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Keywords: Nonlocal elasticity, fractional calculus, wave propagation.

SUMMARY. In this paper, the wave propagation in one-dimensional elastic continua, characterized by nonlocal interactions, is investigated by means of a fractional calculus approach. Derivatives of a non-integer order $1 < \alpha < 2$ with respect to the spatial variable are involved in the governing equation.

1 INTRODUCTION

The elastic behaviour of non-local continua has recently been investigated in a fractional calculus framework [1, 2, 3]. This approach is original in many respects. With reference to classical nonlocal elastic approaches, the novelty is that the departure from local elasticity is obtained by lowering the order of (fractional) derivation of the displacement function in the governing equation. On the other hand, compared with other fractional calculus applications in mechanics, the originality is that fractional derivatives are taken with respect to the spatial variable (and not with respect to the time variable, as it occurs in visco-elasticity). Another important feature is that fractional operators have a clear mechanical meaning, i.e. they describe the interactions between non-adjacent points of the body by means of linear elastic springs whose stiffness decays as a power-law of the distance. Limiting the analysis to a one-dimensional model, i.e. the fractional nonlocal elastic bar, the governing equation is a second order fractional differential equation in the displacement variable, where, beyond the usual second order derivative, a fractional derivative of order α with respect to the spatial variable appears. While in [1] the range $0 < \alpha < 1$ was investigated, in [2, 3] the model was extended to the range $1 < \alpha < 2$. It was also highlighted that only the range $1 < \alpha < 2$ provides a model within Eringen (integral) nonlocal elasticity [4] framework, since it refers to a material whose stress is proportional to the fractional integral of the strain.

Starting from this result, in the present work the phenomenon of elastic wave propagation in a fractional nonlocal elastic bar is investigated for $1 < \alpha < 2$. After deriving the fractional nonlocal wave equation, the problem is analyzed for what concerns finite spatial domains. Fractional finite differences [5] are involved in the discretization process. Eventually, the longitudinal resonant frequencies related to such nonlocal continua are evaluated. Solutions are examined and compared with the classical one, which is recovered by the present model as the order of fractional derivation coincides to the integer value ($\alpha = 2$).

Note that the present study extends the results obtained in [6], where only the range $0 < \alpha < 1$ was investigated.

2 ERINGEN NONLOCAL FRACTIONAL MODEL

According to Eringen nonlocal elasticity [4], the stress $\sigma(x)$ at a given point depends on the strain $\varepsilon(x)$ in a neighborhood of that point by means of a convolution integral. This dependence is described by a proper attenuation function g, which decays along with the distance. In the case of a one-dimensional domain (i.e. a bar), the constitutive law reads:

$$\sigma(x) = E\left[\beta_1\varepsilon(x) + \beta_2\kappa_\alpha \int_a^b \varepsilon(y)g(x-y)\mathrm{d}y\right] \tag{1}$$

where $x = a$ and $x = b$ are the bar extreme coordinates, E the Young's modulus, ε the strain defined as the derivative of the longitudinal displacement u and κ_α is a material constant. The bar length is L ($L = b - a$). The parameters β_1 and β_2, as in the classical nonlocal approach [7], weigh the local and the nonlocal contributions: $\beta_1 + \beta_2 = 1, 0 \le \beta_1, \beta_2 \le 1$.

If the attenuation function g is taken in the form [2, 3]

$$g(\xi) = \frac{1}{2\Gamma(2-\alpha)|\xi|^{\alpha-1}}, \tag{2}$$

with $1 < \alpha < 2$, the constitutive relationship becomes:

$$\sigma(x) = E\left[\beta_1\varepsilon + \beta_2\kappa_\alpha(I_{a,b}^{2-\alpha}\varepsilon)\right], \tag{3}$$

where the operator $I_{a,b}^{\beta}$ is the fractional Riesz integral ($\beta > 0$, [8])

$$I_{a,b}^{\beta} f(x) = \frac{1}{2\Gamma(\beta)} \int_a^b \frac{f(y)}{|x-y|^{1-\beta}}\mathrm{d}y. \tag{4}$$

The constant κ_α has hence anomalous physical dimensions $[\mathbf{L}]^{\alpha-2}$ and the following condition holds, for the sake of completeness: $\kappa_\alpha = 1$ for $\alpha = 2$.

Equation (3) reverts to the classical constitutive relationship for $\alpha = 2$

$$\sigma = (\beta_1 + \beta_2)E\varepsilon = E\varepsilon, \tag{5}$$

while, for $\alpha = 1$, it provides

$$\sigma = \beta_1 E\varepsilon + \frac{\beta_2 E\kappa_\alpha}{2}(u_b - u_a), \tag{6}$$

which describes the behavior of a bar possessing a reduced Young's modulus $\beta_1 E$ with a spring of stiffness $\beta_2 EA\kappa_\alpha/2$ connecting its extremes, A being the bar cross-section.

In order to get the equilibrium equation in terms of the displacement function $u(x)$, we simply need to substitute Eq. (3) into the static equation $\mathrm{d}\sigma/\mathrm{d}x + f(x) = 0$, where $f(x)$ is the longitudinal force per unit volume. By exploiting the definitions of the Riemann-Liouville fractional derivatives ($1 < \beta < 2$, [8, 9]):

$$D_{a+}^{\beta} f(x) = \frac{f(a)}{\Gamma(1-\beta)(x-a)^{\beta}} + \frac{f'(a)}{\Gamma(2-\beta)(x-a)^{\beta-1}} + \frac{1}{\Gamma(2-\beta)} \int_a^x \frac{f''(y)}{(x-y)^{\beta-1}}\mathrm{d}y \tag{7}$$

$$D_{b-}^{\beta} f(x) = \frac{f(b)}{\Gamma(1-\beta)(b-x)^{\beta}} - \frac{f'(b)}{\Gamma(2-\beta)(b-x)^{\beta-1}} + \frac{1}{\Gamma(2-\beta)} \int_x^b \frac{f''(y)}{(y-x)^{\beta-1}}\mathrm{d}y \tag{8}$$

some more analytical manipulations lead to [2, 10]:

$$\beta_1 \frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{\beta_2\kappa_\alpha}{2}\{D_{a+}^{\alpha}[u(x) - u(a)] + D_{b-}^{\alpha}[u(x) - u(b)]\} = -\frac{f(x)}{E}. \tag{9}$$

The term in the curly brackets is equal to $2u''$ when $\alpha = 2$, and vanishes when $\alpha = 1$. Equation (9) is a fractional differential equation, whose solution, obtained by means of fractional finite differences, was provided in [3]. A detailed discussion on the boundary conditions related to Eq. (9) can be found in [2].

Notice that Eq.(9) can be rewritten, by exploiting the definition of the Marchaud fractional derivatives [8], as [2, 3]:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{\kappa_\alpha(\alpha-1)}{2\Gamma(2-\alpha)}\left[\frac{u(x)-u(a)}{(x-a)^\alpha} + \frac{u(x)-u(b)}{(b-x)^\alpha} + \alpha\int_a^b \frac{u(x)-u(y)}{|x-y|^{1+\alpha}}\mathrm{d}y\right] = -\frac{f(x)}{E} \qquad (10)$$

2.1 *Equivalent point-spring model*

On the basis of the analysis presented in [1], a physical interpretation of the governing equation (10) was given in [2]. Let us introduce a partition of the interval $[a, b]$ on the x axis made of n ($n \in \mathrm{N}$) intervals of length $\Delta x = L/n$. The generic point of the partition has the abscissa x_i, with $i = 1, ..., n+1$ and $x_1 = a$, $x_{n+1} = b$; that is, $x_i = a + (i-1)\Delta x$. Hence, for the inner points of the domain ($i = 2, ..., n$), the discrete form of Eq. (10) can be written as

$$k^l_{i,i+1}(u_i - u_{i+1}) + k^l_{i,i-1}(u_i - u_{i-1}) + k^{vs}_{i,1}(u_i - u_1) + k^{vs}_{i,n+1}(u_i - u_{n+1}) + \\ + \sum_{j=1, j\neq i}^{n+1} k^{vv}_{i,j}(u_i - u_j) = f_i A\Delta x \qquad (11)$$

where $u_i \equiv u(x_i)$ and $f_i \equiv f(x_i)$. It is evident how the nonlocal fractional model is equivalent to a point-spring model where three kinds of springs appear: the local springs, ruling the local interactions, whose stiffness is k^l; the springs connecting the inner material points with the bar edges, ruling the volume-surface long-range interactions, with stiffness k^{vs}; the springs connecting the inner material points each other, describing the nonlocal interactions between non-adjacent volumes, whose stiffness is k^{vv}. Provided that the indexes are never equal one to the other, the following expressions for the stiffnesses hold ($i = 1, ..., n+1$):

$$k^l_{i,i+1} = k^l_{i+1,i} = \beta_1 EA/\Delta x \qquad (12)$$

$$k^{vs}_{i,1} = k^{vs}_{1,i} = \beta_2 EA\kappa_\alpha \frac{\alpha-1}{2\Gamma(2-\alpha)}\frac{\Delta x}{(x_i - x_1)^\alpha} \qquad (13)$$

$$k^{vs}_{i,n+1} = k^{vs}_{n+1,i} = \beta_2 EA\kappa_\alpha \frac{\alpha-1}{2\Gamma(2-\alpha)}\frac{\Delta x}{(x_{n+1} - x_i)^\alpha} \qquad (14)$$

$$k^{vv}_{i,j} = k^{vv}_{j,i} = \beta_2 EA\kappa_\alpha \frac{\alpha(\alpha-1)}{2\Gamma(2-\alpha)}\frac{(\Delta x)^2}{|x_i - x_j|^{1+\alpha}} \qquad (15)$$

Furthermore, by exploiting the Principle of Virtual Work to derive the proper either kinematic or static boundary conditions, it is possible to show that a fourth set of springs has to be introduced: it is composed by a unique spring connecting the two bar extremes with the stiffness:

$$k^{ss}_{1,n+1} = k^{ss}_{n+1,1} = \frac{\beta_2 EA\kappa_\alpha}{2\Gamma(2-\alpha)}\frac{1}{(x_{n+1} - x_1)^{\alpha-1}} \qquad (16)$$

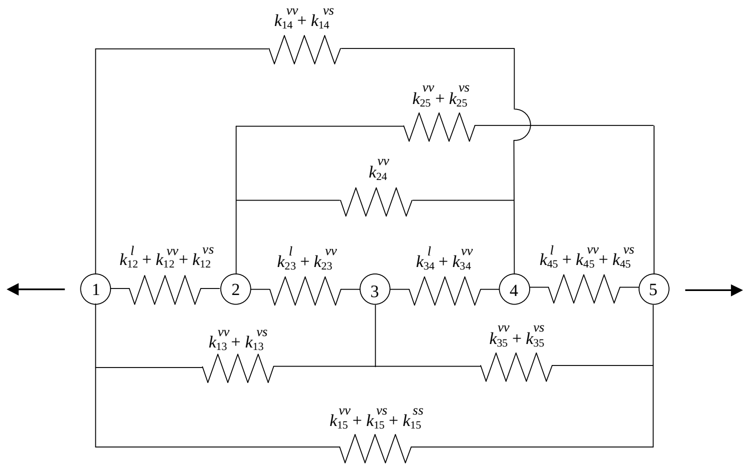

Figure 1: Pont-spring model equivalent to the nonlocal fractional elastic bar $(n = 4)$.

The superscript ss for the stiffness (16) is used since the spring connecting the bar edges can be seen as modelling the interactions between material points lying on the surface, which, in the simple one-dimensional model under examination, reduce to the two points $x = a, b$. Note that the presence of such a spring was implicitly embedded in the constitutive equation (3). However, since it provides a constant stress contribution throughout the bar length, its presence was lost by derivation when inserting the constitutive relationship into the differential equilibrium equation, i.e. when passing from Eq. (3) to Eq. (9).

To summarize, the constitutive fractional relationship (3) is equivalent to a point-spring model with four sets of springs, one local (12) and three nonlocal (13)-(16). Note that their stiffnesses all decay with the distance, although the decaying velocity differs from one kind to the other. The equivalent point-spring model is drawn in Fig.1 for $n = 4$.

For what concerns the limit cases, $\alpha = 2$ corresponds to the classical local elastic bar. In fact, if $\alpha \to 2^-$, since $\Gamma(0^+) = +\infty$, the surface-surface (Eq. (16)) and the volume-surface (Eqs. (13)-(14)) contributions disappear. For what concerns the interactions between inner material points (Eq. (15)), only the interactions between adjacent material points are retained (the Gamma function tends to infinity, but the integral in Eq. (10) diverges). Correspondingly, the additive term in Eq. (3) has the same form as the classical (local) one, the model representing a bar with a Young's modulus equal to $E(\beta_1 + \beta_2) = E$, while the governing equation (10) becomes $u'' = f/E$. On the other hand, if $\alpha \to 1^+$, the volume-volume and the volume-surface spring interactions ruled by Eqs. (13)-(15) vanish, and only the contribution (16) remains (together with the local springs (12)): the nonlocal model corresponds to a classical elastic bar with a reduced Young's modulus $\beta_1 E$ in parallel with a spring of stiffness $EA\kappa_\alpha/2$. The governing equation reverts to: $u'' = f/(\beta_1 E)$.

3 WAVE PROPAGATION

By means of a variational approach, similarly to that performed in [6], the fractional wave equation on a bar of finite length takes the following expression:

$$u_{tt}(x,t) = c_l^2 \left(\beta_1 u_{xx}(x,t) + \frac{\beta_2 \kappa_\alpha}{2} \{ {}_x D_{a+}^\alpha [u(x,t) - u(a,t)] +_x D_{b-}^\alpha [u(x,t) - u(b,t)] \} \right), \tag{17}$$

where t is the time variable, $c_l = \sqrt{E/\rho}$ is the well-known propagation speed of the wave, ρ being the bar volumetric density, and ${}_x D_{a+}^\alpha$ and ${}_x D_{b-}^\alpha$ are the fractional derivatives with respect to the spatial variable x . The conventions $[\cdot]_{tt} = \partial^2/\partial t^2$ and $[\cdot]_{xx} = \partial^2/\partial x^2$ are now adopted, for the sake of simplicity. For $\alpha = 2$, as already discussed, the constitutive relationship is the classical one and Eq. (17) becomes:

$$u_{tt}(x,t) = c_l^2 u_{xx}(x,t). \tag{18}$$

On the other hand, for $\alpha = 1$ the term in the curly brackets vanishes and Eq. (17) provides

$$u_{tt}(x,t) = c_l^2 \beta_1 u_{xx}(x,t) = \frac{\beta_1 E}{\rho} u_{xx}(x,t), \tag{19}$$

i.e., the wave equation on a local bar with a reduced Young's modulus. Consequently, also the wave propagation speed results reduced.

Suitable initial and boundary conditions must be assigned to Eq. (17) (only Dirichlet boundary conditions will be considered in the present analysis, for the sake of simplicity):

$$u(x,0) = \gamma_1(x) \qquad u_t(x,0) = \gamma_2(x),$$

$$u(a,t) = g_1(t) \qquad u(b,t) = g_2(t).$$

Analytical solutions of the fractional wave equation can be obtained through the Laplace-Fourier transforms [10], for what concerns infinite space domains. On the other hand, if a bar of finite length is taken into account, as in the present case, the problem can be faced, to the authors' best knowledge, only by numerical schemes. Since the order of derivation is comprised between 1 and 2, we chose to implement the so-called L2 algorithm firstly proposed by Oldham and Spanier [5] and later applied to fractional diffusion equations by Yang et al. [11]. The L2 algorithm is based on the formulae (7) and (8). By approximating the first and the second order derivatives by means of the usual finite differences and evaluating analytically the remaining part of the integrals in Eqs. (7-8), we get the following approximate discrete expressions of the fractional derivatives in the internal points of the domain $[a,b]$, i.e. for $i = 2, ..., n$:

$$\begin{aligned} {}_x D_{a+}^\alpha f(x_i,t_j) \approx \frac{(\Delta x)^{-(\alpha)}}{\Gamma(3-\alpha)} \{ &\frac{(1-\alpha)(2-\alpha)}{(i-1)^\alpha} f_{1,j} + \frac{2-\alpha}{(i-1)^{\alpha-1}} (f_{2,j} - f_{1,j}) + \\ &+ \sum_{k=0}^{i-2} (f_{i-k+1,j} - 2f_{i-k,j} + f_{i-k-1,j})[(k+1)^{2-\alpha} - k^{2-\alpha}] \} \end{aligned} \tag{20}$$

$$\begin{aligned} {}_x D_{b-}^\alpha f(x_i,t_j) \approx \frac{(\Delta x)^{-(\alpha)}}{\Gamma(3-\alpha)} \{ &\frac{(1-\alpha)(2-\alpha)}{(n-i+1)^\alpha} f_{n+1,j} - \frac{2-\alpha}{(n-i+1)^{\alpha-1}} (f_{n+1,j} - f_{n,j}) + \\ &+ \sum_{k=0}^{n-i} (f_{i+k+1,j} - 2f_{i+k,j} + f_{i+k-1,j})[(k+1)^{2-\alpha} - k^{2-\alpha}] \} \end{aligned} \tag{21}$$

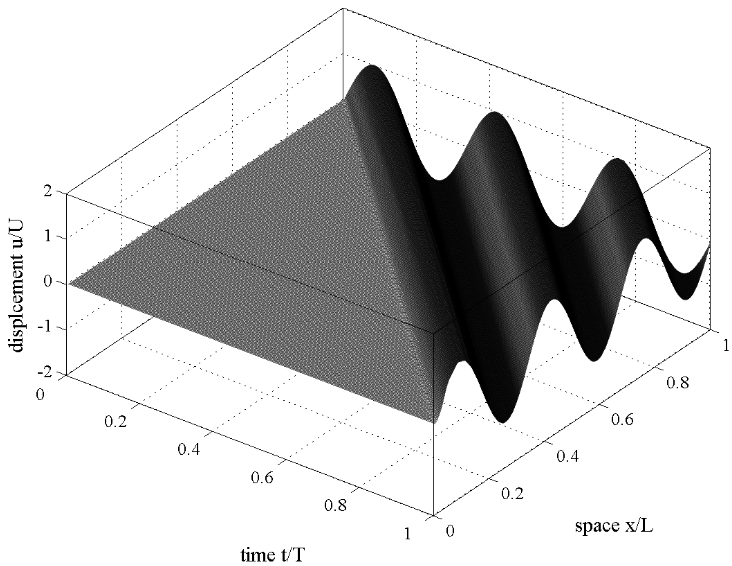

Figure 2: Wave propagation in a local bar: $\alpha = 2$.

Notice that $t_j$is equal to $j\Delta t$, with $j = 1, ..., m + 1$, where $\Delta t = T/m$ represents the discretization step of the time domain $[0, T]$.

The final algorithm to solve Eq. (17) can be written starting from that proposed for the classical wave equation ($\alpha = 2$), by properly taking into account the contributions provided by Eqs. (20-21).

We have applied the developed fractional nonlocal model to investigate the wave propagation in a clamped bar of length L ($a = 0, b = L$) subjected to a prescribed sinusoidal displacement at the right extreme ($b = L$). The following conditions are assigned to Eq.(17):

$$\gamma_1 = \gamma_2 = 0, \qquad g_1 = 0, \quad g_2 = U_f \sin(\omega_F t),$$

where U_f and ω_f are the amplitude and the frequency of the forcing term, respectively.

The parameters used for computations are: $L = 5$m, $A = 0.1$m^2, $M = 5$Kg, (thus, $\rho = M/(AL) = 10$Kg/m^3), $E = 10$N/m^2, $\beta_1 = 0.1$, $\kappa_\alpha = 1$ m$^{\alpha-2}$, $T = 5$s, $U_f = 0.001$m and $\omega_F = \pi/50$ 1/s. Moreover, n and m are chosen equal to 201 and 301, respectively: thus $\lambda = \Delta t/\Delta x < 1$, and the numerical scheme stability is guaranteed.

Results are presented in Fig. 2,3 and 4 for what concerns α=2, 1.5 and 1, respectively. For non-integer orders of derivation (Fig. 3), the wave shape is not so defined as it is in the local case . This is imputable to the presence of long-range interactions (Eqs.(13)-(16)), which prevent the formation of a marked wavefront. Notice also that the wave propagation speed, in the case $\alpha = 1$ (Fig. 4), results consistently reduced by a factor $\sqrt{\beta_1 = 0.1} \approx 0.32$ with respect to the classical case (Fig. 4, $\alpha = 2$).

3.1 Resonant frequencies

As the wave approaches the fixed left end, it starts to reflect back in the opposite direction along the bar and to interfere with the incoming wave. If the forcing frequency coincides with one of

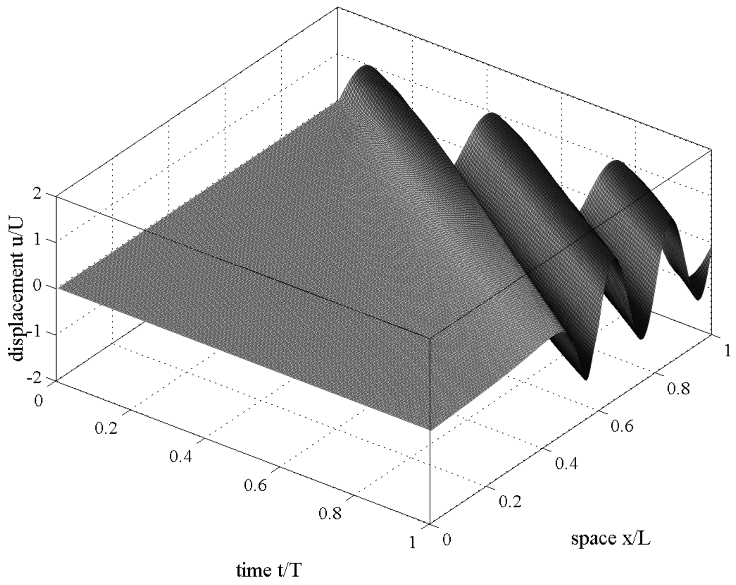

Figure 3: Wave propagation in a nonlocal bar: $\alpha = 1.5$.

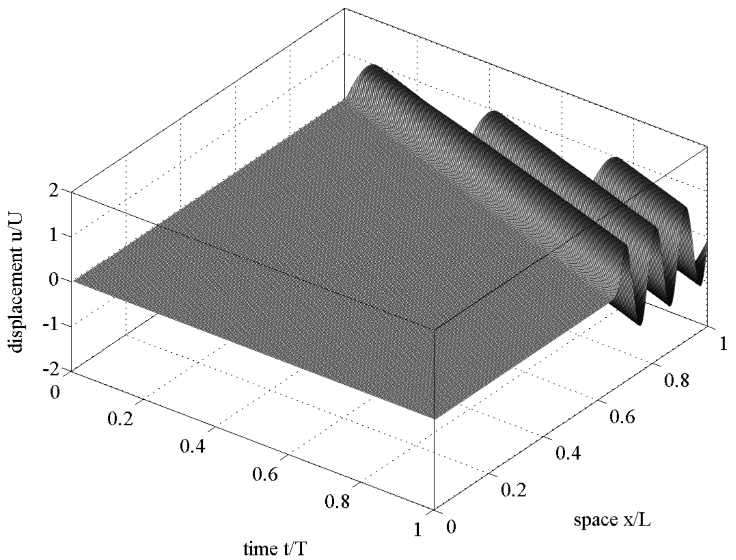

Figure 4: Wave propagation in a nonlocal bar: $\alpha = 1$.

the resonant frequencies of the structure, it is legitimate to expect that the asymptotic condition, for $T \to \infty$, is a standing elastic wave, i.e., a wave that remains in a constant configuration.

In the present section, the resonant natural frequencies ω of a nonlocal bar are evaluated. The analysis reduces to the solution of the well-known eigenvalue problem:

$$\mathrm{Det}(\mathbf{K} - \omega^2 \mathbf{M}) = 0, \tag{22}$$

$\mathbf{K}$ and $\mathbf{M}$ being the stiffness and mass matrices, respectively.

The stiffness matrix $\mathbf{K}$, according to what has been presented in Section 2.1, is the sum of four stiffness matrices:

$$\mathbf{K} = \mathbf{K}^l + \mathbf{K}^{vv} + \mathbf{K}^{vs} + \mathbf{K}^{ss} \tag{23}$$

whose non-diagonal terms are provided by the opposite of the corresponding stiffnesses (12-16). Furthermore, the diagonal terms $k_{i,i}$ of each matrix are given by the relationship:

$$k_{i,i} = \sum_{j=1, j\neq 1}^{n+1} k_{i,j} \qquad i = 1, ..., n+1 \tag{24}$$

Note that all the four matrices are symmetrical, with positive elements on the diagonal and negative outside. More in detail, the local matrix $\mathbf{K}^l$ is tridiagonal; the nonlocal matrix $\mathbf{K}^{vv}$ ruling the long-range interactions between inner points is fully populated; the nonlocal matrix related to the inner-outer interactions $\mathbf{K}^{vs}$ has only border and diagonal elements different from zero; finally, the nonlocal matrix $\mathbf{K}^{ss}$ describing the interaction between the bar edges is empty except for the four corner elements.

On the other hand, $\mathbf{M}$ is a diagonal matrix, whose elements are all equal to M/n.

Let us start by considering a double clamped bar. The constraint conditions can be expressed by deleting the first and the last rows and columns in the matrices $\mathbf{K}$ and $\mathbf{M}$. Thus, in the present case, $\mathbf{K}^{ss}$ provides no contributions. The first ten natural frequencies obtained by solving Eq.(22), for different fractional orders, are plotted in Fig. 5. If $\alpha = 2$ (Eq.(5)), as in the classical case, it is found that:

$$\omega_{r,\alpha=2} = r\frac{\pi}{L}\sqrt{\frac{E}{\rho}}, \qquad r = 1, 2, 3... \tag{25}$$

On the other hand, for $\alpha = 1$, we get :

$$\omega_{r,\alpha=1} = r\frac{\pi}{L}\sqrt{\frac{\beta_1 E}{\rho}}, \qquad r = 1, 2, 3... \tag{26}$$

since the second contribution in Eq.(6) vanishes. For non-integer values of α, the relation $\omega_{r,\alpha}$ vs. r is not linear: the resonant frequencies are comprised, at least for the higher modes, between those evaluated through Eqs.(25) and (26).

Eventually, an analogous situation is recovered if a single-clamped bar is considered. Results are presented in Fig. 6. While Eq. (25) still holds, provided that r is substituted by $r - 0.5$, a generalization of Eq. (26) is not so straightforward, since the matrix $\mathbf{K}^{ss}$ contribution is different from zero (see Eq. (6))

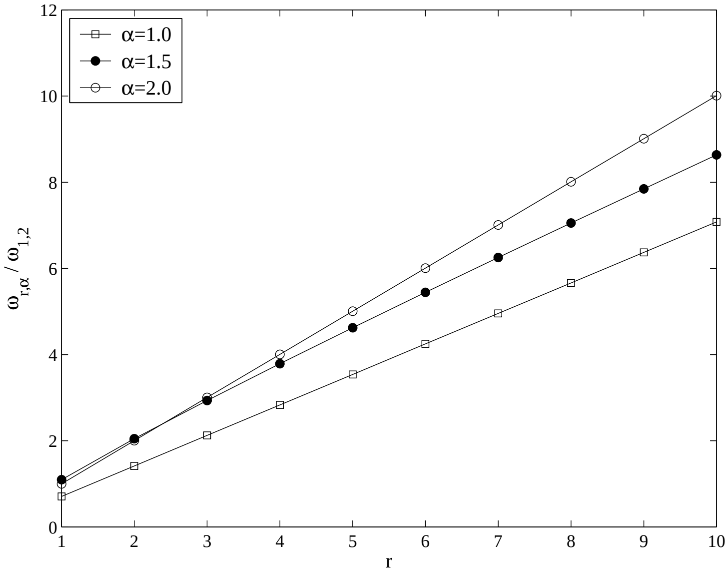

Figure 5: First ten resonant frequencies of a double clamped nonlocal bar, for different fractional orders α.

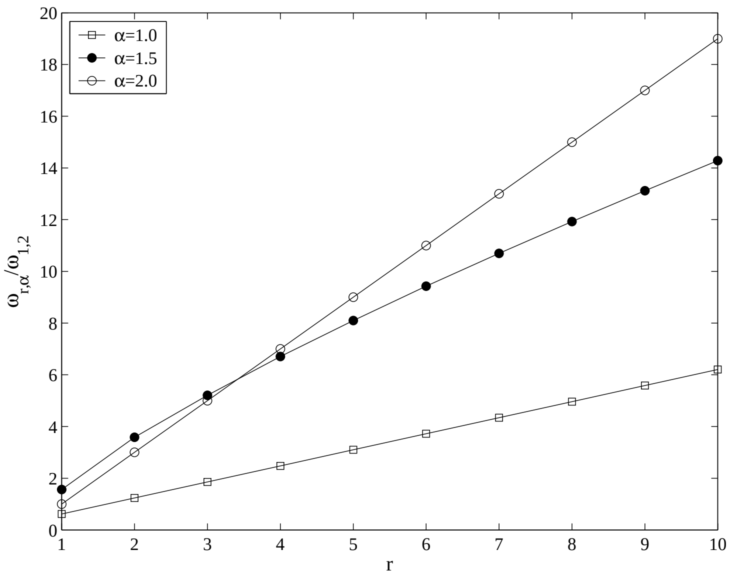

Figure 6: First ten resonant frequencies of a single clamped nonlocal bar, for different fractional orders α.

Acknowledgments
The financial support of the Italian Ministry of Education, University and Research (MIUR) to the Project ”Advanced applications of Fracture Mechanics for the study of integrity and durability of materials and structures” within the ”Programmi di ricerca scientifica di rilevante interesse nazionale (PRIN)” program for the year 2008 is gratefully acknowledged.

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