A Finite Fracture Mechanics approach to the asymptotic behaviour of U-notched structures

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ABSTRACT. A Finite Fracture Mechanics criterion is put forward to predict the critical failure loads of brittle U-notched specimens, subjected to mode I loading. The criterion, recently applied to investigate V-notched structures, requires the contemporaneous fulfilment of stress requirements and energy conditions for fracture to propagate: both the stress field function at the notch tip and the stress intensity factor function related to a crack stemming from the tip are involved. The finite crack advancement results to be a structural parameter, depending on the material properties as well as on the notch radius. Eventually, the predictions are compared with those provided by other theoretical approaches and with experimental data available in the Literature, showing a general good agreement.

KEYWORDS. Finite Fracture Mechanics, blunted notches, brittle failure.

INTRODUCTION

A coupled Finite Fracture Mechanics (FFM) criterion has been recently proposed to predict the failure load of brittle V-notched structures [1-3]. The criterion is based on the hypothesis of a finite crack advance Δ and assumes a contemporaneous fulfilment of stress requirement and energy balance for crack propagation:

$$\begin{cases} \int_0^{\Delta} \sigma_y(x)\, dx \geq \sigma_u\, \Delta \\ \int_0^{\Delta} K_I^2(c)\, dc \geq K_{Ic}^2\, \Delta \end{cases}, \quad (1)$$

where $\sigma_y(x)$ is the singular stress field ahead of the notch tip, $K_I(c)$ is the stress intensity factor related to a crack of length c stemming from the notch root, σ_u and K_{Ic} are the material tensile strength and fracture toughness, respectively.
The former inequality requires that the average stress upon the crack advance Δ is higher than material tensile strength; the latter one ensures that the energy available for a crack increment Δ is higher than the energy necessary to create the new fracture surface. For positive geometries, the failure load (i.e. the lowest load satisfying Eq. (1)) is attained when the inequalities are strictly verified, i.e. they are replaced by two equations. In such a case Eq. (1) becomes a system of two equations in two unknowns: the crack advancement and the failure load.
The coupled FFM criterion was introduced to remove some inconsistencies related to the criteria previously introduced [4-6], according to which either stress or energetic considerations result violated (see, for instance, [2]). Notice that a similar approach to Eq. (1), but based on a point-wise stress requirement, was proposed in [7].
In the present work, the coupled criterion (1) is applied to estimate the failure loads of U-notched brittle structures, subjected to mode I loading. The analysis involves the characterization of the stress field and the stress intensity factor functions to be inserted into Eq. (1), which leads to the solution of non-linear equations. After considering the general case of an elliptical hole in an infinite plate under remote tension, results related to three or four point bending tests on

ceramic U-notched samples [8-9] are taken into account. Predictions by the FFM coupled criterion are in good agreement both with experimental data and with those provided by other theoretical approaches [10-12].

ELLIPTICAL HOLE IN A PLATE SUBJECTED TO TENSION

Let us consider an infinite plate subjected to a tensile traction σ in the y direction. This uniaxial condition is disturbed by an elliptical hole, whose 2a-long major axis is directed along the x-axis and the 2b-long minor axis along the y-axis (Fig. 1a). This geometry was analyzed by Inglis in 1913 [13] (see also [14]).

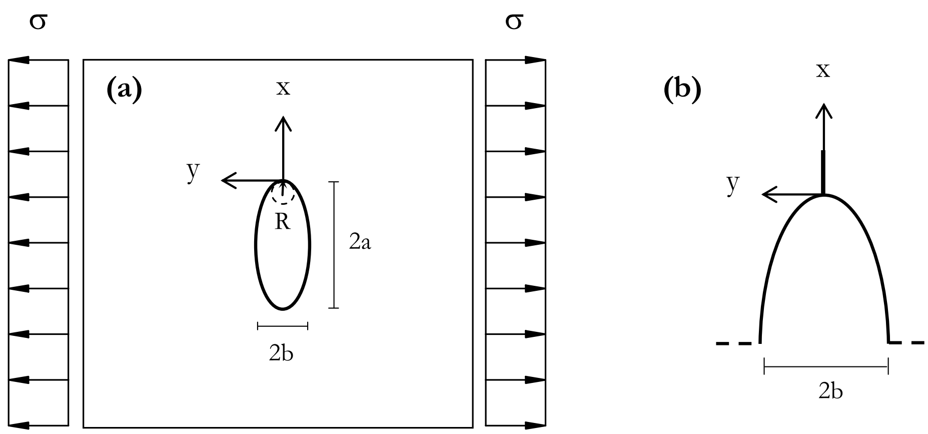

Figure 1: (a) Infinite plate containing an elliptical hole, subjected to uniaxial applied tension σ. (b) Crack of length c stemming from the notch tip.

By denoting with k the ratio between the axis lengths (k=b/a), the stress field $\sigma_y(x)$ ahead of the notch tip reads, in cartesian coordinates, as:

$$\sigma_y(x,0) = \frac{k^2 + (1-2k)\mathrm{cotgh}(\xi) + (k^2/(1+k))\cosh(\xi)/\sinh^3(\xi)}{(1-k)^2}\sigma, \tag{2}$$

with

$$\xi = \mathrm{arccosh}\frac{(x+a)/a}{\sqrt{1-k^2}}, \qquad x \geq 0. \tag{3}$$

The stress field described by Eq. (2) acts also on the opposite side of the elliptic hole, due to symmetry. For the sake of simplicity, the present analysis will only refer to the domain x≥0. Notice that Eq. (2) could be rewritten equivalently in terms of the root radius $R=b^2/a=k^2a$ and that evaluating Eq. (2) at x=0 yields:

$$\frac{\sigma_{max}}{\sigma} = 1 + \frac{2}{k}, \tag{4}$$

which represents the well-known stress concentration factor.
For sufficiently slender notches (k<< 1), Eq. (2) can be approximated by the Creager-Paris' expression [15]:

$$\sigma_y(x,0) = \frac{2K_I^U}{\sqrt{\pi}} \frac{x+R}{(2x+R)^{3/2}}, \tag{5}$$

where K_I^U is the apparent stress intensity factor (SIF)

$$K_I^U = \sigma\sqrt{\pi a}. \tag{6}$$

It is simple to show that, according to Eq. (5), the singular asymptotic stress field ahead of a crack tip is recovered for $R \to 0$.
Before proceeding, it is worthwhile to mention that also Eq. (2) could be expressed through the apparent SIF K_I^U (Eq. (6)), as it will be done in the next sections.

Stress intensity factor for a crack at the notch root

In order to apply the coupled FFM criterion (Eq. 1), the SIF $K_I(c)$ related to a short crack of length c emanating from the notch tip is needed (Fig. 1b). For many notch-crack problems, $K_I(c)$ could be generally expressed as

$$K_I(c) = F\sigma\sqrt{\pi c}, \tag{7}$$

F being a geometric factor [16]. Different analytical expressions for F have been proposed in the Literature [17]. Lukas, for instance, proposed the following one [18]:

$$F = 1.12\left(1+\frac{2}{k}\right)\sqrt{\frac{R}{4.5c+R}}, \tag{8}$$

which provides errors within 5% provided that $c/R \le 0.2/k$. For $k \to 0$, Eq. (9) is thus valid for any ratio c/R.
Substituting Eq. (8) into Eq. (7), with the help of Eqs. (4) and (6), yields:

$$K_I(c) = 1.12(2+k)\sqrt{\frac{c}{4.5c+R}}K_I^U. \tag{9}$$

Equation (9) provides coherently $K_I(c)=1.12\sigma\sqrt{\pi c}$, i.e. the stress intensity factor for an edge crack in a semi-infinite plate under remote tension, as $k\to\infty$. On the other hand, for $k\to 0$ (i.e. the crack case), it is found that $K_I(c) \approx 1.056 K_I^U$.

Result analysis by the coupled FFM criterion

The coupled FFM criterion can be now be implemented. In critical conditions, by inserting Eqs. (2) and (9) into Eq. (1) and integrating, it is found that:

$$\begin{cases} \dfrac{K_{Ic}^U}{\sigma_u\sqrt{R}} = \dfrac{1}{\tilde{\sigma}(\mu_c)} \\ \dfrac{K_{Ic}^U}{K_{Ic}} = \sqrt{\dfrac{1}{\tilde{g}(\mu_c)}} \end{cases}, \tag{10}$$

where the two functions $\tilde{\sigma}(\mu_c)$ and $\tilde{g}(\mu_c)$ are expressed as:

$$\tilde{\sigma}(\mu_c) = \frac{\sqrt{k^2-1}\left[k^2\sinh(\xi_c)+(1-2k)\cosh(\xi_c)+\dfrac{k^2}{1+k}\dfrac{1}{\cosh(\xi_c)}\right]}{\sqrt{\pi}(k-1)^2 k\mu_c}, \tag{11}$$

where

$$\xi_c = \operatorname{arcsinh}\frac{1+\mu_c k^2}{\sqrt{k^2-1}}, \tag{12}$$

and

$$\tilde{g}(\mu_c) = \frac{1.12^2}{4.5}(k+2)^2\left[1-\frac{1}{4.5\mu_c}\ln\left(1+4.5\mu_c\right)\right]. \tag{13}$$

Notice that the same terminology used in [11] is here adopted. The two unknowns in (10) are the dimensionless critical crack advancement $\mu_c=\Delta_c/R$ and the dimensionless apparent fracture toughness $\overline{K}_{Ic}^U = K_{Ic}^U / K_{Ic}$ (i.e. the dimensionless failure load, Eq. (6)). System (10) could be rewritten as:

$$\begin{cases} \sqrt{\overline{R}} = \dfrac{\tilde{\sigma}(\mu_c)}{\sqrt{\tilde{g}(\mu_c)}} \\ \overline{K}_{Ic}^U = \sqrt{\dfrac{1}{\tilde{g}(\mu_c)}} \end{cases}. \tag{14}$$

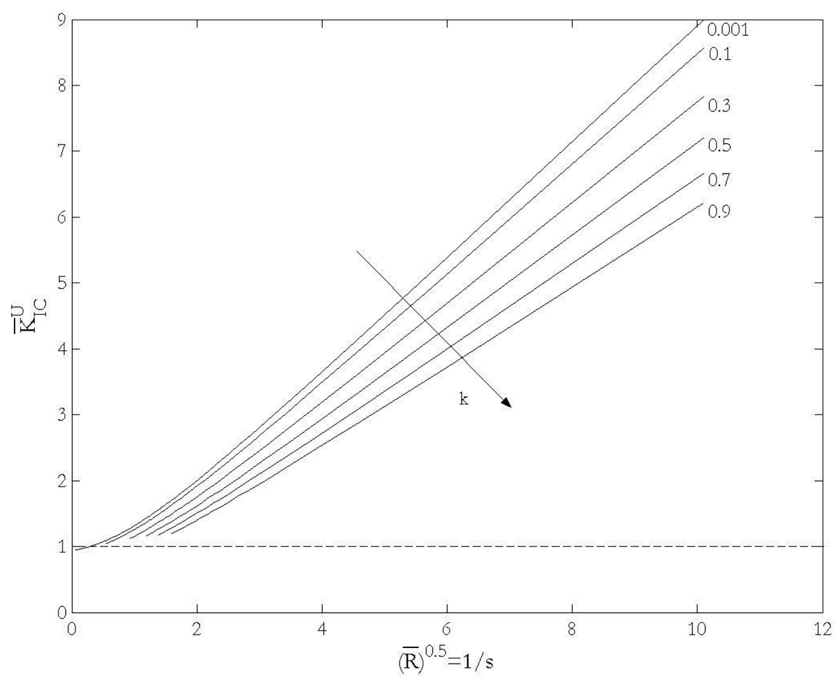

Figure 2: Dimensionless apparent fracture toughness vs. dimensionless square root of the notch-root radius for different k-ratios, according to the coupled FFM criterion.

Once μ_c is known, the first equation in (14) provides the dimensionless square root of the notch-root radius $\sqrt{\overline{R}} = \sigma_u\sqrt{R} / K_{Ic} = 1/s$, where s is the brittleness number [19-20]: brittle structural behaviours are generally expected

for low brittleness numbers. On the other hand, the dimensionless apparent fracture toughness is provided by the second equation of system (14). In Fig. (2) the curves $\overline{K}_{Ic}^{U}$ vs. $\sqrt{\overline{R}}$ for different k-ratios are reported.
How it can be seen, for a fixed k, the apparent fracture toughness decreases as the radius decreases. For low k-values the hole becomes more sharp: as a consequence, as $\sqrt{\overline{R}} \to 0$, the curve approaches the unit value, that is the apparent fracture toughness approaches the real fracture toughness of the material.
The crack advance in critical conditions Δ_c, representing the second output of Eq. (14), is not plotted here, but it can be shown that: (1) it decreases as s decreases (2) different k values give almost identical curves. The dependence on the brittleness number shows that the crack advance is a structural parameter, depending not only on the material properties but also on the geometric dimensions, through the notch-root radius R.

EXPERIMENTAL VALIDATION

In this Section, the theoretical predictions from the FFM coupled criterion are compared with some experimental results available in the Literature. Four data sets are taken into account: they all refer to three or four point bending tests on single edge notched ceramic specimens [8-9]. Details of the sample geometry can be found in the quoted references, while the material properties have been reported in Table 1: the tensile strength values related to the tests carried out in [9] are not provided in the referenced paper and they are taken from a best fit procedure over a range of admissible values.

Material	**Ref.**	**σ_u** [MPa]	**K_{Ic}** [MPa m$^{1/2}$]
Alumina, Al_2O_3	[8]	290	3.81
Alumina, Al_2O_3	[9]	300	2.80
Silicon carbide, SiC	[9]	400	2.40
Silicon nitride, Si_3N_4	[9]	700	5.40

Table 1: Tensile strength and fracture toughness of the ceramic materials considered in this paper.

Since in the tested specimens the notch-root radius is much smaller than the other geometric dimensions, the Creager-Paris expression (5) results to be sufficient to describe the stress field ahead of the notch tip (Fig. 1). In such a case, by substituting Eq. (5) and (9) into Eq. (1), the dimensionless apparent fracture toughness becomes the solution of the following non-linear equation:

$$\overline{K}_{Ic}^{U} - \frac{0.947}{\sqrt{1 - \frac{1.3955\overline{R}}{\left(2\overline{K}_{Ic}^{U}\right)^2 - \pi\overline{R}} \log\left(2.866\frac{\left(\overline{K}_{Ic}^{U}\right)^2}{\overline{R}} - 1.25\right)}} = 0 \tag{15}$$

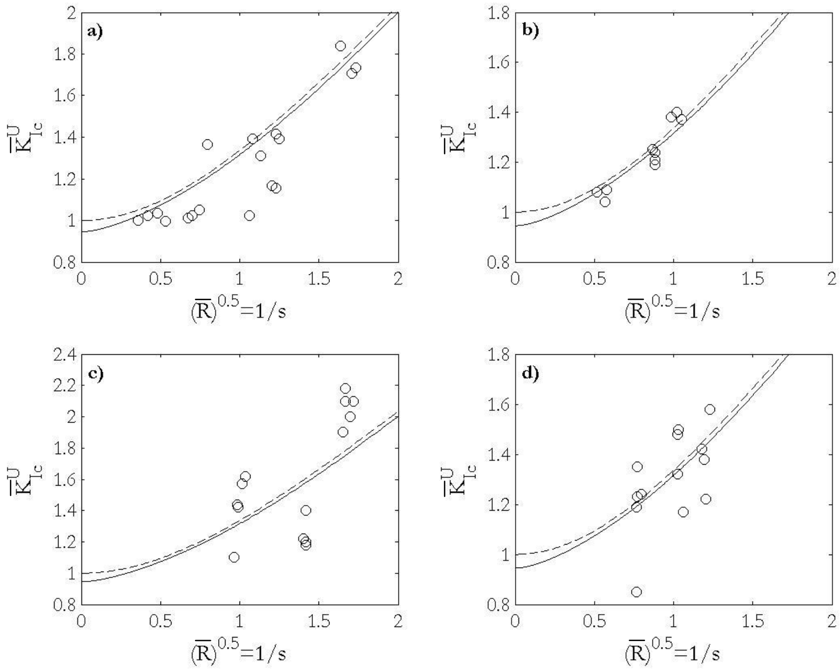

Figure 3: Dimensionless apparent fracture toughness vs. dimensionless square root of the notch root radius according to the coupled FFM criterion (continuous line) and the averaged stress criterion (dashed line). Experimental data (circles) refer to: a) alumina [8], b) alumina [9], c) silicon carbide [9], d) silicon nitride [9].

Predictions according to Eq. (15) and experimental data have been reported in Fig. 3, showing a general good agreement. For the sake of completeness, the results obtained by the mean stress criterion [10] have been also plotted. This criterion is based just on the former inequality in system (1), with $\Delta=(2/\pi)(K_{Ic}/\sigma_u)^2$. In formulae, it provides:

$$\overline{K}_{Ic}^{U} = \sqrt{1+\pi\overline{R}}\ . \tag{16}$$

From Fig. 3 it can be observed that the only significant difference (≈5%) between the predictions by the two criteria (Eqs. (15) and (16), respectively) is related to the behaviour near $\overline{R}=0$, due to the approximation introduced in [18]. In order to overcome this drawback, numerical simulations to estimate the real fracture energy can be carried out, as done in [11]. However, it should be added that the related solutions do no differ so much over the radius range of practical interest. Eventually, the analysis performed in [12] suggests that the results according to Eq. (15) will be very close also to those obtained through other theoretical approaches [5-6, 11].

CONCLUSIONS

In the present paper, the coupled stress and energy FFM criterion introduced in [1-3] has been applied to investigate the brittle fracture behaviour of blunt-notched structures. The predictions are in good agreement both with experimental data and with those provided by the simple average stress criterion [10], which can thus be applied to get sufficiently accurate results.

ACKNOWLEDGMENTS

The financial support of the Italian Ministry of Education, University and Research (MIUR) to the Project "Advanced applications of Fracture Mechanics for the study of integrity and durability of materials and structures" within the "Programmi di ricerca scientifica di rilevante interesse nazionale (PRIN)" program for the year 2008 is gratefully acknowledged.

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