2})$, $\beta = 7$.

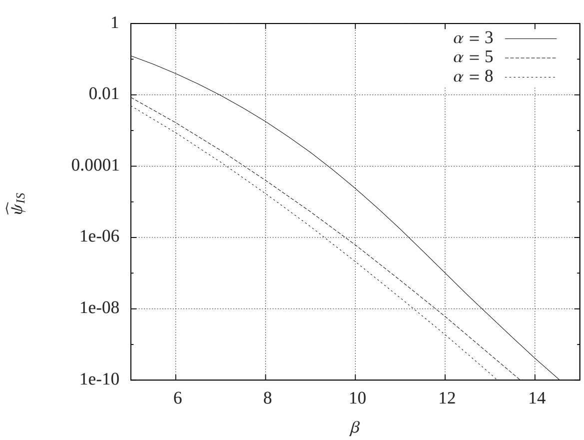

Fig. 3. Constant signals: plots of the estimated tail of the interference for different values of α.

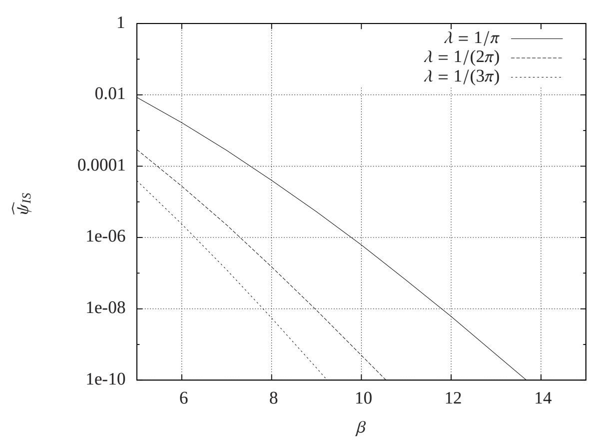

Fig. 4. Constant signals: plots of the estimated tail of the interference for different values of λ.

2. More precisely, the tables report the estimates $\widehat{\psi}_{CMC}$ and $\widehat{\psi}_{IS}$ as well as the corresponding 99% confidence intervals half-width, denoted by hw_{CMC} and hw_{IS}, respectively. Note that the IS estimator provides predictions that are much more accurate than those one given by the CMC estimator, for any choice of N and β. More particularly, for $\beta = 5$ the estimates $\widehat{\psi}_{IS}$ are about three times more accurate than the estimates $\widehat{\psi}_{CMC}$ (i.e. with a 99% confidence interval that is about three times narrower.) For $\beta = 7$ the degree of accuracy of $\widehat{\psi}_{IS}$ with respect to $\widehat{\psi}_{CMC}$ increases to an order of magnitude. These are all consequences of the fact that the selected importance sampling law is asymptotically efficient.

The impact of the system parameters α and λ on the tail of the interference is evaluated in Figures 3 and 4. More precisely, in Figure 3 we plot $\widehat{\psi}_{IS}$ as a function of $\beta > 3$, for $\alpha = 3, 5, 8$ (M, R, λ, b and $t(\cdot)$ are chosen as in Figure 1). In Figure 4 we plot $\widehat{\psi}_{IS}$ as a function of $\beta > 3$, for $\lambda = 1/\pi, 1/2\pi, 1/3\pi$ (M, R, α, b and $t(\cdot)$ are chosen as in Figure 1). Note that the tail of the interference exhibits a significant dependence on α and λ. Indeed, by increasing α, the tail decreases since the received signal power from an interfering node at distance greater than 1 becomes more and more smaller. Similarly, by decreasing λ, the tail of the interference decreases since the distance of all the interfering nodes from the origin increases as $\lambda^{-1/2}$.

The impact of the choice of M on the accuracy of the estimate of $\psi(\beta)$ by the importance sampling estimator $\widehat{\psi}_{IS}(\beta, t(\beta), M)$ is gauged in Figure 5. More precisely, for $\lambda = 1/\pi$, $R = 1$, $\alpha = 2.5$, $b = 1$ and $t(\beta) = \log(\beta(\log\beta)^{0.2})$, $\beta > 3$, in Figure 5 we report the values of $\widehat{\psi}_{IS}(\beta, t(\beta), M)$ for different choices of M (ranging from 5 to 80). Note that, as α decreases, the truncation induced by the choice of M becomes potentially more critical. Curves are hardly distinguishable for $M \geq 20$. Thus, we can conclude that the choice of M is not critical, unless we select a value for α very close to 2.

In Figure 6 we report the ratio between the number of times we found $r_\beta > M$ (on N replica) and the total number of replica N (shortly, the fraction of samples for which the value of M has been reached). We denoted such ratio by $F_M(\beta)$. More precisely, we considered different choices for α and λ and $M = 80$, $R = 1$, $b = 1$, $t(\beta) = \log(\beta(\log\beta)^{0.2})$, $\beta > 3$. Note that in all cases the ratio decreases as β increases. This is a direct consequence of the fact that the simulation law $P_{t(\beta)}$ is asymptotically admissible.

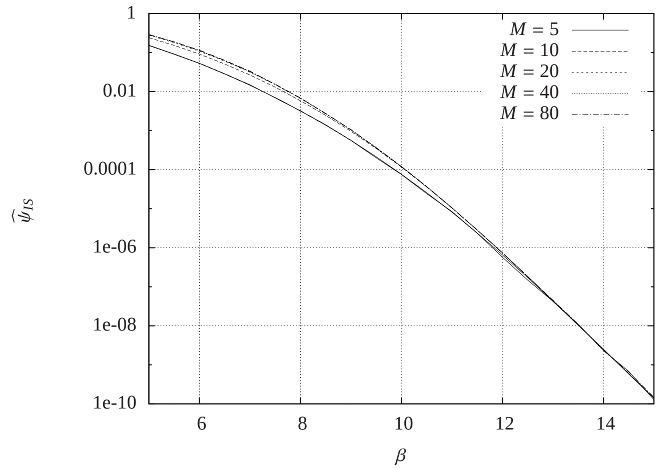

Fig. 5. Plot of the function $\widehat{\psi}_{IS}(\cdot, t(\cdot), M)$ for $M \in \{5, 10, 20, 40, 80\}$, $\lambda = 1/\pi$, $R = 1$, $\alpha = 2.5$, $b = 1$, $t(\beta) = \log(\beta(\log\beta)^{0.2})$.

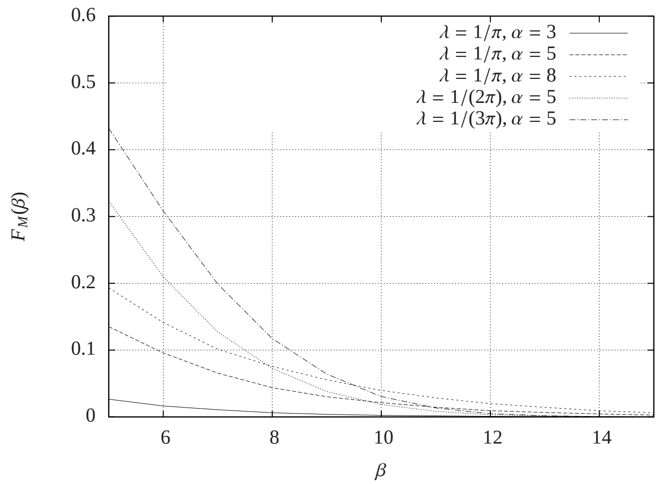

Fig. 6. Fraction of samples for which the value M has been reached.

Finally, in Figure 7 we report the ratio between $\log(\widehat{\psi}_{IS}(\beta, t(\beta), M))$ and $-(R^\alpha/b)\beta\log\beta$, i.e. the asymptotic expression of $\log\psi(\beta)$ (see [15]). We considered $M = 80$, $R = 1$, $b = 1$, $t(\beta) = \log(\beta(\log\beta)^{0.2})$, $\beta > 3$, and different choices of α and λ. Note that $-\log\widehat{\psi}_{IS}(\beta, t(\beta), M)/[(R^\alpha/b)\beta\log\beta]$ significantly differs from 1, for the values of α and λ considered. This is not surprising since the quantity $(R^\alpha/b)\beta\log\beta = \beta\log\beta$ does not depend on α and λ, while $\widehat{\psi}_{IS}$ significantly depends on α and λ (see Figures 3 and 4). We conclude that the asymptotic approximation $-(R^\alpha/b)\beta\log\beta$ may be too crude to provide any insights on the behavior of $\psi(\beta)$ for significant values of β, in several cases.

Exponential signals (Continued)

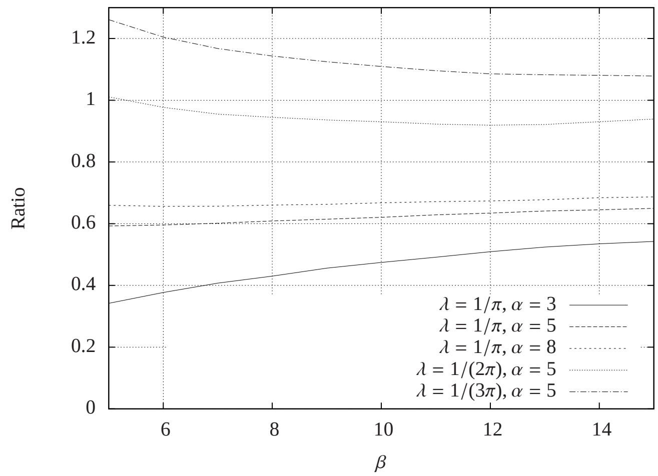

Fig. 7. Plot of the function $\beta \mapsto -\log \widehat{\psi}_{IS}(\beta, t(\beta), M)/[(R^\alpha/b)\beta \log \beta]$ for different choices of the parameters λ and α and $M = 80$, $R = 1$, $b = 1$, $t(\beta) = \log(\beta(\log \beta)^{0.2})$.

Suppose that the signals Y_k are exponentially distributed with mean γ_3^{-1}. This choice corresponds to the classical Rayleigh fading, which is widely accepted as reasonable simple model of propagation effects, under non line of sight conditions. For instance, Rayleigh fading captures pretty well the effect of heavily built-up urban environments on radio signals [14].

Applying Theorem 3.2 we have that, for any $t \in (0, \gamma_3 R^\alpha)$, under P_t, $\{X_k\}_{k \geq 1}$ is a non-homogeneous Poisson process with intensity function

$$\Lambda_t(x) = \frac{\lambda \gamma_3}{\gamma_3 - t \max(R, \|x\|)^{-\alpha}}$$

and, given $\{X_k\}_{k \geq 1}$, the signals are mutually independent and the law of $Y_k \,|\, X_k = x$ is Exponential with mean $(\gamma_3 - t \max(R, \|x\|)^{-\alpha})^{-1}$; indeed

$$\mathrm{d}P_t^{(Y_k \,|\, X_k = x)}(y) = \frac{\mathrm{e}^{t\ell(\|x\|)y}}{\mathrm{E}[\mathrm{e}^{t\ell(\|x\|)Y_1}]} \,\mathrm{d}P^{(Y_1)}(y) = (\gamma_3 - t \max(R, \|x\|)^{-\alpha}) \mathrm{e}^{-(\gamma_3 - t \max(R, \|x\|)^{-\alpha})y} \,\mathrm{d}y.$$

In Figure 8 we compare the numerical estimates of $\psi(\beta)$ given by the crude Monte Carlo estimator $\widehat{\psi}_{CMC}$ and the importance sampling estimator $\widehat{\psi}_{IS}$. More precisely, we compare such estimates, as $\beta > 1$ varies, setting $M = 80$, $R = 1$, $\lambda = 1/\pi$, $\gamma_3 = 1$, $\alpha = 5$ and considering the asymptotically efficient law defined by $t(\beta) = \gamma_3 R^\alpha - \beta^{-\frac{2}{3}} = 1 - \beta^{-\frac{2}{3}}$, $\beta > (\gamma_3 R^\alpha)^{-3/2} = 1$. For both the estimators $N = 10^5$ samples have been simulated. As for the case of constant signals, the importance sampling technique allows to obtain numerical estimates of $\psi(\beta)$ which are dramatically more accurate than those one obtained with a classical Monte Carlo approach (see the 99% confidence intervals represented on the plots.) Here again, for $\beta > 15$ the crude Monte Carlo estimator is unable to provide even rough estimates of $\psi(\beta, M)$, since no samples of the interference above the threshold have been observed.

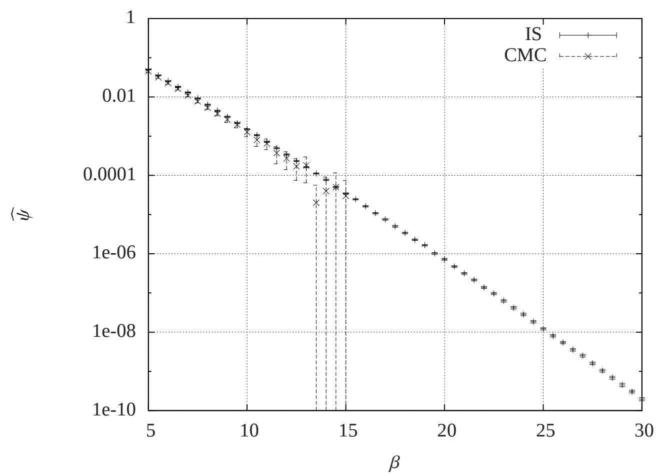

Fig. 8. Exponential signals: comparison between the CMC estimator and the IS estimator for the following choice of the parameters: $M = 80$, $R = 1$, $\lambda = \frac{1}{\pi}$, $\gamma_3 = 1$, $\alpha = 5$, $t(\beta) = 1 - \beta^{-\frac{2}{3}}$.

Weibull superexponential signals (Continued)

Suppose that the signals Y_k are Weibull distributed with parameters $\gamma_1 = \frac{1}{2\sigma^2}$, $\sigma > 0$, and $\gamma_2 = 2$, i.e. the signals follow the standard Rayleigh distribution with tail function $P(Y_1 > y) = \mathrm{e}^{\frac{-y^2}{2\sigma^2}}$. We emphasize that Weibull distributions have been recently shown [27] to fit well to experimental fading channel measurements, for both indoor and outdoor environments.

Applying Theorem 3.2 we have that, under P_t, $t > 0$, $\{X_k\}_{k\geq 1}$ is a non-homogeneous Poisson process with intensity function $\Lambda_t(x) = \lambda\gamma(t, x)$, where

$$\gamma(t,x) := 1 + \sigma t \max(R, \|x\|)^{-\alpha}\, \mathrm{e}^{\sigma^2(t\max(R,\|x\|)^{-\alpha})^2/2} \sqrt{\frac{\pi}{2}} \left(\mathrm{erf}\left(\frac{\sigma t \max(R, \|x\|)^{-\alpha}}{\sqrt{2}} \right) + 1 \right)$$

and $\mathrm{erf}(z) := (2/\sqrt{\pi}) \int_0^z \mathrm{e}^{-t^2}\, \mathrm{d}t$ is the error function. Moreover, given $\{X_k\}_{k\geq 1}$, under P_t the signals are mutually independent and the law of $Y_k \mid X_k = x$ is

$$\mathrm{d}P_t^{(Y_k \mid X_k = x)}(y) = \frac{y\mathrm{e}^{t\max(R,\|x\|)^{-\alpha}y}}{\sigma^2\gamma(t,x)}\,\mathrm{e}^{-(y^2/2\sigma^2)}\mathrm{d}y$$

$$= \frac{y\mathrm{e}^{[(\sigma/\sqrt{2})t\max(R,\|x\|)^{-\alpha}]^2}}{\sigma^2\gamma(t,x)}\,\mathrm{e}^{-[y/\sqrt{2\sigma^2}-(\sigma/\sqrt{2})t\max(R,\|x\|)^{-\alpha}]^2}\mathrm{d}y.$$

To sample from the law $\mathrm{d}P_t^{(Y_k \mid X_k = x)}(y)$ we use a composition method [28] exploiting the trivial identity

$$Y_k = Y_k\mathbf{1}\{Y_k < \sigma^2 t\max(R,\|x\|)^{-\alpha}\} + Y_k\mathbf{1}\{Y_k \geq \sigma^2 t\max(R,\|x\|)^{-\alpha}\}.$$

Here, we limit ourselves to say that, given the event $\{Y_k < \sigma^2 t\max(R,\|x\|)^{-\alpha}\}$, Y_k is generated using the acceptance/rejection method, where we leverage on the inequality

$$\frac{y}{\sigma^2}\mathrm{e}^{-[y/\sqrt{2\sigma^2}-(\sigma/\sqrt{2})t\max(R,\|x\|)^{-\alpha}]^2} \leq \frac{y}{\sigma^2}, \quad \forall\ y \in [0, \sigma^2 t\max(R,\|x\|)^{-\alpha}].$$

Given the event $\{Y_k \geq \sigma^2 t\max(R,\|x\|)^{-\alpha}\}$, Y_k is generated using again a composition method. Indeed, given $\{Y_k \geq \sigma^2 t\max(R,\|x\|)^{-\alpha}\}$, the density of $Y_k - \sigma^2 t\max(R,\|x\|)^{-\alpha}$ can be expressed as a mixture between the densities of a Rayleigh and a Gaussian distribution.

In Figure 9 we compare the numerical estimates of $\psi(\beta)$ given by the crude Monte Carlo estimator $\widehat{\psi}_{CMC}$ and the importance sampling estimator $\widehat{\psi}_{IS}$. More precisely, we compare such estimates, as $\beta > 1$ varies, setting $M = 80$, $R = 1$, $\lambda = 1/\pi$, $\sigma = \left(\frac{2}{\pi}\right)^{1/2}$, $\alpha = 5$ and considering the asymptotically efficient law defined by $t(\beta) = [1+(\pi/4)]\log^{\frac{1}{2}}\beta$. For both the estimators $N = 10^5$ samples have been simulated. As for the previous cases, the importance sampling technique allows to obtain numerical estimates of $\psi(\beta)$ which are extremely more accurate than those one obtained with a classical Monte Carlo method (see the 99$\%$ confidence intervals represented on the plots.) Here also, for $\beta \geq 12$ we found that the crude Monte Carlo estimator is unable to provide estimates of $\psi(\beta, M)$, since no samples of the interference above the threshold have been observed.

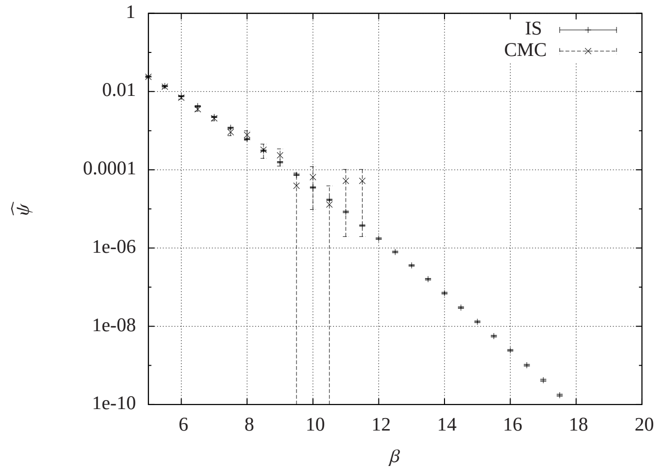

Fig. 9. Rayleigh signals: comparison between the CMC estimator and the IS estimator for the following choice of the parameters: $M = 80$, $R = 1$, $\lambda = \frac{1}{\pi}$, $\sigma = \sqrt{\frac{2}{\pi}}$, $\alpha = 5$, $t(\beta) = [1 + (\pi/4)] \log^{\frac{1}{2}} \beta$.

VII. Conclusions

In this paper we have presented a new provably efficient simulation procedure, based on state-dependent importance sampling, to estimate the tail of the interference in wireless scenarios where interfering nodes are placed according to a Poisson process. An extensive set of numerical results illustrate the features of the proposed algorithm. **We remark that even if we analyzed the ideal Hertzian propagation model, up to minor modifications, the algorithm may be used to estimate the tail of the interference in Poisson network models with attenuation functions of the form $L(x) := \ell(\|x\|)$ with $\ell : [0,\infty) \to (0,\infty)$, continuous, non-increasing and such that:**

$$\exists\ c > 0,\ \alpha > 2\text{: } \ell(r) \leq c r^{-\alpha}, \text{ for all } r \text{ sufficiently large.}$$

Note that in such models the tail of the interference has the same asymptotic behavior as in the ideal Hertzian propagation model (see Section VI in [15].)

References

[1] Abate, J. (1995). Numerical inversion of Laplace transforms of probability distributions. *ORSA Journal on Computing*, **7**, 36–43.

[2] Ali, O.B., Cardinal, C. and Gagnon, F. (2010). Performance of optimum combining in a Poisson field of interferes and Rayleigh fading channels. *IEEE Transactions on Wireless Communications* **9**, 2461–2467.

[3] Asmussen, S. and Glynn, P. (2007). *Stochastic Simulation*, Springer, New York.

[4] Baccelli, F. and Błaszczyszyn, B. (2001). On a coverage process ranging from the Boolean model to the Poisson-Voronoi tessellation with applications to wireless communications. *Advances in Applied Probability* **33**, 293–323.

[5] Baccelli, F. and Blaszczyszyn, B. (2009). *Stochastic Geometry and Wireless Networks*, Foundations and Trends in Networking, NoW Publishers, Part I: Theory.

[6] Baccelli, F. and Blaszczyszyn, B. (2009). *Stochastic Geometry and Wireless Networks*, Foundations and Trends in Networking, NoW Publishers, Part 2: Applications.

[7] Baccelli, F., Blaszczyszyn., B. and Muhlethaler, P. (2006). An Aloha protocol for multihop mobile wireless networks. *IEEE Transactions on Information Theory* **52**, 421–436.

[8] Bucklew, J.A. (2004). *Introduction to Rare Event Simulation*, Springer, New York.

[9] Cohen, A.M. (2007). *Numerical methods for Laplace transform inversion*, Springer, New York.

[10] Daley, D.J. and Vere-Jones, D. (2003). *An Introduction to the Theory of Point Processes*, Springer, New York.

[11] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications (2nd edition)*, Springer, New York.

[12] Dousse, O., Baccelli, F. and Thiran, P. (2005). Impact of interferences on connectivity in ad hoc networks. *IEEE/ACM Transaction on Networking* **13**, 425–436.

[13] Dousse, O., Franceschetti, M., Macris, N., Meester, R. and Thiran, P. (2006). Percolation in the signal to interference ratio graph. *Journal of Applied Probability* **43**, 552–562.

[14] Foschini G. J. and Gans M. J. (1998). On Limits of Wireless Communications in a Fading Environment when Using Multiple Antennas *Wireless Personal Communications* **6**, 311–335.

[15] Ganesh, A. and Torrisi, G.L. (2008). Large deviations of the interference in a wireless communication model. *IEEE Transactions on Information Theory* **54**, 3505–3517.

[16] Ganti, R.K. and Haenggi, M. (2009). Interference and outage in clustered wireless ad hoc networks. *IEEE Transactions on Information Theory* **55**, 4067–4086.

[17] Ganti, R.K., Andrews, J.G. and Haenggi, M. (2011). High-SIR transmission capacity of wireless networks with general fading and node distribution. *IEEE Transactions on Information Theory* **57**, 3100–3116.

[18] Giacomelli, R., Ganti, R.K. and Haenggi, M. (2011). Outage probability of general ad hoc networks in the high-reliability regime. *IEEE/ACM Transactions on Networking* **19**, 1151–1163.

[19] Gupta, P. and Kumar, P.R. (2000). The capacity of wireless networks. *IEEE Transactions on Information Theory* **46**, 388–404.

[20] Haenggi, M. and Ganti, R. K., (2008) *Interference in Large Wireless Networks*, Foundations and Trends in Networking, NoW Publishers.

[21] Hunter, A. Andrews, J. and Weber, S. (2008). Transmission capacity of ad hoc networks with spatial diversity. *IEEE Transactions on Wireless Communications* **7**, 5058–5071.

[22] Murray, J.D. (1984). *Asymptotic Analysis*, Springer, New York.

[23] Ozgur, A., Leveque, O. and Tse, D. (2002). Hierarchical cooperation achieves optimal capacity scaling in ad hoc networks. *IEEE Transactions on Information Theory* **53**, 3549–3572.

[24] Penrose, M. (2003). *Random Geometric Graphs*, Oxford University Press, New York.

[25] Privault, N. and Torrisi, G.L. (2011). Density estimation of functionals of spatial point processes with application to wireless networks. *SIAM Journal on Mathematical Analysis*, **43**, 1311-1344.

[26] Rolski, T., Schmidli, H., Schmidt, V. and Teugels, J. (1999). *Stochastic Processes for Insurance and Finance*, Wiley, Chichester.

[27] Sagias, N. and Karagiannis, G. (2005). A Gaussian class of multivariate Weibull distributions: theory and applications in fading channels. *IEEE Transactions on Information Theory* **51**, 3608–3619.

[28] Stewart, W.J. (2009). *Probability, Markov Chains, Queues and Simulations*, Princeton University Press, New Jersey.

[29] Valkó, P.P. and Abate, J. (2004). Comparison of sequence accelerators for the Gaver method of numerical Laplace transform inversion. *Computers & Mathematics with Applications* **48**, 629–636.

[30] Weber, S., Andrews, J.G. and Jindal, N. (2007). The effect of fading, channel inversion, and threshold scheduling on ad hoc networks. *IEEE Transactions on Information Theory* **53**, 4127–4149.

[31] Weber, S., Yang, X., Andrews, J.G. and de Veciana, G. (2005). Transmission capacity of wireless ad hoc networks with outage constraints. *IEEE Transactions on Information Theory* **51**, 4091–4102.

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