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Research

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Abstract: We prove the existence of an unbounded connected branch of nontrivial homoclinic trajectories of a family of discrete nonautonomous asymptotically hyperbolic systems parametrized by a circle under assumptions involving the topological properties of the asymptotic stable bundles.

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1. Introduction

It was shown in [23] that the different twisting of the asymptotic stable bundles at plus and minus infinity of a family of discrete, nonautonomous, asymptotically hyperbolic systems parametrized by a circle, leads to the appearance of homoclinic trajectories bifurcating from the trivial branch of stationary solutions.

In [23], only small homoclinic trajectories close to the stationary branch were found. Here we will improve our previous results for the problem. It turns out that the same topological condition supplemented with other listed below, which ensure the properness of the nonlinear operator naturally associated to the problem, allows us to establish the existence of homoclinic trajectories of arbitrarelly large norm.

What is more, using the global bifurcation theory of [18], we show the existence of connected branches of nontrivial homoclinics going from the stationary branch to infinity.

Much as in our previous paper we will translate the appearance of homoclinic trajectories into a problem of

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bifurcation of zeros of a parametrized family of C^1-Fredholm maps $G\colon S^1 \times X \to X$, where X is a function space naturally associated to the problem. Our approach uses a peculiarity of the topological degree for proper Fredholm maps of index zero. Namely, that it is preserved only up to sign under homotopies of Fredholm maps ([12, 18]). As a matter of fact, that the degree can change sign along a homotopy will be central to our arguments. To say it shortly, we turn the lack of homotopy invariance of the degree of Fredholm maps into an useful instrument for the analysis of bifurcation phenomena.

We refer to [24, 25, 27, 28] for other bifurcation results for bounded solutions of difference equations, however to our best knowledge both the results and the methods of this paper are novel. The relation between the topological properties of the asymptotic bundles and bifurcation of homoclinics is far more subtle than the classical spectral analysis at the potential bifurcation points. Even for the simplest topologically nontrivial parameters space S^1 it requires topological instruments which at a first glance may appear unfamiliar to many. However we believe that the interest of the result and the generality of the method in proof provides enough reasons for its introduction.

The paper is organized as follows. In Section 2 we introduce the problem and the basic invariant measuring the topological nontriviality of the asymptotic bundles. Then we state our main theorem about the existence of a connected branch of nontrivial homoclinic solutions which connects the stationary branch to infinity through homoclinics of arbitrarily large norm.

In Section 3 we state and prove an index theorem for families of linear Fredholm operators which will be used in order to show that the twisting of asymptotic bundles forces the appearance of homoclinic trajectories. The proof is similar to the one given in [23] except for the fact that (in order to ensure properness of the relevant map) we have to work in a different function space. Section 4 is devoted to show that G is a continuous family of C^1-Fredholm maps. The continuity and smoothness of G involves only standard arguments, many of them taken from [26, 27]. The Fredholm property is derived from the asymptotic hyperbolicity of the linearization at the stationary solution. In order to apply the global bifurcation theory for C^1-Fredholm maps the map G has to be proper on closed bounded sets. Using ideas from [30] we will prove properness of G in Section 5. In Section 6 we discuss the generalized homotopy property of the topological degree constructed in [18] and we prove our main theorem using the computation of the index bundle from Section 3. Section 7 contains a nontrivial example illustrating our result.

2. The main theorem

All considered topological spaces are metric and all single-valued maps between spaces are continuous. Given a normed space $(\mathbb{E}, \|\cdot\|)$, $\bar{B}(x,r)$, and $B(x,r)$, will denote the closed and open disk centered at x of radius r respectively. The Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$; $\bar{B}_d(x,r)$ (resp. $B_d(x,r)$) is the closed (resp. open) disk centered at $x \in \mathbb{R}^d$; $d \geq 1$, of radius r. Additionally, throughout the article, a norm of a matrix M will be denoted by $|M|$.

A nonautonomous discrete dynamical system on $\mathbb{R}^d$ is defined by a doubly infinite sequence of maps

$$\mathbf{f} = \{f_n \colon \mathbb{R}^d \to \mathbb{R}^d \mid n \in \mathbb{Z}\}. \tag{1}$$

A trajectory of the system $\mathbf{f} \colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ is a sequence $\mathbf{x} = (x_n)$ such that

$$x_{n+1} = f_n(x_n). \tag{2}$$

A constant trajectory of $\mathbf{f}$ is called *stationary*. In the terminology of [24] (2) is a nonautonomous difference equation whose solutions are trajectories of the corresponding dynamical system.

Let us assume that each f_n is C^1 and that $f_n(0) = 0$. Hence, the sequence $\mathbf{0} = (0_n)$ is a stationary trajectory of $\mathbf{f}$. A trajectory $\mathbf{x} = (x_n)$ of $\mathbf{f}$ is called *homoclinic* to $\mathbf{0}$, or simply a homoclinic trajectory, if $\lim\limits_{n\to\pm\infty} x_n = 0$. Any stationary trajectory is trivially homoclinic to itself. Here we will be interested in nontrivial trajectories homoclinic to $\mathbf{0}$. Observe that a homoclinic trajectory of $\mathbf{f}$ is naturally an element of

$$\mathbf{c}(\mathbb{R}^d) := \left\{ \mathbf{x} \colon \mathbb{Z} \to \mathbb{R}^d \;\middle|\; \lim_{|n|\to\infty} x_n = 0 \right\}$$

equipped with the norm $\|\mathbf{x}\|_\infty := \sup\limits_{k\in\mathbb{Z}} |x_n|$ (see [23]). However in this paper we restrict yourself to the following space

$$\mathbf{l}^2 := \left\{ \mathbf{x} \colon \mathbb{Z} \to \mathbb{R}^d \;\middle|\; \|\mathbf{x}\| := \left(\sum_{n\in\mathbb{Z}} |x_n|^2 \right)^{1/2} < \infty \right\} \subset \mathbf{c}(\mathbb{R}^d).$$

The value $\mathbf{x}(n) = x_n$ of an element $\mathbf{x} \in \mathbf{l}^2$ in $n \in \mathbb{Z}$ will be denoted also by $p_n(\mathbf{x})$ according to the convenience.

We will show below that, under appropriate assumptions on the dynamical system $\mathbf{f}$, the Nemytskii (substitution) operator $F \colon \mathbf{l}^2 \to \mathbf{l}^2$ given by

$$F(\mathbf{x}) = (f_n(x_n)) \tag{3}$$

is a well defined C^1-map verifying $F(\mathbf{0}) = \mathbf{0}$. In this way nontrivial homoclinic trajectories become the nontrivial solutions of the equation $S\mathbf{x} - F(\mathbf{x}) = \mathbf{0}$, where $S \colon \mathbf{l}^2 \to \mathbf{l}^2$ is the shift operator given by

$$S\mathbf{x} = (x_{n+1}). \tag{4}$$

It should be noted that we have considered the space $\mathbf{l}^2$ here because on this space we are able to provide simple conditions which are necessary and sufficient for $S - F$ to be proper on closed bounded subsets of $\mathbf{l}^2$.

The linearization of the system $\mathbf{f}$ at the stationary solution $\mathbf{0}$ is the nonautonomous linear dynamical system $\mathbf{a} \colon \mathbb{Z} \times \mathbb{R}^d \to \mathbb{R}^d$ defined by the sequence of matrices $(a_n) \in \mathbb{R}^{d\times d}$, with $a_n = Df_n(0)$. The corresponding linear difference equation is defined by

$$x_{n+1} = a_n x_n. \tag{5}$$

We will deal only with discrete nonautonomous dynamical systems whose linearization at $\mathbf{0}$ is asymptotic for $n \to \pm\infty$ to an autonomous linear hyperbolic dynamical system $\mathbf{a}$ (i.e., verifying $a_n = a$ for all $n \in \mathbb{Z}$, where a is a hyperbolic matrix). We will call systems with the above property *asymptotically hyperbolic.*

Let us recall that an invertible matrix a is called *hyperbolic* if a has no eigenvalues of norm one, i.e., $\sigma(a) \cap \{|z| = 1\} = \emptyset$. The spectrum $\sigma(a)$ of a hyperbolic matrix a consists of two disjoint closed subsets $\sigma(a) \cap \{|z| < 1\}$ and $\sigma(a) \cap \{|z| > 1\}$, so $\mathbb{R}^d$ has the a-invariant spectral decomposition $\mathbb{R}^d = E^s(a) \oplus E^u(a)$, where $E^s(a)$ (respectively $E^u(a)$) is the direct sum real parts of the generalized eigenspaces corresponding to eigenvalues of a inside of the unit disk (respectively outside of the unit disk).

It is easy to see that $\zeta \in E^s(a)$ if and only if $\lim_{n\to\infty} a^n\zeta = 0$. The unstable subspace $E^u(a)$ has a similar characterization, i.e., $\zeta \in E^u(a)$ if and only if $\lim_{n\to\infty} a^{-n}\zeta = 0$. Let us describe precisely our setting and assumptions.

A C^1*-family of dynamical systems parametrized by the unit circle* S^1 is defined by a sequence of maps

$$\mathbf{f} = \left\{ f_n \colon S^1 \times \mathbb{R}^d \to \mathbb{R}^d \mid n \in \mathbb{Z} \right\} \tag{6}$$

such that f_n is C^1, for all $n \in \mathbb{Z}$.

In what follows we will also assume everywhere that $f_n(\lambda, 0) = 0$, for all $\lambda \in S^1$ and $n \in \mathbb{Z}$.

We will use $\mathbf{f}_\lambda$ to denote the dynamical system corresponding to the parameter value λ. We will use $\mathbf{f}_\lambda$ to denote the dynamical system corresponding to the parameter value λ.

Alternatively one can think of $\mathbf{f}$ as a double infinite sequence of C^1-maps $f_n \colon [a,b] \times \mathbb{R}^d \to \mathbb{R}^d$ such that $f_n(a,-)$ coincides with $f_n(b,-)$ up to the first order.

We will say that $(\lambda, \mathbf{x})$ is a *homoclinic solution* for the family $\mathbf{f}$ if $(\lambda, \mathbf{x})$ solves the parameter-dependent difference equation:

$$x_{n+1} = f_n(\lambda, x_n), \quad \text{for all } n \in \mathbb{Z}, \tag{7}$$

or equivalently, if $\mathbf{x} = (x_n)$ is a homoclinic trajectory of the dynamical system $\mathbf{f}_\lambda$. Homoclinic solutions of (7) of the form $(\lambda, \mathbf{0})$ are called trivial and the set $S^1 \times \{\mathbf{0}\}$ is called the *trivial or stationary branch.*

Our aim is to show how the topology of the simplest topologically nontrivial parameter space S^1 on which our dynamical system depends forces the appearance of branches of homoclinic trajectories connecting small homoclinic trajectories to the arbitrarily large ones. For this we will apply the global bifurcation theory for families of C^1-Fredholm maps established in [18] to the family defined by

$$G(\lambda, \mathbf{x}) = S\mathbf{x} - F(\lambda, \mathbf{x}), \tag{8}$$

where $F \colon S^1 \times \mathbf{l}^2 \to \mathbf{l}^2$ is the parametrized substitution (Nemytskii) operator $F(\lambda, \mathbf{x}) := (f_n(\lambda, x_n))$.

We will assume that the family of discrete dynamical systems $\mathbf{f} \colon \mathbb{Z} \times S^1 \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following conditions:

(A1) For any $M > 0$ and $\varepsilon > 0$ there exists $0 < \delta < M$ such that for all $(\lambda_1, x_1), (\lambda_2, x_2) \in S^1 \times \bar{B}_d(0, M)$ with $d\big((\lambda_1, x_1), (\lambda_2, x_2)\big) < \delta$ one has

$$\sup_{n\in\mathbb{Z}} \left| \frac{\partial f_n(\lambda_1, x_1)}{\partial x} - \frac{\partial f_n(\lambda_2, x_2)}{\partial x} \right| < \varepsilon \text{ and } \sup_{n\in\mathbb{Z}} \left| \frac{\partial f_n(\lambda_1, x_1)}{\partial \lambda} - \frac{\partial f_n(\lambda_2, x_2)}{\partial \lambda} \right| < \varepsilon,$$

where d is the product distance in the metric space $S^1 \times \mathbb{R}^d \subset \mathbb{R}^2 \times \mathbb{R}^d$.

(A2) The family of matrices

$$a_n(\lambda) := \frac{\partial f_n(\lambda, 0)}{\partial x} \xrightarrow[n\to\pm\infty]{} a(\lambda, \pm\infty)$$

(uniformly with respect to $\lambda \in S^1$), where $a(\lambda, \pm\infty)$ is a hyperbolic matrix. Moreover, we assume that for some (and hence for all) $(\lambda, 0) \in S^1 \times \mathbb{R}^d$, the limits $a(\lambda_0, +\infty)$ and $a(\lambda_0, -\infty)$ have the same number of eigenvalues (counting algebraic multiplicities) inside of the unit disk.

(A3) There exists $\lambda_0 \in S^1$ (say $\lambda_0 = 1$) such that the following two difference equations

$$x_{n+1} = f_n(1, x_n) \quad \text{and} \quad x_{n+1} = a_n(1)x_n$$

admit only the trivial solution $(x_n = 0)_{n\in\mathbb{Z}}$. Equivalently, $\mathbf{f}(1)$ and $\mathbf{a}(1)$ have no nontrivial homoclinic trajectories.

(A4) For any $x \in \mathbb{R}^d$ and $\lambda \in S^1$,

$$f_n(\lambda, x) \xrightarrow[n\to\pm\infty]{} f^{\infty}_{\pm}(\lambda, x)$$

(uniformly with respect to any bounded set $B \subset \mathbb{R}^d$) and the following two difference equations

$$x_{n+1} = f^{\infty}_{+}(\lambda, x_n) \quad \text{and} \quad x_{n+1} = f^{\infty}_{-}(\lambda, x_n)$$

admit, for any $\lambda \in S^1$, only the trivial solution $(x_n = 0)_{n\in\mathbb{Z}}$.

By $(A2)$ the map $\lambda \to a(\lambda, \pm\infty)$ is a continuous family of hyperbolic matrices. Since there are no eigenvalues of $a(\lambda, \pm\infty)$ on the unit circle, the projectors to the spectral subspaces corresponding to the spectrum inside and outside the unit disk depend continuously on the parameter λ (see [15]). It is well known that the images of a continuous family of projectors form a vector bundle over the parameter space [11]. Therefore, the vector spaces $E^s(\lambda, \pm\infty)$ and $E^u(\lambda, \pm\infty)$ whose elements are the generalized real eigenvectors of $a(\lambda, \pm\infty)$ corresponding to the eigenvalues with absolute value smaller (respectively greater) than 1 are fibers of a pair of vector bundles $E^s(\pm\infty)$ and $E^u(\pm\infty)$ over S^1 which decompose the trivial bundle $\Theta(\mathbb{R}^d)$ with fiber $\mathbb{R}^d$ into a direct sum:

$$E^s(\pm\infty) \oplus E^u(\pm\infty) = \Theta(\mathbb{R}^d). \tag{9}$$

In what follows $E^s(\pm\infty)$ and $E^u(\pm\infty)$ will be called *stable* and *unstable* asymptotic bundles at $\pm\infty$. Our main theorem relates the appearance of homoclinic solutions to the topology of the asymptotic stable bundles $E^s(\pm\infty)$. Due to relation (9) the consideration of the unstable bundles would give the same result. In what follows, for notational reasons, it will be convenient for us to work with the multiplicative group $\mathbb{Z}_2 = \{1, -1\}$ instead of the standard additive $\mathbb{Z}_2 = \{0, 1\}$. A vector bundle over S^1 is orientable if and only if it is trivial, i.e., isomorphic to a product $S^1 \times \mathbb{R}^k$. Moreover, whether a given vector bundle E over S^1 is trivial or is not is determined by a topological invariant $w_1(E) \in \mathbb{Z}_2$.

In order to define $w_1(E)$ let us identify S^1 with the quotient of an interval $I = [a, b]$ by its boundary $\partial I = \{a, b\}$. If $p\colon [a, b] \to S^1 = I/\partial I$ is the projection, the pullback bundle $p^*E = E'$ is the vector bundle over I with fibers $E'_t = E_{p(t)}$. Since I is contractible to a point, E' is trivial and the choice of an isomorphism between E' and the product bundle provides E' with a frame, i.e., a basis $\{e_1(t), ..., e_k(t)\}$ of E'_t continuously depending on t. Since $E'_a = E_{p(a)} = E_{p(b)} = E'_b$, $\{e_i(a) \mid 1 \le i \le k\}$ and $\{e_i(b) \mid 1 \le i \le k\}$ are two bases of the same vector space. We define $w_1(E) \in \mathbb{Z}_2$ by

$$w_1(E) := \operatorname{sign} \det C, \tag{10}$$

where C is the matrix expressing the basis $\{e_i(b) \mid 1 \le i \le k\}$ in terms of the basis $\{e_i(a) \mid 1 \le i \le k\}$. It is easy to see that $w_1(E)$ is independent from the choice of the frame. Clearly $w_1(E) = 1$ if and only if E is trivial. Indeed, if E is a trivial bundle, then by definition, $w_1(E) = 1$. On the other hand, if $w_1(E) = 1$, $\det C > 0$, and there exists a path $C(t)$ with $C(a) = C$ and $C(b) = \mathrm{Id}$. Then $f_i(t) = C(t)e_i(t)$ is a frame such that $f_i(a) = f_i(b)$ and hence $\Phi(t, x_1, \dots, x_k) = (t, \sum x_i f_i(t))$ is an isomorphism between $S^1 \times \mathbb{R}^k$ and E.

Remark 2.1.
Under the isomorphism $H^1(S^1; \mathbb{Z}_2) \cong \mathbb{Z}_2$, $w_1(E)$ can be identified with the first Stiefel-Whitney class of E.

Let us recall from [23] that a point $\lambda_* \in S^1$ is a *bifurcation point* for homoclinic solutions of (7) from the trivial branch of stationary solutions $\mathcal{T}_0 = \{(\lambda, \mathbf{0}) \mid \lambda_0 \in S^1\}$ if in every neighborhood of $(\lambda_*, \mathbf{0})$ there is a point $(\lambda, \mathbf{x})$ such that $\mathbf{x}$ is a nontrivial homoclinic solution of $x_{n+1} = f_n(\lambda, x_n)$.

Bifurcation points from infinity are defined in a similar way. Namely, $\lambda_* \in S^1$ is a *bifurcation point from infinity* for homoclinic solutions of (7) if there is a sequence $(\lambda_n, \mathbf{x}_n)$ of homoclinic solutions of (7) with $\lambda_n \xrightarrow[n\to\infty]{} \lambda_*$ and $\|\mathbf{x}_n\| \xrightarrow[n\to\infty]{} \infty$. Due to the compactness of S^1, any unbounded sequence of solutions contains as subsequence $(\lambda_n, \mathbf{x}_n)$ such that λ_n converges to a bifurcation point from infinity. By B_0 (resp. B_∞) we will denote the set of all bifurcation points of (7) from the trivial branch of stationary solutions (resp. the set of all bifurcation points of (7) from infinity).

In order to state our result in a more symmetric form we will introduce the trivial branch at infinity. The one point boundification of a normed space E is the topological space $E^+ := E \cup \{\infty\}$ with a base of neighborhoods of $\{\infty\}$ given by $D \cup \{\infty\}$, where D is a complement of a closed bounded subset of E. Notice that if $A \subset E$ is a closed and locally compact subset of E, then its closure $\bar{A} = A \cup \{\infty\}$ in E^+ is the one-point compactification

A^+ of A. A sequence in $x_n \in E$ such that $\|x_n\| \xrightarrow[n\to\infty]{} \infty$ converges to the point $\{\infty\}$ in E^+. By definition the subset $\mathcal{T}_\infty = \{(\lambda, \infty) \mid \lambda \in S^1\}$ of $S^1 \times \mathbf{l}^{2+}$ is the *trivial branch at* $\{\infty\}$.

The main result of this paper reads as follows:

Theorem 2.1.
If the system (6) *verifies* (A1)–(A4) *and if*

$$w_1(E^s(+\infty)) \neq w_1(E^s(-\infty)), \tag{11}$$

then:

[i] *The connected component $\mathcal{C}_0$ of $\mathcal{T}_0$ in the set $\mathcal{S} \subset S^1 \times \mathbf{l}^2$ of all homoclinic solutions of* (7) *is unbounded. In particular, both B_0 and B_∞ are nonempty.*

[ii] *The set $\mathcal{S}_0 = \mathcal{S} - \mathcal{T}_0$ of all nontrivial homoclinic solutions of* (7) *contains a continuum* (*i.e., closed connected subset* $\mathcal{C}$) *whose closure $\bar{\mathcal{C}}$ in $S^1 \times \mathbf{l}^{2+}$ intersects both $\mathcal{T}_0$ and $\mathcal{T}_\infty$.*

Therefore, not only B_0 and B_∞ are not empty but there is a connected branch of nontrivial homoclinic solutions of (7) connecting $\mathcal{B}_0 = B_0 \times \{\mathbf{0}\}$ to $\mathcal{B}_\infty = B_\infty \times \{\infty\}$. The theorem will be proved in Section 6. The main ingredients of the proof are the computation of the index bundle of the family of linearized equations at the trivial branch in terms of the asymptotic stable bundles at $\pm\infty$ and the generalized homotopy property of the base point degree of the family of induced Fredholm maps. The next section is entirely devoted to the first of the above mentioned tools.

3. The index bundle

Our goal here is to establish the Fredholm property of operators induced on functional spaces by a linear asymptotically hyperbolic systems and to compute the index bundle of a parametrized family of such operators.

Firstly, let us shortly recall the concept of the index bundle of a family of Fredholm operators. For a more complete presentation see [23]. A bounded operator $T \in \mathcal{L}(X, Y)$ ([1]) is Fredholm if it has finite dimensional kernel and cokernel. The index of a Fredholm operator is by definition $\operatorname{ind} T := \dim \operatorname{Ker} T - \dim \operatorname{Coker} T$. The space of all Fredholm operators will be denoted by $\Phi(X, Y)$ and those of index k by $\Phi_k(X, Y)$. For each k, $\Phi_k(X, Y)$ is an open subset of $\mathcal{L}(X, Y)$.

The index bundle generalizes to the case of families of Fredholm operators the concept of index of a single Fredholm operator. If a family L_λ of Fredholm operators depends continuously on a parameter λ belonging to some compact topological space Λ and if the kernels $\operatorname{Ker} L_\lambda$ and cokernels $\operatorname{Coker} L_\lambda$ form two vector bundles $\operatorname{Ker} L$ and $\operatorname{Coker} L$ over Λ, then, roughly speaking, the index bundle is the difference $\operatorname{Ker} L - \operatorname{Coker} L$, where one gives a meaning

[1] *By $\mathcal{L}(X, Y)$ (resp. $\mathcal{L}(X)$) we will denote the space of bounded linear operators between two Banach spaces X and Y (resp. from X into itself).*

to the difference by working in the Grothendieck group $KO(\Lambda)$, which by definition is the group completion of the abelian semigroup $\mathrm{Vect}(\Lambda)$ of all isomorphisms classes of real vector bundles over Λ. The elements of $KO(\Lambda)$ are called virtual bundles. Each virtual bundle is a difference $[E]-[F]$, where E, F are vector bundles over Λ and $[E]$ denotes the corresponding element of $KO(\Lambda)$. One can show that $[E]-[F]=0$ in $KO(\Lambda)$ if and only if the two vector bundles become isomorphic after the addition of a trivial vector bundle to both sides. Taking complex vector bundles instead of the real ones leads to the complex Grothendieck group denoted by $K(\Lambda)$. In what follows the trivial bundle with fiber $\Lambda \times V$ will be denoted by $\Theta(V)$ and $\Theta(\mathbb{R}^d)$ will be simplified to Θ^d.

Let $X,\ Y$ be real Banach spaces and let $L\colon \Lambda \to \Phi(X,Y)$ be a continuous family of Fredholm operators. $L_\lambda \in \Phi(X,Y)$ will denote the value of L at the point $\lambda \in \Lambda$. In general neither the kernels nor cokernels of L_λ will form a vector bundle. However, since $\operatorname{Coker} L_\lambda$ is finite dimensional, using compactness of Λ, one can find a finite dimensional subspace V of Y such that

$$\operatorname{Im} L_\lambda + V = Y \text{ for all } \lambda \in \Lambda. \tag{12}$$

Because of the transversality condition (12) the family of finite dimensional subspaces $E_\lambda = L_\lambda^{-1}(V)$ defines a vector bundle over Λ (see [23]) with total space

$$E = \bigcup_{\lambda \in \Lambda} \{\lambda\} \times E_\lambda.$$

By definition, the *index bundle* is the virtual bundle:

$$\operatorname{Ind} L = [E] - [\Theta(V)] \in KO(\Lambda). \tag{13}$$

The index bundle enjoys the same nice properties of the ordinary index. Namely, homotopy invariance, additivity with respect to directs sums, logarithmic property under composition of operators. Clearly it vanishes if L is a family of isomorphisms. We will mainly use in the sequel:

(*i*) *Homotopy invariance:* Let $H\colon [0,1]\times\Lambda \to \Phi(X,Y)$ be a homotopy, then $\operatorname{Ind} H_0 = \operatorname{Ind} H_1$. In particular, $\operatorname{Ind}(L+K) = \operatorname{Ind} L$, if K is a family of compact operators.

(*ii*) *Logarithmic property:* $\operatorname{Ind}\big(LM\big) = \operatorname{Ind} L + \operatorname{Ind} M$.

We will mostly work with families of Fredholm operators of index 0. The index bundle of a family of Fredholm operators of index 0 belongs to the reduced Grothendieck group $\widetilde{KO}(\Lambda)$, i.e., the subgroup generated by elements $[E]-[F]$ such that $\dim E_\lambda = \dim F_\lambda$. It can be shown that any element $\eta \in \widetilde{KO}(\Lambda)$ can be written as $[E]-[\Theta^N]$. Moreover, $[E]-[\Theta^N] = [E']-[\Theta^M]$ in $\widetilde{KO}(\Lambda)$ if and only if there exist two trivial bundles Θ and Θ' such that $E\oplus\Theta$ is isomorphic to $E'\oplus\Theta'$, (see [14, Theorem 3.8]).

Now let us compute the index bundle of the family of operators associated to a family of linear asymptotically hyperbolic systems. Denoting with $GL(d)$ the set of all invertible matrices in $\mathbb{R}^{d\times d}$, let $\mathbf{a}\colon \mathbb{Z}\times S^1 \to GL(d)$ be a family of linear asymptotically hyperbolic systems. This means:

(a) As $n \to \pm\infty$ the sequence $\mathbf{a}(\lambda) = (a_n(\lambda))$ converges uniformly with respect to $\lambda \in S^1$ to a family of matrices $a(\lambda, \pm\infty)$.

(b) $a(\lambda, \pm\infty) \in GL(d)$ is hyperbolic for all $\lambda \in S^1$.

It is easy to see that (a) implies that $\mathbf{a}_\pm \colon S^1 \to GL(d)$ given by $\mathbf{a}_\pm(\lambda) := a(\lambda, \pm\infty)$ are continuous functions of λ. We associate to the family $\mathbf{a} \colon \mathbb{Z} \times S^1 \to GL(d)$ the family of linear operators

$$L = \left\{ L_\lambda \colon \mathbf{l}^2 \to \mathbf{l}^2 \mid \lambda \in S^1 \right\}$$

defined by $L_\lambda = S - A_\lambda$, where S is the shift operator and $A_\lambda \colon \mathbf{l}^2 \to \mathbf{l}^2$ is the substitution operator $A_\lambda \mathbf{x} := (a_n(\lambda)x_n)$. Since the sequence $(a_n(\lambda))$ converges, it is bounded, from which follows immediately that A_λ and L_λ are well defined bounded operators.

Lemma 3.1.
The map $A \colon S^1 \to \mathcal{L}(\mathbf{l}^2)$ defined by $A(\lambda) := A_\lambda$ is continuous with respect to the norm topology of $\mathcal{L}(\mathbf{l}^2)$. Hence the same holds for the family L.

Proof. Fix $\mathbf{x} = (x_n) \in \mathbf{l}^2$. Then

$$\|(A(\lambda) - A(\mu))\mathbf{x}\|^2 = \sum_{n \in \mathbb{Z}} |(a_n(\lambda) - a_n(\mu))x_n|^2 \le \sum_{n \in \mathbb{Z}} |a_n(\lambda) - a_n(\mu)|^2 \, |x_n|^2.$$

Furthermore, $|a_n(\lambda) - a_n(\mu)| \le |a_n(\lambda) - \mathbf{a}_\pm(\lambda)| + |a_n(\mu) - \mathbf{a}_\pm(\mu)| + |\mathbf{a}_\pm(\lambda) - \mathbf{a}_\pm(\mu)|$. Fix $\varepsilon > 0$. Then Assumption (a) implies that there exists $n_0 > 0$ such that

$$|a_n(\lambda) - \mathbf{a}_+(\lambda)| < \varepsilon/3 \quad \text{for all } n \ge n_0 \text{ and for all } \lambda \in S^1,$$
$$|a_n(\lambda) - \mathbf{a}_-(\lambda)| < \varepsilon/3 \quad \text{for all } n \le -n_0 \text{ and for all } \lambda \in S^1.$$

Moreover, there exists $\delta > 0$ such that if $d((\lambda, 0), (\mu, 0)) < \delta$, $\lambda, \mu \in S^1$, then

$$|\mathbf{a}_\pm(\lambda) - \mathbf{a}_\pm(\mu)| \le \varepsilon/3 \text{ and } |a_k(\lambda) - a_k(\mu)| \le \varepsilon/3 \text{ for all } -n_0 < k < n_0.$$

Finally, taking into account the above considerations, one obtains that

$$\|(A(\lambda) - A(\mu))\mathbf{x}\|^2 \le \sum_{n \in \mathbb{Z}} |a_n(\lambda) - a_n(\mu)|^2 \, |x_n|^2 \le \sum_{n \in \mathbb{Z}} \varepsilon^2 |x_n|^2 = \varepsilon^2 \|\mathbf{x}\|^2$$

provided $d((\lambda, 0), (\mu, 0)) < \delta$, which implies that A is continuous with respect to the norm topology of $\mathcal{L}(\mathbf{l}^2)$. $\square$

Clearly, $\mathbf{x} = (x_n) \in \mathbf{l}^2$ verifies a linear difference equation $x_{n+1} = a_n(\lambda)x_n$ if and only if $L_\lambda \mathbf{x} = 0$. By the discussion in the previous section the families $a(\lambda, \pm\infty) \in GL(d)$ define two vector bundles $E^s(\pm\infty)$ over S^1. The next theorem relates the index bundle of the family L to $E^s(\pm\infty)$.

Theorem 3.1.
Let $\mathbf{a}\colon \mathbb{Z}\times S^1 \to GL(d)$ be a continuous map verifying (a) and (b). Then the family $L\colon S^1 \to \mathcal{L}\left(\mathbf{l}^2\right)$ verifies:

(i) L_λ is a Fredholm operator for all $\lambda \in S^1$.

(ii) $\operatorname{Ind} L = [E^s(+\infty)] - [E^s(-\infty)] \in KO(S^1)$.

In particular, applying to $\operatorname{Ind} L$ the rank homomorphism $rk\colon KO(S^1) \to \mathbb{Z}$, $\operatorname{rk}([E]-[F]) = \dim E_\lambda - \dim F_\lambda$, we obtain

$$\operatorname{ind} L_\lambda = \dim E^s(+\infty) - \dim E^s(-\infty). \tag{14}$$

Proof. The proof is similar to the proof of Theorem 4.1 in [23]. However, since here we are working on a proper subspace $\mathbf{l}^2$ of $\mathbf{c}(\mathbb{R}^d)$ we have to check carefully that all constructed elements belong to this subspace. Let $\bar{\mathbf{a}}\colon \mathbb{Z}\times S^1 \to GL(d)$ be defined by

$$\bar{\mathbf{a}}(n,\lambda) = (\bar{a}_n(\lambda)) = \begin{cases} a(\lambda,+\infty) & \text{if} \quad n \geq 0, \\ a(\lambda,-\infty) & \text{if} \quad n < 0. \end{cases} \tag{15}$$

Fix $\lambda \in S^1$ and denote by $\bar{A}_\lambda \in \mathcal{L}(X)$ the operator associated to $\bar{\mathbf{a}}_\lambda$, where $X := \mathbf{l}^2$. We claim that the operator $K_\lambda = A_\lambda - \bar{A}_\lambda$ is a compact operator. To this end, we will show that K_λ is the limit (in the norm topology of $\mathcal{L}(X)$) of a sequence of operators $\tilde{K}_\lambda^m$ with finite dimensional range. We observe that K_λ is defined by $K_\lambda \mathbf{x} = (k_n(\lambda)x_n)$, where $k_n(\lambda) = a_n(\lambda) - \bar{a}_n(\lambda)$ and define

$$\tilde{K}_\lambda^m \mathbf{x} = \begin{cases} k_n(\lambda)x_n & \text{if} \quad |n| \leq m, \\ 0 & \text{if} \quad |n| > m. \end{cases} \tag{16}$$

Clearly $\operatorname{Im} \tilde{K}_\lambda^m$ is finite dimensional. We are to prove that

$$\sup_{\|\mathbf{x}\|=1} \|(K_\lambda - \tilde{K}_\lambda^m)\mathbf{x}\| \xrightarrow[m\to\infty]{} 0, \tag{17}$$

for $\mathbf{x} \in X$. Observe that

$$\|(K_\lambda - \tilde{K}_\lambda^m)\mathbf{x}\| = \sum_{|n|>m} |k_n(\lambda)x_n|^2 \geq \sum_{|n|>m+1} |k_n(\lambda)x_n|^2 = \|(K_\lambda - \tilde{K}_\lambda^{m+1})\mathbf{x}\|, \tag{18}$$

for all $m \in \mathbb{N}$. Since $\lim\limits_{|n|\to\infty} k_n(\lambda) = 0$, we infer that for all $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $|n| > n_0$ and $\|\mathbf{x}\| = 1$ one has $|k_n(\lambda)x_n| < \varepsilon$. Consequently, for all $\varepsilon > 0$ there exists $n_0 > 0$ such that

$$\sup_{\|\mathbf{x}\|=1} \|(K_\lambda - \tilde{K}_\lambda^{n_0})\mathbf{x}\| \leq \varepsilon. \tag{19}$$

Now taking into account (18) and (19), we deduce that for all $\varepsilon > 0$ there exists $n_0 > 0$ such that for all $m \geq n_0$ one has

$$\sup_{\|\mathbf{x}\|=1} \|(K_\lambda - \tilde{K}^m_\lambda)\mathbf{x}\| \leq \varepsilon, \tag{20}$$

which proves (17) and the compactness of the operator K_λ. Let $\bar{L}_\lambda = S - \bar{A}_\lambda$. Then $L_\lambda - \bar{L}_\lambda = K_\lambda$ and hence the family L differs from the family $\bar{L}$ by a family of compact operators. Therefore L_λ is Fredholm if and only if $\bar{L}_\lambda$ is Fredholm and moreover the homotopy invariance of the index bundle applied to the homotopy $H(\lambda, t) = \bar{L}_\lambda + tK_\lambda$ shows that $\operatorname{Ind} \bar{L} = \operatorname{Ind} L$. Hence in order to prove the theorem we can assume without loss of generality that $\mathbf{a}$ has already the special form of (15), which we will do from now on. Let

$$\mathbf{l}^+_k := \{\mathbf{x} \in \mathbf{l}^2 \mid x_n = 0 \ \text{ for } \ n < k\}, \ \ \mathbf{l}^-_k := \{\mathbf{x} \in \mathbf{l}^2 \mid x_n = 0 \ \text{ for } \ n > k\}.$$

Both $\mathbf{l}^\pm_k$ are closed subspaces of $\mathbf{l}^2$. Put $X^+ = Y^+ = \mathbf{l}^+_0$ and $X^- = \mathbf{l}^-_0$, $Y^- = \mathbf{l}^-_{-1}$. Let us consider four linear operators $I\colon Y^- \oplus Y^+ \to X$, $J\colon X \to X^- \oplus X^+$, $L^+_\lambda\colon X^+ \to Y^+$ and $L^-_\lambda\colon X^- \to Y^-$ defined respectively by

$$I(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y},$$

$$J(\mathbf{x})(n) = \begin{cases} (x_0, x_0) & \text{if } \ n = 0, \\ (x_n, 0) & \text{if } \ n < 0, \\ (0, x_n) & \text{if } \ n > 0, \end{cases}$$

$$(L^+_\lambda \mathbf{x})(n) = \begin{cases} x_{n+1} - a(\lambda, +\infty)x_n & \text{if } \ n \geq 0, \\ 0 & \text{if } \ n < 0, \end{cases}$$

$$(L^-_\lambda \mathbf{x})(n) = \begin{cases} 0 & \text{if } \ n > -1, \\ x_{n+1} - a(\lambda, -\infty)x_n & \text{if } \ n \leq -1. \end{cases}$$

We decompose $L_\lambda\colon X \to X$ via the following commutative diagram:

$$\begin{tikzcd} X^- \oplus X^+ \arrow[r, "L^-_\lambda \oplus L^+_\lambda"] & Y^- \oplus Y^+ \arrow[d, "I"] \\ X \arrow[u, "J"] \arrow[r, "L_\lambda"] & X. \end{tikzcd} \tag{21}$$

The commutativity of diagram (21) is easy to check. Indeed, one has

$$I(L^-_\lambda \oplus L^+_\lambda)J\mathbf{x}(n) = L^-_\lambda J\mathbf{x}(n) + L^+_\lambda J\mathbf{x}(n) = \begin{cases} (L^+_\lambda \mathbf{x})(n) & \text{if } \ n \geq 0, \\ (L^-_\lambda \mathbf{x})(n) & \text{if } \ n < 0, \end{cases}$$

which is the same as

$$(L_\lambda \mathbf{x})(n) = \begin{cases} x_{n+1} - a(\lambda,+\infty)x_n & \text{if } \ n \geq 0, \\ x_{n+1} - a(\lambda,-\infty)x_n & \text{if } \ n < 0. \end{cases} \tag{22}$$

Next, we will show that $L^{\pm}_\lambda \colon X^{\pm} \to Y^{\pm}$ are Fredholm and we will compute the index bundles of $L^{\pm}$. For L^+_λ this is the content of the following Lemma:

Lemma 3.2.
Let $a \in GL(d)$ be a hyperbolic matrix. Then the operator $S - A \colon \mathbf{l}^+_0 \to \mathbf{l}^+_0$ defined by

$$((S-A)\mathbf{x})(n) = \begin{cases} x_{n+1} - ax_n & \text{if } \ n \geq 0, \\ 0 & \text{if } \ n < 0, \end{cases}$$

is surjective with $\ker(S-A) = \{\mathbf{x} \in \mathbf{l}^+_0 \mid x_{n+1} = a^n x_0$ *for all* $n \geq 0$ *and* $x_0 \in E^s(a)\}$.

This lemma was proved in [2, Lemma 2.1] for the operator induced in $\mathbf{c}^+_0 := \{\mathbf{x} \in \mathbf{c}(\mathbb{R}^d) \mid x_i = 0 \ \text{for} \ i < 0\}$ by constructing an explicit right inverse R to the operator $S - A \colon \mathbf{c}^+_0 \to \mathbf{c}^+_0$, via the convolution with the matrix function $g(n) = a^{n-1}(\mathbb{1}_{Z^+}(n)\mathrm{Id}_{\mathbb{R}^d} - P^u)$, where P^u is the projector on the unstable subspace and $Z^+ = \{1, 2, ...\}$. They prove that function g belongs to $l^1(\mathbb{Z}, \mathbb{R}^{d\times d})$. But the convolution with a matrix function in $l^1(\mathbb{Z}, \mathbb{R}^{d\times d})$ sends $\mathbf{l}^2$ into itself. Hence the assertion of this lemma is also true for $S - A \colon \mathbf{l}^+_0 \to \mathbf{l}^+_0$. For the assertion regarding the kernel is enough to observe that if $\mathbf{x} = (x_n) \in \mathbf{c}^+_0$ and $x_{n+1} = a^n x_0$ for all $n \geq 0$ and $x_0 \in E^s(a)$, then the spectral radius theorem guarantees that $\mathbf{x} \in \mathbf{l}^+_0$. □

By Lemma 3.2

$$\mathrm{Ker}\, L^+_\lambda = \{\mathbf{x} \in X^+ \mid x_n = a(\lambda,+\infty)^n x_0 \ \text{ and } \ x_0 \in E^s(\lambda,+\infty)\}. \tag{23}$$

Hence the transformation $\mathbf{x} \mapsto x_0$ defines an isomorphism between $\mathrm{Ker}\, L^+$ and $E^s(\lambda,+\infty)$, which is finite dimensional. Being $\mathrm{Coker}\, L_\lambda = \{0\}$, L^+_λ is Fredholm with $\mathrm{ind}\, L^+_\lambda = \dim E^s(\lambda,+\infty)$. Clearly the index bundle $\mathrm{Ind}\, L^+ = [E^s(+\infty)]$. We will reduce the calculation of $\mathrm{Ind}\, L^-$ to Lemma 3.2 as follows. Put $Y^- := \mathbf{l}^-_{-1}$ and $X^- := \mathbf{l}^-_0$ and consider the family of isomorphisms $B = \{B_\lambda \colon Y^- \to Y^- \mid \lambda \in S^1\}$ defined by

$$(B_\lambda \mathbf{x})(n) = \begin{cases} 0 & \text{if } \ n > -1, \\ -a^{-1}(\lambda,-\infty)x_n & \text{if } \ n \leq -1. \end{cases}$$

We compose $L^-_\lambda \colon X^- \to Y^-$ on the right with the isomorphism $B_\lambda \colon Y^- \to Y^-$ followed by the negative shift S^{-1} viewed as an operator from Y^- to X^-. Since both operators are isomorphisms, the composition does not affect the Fredholm property. On the other hand, considering S^{-1} as a constant family of isomorphisms, by logarithmic property of the index bundle, $\mathrm{Ind}\, S^{-1}BL^- = \mathrm{Ind}\, L^-$. Hence the index bundle of L^- coincides with the index bundle of the family $D = S^{-1}BL^-$. Observe now that if $\mathbf{x} \in Y^-$, then

$$(B_\lambda L^-_\lambda)\mathbf{x})(n) = \begin{cases} 0 & \text{if } \ n > -1, \\ x_n - a^{-1}(\lambda,-\infty)x_{n+1} & \text{if } \ n \leq -1. \end{cases}$$

But since $S^{-1}\mathbf{x} = (x_{n-1})$, one obtains

$$(D_\lambda \mathbf{x})(n) = \begin{cases} 0 & \text{if } n > 0, \\ x_{n-1} - a^{-1}(\lambda, -\infty)x_n & \text{if } n \leq 0. \end{cases}$$

Thus $D_\lambda \colon X^- \to X^-$ is the same type of operator as L^+_λ but with n going from 0 to $-\infty$. By Lemma 3.2, each D_λ is surjective. Moreover,

$$\operatorname{Ker} D_\lambda = \operatorname{Ker} L^-_\lambda = \{\mathbf{x} \in X^- \mid x_n = a(\lambda, -\infty)^n x_0 \text{ and } x_0 \in E^u(\lambda, -\infty))\} \tag{24}$$

is isomorphic to $E^u(\lambda, -\infty)$. Summing up, we have obtained that $\operatorname{Ind} L^+ = [E^s(+\infty)]$ and $\operatorname{Ind} L^- = [E^u(-\infty)]$. In particular we have

$$\operatorname{ind} L^+_\lambda = \dim E^s(\lambda, +\infty) \quad \text{and} \quad \operatorname{ind} L^-_\lambda = \dim E^u(\lambda, -\infty). \tag{25}$$

With this at hand we can compute the index bundle of L completing the proof of the theorem. Let us notice firstly that I and J are Fredholm operators. Indeed, $I \colon Y^- \oplus Y^+ \to X$ is clearly an isomorphism, and the map $J \colon X \to X^- \oplus X^+$ is a monomorphism whose image is given by $\operatorname{Im} J = \{(\mathbf{a}, \mathbf{b}) \in X^- \oplus X^+ \mid a_0 = b_0\}$. Putting $P \colon X^- \oplus X^+ \to \mathbb{R}^d$ by $P(\mathbf{a}, \mathbf{b}) := a_0 - b_0$, for $\mathbf{a} \in X^-$ and $\mathbf{b} \in X^+$, one obtains that $\operatorname{Im} J = \operatorname{Ker} P$. But since P is an epimorphism, we deduce that $\operatorname{Coker} J = X^- \oplus X^+/\operatorname{Ker} P \simeq \mathbb{R}^d$ and therefore J is Fredholm of index $-d$. From the commutativity of diagram (21) and (25) it follows that $L_\lambda = I(L^-_\lambda \oplus L^+_\lambda)J$ is Fredholm and

$$\begin{aligned} \operatorname{ind}(L_\lambda) &= \operatorname{ind}(I) + \operatorname{ind}(L^-_\lambda \oplus L^+_\lambda) + \operatorname{ind}(J) = \dim E^s(\lambda, +\infty) + \dim E^u(\lambda, -\infty) - d = \\ &\dim E^s(\lambda, +\infty) - \dim E^s(\lambda, -\infty). \end{aligned} \tag{26}$$

As for (ii), considering I and J as constant families of Fredholm operators, $\operatorname{Ind} I = 0$, $\operatorname{Ind} J = -[\Theta(\mathbb{R}^d)]$. Using the logarithmic and direct sum properties of the index bundle together with (9), we obtain

$$\operatorname{Ind} L = [E^u(-\infty)] + [E^s(+\infty)] - [\Theta(\mathbb{R}^d)] = [E^s(+\infty)] - [E^s(-\infty)],$$

which proves (ii). □

Remark 3.1.
Notice that from (23), (24) and (22) it follows that in the case of systems of the special form (15) elements of $\operatorname{Ker} L_\lambda$ are sequences $(x_n) \in X$ such that $x_0 \in E^s(\lambda, +\infty) \cap E^u(\lambda, -\infty)$ and $x_n = a(\lambda, +\infty)^n x_0$, for $n \geq 0$ and $x_n = a(\lambda, -\infty)^n x_0$, for $n \leq 0$.

The obstruction $w_1(E)$ to the triviality of vector bundle E over S^1 defined in Section 2 induces a well defined isomorphism (see [23]) $w_1 \colon \widetilde{KO}(S^1) \to \mathbb{Z}_2$ by putting

$$w_1([E] - [F]) = w_1(E)w_1(F). \tag{27}$$

From this and Theorem 3.1 we obtain:

Corollary 3.1.

$$w_1(\operatorname{Ind} L) = w_1(E^s(+\infty) w_1(E^s(-\infty)). \tag{28}$$

4. The continuity smoothness and the Fredholm property of the family $G(\lambda, \mathbf{x}) = S\mathbf{x} - F(\lambda, x)$

In this section we will study the differentiable properties of a nonlinear operator G induced by a discrete nonautonomous system (7) parametrized by a parameter space S^1 and Fredholmness of an operator $D_x G$. We keep the notations and assumptions from Section 2. For any $\mathbf{x} \in \mathbf{l}^2$ define

$$F(\lambda, \mathbf{x}) = (f_n(\lambda, x_n)), \;\; F^{\infty}_{\pm}(\lambda, \mathbf{x}) = (f^{\infty}_{\pm}(\lambda, x_n)), \;\; G(\lambda, \mathbf{x}) = S\mathbf{x} - F(\lambda, \mathbf{x}), \;\; G^{\infty}_{\pm}(\lambda, \mathbf{x}) = S\mathbf{x} - F^{\infty}_{\pm}(\lambda, \mathbf{x}). \tag{29}$$

Proposition 4.1.
Under Assumptions (A1)–(A4), $F(\lambda, \mathbf{x})$, $F^{\infty}_{\pm}(\lambda, \mathbf{x})$, $G(\lambda, \mathbf{x})$ and $G^{\infty}_{\pm}(\lambda, \mathbf{x})$ belong to $\mathbf{l}^2$, for each $\lambda \in S^1$ and $\mathbf{x} \in \mathbf{l}^2$.

Proof. First of all we will need the following lemma, which will be used repeatedly in what follows. Working on any coordinate chart of S^1 we will denote with λ also the coordinate of the point $\lambda \in S^1$.

Lemma 4.1.
Assumptions (A1)–(A2) imply that

$$\sup_{(n,\lambda,y)\in\mathbb{Z}\times S^1\times\bar{B}_d(0,M)} \left|\frac{\partial f_n(\lambda, y)}{\partial x}\right| < \infty \quad \text{and} \quad \sup_{(n,\mu,y)\in\mathbb{Z}\times S^1\times\bar{B}_d(0,M)} \left|\frac{\partial f_n(\mu, y)}{\partial \lambda}\right| < \infty$$

for any $M > 0$.

Proof. Let us observe that from Assumption (A2) it follows easily that

$$C_0 := \sup_{(n,\lambda)\in\mathbb{Z}\times S^1} \left|\frac{\partial f_n(\lambda, 0)}{\partial x}\right| < \infty. \tag{30}$$

Fix $M > 0$ and $\varepsilon > 0$. Let $\delta > 0$ be as in Assumption (A1). Take $(n, \lambda, y) \in \mathbb{Z} \times S^1 \times \bar{B}_d(0, M)$. Then there exists $n_0 > 0$ such that $n_0 \leq M/\delta < n_0 + 1$. Furthermore, there exist $0 < k \leq n_0 + 1$ and points $y_0 = 0, y_1, ..., y_{k-1}, y_k = y \in \bar{B}_d(0, M)$ such that $|y_i - y_{i+1}| < \delta$, for $i = 0, ..., k-1$. Thus

$$\begin{aligned}\left|\frac{\partial f_n(\lambda, y)}{\partial x}\right| &\leq \left|\frac{\partial f_n(\lambda, 0)}{\partial x}\right| + \left|\frac{\partial f_n(\lambda, y_1)}{\partial x} - \frac{\partial f_n(\lambda, 0)}{\partial x}\right| + \left|\frac{\partial f_n(\lambda, y_2)}{\partial x} - \frac{\partial f_n(\lambda, y_1)}{\partial x}\right| + \ldots + \\ &\left|\frac{\partial f_n(\lambda, y_{k-1})}{\partial x} - \frac{\partial f_n(\lambda, y_{k-2})}{\partial x}\right| + \left|\frac{\partial f_n(\lambda, y)}{\partial x} - \frac{\partial f_n(\lambda, y_{k-1})}{\partial x}\right| \leq C_0 + k\varepsilon \leq C_0 + (n_0+1)\varepsilon,\end{aligned}$$

where C_0 is as in (30). Observe that $\dfrac{\partial f_n(\mu, 0)}{\partial \lambda} = 0$, for all $\mu \in S^1$ and $n \in \mathbb{Z}$. It follows from the fact that $f_n(\mu, 0) = 0$, for all $\mu \in S^1$ and $n \in \mathbb{Z}$. Consequently, by the same reasoning as above, one can conclude the second part of the assertion of the lemma. □

Now fix $\mathbf{x} \in \mathbf{l}^2$ and $\lambda \in S^1$. Let $M := \sup_{n\in\mathbb{Z}} |\mathbf{x}(n)|$. Then Lemma 4.1 implies that

$$C := \sup_{(n,y)\in\mathbb{Z}\times\bar{B}_d(0,M)} \left|\frac{\partial f_n(\lambda,y)}{\partial x}\right| < \infty.$$

Hence, using the mean value theorem, we get

$$|f_n(\lambda,y)| = |f_n(\lambda,y) - f_n(\lambda,0)| \leq \left(\sup_{s\in[0,1]}\left|\frac{\partial f_n(\lambda,sy)}{\partial x}\right|\right)|y| \leq C|y|. \tag{31}$$

Consequently,

$$\|F(\lambda,\mathbf{x})\| = \|(f_n(\lambda,\mathbf{x}(n)))\| \leq C\|(\mathbf{x}(n))\| = C\|\mathbf{x}\|,$$

which implies that $F(\lambda,\mathbf{x})$ and hence also $G(\lambda,\mathbf{x})$ belong to $\mathbf{l}^2$.

On the other hand, $|f_n(\lambda,y)| \xrightarrow[n\to\pm\infty]{} |f^{\infty}_{\pm}(\lambda,y)|$. Thus, after passing to the limit in (31) as $n \to \pm\infty$, we get $|f^{\infty}_{\pm}(\lambda,y)| \leq C|y|$, for all $y \in \bar{B}_d(0,M)$. From which it follows that $\|F^{\infty}_{\pm}(\lambda,\mathbf{x})\| \leq C\|\mathbf{x}\|$. Therefore, $F^{\infty}_{\pm}(\lambda,\mathbf{x})$, and $G^{\infty}_{\pm}(\lambda,\mathbf{x})$ belong to $\mathbf{l}^2$. □

Using once again Lemma 4.1, we define two families of linear bounded operators $T\colon S^1\times\mathbf{l}^2 \to \mathcal{L}(\mathbf{l}^2)$ and $\tilde{T}\colon S^1 \times \mathbf{l}^2 \to \mathcal{L}(\mathbb{R},\mathbf{l}^2)$ by

$$T(\lambda,\mathbf{x})\mathbf{y} := \left(\frac{\partial f_n(\lambda,x_n)}{\partial x}y_n\right) \quad\text{and}\quad \tilde{T}(\lambda,\mathbf{x})z := \left(\frac{\partial f_n(\lambda,x_n)}{\partial \lambda}z\right) \tag{32}$$

for $\mathbf{x} = (x_n), \mathbf{y} = (y_n) \in \mathbf{l}^2$, $\lambda \in S^1$ and $z \in \mathbb{R}$.

Proposition 4.2.
The map $F\colon S^1\times\mathbf{l}^2 \to \mathbf{l}^2$ defined in (29) is C^1. Moreover, $D_xF(\lambda,\mathbf{x}) = T(\lambda,\mathbf{x})$ and $D_\lambda F(\lambda,\mathbf{x}) = \tilde{T}(\lambda,\mathbf{x})$.

Proof. Observe that it is suffices to prove that D_xF and $D_\lambda F$ exist and are continuous on $S^1\times\mathbf{l}^2$. Firstly we prove that D_xF exists and $D_xF(\lambda,\mathbf{x}) = T(\lambda,\mathbf{x})$. Fix $\mathbf{x}\in\mathbf{l}^2$ and $\lambda\in S^1$. Then

$$\begin{aligned} R(\mathbf{x},\mathbf{h};\lambda) &:= \|F(\lambda,\mathbf{x}+\mathbf{h}) - F(\lambda,\mathbf{x}) - T(\lambda,\mathbf{x})\mathbf{h}\| = \\ &\left(\sum_{n\in\mathbb{Z}}\left|f_n(\lambda,x_n+h_n) - f_n(\lambda,x_n) - \frac{\partial f_n(\lambda,x_n)}{\partial x}h_n\right|^2\right)^{1/2}, \end{aligned} \tag{33}$$

where $\mathbf{h}\in\mathbf{l}^2$ and $\lambda\in S^1$. We are to show that $R(\mathbf{x},\mathbf{h};\lambda)\|\mathbf{h}\|^{-1} \to 0$ as $\|\mathbf{h}\| \to 0$. Let

$$c_n(\mathbf{h};\lambda) := \sup_{s\in[0,1]}\left|\frac{\partial f_n(\lambda,x_n+sh_n)}{\partial x} - \frac{\partial f_n(\lambda,x_n)}{\partial x}\right|,$$

for $n \in \mathbb{Z}$. Then Assumption $(A1)$ implies that

$$\sup_{n\in\mathbb{Z}} c_n(\mathbf{h};\lambda) \xrightarrow[\|\mathbf{h}\|\to 0]{} 0.$$

Then

$$\left| f_n(\lambda, x_n + h_n) - f_n(\lambda, x_n) - \frac{\partial f_n(\lambda, x_n)}{\partial x} h_n \right| = \left| \int_0^1 \frac{\partial f_n(\lambda, x_n + s h_n)}{\partial x} h_n ds - \frac{\partial f_n(\lambda, x_n)}{\partial x} h_n \right| \leq$$
$$|h_n| \int_0^1 \left| \frac{\partial f_n(\lambda, x_n + s h_n)}{\partial x} - \frac{\partial f_n(\lambda, x_n)}{\partial x} \right| ds \leq |h_n| \int_0^1 c_n(\mathbf{h};\lambda) ds \leq |h_n| \sup_{n\in\mathbb{Z}} c_n(\mathbf{h};\lambda).$$

Hence, taking into account (33), we infer that

$$0 \leq R(\mathbf{x}, \mathbf{h}; \lambda) \leq \|\mathbf{h}\| \sup_{n\in\mathbb{Z}} c_n(\mathbf{h};\lambda), \tag{34}$$

which implies that $R(\mathbf{x}, \mathbf{h}; \lambda)\|\mathbf{h}\|^{-1} \to 0$ as $\|\mathbf{h}\| \to 0$. Now we will show that $T\colon S^1 \times \mathbf{l}^2 \to \mathcal{L}(\mathbf{l}^2)$ is continuous. To this end, observe that

$$\begin{aligned} &\|(T(\lambda, \mathbf{x}) - T(\mu, \mathbf{y}))\mathbf{z}\|^2 = \\ &\sum_{n\in\mathbb{Z}} \left| \left(\frac{\partial f_n(\lambda, \mathbf{x}(n))}{\partial x} - \frac{\partial f_n(\mu, \mathbf{y}(n))}{\partial x} \right) \mathbf{z}(n) \right|^2 \leq \sum_{n\in\mathbb{Z}} \left| \frac{\partial f_n(\lambda, \mathbf{x}(n))}{\partial x} - \frac{\partial f_n(\mu, \mathbf{y}(n))}{\partial x} \right|^2 |\mathbf{z}(n)|^2. \end{aligned} \tag{35}$$

Assumption $(A1)$ implies that for any $M > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such for all $(\lambda_1, x_1), (\lambda_2, x_2) \in S^1 \times \mathbb{R}^d$ with $d\big((\lambda_1, x_1), (\lambda_2, x_2)\big) < \delta$, one has

$$\sup_{n\in\mathbb{Z}} \left| \frac{\partial f_n(\lambda_1, x_1)}{\partial x} - \frac{\partial f_n(\lambda_2, x_2)}{\partial x} \right| < \varepsilon.$$

Fix $\mathbf{x} \in \mathbf{l}^2$ and $\varepsilon > 0$ and take $\delta > 0$ as above (for $M := 2\|\mathbf{x}\|$). Let $d((\lambda, 0), (\mu, 0)) < \min\{\delta/4, \|\mathbf{x}\|\}$ and $\|\mathbf{x} - \mathbf{y}\| < \min\{\delta/4, \|\mathbf{x}\|\}$. Then for any $k \in \mathbb{Z}$ one has $|\mathbf{x}(k) - \mathbf{y}(k)| \leq \|\mathbf{x} - \mathbf{y}\|$ and

$$\left| \frac{\partial f_k(\lambda, \mathbf{x}(k))}{\partial x} - \frac{\partial f_k(\mu, \mathbf{y}(k))}{\partial x} \right| \leq \sup_{n\in\mathbb{Z}} \left| \frac{\partial f_n(\lambda, \mathbf{x}(k))}{\partial x} - \frac{\partial f_n(\mu, \mathbf{y}(k))}{\partial x} \right| < \varepsilon. \tag{36}$$

Thus, taking into account (35) and (36), we infer that

$$\|(T(\lambda, \mathbf{x}) - T(\mu, \mathbf{y}))\mathbf{z}\|^2 \leq \sum_{n\in\mathbb{Z}} \varepsilon^2 |\mathbf{z}(n)|^2 = \varepsilon^2 \|\mathbf{z}\|^2$$

provided $d((\lambda, 0), (\mu, 0)) < \min\{\delta/4, \|\mathbf{x}\|\}$ and $\|\mathbf{x} - \mathbf{y}\| < \min\{\delta/4, \|\mathbf{x}\|\}$. Consequently, we deduce that T is continuous (with respect to the norm topology of $\mathcal{L}(\mathbf{l}^2)$).

Finally, it is not hard to see that the same reasoning as above implies that $D_\lambda F(\lambda, \mathbf{x}) = \tilde{T}(\lambda, \mathbf{x})$ and that $D_\lambda F$ is continuous on $S^1 \times \mathbf{l}^2$. This completes the proof. $\square$

Now we will show that G is C^1 such that $D_xG(\lambda,\mathbf{x})$ is a Fredholm operator of index 0. For this purpose we need to prove the following lemma.

Lemma 4.2.

Under Assumptions $(A1)$–$(A2)$*, for any* $\mathbf{x}=(x_n)\in\mathbf{c}(\mathbb{R}^d)$*, one has*

$$\frac{\partial f_n(\lambda,x_n)}{\partial x}\xrightarrow[n\to\pm\infty]{} a(\lambda,\pm\infty)\ (\textit{uniformly with respect to } \lambda\in S^1).$$

Proof. Fix $\mathbf{x}\in\mathbf{c}(\mathbb{R}^d)$ and $\varepsilon>0$. Then Assumption $(A1)$ implies that there exists $\delta>0$ (for $M:=2\|\mathbf{x}\|_\infty$) such that

$$\forall_{k\in\mathbb{Z}}\,\forall_{|y|\le\delta}\,\forall_{\lambda\in S^1}\left|\frac{\partial f_k(\lambda,y)}{\partial x}-\frac{\partial f_k(\lambda,0)}{\partial x}\right|<\varepsilon.$$

Since $x_n\xrightarrow[n\to\pm\infty]{}0$, it follows that there exists $n_0>0$ such that $|x_n|\le\delta$ for $|n|\ge n_0$. Hence

$$\forall_{|k|\ge n_0}\,\forall_{\lambda\in S^1}\left|\frac{\partial f_k(\lambda,x_k)}{\partial x}-\frac{\partial f_k(\lambda,0)}{\partial x}\right|<\varepsilon.$$

Now the assertion of lemma follows from Assumption $(A2)$. □

Theorem 4.1.

Under Assumptions $(A1)$–$(A2)$*, the map G is C^1. Moreover, for any $\lambda\in S^1$ the map $G_\lambda\colon\mathbf{l}^2\to\mathbf{l}^2$ is a Fredholm map of index* 0*.*

Proof. From Proposition 4.2 it follows directly that the map $G(\lambda,\mathbf{x}):=S\mathbf{x}-F(\lambda,\mathbf{x})$ is C^1. Fix $\mathbf{x}\in\mathbf{l}^2$ and $\lambda\in S^1$. Let $a_n(\lambda,x_n):=\dfrac{\partial f_n(\lambda,x_n)}{\partial x}$. From Proposition 4.2 it follows that $D_xG(\lambda,\mathbf{x})$ is the operator $L_\lambda\colon\mathbf{l}^2\to\mathbf{l}^2$ defined by

$$L_\lambda\mathbf{y}=(y_{n+1}-a_n(\lambda,x_n)y_n). \tag{37}$$

Assumption $(A2)$ and Lemma 4.2 imply that $\mathbf{a}=(a_n(\lambda,x_n))$ is asymptotically hyperbolic. Consequently by Theorem 3.1, the operator L_λ is Fredholm with index given by (26). Thus $\operatorname{ind}L_\lambda=0$, since by $(A2)$ the stable subspaces at $\pm\infty$ have the same dimension. □

Lemma 4.3.

For any bounded sequence $(\mathbf{x}_n)\subset\mathbf{l}^2$ the family of functions $\{G(\cdot,\mathbf{x}_n)\colon S^1\to\mathbf{l}^2\}_{n\in\mathbb{Z}}$ is equicontinuous.

Proof. First observe that there exists $M>0$ such that $|\mathbf{x}_n(k)|\le\|\mathbf{x}_n\|\le M$ for $n\in\mathbb{N}$ and $k\in\mathbb{Z}$. From Lemmas 4.1 and 4.2 it follows that

$$L_M:=\sup_{(\mu,\mathbf{x})\in[a,b]\times\bar{B}(0,M)}\|D_\lambda G(\mu,\mathbf{x})\|<\infty.$$

Integrating $D_\lambda G(\mu,\mathbf{x})$ over an arc of the circle joining λ_1 with λ_2 we get

$$\|G(\lambda_2,\mathbf{x}_n)-G(\lambda_1,\mathbf{x}_n)\|\le L\operatorname{dist}(\lambda_2,\lambda_1)$$

which implies the equicontinuity of the family $\left\{G(\cdot,\mathbf{x}_n)\colon S^1\to\mathbf{l}^2\right\}_{n\in\mathbb{Z}}$. □

5. Properness

We are going to discuss a properness criterion for the map G adapting to our framework the approach used in [30].

Definition 5.1 ([30]).
We say that a sequence $(\mathbf{x}_n)$ in $\mathbf{l}^2$ vanishes uniformly at infinity if, for all $\varepsilon > 0$, there exist $n_0 \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ such that $|\mathbf{x}_n(m)| \leq \varepsilon$ for all $n \geq n_0$ and for all $|m| \geq m_0$

Lemma 5.1.
Let $\mathbf{x} \in \mathbf{l}^2$ and let $(\mathbf{x}_n) \subset \mathbf{l}^2$. Then $\mathbf{x}_n \xrightarrow[n\to\infty]{} \mathbf{x}$ weakly in $\mathbf{l}^2$ if and only if $(\mathbf{x}_n)$ is norm bounded in $\mathbf{l}^2$ and $p_k(\mathbf{x}_n) \xrightarrow[n\to\infty]{} p_k(\mathbf{x})$, for all $k \in \mathbb{Z}$, where $p_k \colon \mathbf{l}^2 \to \mathbb{R}$ are the canonical projections.

Proof. This is proved in [6, Theorem 14.4]. □

Lemma 5.2.
Let $(h_n) \subset \mathbb{Z}$ be a sequence such that $\lim\limits_{n\to\infty} |h_n| = \infty$ and let $\mathbf{x} \in \mathbf{l}^2$. Define the sequence $(\widetilde{\mathbf{x}}_n)$ by $\widetilde{\mathbf{x}}_n(m) := \mathbf{x}(m + h_n)$ for $m \in \mathbb{Z}$, then $\widetilde{\mathbf{x}}_n \xrightarrow[n\to\infty]{} \mathbf{0}$ weakly.

Proof. It is a straightforward from Lemma 5.1. □

Lemma 5.3.
Let $(\mathbf{x}_n)$ be a bounded sequence in $\mathbf{l}^2$ and let $\mathbf{x} \in \mathbf{l}^2$. The following statements are equivalent:

(i) $\|\mathbf{x}_n - \mathbf{x}\|_\infty \xrightarrow[n\to\infty]{} 0$.

(ii) $\mathbf{x}_n \xrightarrow[n\to\infty]{} \mathbf{x}$ *in* $\mathbf{l}^2$ *and* $(\mathbf{x}_n)$ *vanishes uniformly at infinity.*

Proof. First, observe that the implication $(i) \implies (ii)$ is obvious. We are to show that (ii) implies (i). Fix $\varepsilon > 0$. Then there exist $m_0 \in \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $|\mathbf{x}_n(m)| < \varepsilon$ and $|\mathbf{x}(m)| < \varepsilon$, for all $|m| \geq m_0$ and $n \geq n_0$. Hence $|\mathbf{x}_n(m) - \mathbf{x}(m)| < 2\varepsilon$, for all $|m| \geq m_0$ and $n \geq n_0$. Lemma 5.1 implies that there exists $n_1 \in \mathbb{N}$ such that $|\mathbf{x}_n(m) - \mathbf{x}(m)| < \varepsilon$, for all $n \geq n_1$ and $|m| < m_0$. Thus we deduce that $\|\mathbf{x}_n - \mathbf{x}\|_\infty \leq 2\varepsilon$, for all $n \geq \max\{n_0, n_1\}$, which implies that $\|\mathbf{x}_n - \mathbf{x}\|_\infty \xrightarrow[n\to\infty]{} 0$. □

The following lemma will play a crucial role in the proof the properness.

Lemma 5.4 (Shifted subsequence lemma).
Let $(\mathbf{x}_n) \subset \mathbf{l}^2$ be a bounded sequence. Then at least one of the following properties must hold.

(i) $(\mathbf{x}_n)$ *vanishes uniformly at infinity.*

(ii) *There is a sequence $(l_k) \subset \mathbb{Z}$ with $\lim\limits_{k\to\infty} l_k = \infty$ and a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ such that a sequence $(\widetilde{\mathbf{x}}_k)$ defined by $\widetilde{\mathbf{x}}_k(m) := \mathbf{x}_{n_k}(m + l_k)$, for $m \in \mathbb{Z}$, converges weakly in $\mathbf{l}^2$ to $\widetilde{\mathbf{x}} \neq \mathbf{0}$.*

(iii) *There is a sequence $(l_k) \subset \mathbb{Z}$ with $\lim\limits_{k\to\infty} l_k = -\infty$ and a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ such that a sequence $(\widetilde{\mathbf{x}}_k)$ defined by $\widetilde{\mathbf{x}}_k(m) := \mathbf{x}_{n_k}(m + l_k)$, for $m \in \mathbb{Z}$, converges weakly to $\widetilde{\mathbf{x}} \neq \mathbf{0}$.*

Proof. Assume that $(\mathbf{x}_n)$ does not satisfy (i). Then there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ there exists $m_k \in \mathbb{Z}$ with $|m_k| \geq k$ and there exists $n_k \geq k$ such that $|\mathbf{x}_{n_k}(m_k)| \geq \varepsilon$. By passing to a subsequence, if necessary, we may suppose that (m_k) diverges either to ∞ or to $-\infty$. Suppose that $\lim\limits_{k\to\infty} m_k = \infty$. Let $l_k \in \mathbb{Z}$ be defined by $l_k := m_k$. Let $\widetilde{\mathbf{x}}_k := (\mathbf{x}_{n_k}(n + l_k))$. It is clear that $\|\widetilde{\mathbf{x}}_k\| = \|\mathbf{x}_{n_k}\|$. Since $(\tilde{\mathbf{x}}_k)$ is bounded, i.e., there exists $K > 0$ such that $\|\widetilde{\mathbf{x}}_k\| \leq K$ for all $k \in \mathbb{N}$, we can assume $(\widetilde{\mathbf{x}}_k)$ converges weakly in $\mathbf{l}^2$ to some element $\widetilde{\mathbf{x}}$. We will show that $\tilde{\mathbf{x}} \neq \mathbf{0}$. Observe that $\varepsilon \leq |\widetilde{\mathbf{x}}_k(0)| \leq K$. Hence $\lim\limits_{k\to\infty} \widetilde{\mathbf{x}}_k(0) = \widetilde{\mathbf{x}}(0) \neq 0$, since $\widetilde{\mathbf{x}}_k \rightharpoonup \widetilde{\mathbf{x}}$ in $\mathbf{l}^2$ (see Lemma 5.1). The same reasoning shows that (iii) holds if $\lim\limits_{k\to\infty} l_k = -\infty$. □

In the remaining part of this section we will study the properties of the maps $F_\lambda(\mathbf{x}) = F(\lambda, \mathbf{x})$ and $G_\lambda(\mathbf{x}) = S\mathbf{x} - F_\lambda(\mathbf{x})$ for a fixed value of parameter $\lambda \in S^1$. We will consider our assumptions $(A1)$–$(A4)$ to hold for the constant family $\mathbf{f}(\lambda, \mathbf{x}) = \mathbf{f}(\mathbf{x})$ and drop λ everywhere from the notations. For example, the derivative of G with respect to the second variable will be denoted by $DG(\mathbf{x})$ instead of $D_xG(\lambda, \mathbf{x})$).

Lemma 5.5.
$F\colon \mathbf{l}^2 \to \mathbf{l}^2$, $F^\infty_\pm\colon \mathbf{l}^2 \to \mathbf{l}^2$, $G\colon \mathbf{l}^2 \to \mathbf{l}^2$ and $G^\infty_\pm\colon \mathbf{l}^2 \to \mathbf{l}^2$ are weakly continuous.

Proof. Fix $\mathbf{x} \in \mathbf{l}^2$. Let $\mathbf{x}_k \xrightharpoonup[k\to\infty]{} \mathbf{x}$. We will show that $F^\infty_\pm(\mathbf{x}_k) \xrightharpoonup[k\to\infty]{} F^\infty_\pm(\mathbf{x})$. To this end, by Lemma 5.1 it suffices to show that $(F^\infty_\pm(\mathbf{x}_k))$ is norm bounded in $\mathbf{l}^2$ and $p_n(F^\infty_\pm(\mathbf{x}_k)) \xrightarrow[k\to\infty]{} p_n(F^\infty_\pm(\mathbf{x}))$, for all $n \in \mathbb{Z}$, where $p_n\colon \mathbf{l}^2 \to \mathbb{R}$ are the canonical projections. First observe that there exists $M > 0$ such that $\|\mathbf{x}_k\| < M$ for all $k \in \mathbb{N}$ and hence $|\mathbf{x}_k(n)| < M$ for $k \in \mathbb{N}$ and $n \in \mathbb{Z}$. From Lemma 4.1 we infer that

$$C := \sup_{(n,y)\in\mathbb{Z}\times\bar{B}_d(0,M)} |Df_n(y)| < \infty.$$

Thus, reasoning as in the proof of Proposition 4.1, we get $\|F^\infty_\pm(\mathbf{x}_k)\| \leq C\|\mathbf{x}_k\| < CM$, for all $k \in \mathbb{N}$. On the other hand, since $f_n \xrightarrow[n\to\pm\infty]{} f^\infty_\pm$, uniformly on bounded subsets of $\mathbb{R}^d$, it follows that the map $f^\infty_\pm\colon \mathbb{R}^d \to \mathbb{R}^d$ is continuous.

Since $\mathbf{x}_k(n) \xrightarrow[k\to\infty]{} \mathbf{x}(n)$, we deduce that

$$p_n(F^\infty_\pm(\mathbf{x}_k)) = f^\infty_\pm(\mathbf{x}_k(n)) \xrightarrow[k\to\infty]{} f^\infty_\pm(\mathbf{x}(n)) = p_n(F^\infty_\pm(\mathbf{x})),$$

which completes the proof that $F^\infty_\pm$ is weakly continuous. The same proof shows that F is also weakly continuous which, on its turn implies that both G and $G^\infty_\pm$ are weakly continuous. This completes the proof. □

Lemma 5.6.
If $G\colon \mathbf{l}^2 \to \mathbf{l}^2$ is a Fredholm map, then the following statements are equivalent:

(a) *The restricted map $G_{|D}$ is proper for each closed and bounded subset D of $\mathbf{l}^2$.*

(b) *If $(\mathbf{x}_n)$ is a bounded sequence in $\mathbf{l}^2$ such that $(G(\mathbf{x}_n))$ is convergent in $\mathbf{l}^2$, then $(\mathbf{x}_n)$ has a convergent subsequence in $\mathbf{c}(\mathbb{R}^d)$.*

Proof. A map G is proper on closed bounded subsets if and only if any bounded sequence $(\mathbf{x}_n)$ such that $G(\mathbf{x}_n)$ is convergent has a subsequence convergent to some point of the set.

Hence, that (a) implies (b) follows plainly from the continuity of the embedding $\mathbf{l}^2 \hookrightarrow \mathbf{c}(\mathbb{R}^d)$.

In order to show that (b) implies (a), let $(\mathbf{x}_n)$ be a bounded sequence (i.e., there exists $C > 0$ such that $\|\mathbf{x}_n\| \leq C$ for all $n \in \mathbb{N}$) such that

$$\|G(\mathbf{x}_n) - \mathbf{y}\| \xrightarrow[n\to\infty]{} 0, \tag{38}$$

where $\mathbf{y} \in \mathbf{l}^2$. Since the ball $\bar{B}(\mathbf{0}, C)$ in $\mathbf{l}^2$ is weakly-compact we can assume that $\mathbf{x}_n \rightharpoonup \mathbf{x}$ in $\mathbf{l}^2$ (and hence $\|\mathbf{x}\| \leq C$). By (b), passing to a subsequence if necessary, we can assume that $\|\mathbf{x}_n - \mathbf{x}\|_\infty \to 0$ as $n \to \infty$. Since, by Lemma 5.5, G is weakly continuous, we have $G(\mathbf{x}_n) \xrightarrow[n\to\infty]{} G(\mathbf{x})$, and consequently $\mathbf{y} = G(\mathbf{x})$.

Lemma 5.7.

$$\|G(\mathbf{x}_n) - G(\mathbf{x}) - DG(\mathbf{x})(\mathbf{x}_n - \mathbf{x})\| \xrightarrow[n\to\infty]{} 0. \tag{39}$$

Proof. The assertion (39) is equivalent to

$$\|F(\mathbf{x}_n) - F(\mathbf{x}) - DF(\mathbf{x})(\mathbf{x}_n - \mathbf{x})\| \xrightarrow[n\to\infty]{} 0. \tag{40}$$

Fix $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $\|\tilde{\mathbf{x}} - \mathbf{x}\|_\infty < \delta$, then

$$\sup_{k\in\mathbb{Z}} |Df_k(\tilde{\mathbf{x}}(k)) - Df_k(\mathbf{x}(k))| < \varepsilon \text{ (see Assumption (A1))}. \tag{41}$$

Let $n_0 > 0$ be such that $\|\mathbf{x}_n - \mathbf{x}\|_\infty < \delta$, for all $n \geq n_0$. Fix $n \geq n_0$ and $k \in \mathbb{Z}$. Then

$$\begin{aligned}
&f_k(\mathbf{x}_n(k)) - f_k(\mathbf{x}(k)) - Df_k(\mathbf{x}(k))(\mathbf{x}_n(k) - \mathbf{x}(k)) = \\
&\int_0^1 Df_k\big(\mathbf{x}_n(k) - s[\mathbf{x}_n(k) - \mathbf{x}(k)]\big)\big(\mathbf{x}_n(k) - \mathbf{x}(k)\big)ds - \int_0^1 Df_k(\mathbf{x}(k))(\mathbf{x}_n(k) - \mathbf{x}(k))ds = \\
&\int_0^1 \Big(Df_k\big(\mathbf{x}_n(k) - s[\mathbf{x}_n(k) - \mathbf{x}(k)]\big) - Df_k(\mathbf{x}(k))\Big)\left(\mathbf{x}_n(k) - \mathbf{x}(k)\right) ds.
\end{aligned}$$

Hence

$$\begin{aligned}
&|f_k(\mathbf{x}_n(k)) - f_k(\mathbf{x}(k)) - Df_k(\mathbf{x}(k))(\mathbf{x}_n(k) - \mathbf{x}(k))| = \\
&\left|\int_0^1 \Big(Df_k\big(\mathbf{x}_n(k) - s[\mathbf{x}_n(k) - \mathbf{x}(k)]\big) - Df_k(\mathbf{x}(k))\Big)\big(\mathbf{x}_n(k) - \mathbf{x}(k)\big)ds\right| \leq \\
&\int_0^1 \big|Df_k\big(\mathbf{x}_n(k) - s[\mathbf{x}_n(k) - \mathbf{x}(k)]\big) - Df_k(\mathbf{x}(k))\big|\, |\mathbf{x}_n(k) - \mathbf{x}(k)|ds.
\end{aligned}$$

Taking into account (41), we obtain

$$\int_0^1 \big|Df_k\big(\mathbf{x}_n(k) - s[\mathbf{x}_n(k) - \mathbf{x}(k)]\big) - Df_k(\mathbf{x}(k))\big|\, |\mathbf{x}_n(k) - \mathbf{x}(k)|ds \leq \int_0^1 \varepsilon\, |\mathbf{x}_n(k) - \mathbf{x}(k)|\, ds = \varepsilon\, |\mathbf{x}_n(k) - \mathbf{x}(k)|\,.$$

Thus, we arrive at

$$\|F(\mathbf{x}_n) - F(\mathbf{x}) - DF(\mathbf{x})(\mathbf{x}_n - \mathbf{x})\|^2 = \sum_{k\in\mathbb{Z}} |f_k(\mathbf{x}_n(k)) - f_k(\mathbf{x}(k)) - Df_k(\mathbf{x}(k))(\mathbf{x}_n(k) - \mathbf{x}(k))|^2$$
$$\leq \sum_{k\in\mathbb{Z}} \varepsilon^2 |\mathbf{x}_n(k) - \mathbf{x}(k)|^2 = \varepsilon^2 \|\mathbf{x}_n - \mathbf{x}\|^2 \leq \varepsilon^2 (2C)^2,$$

for $n \geq n_0$. This proves that

$$\|F(\mathbf{x}_n) - F(\mathbf{x}) - DF(\mathbf{x})(\mathbf{x}_n - \mathbf{x})\| \xrightarrow[n\to\infty]{} 0 \tag{42}$$

and the lemma. □

By the above lemma and (38) we have $\|DG(\mathbf{x})(\mathbf{x}_n - \mathbf{x})\| \xrightarrow[n\to\infty]{} 0$. By Riesz criterion, Fredholm operators are invertible modulo compact operators. Therefore, there exist a bounded operator $B\colon \mathbf{l}^2 \to \mathbf{l}^2$ and a compact operator $K\colon \mathbf{l}^2 \to \mathbf{l}^2$ such that $B \circ DG(\mathbf{x}) = I + K$. In turn this implies that

$$\begin{aligned} &\|\mathbf{x}_n - \mathbf{x}\| = \|(B \circ DG(\mathbf{x}) - K)(\mathbf{x}_n - \mathbf{x})\| \leq \|\big(B \circ DG(\mathbf{x})\big)(\mathbf{x}_n - \mathbf{x})\| + \|K(\mathbf{x}_n - \mathbf{x})\| \leq \\ &\|B\|\|DG(\mathbf{x})(\mathbf{x}_n - \mathbf{x})\| + \|K(\mathbf{x}_n - \mathbf{x})\|. \end{aligned} \tag{43}$$

Since a compact operator $K\colon \mathbf{l}^2 \to \mathbf{l}^2$ maps weakly convergent sequences onto norm convergent sequences, we infer that

$$\|K(\mathbf{x}_n - \mathbf{x})\| \xrightarrow[n\to\infty]{} 0. \tag{44}$$

Thus, in view of (42)–(44), one obtains $\|\mathbf{x}_n - \mathbf{x}\| \xrightarrow[n\to\infty]{} 0$, which completes the proof. □

Given $m \in \mathbb{Z}$, by $S_m\colon \mathbf{l}^2 \to \mathbf{l}^2$ we will denote the m-shift operator defined by

$$S_m\mathbf{x} := S^m(\mathbf{x}) = (x_{n+m}). \tag{45}$$

Lemma 5.8.
For any $\mathbf{x} \in \mathbf{l}^2$ and $m \in \mathbb{Z}$, $p_m\left(S_kG(\mathbf{x}) - S_kG^\infty_\pm(\mathbf{x})\right) \xrightarrow[k\to\pm\infty]{} 0$ (uniformly with respect to any bounded set $B \subset \mathbf{l}^2$).

Proof. Let $B \subset \mathbf{l}^2$ be a bounded subset. Then there exists a constant C such that $|\mathbf{x}(n)| \leq C$, for all $n \in \mathbb{Z}$ and $\mathbf{x} \in B$. Assumption $(A4)$ implies that there exists a positive integer $\tilde{n}_0 = n(\varepsilon, B)$ such that $|f_{\pm k}(x) - f^\infty_\pm(x)| < \varepsilon$, for $k \geq \tilde{n}_0$ and x with $|x| \leq C$. Consequently,

$$|f_{\pm k}(\mathbf{x}(\pm k)) - f^\infty_\pm(\mathbf{x}(\pm k))| < \varepsilon, \tag{46}$$

for $k \geq \tilde{n}_0$ and $\mathbf{x} \in B$. Finally, taking into account (46), we deduce that for any $k \geq n_0 := \tilde{n}_0 + |m|$ one has

$$|p_{m\pm k}(G(\mathbf{x}) - G^\infty_\pm(\mathbf{x}))| = |p_{m\pm k}(F(\mathbf{x}) - F^\infty_\pm(\mathbf{x}))| = |f_{m\pm k}(\mathbf{x}(m \pm k)) - f^\infty_\pm(\mathbf{x}(m \pm k))| < \varepsilon,$$

for all $\mathbf{x} \in B$. Finally, it suffices to observe that

$$|p_m\left(S_l G(\mathbf{x}) - S_l G^{\infty}_{\pm}(\mathbf{x})\right)| = |p_{m+l}(G(\mathbf{x}) - G^{\infty}_{\pm}(\mathbf{x}))|.$$

This completes the proof. $\square$

Theorem 5.1.
Under Assumptions (A1)–(A2) and (A4), $G\colon \mathbf{l}^2 \to \mathbf{l}^2$ is proper on the closed bounded subsets of $\mathbf{l}^2$.

Proof. In view of Lemma 5.3 and Lemma 5.6 it suffices to show that any bounded sequence $(\mathbf{x}_n)$ in $\mathbf{l}^2$ such that $\|G(\mathbf{x}_n) - \mathbf{y}\| \xrightarrow[n\to\infty]{} 0$ for some $\mathbf{y} \in \mathbf{l}^2$ has a weakly convergent subsequence which vanishes uniformly at infinity. Since $(\mathbf{x}_n)$ is bounded, we may assume without loss of generality that $\mathbf{x}_n \xrightarrow[n\to\infty]{} \mathbf{x}$ weakly in $\mathbf{l}^2$ for some $\mathbf{x} \in \mathbf{l}^2$. If the alternative (ii) of Lemma 5.4 holds, $(\mathbf{x}_n)$ has a subsequence x_{n_k} whose translates $\widetilde{\mathbf{x}}_k(n) := \mathbf{x}_{n_k}(n + l_k)$ converge weakly to $\widetilde{\mathbf{x}} \neq \mathbf{0}$.
Observe that $\|G(\mathbf{x}_n) - \mathbf{y}\| = \|S_{l_k} G(\mathbf{x}_n) - \widetilde{\mathbf{y}}_k\|$, where $\widetilde{\mathbf{y}}_k := S_{l_k}\mathbf{y}$, and therefore

$$\|S_{l_k} G(\mathbf{x}_{n_k}) - \widetilde{\mathbf{y}}_k\| \leq \|S_{l_k} G(\mathbf{x}_{n_k}) - S_{l_k} G(\mathbf{x}_n)\| + \|S_{l_k} G(\mathbf{x}_n) - \widetilde{\mathbf{y}}_k\| = \|G(\mathbf{x}_{n_k}) - G(\mathbf{x}_n)\| + \|G(\mathbf{x}_n) - \mathbf{y}\|,$$

which shows that

$$\|S_{l_k} G(\mathbf{x}_{n_k}) - \widetilde{\mathbf{y}}_k\| \xrightarrow[k\to\infty]{} 0. \tag{47}$$

Now let us fix $m \in \mathbb{Z}$. By Lemma 5.8

$$|p_m\left(S_{l_k} G(\mathbf{x}_{n_k}) - S_{l_k} G^{\infty}_{+}(\mathbf{x}_{n_k})\right)| \xrightarrow[k\to\infty]{} 0 \tag{48}$$

and hence $|p_m(S_{l_k} G^{\infty}_{+}(\mathbf{x}_{n_k}) - \widetilde{\mathbf{y}}_k)| \xrightarrow[k\to\infty]{} 0$ as well. Since

$$S_{l_k}(G^{\infty}_{+}(\mathbf{x}_{n_k})) = S(\mathbf{x}_{n_k}(n + l_k)) - (f^{\infty}_{+}(\mathbf{x}_{n_k}(n + l_k))) = S(\widetilde{\mathbf{x}}_k(n)) - (f^{\infty}_{+}(\widetilde{\mathbf{x}}_k(n))) = G^{\infty}_{+}(\widetilde{\mathbf{x}}_k),$$

we deduce that

$$|p_m(G^{\infty}_{+}(\widetilde{\mathbf{x}}_k) - \widetilde{\mathbf{y}}_k)| \xrightarrow[k\to\infty]{} 0.$$

Since the sequence $(G^{\infty}_{+}(\widetilde{\mathbf{x}}_k) - \widetilde{\mathbf{y}}_k)$ is bounded in $\mathbf{l}^2$, it follows from Lemma 5.1 that $G^{\infty}_{+}(\widetilde{\mathbf{x}}_k) - \widetilde{\mathbf{y}}_k \xrightarrow[k\to\infty]{} \mathbf{0}$ in $\mathbf{l}^2$. But Lemma 5.2 implies that $\widetilde{\mathbf{y}}_k \xrightarrow[k\to\infty]{} \mathbf{0}$ in $\mathbf{l}^2$. Hence we get that $G^{\infty}_{+}(\widetilde{\mathbf{x}}_k) \xrightarrow[k\to\infty]{} \mathbf{0}$ in $\mathbf{l}^2$. However, the weak sequential continuity of $G^{\infty}_{+}\colon \mathbf{l}^2 \to \mathbf{l}^2$ implies that $G^{\infty}_{+}(\widetilde{\mathbf{x}}_k) \xrightarrow[k\to\infty]{} G^{\infty}_{+}(\widetilde{\mathbf{x}})$ weakly in $\mathbf{l}^2$, which implies that $G^{\infty}_{+}(\widetilde{\mathbf{x}}) = \mathbf{0}$, contradicting Assumption (A4). This shows that the sequence $(\mathbf{x}_n)$ cannot have the property (ii) of Lemma 5.4. By a similar arguments we can exclude the property (iii) in Lemma 5.4. This completes the proof. $\square$

6. Proof of the main theorem

For the proof we will use an extension of Leray-Schauder degree to proper Fredholm maps of index 0 introduced in [18] under the name of *base point degree.* What is of interest for us is a very special form of the homotopy principle for this degree. As a consequence of Kuiper's theorem about the contractibility of the linear group of a Hilbert space, only the absolute value of any degree theory for general Fredholm maps extending the Leray-Schauder degree can be homotopy invariant. The most interesting characteristic of the base point degree consists in that the change in sign of the degree along a homotopy can be determined using an invariant of paths of linear Fredholm operators of index zero called *parity.*

The parity is defined as follows: Let $L\colon [a,b] \to \Phi_0(X,Y)$ be a path of Fredholm operators such that both L_a and L_b are invertible. It can be shown [11] that there exists a path of invertible operators $P\colon [a,b] \to GL(Y,X)$ such that $L_tP_t = \mathrm{Id}_Y - K_t$, where K_t is a family of operators with $\operatorname{Im} K_t$ contained in a finite dimensional subspace V of Y. Such a path P is called a (regular) parametrix. If P is a parametrix, being L_aP_a and L_bP_b invertible so are their restrictions $C_a, C_b\colon V \to V$ to the subspace V containing the images of K_a, K_b. The *parity* of the path L is the element $\sigma(L) \in \mathbb{Z}_2 = \{1,-1\}$ defined by

$$\sigma(L) = \operatorname{sign}\det C(a)\operatorname{sign}\det C(b).$$

The above definition is independent of the choices involved. The parity is multiplicative and invariant under homotopies of paths with invertible end points. If the path L is closed, i.e., $L_a = L_b$, then, via the identification $S^1 \simeq [a,b]/\{a,b\}$ we can consider the path L as a map $L\colon S^1 \to \Phi_0(X,Y)$ and relate the parity of a closed path with the obstruction to triviality $w_1\colon \widetilde{KO}(S^1) \to \mathbb{Z}_2$.

Lemma 6.1 ([11], Proposition 1.6.4 or [23], Proposition 3.1).
Under the above assumptions,

$$\sigma(L) = w_1(\operatorname{Ind} L). \tag{49}$$

Now let us recall the construction of the degree. A C^1-Fredholm map of index 0 is by definition a C^1-map $f\colon \mathcal{O} \to Y$ such that the Fréchet derivative $Df(x)$ of f at x is a Fredholm operator of index 0.

Let $\mathcal{O} \subset X$ be an open simply connected set. An *admissible triple* (f,Ω,y) is defined by a C^1-Fredholm map of index 0, $f\colon \mathcal{O} \to Y$ which is proper on closed bounded subsets of $\mathcal{O}$, an open bounded set Ω whose closure is contained in $\mathcal{O}$ and a point $y \in Y$ such that $y \notin f(\partial\Omega)$. The construction of [18] associates to each admissible triple (f,Ω,y) and each point $b \in \mathcal{O}$, called *base point*, an integral number $\deg_b(f,\Omega,y) \in \mathbb{Z}$ called *base point degree.* A *regular base point* is a point $b \in \mathcal{O}$ which is a regular point of the map f (i.e., $Df(b)$ is an isomorphism). If b is a regular base point and y is a regular value of the restriction of f to Ω, then the base point degree is defined by

$$\deg_b(f,\Omega,0) = \sum_{x\in f^{-1}(0)} \sigma(Df \circ \gamma), \tag{50}$$

where γ is any path in $\mathcal{O}$ joining b to x.

In order to define the degree for any $y \in Y$ it is used an approximation result by regular values (see [18] for details). If b is a singular point of f, then by definition $\deg_b(f, \Omega, y) = 0$. The degree such defined has the usual additivity, excision and normalization properties. For C^1-maps that are compact perturbations of the identity it coincides with the Leray-Schauder degree. However the homotopy property requires a different formulation (for the sake of definiteness we will take $y = 0$):

Lemma 6.2.
Let $h\colon [0,1] \times \mathcal{O} \to Y$ be a continuous map that is proper on closed bounded subsets and such that each h_t is a C^1-Fredholm map. Let Ω be an open bounded subset of X such that $0 \notin h([0,1] \times \partial\Omega)$. If $b_i \in \mathcal{O}$ is a base point for $h_i := h(i, \cdot), i = 0, 1,$ then

$$\deg_{b_0}(h_0, \Omega, 0) = \sigma(M)\deg_{b_1}(h_1, \Omega, 0),$$

where $M\colon [0,1] \to \Phi_0(X,Y)$ is the path $L \circ \gamma$, where $L(t,x) := D_x h(t,x)$ and γ is any path joining $(0, b_0)$ to $(1, b_1)$ in $[0,1] \times \mathcal{O}$.

Notice that $\sigma(M)$ is independent of the choice of the path γ because $[0,1] \times \mathcal{O}$ is simply connected. The proof of the lemma 6.2 can be found in ([22, Lemma 2.3.1]). Here we will need a minor generalization of the above property.

Lemma 6.3 (Generalized Homotopy Property).
Let $h\colon [0,1] \times \mathcal{O} \to Y$ be a continuous map that is proper on closed bounded subsets and such that each h_t is a C^1-Fredholm map. Let Ω be an open and bounded set whose closure is contained in $[0,1] \times \mathcal{O}$ such that $0 \notin h(\partial\Omega)$. If $b_i \in \mathcal{O}$ is a base point for $h_i := h(i, \cdot), i = 0, 1,$ then

$$\deg_{b_0}(h_0, \Omega_0, 0) = \sigma(M)\deg_{b_1}(h_1, \Omega_1, 0), \tag{51}$$

where M is as above and $\Omega_t := \{x \in X \mid (t,x) \in \Omega\}$, for $t \in [0,1]$.

Proof. We will prove the lemma assuming that $\deg_{b_0}(h_0, \Omega_0, 0) \neq 0$, which is the only case that we will need in the sequel and leave to the reader the completion of the proof in the general case. Since the degree of a map without regular points vanishes, being the absolute value of the degree a homotopy invariant, it follows from our assumption that for all $\tau \in [0,1]$ there exists a regular base point for h_τ. Let $C(t) := \{x \in X \mid h(t,x) = 0\} \cap \Omega = \{x \in X \mid h(t,x) = 0\} \cap \bar{\Omega}$. Now we will prove that the map $[0,1] \ni t \mapsto C(t) \subset Y$ is upper semicontinuous, i.e., for any point $t_0 \in [0,1]$ and any open neighborhood $V \subset X$ such that $C(t_0) \subset V$ there exists an open neighborhood U_{t_0} of t_0 in $[0,1]$ such that $C(t) \subset V$ for all $t \in U_{t_0}$. Indeed, let t_0 and V be as above. Assume on the contrary that for any $\varepsilon > 0$ there exists $t_\varepsilon \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap [0,1]$ such that $C(t_\varepsilon) \cap (X \setminus V) \neq \emptyset$. Thus there exists a sequence $(t_n, x_n) \in \bar{\Omega}$ such that $t_n \xrightarrow[n\to\infty]{} t_0$, $h(t_n, x_n) = 0$ and $x_n \in X \setminus V$. Since $h^{-1}(0) \cap \bar{\Omega}$ is compact, we can assume that $x_n \xrightarrow[n\to\infty]{} x_0$ for some $x_0 \in X \setminus V$. Furthermore, the continuity of h implies that $h(t_0, x_0) = 0$. Since $(t_0, x_0) \in \bar{\Omega}$, it follows that $x_0 \in C(t_0) \subset V$, which contradicts the fact that $x_0 \in X \setminus V$.

Thus, given any point $t \in I = [0,1]$ and an open neighborhood $V_t \subset \Omega_t$ of $C(t)$ we can find an open neighborhood U_t of t in I such that $C(t') \subset V_t$ for all $t' \in U_t$. Let $\delta > 0$ be the Lebesgue number of the covering $\{U_t \mid t \in I\}$

and let $0 = t_0 < t_1 < \cdots < t_n = 1$ be a partition of I with mesh less than δ. Then any subinterval $I_i = [t_{i-1}, t_i]$ is contained in some element U_{τ_i} of the covering and therefore $C(I_i) \subset V_{\tau_i}$. This means that the graph of $C_{|I_i}$, which, by its very definition, is the set $\{(t,x) \in \Omega \mid t \in I_i, h(t,x) = 0\}$, must be contained in $I_i \times V_{\tau_i}$.

Since $0 \notin h(\bar{\Omega} \cap (I_i \times X) \setminus (I_i \times V_{\tau_i}))$, choosing any regular base point b_i of h_{t_i}, we can apply Lemma 6.2 to the map $h\colon I_i \times \bar{V}_{\tau_i} \to Y$ and use the excision property of the degree in order to obtain

$$\deg_{b_{i-1}}(h_{t_{i-1}}, \Omega_{t_{i-1}}, 0) = \deg_{b_{i-1}}(h_{t_{i-1}}, V_{\tau_i}, 0) = \sigma(M_i)\deg_{b_i}(h_{t_i}, V_{\tau_i}, 0) = \sigma(M_i)\deg_{b_i}(h_{t_i}, \Omega_{t_i}, 0),$$

where $M_i(t) = D_x h(\gamma_i(t))$ and $\gamma_i\colon I_i \to [0,1] \times \mathcal{O}$ is any path joining (t_{i-1}, b_{i-1}) to (t_i, b_i).

Now (51) follows from the above identities, because, by the multiplicative property of the parity, $\prod_{i=1}^n \sigma(M_i) = \sigma(M)$, where $M(t) = D_x h(\gamma(t))$ and γ is the concatenation of all paths γ_i, $1 \leq i \leq n$. This completes the proof. $\square$

We will need also the following result. Recall that nonempty subsets A, B of a space X are separated (in X) if there exists open (and hence closed) neighborhoods $U_A \supset A$, $U_B \supset B$ in X such that $U_A \cap U_B = \emptyset$ and $U_A \cup U_B = X$. Two sets are connected (to each other) in X if there is a connected set $Y \subset X$ with $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$. Whyburn's Lemma (see [3]) says that if A, B are closed subsets of a compact space X that are not connected to each other, then they are separated in X.

Now let us prove our main Theorem 2.1.

Proof. We first prove [i]. Let $X = \mathbf{l}^2$ and let $\mathcal{S} = G^{-1}(\mathbf{0}) \subset S^1 \times X$. $\mathcal{S}$ is a locally compact space and in fact σ-compact, since $\mathcal{S} = \bigcup_{k \in \mathbb{N}} \mathcal{S} \cap \left(S^1 \times \bar{B}(\mathbf{0}, k)\right)$. Let $\mathcal{C}_0$ be the connected component of $\mathcal{T}_0$ in $\mathcal{S}$. Suppose that $\mathcal{C}_0$ is bounded and let W be any bounded closed neighborhood of $\mathcal{C}_0$. Since $\mathcal{C}_0$ is a maximal connected set, $A := \mathcal{S} \cap \partial W$ is not connected with $\mathcal{C}_0$ in the compact space $\mathcal{S} \cap W$. Therefore there exist two compact subsets K_0, K_1 of $\mathcal{S} \cap W$ separating the component $\mathcal{C}_0$ from A. Let $d = \operatorname{dist}(K_0, K_1) > 0$, and let $\Omega := \{(\lambda, \mathbf{x}) \in S^1 \times X \mid d((\lambda, \mathbf{x}), K_0) < 1/2d\}$. Then Ω is an open bounded neighborhood of $\mathcal{C}_0$ in $S^1 \times X$ such that $G(\lambda, \mathbf{x}) \neq \mathbf{0}$ on $\partial\Omega$. For simplicity, we can assume that λ_0 satisfying Assumption $(A3)$ equals 1. Let $q\colon [0,1] \to S^1$ be the identification map taking $0, 1$ into $1 \in S^1$. Let us consider the homotopy $H\colon [0,1] \times X \to X$ defined by $H(t,x) = G(q(t), x)$. Clearly H is a continuous family of C^1-Fredholm maps. Put $\Omega' := p^{-1}(\Omega)$. By construction Ω' is an open bounded subset of $[0,1] \times X$ and H has no zeros on the boundary of Ω'. We will apply the generalized homotopy principle to H on Ω' in order to obtain a contradiction. For this we need to show:

Lemma 6.4.
The restriction of H to any closed bounded subset of $[0,1] \times X$ is proper.

Proof. Let K be a compact subset of X and let D be a closed bounded subset of $[0,1] \times X$. We have to show that $(H_{|D})^{-1}(K)$ is compact. To this end, take any sequence $(t_n, \mathbf{x}_n) \in (H_{|D})^{-1}(K)$. Without loss of generality we can assume that there exist $t_0 \in S^1$ and $\mathbf{y} \in X$ such that

$$t_n \xrightarrow[n\to\infty]{} t_0 \in [0,1] \text{ and } \|H(t_n,\mathbf{x}_n) - \mathbf{y}\| \xrightarrow[n\to\infty]{} 0.$$

Since, by Lemma 4.3, the family of functions $\{G(\cdot,\mathbf{x}_n)\colon S^1 \to \mathbf{l}^2\}_{n\in\mathbb{Z}}$ is equicontinuous and $H(\cdot,\mathbf{x}_n) = G(q(\cdot),\mathbf{x}_n)$, we infer that the family $\{H(\cdot,\mathbf{x}_n)\colon S^1 \to \mathbf{l}^2\}_{n\in\mathbb{Z}}$ is also equicontinuous. Now we will show that

$$\|H(t_0,\mathbf{x}_n) - \mathbf{y}\| \xrightarrow[n\to\infty]{} 0. \tag{52}$$

For this purpose, fix $\varepsilon > 0$. Then there exists $k_0 > 0$ such that

$$\|H(t_m,\mathbf{x}_n) - H(t_0,\mathbf{x}_n)\| < \varepsilon/2 \text{ for } m \geq k_0 \text{ and for all } n \in \mathbb{N},$$
$$\|H(t_n,\mathbf{x}_n) - y\| < \varepsilon/2 \text{ for } n \geq k_0.$$

Hence

$$\|H(t_0,\mathbf{x}_n) - \mathbf{y}\| \leq \|H(t_n,\mathbf{x}_n) - \mathbf{y}\| + \|H(t_n\mathbf{x}_n) - H(t_0,\mathbf{x}_n)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

for $n \geq k_0$, which proves (52).

From Theorem 5.1 it follows that $H_{t_0}\colon X \to X$ is proper on the closed and bounded subsets of X and therefore there exists a subsequence $(\mathbf{x}_{n_k})$ of $(\mathbf{x}_n)$ and $\mathbf{x} \in X$ such that $\|\mathbf{x}_{n_k} - \mathbf{x}\| \xrightarrow[k\to\infty]{} 0$. Therefore, $(t_{n_k},\mathbf{x}_{n_k}) \xrightarrow[k\to\infty]{}$ $(t_0,\mathbf{x})$ in D. Thus we conclude that $(H_{|D})^{-1}(K)$ is compact, which completes the proof of lemma. □

By the above lemma H is an admissible homotopy with $H_0 := H(0,\cdot) = H(1,\cdot) =: H_1$. Furthermore, we can take $b = \mathbf{0}$ as the base point for both H_0 and H_1 since, by $(A3)$, $DH_i(,\mathbf{0}) = DG_1(\mathbf{0})$ is an isomorphism. Now let us apply Lemma 6.3 choosing as path joining $(0,\mathbf{0})$ with $(1,\mathbf{0})$ the path $\gamma(t) = (t,\mathbf{0})$. It follows then, that

$$\deg_{\mathbf{0}}(H_0,\Omega_0',\mathbf{0}) = \sigma(M)\deg_{\mathbf{0}}(H_1,\Omega_1',\mathbf{0}), \tag{53}$$

where M is the closed path of Fredholm operators given by $M(t) := D_xH(t,\mathbf{0}) = DH_t(\mathbf{0})$. On the other hand, Assumption $(A3)$ implies that $\mathbf{0}$ is the only solution of $H_i(\mathbf{x})$; $i = 0,1$ which is a regular point of H_i since $DH_i(,\mathbf{0})$ is an isomorphism. By definition of the base point degree ([18]) and Assumption $(A3)$, one has

$$\deg_{\mathbf{0}}(H_0,\Omega_0',\mathbf{0}) = \deg_{\mathbf{0}}(H_1,\Omega_1',\mathbf{0}) = 1, \tag{54}$$

which in turn implies, by (53), that $\sigma(M) = 1$. But, by Lemma 6.1 and (28), this contradicts our assumption (11).

In order to prove $[ii]$ we observe that $[i]$ implies that the closure of $\mathcal{S}$ in $S^1 \times X^+$ is a compact space and the closure of $\mathcal{C}_0$ in this space is a connected set intersecting both $\mathcal{B}_0$ and $\mathcal{B}_\infty$. In order to conclude the proof of $[ii]$ it is enough to use the following slightly improved version of Whyburn's lemma:

Proposition 6.1 ([3], Proposition 5).
Suppose A and B are closed and not separated in a compact space X. Then there exists a connected set $D \subset X \setminus (A \cup B)$ such that $\bar{D} \cap A \neq \emptyset$ and $\bar{D} \cap B \neq \emptyset$.

□

7. Example

Now we are going to illustrate the content of Theorem 2.1 formulated in Section 2 and the techniques developed in this paper. Fix $0 < \alpha < 1$ and $\beta > 1$. For $\lambda = \exp(i\theta)$, $0 \le \theta \le 2\pi$, we define $a\colon S^1 \to GL(2)$ as follows

$$a(\lambda) = a(\exp i\theta) := \begin{pmatrix} \alpha + (\beta - \alpha)\sin^2\left(\frac{\theta}{2}\right) & \frac{\alpha - \beta}{2}\sin(\theta) \\ \frac{\alpha - \beta}{2}\sin(\theta) & \alpha + (\beta - \alpha)\cos^2\left(\frac{\theta}{2}\right) \end{pmatrix}.$$

Then we can consider the linear nonautonomous system $\mathbf{a} = (a_n(\lambda))\colon \mathbb{Z} \times S^1 \to GL(2)$ defined by

$$a_n(\lambda) = \begin{cases} a(\lambda) & \text{if} \quad n \ge 0, \\ a(1) & \text{if} \quad n < 0. \end{cases} \tag{55}$$

Since independently of $\lambda \in S^1$ the matrix $a(\lambda)$ has two eigenvalues $\alpha \in (0,1)$ and $\beta \in (1,\infty)$, the system $\mathbf{a}\colon \mathbb{Z} \times S^1 \to GL(2)$ is asymptotically hyperbolic. We will apply our results to nonlinear perturbations of $\mathbf{a}\colon \mathbb{Z} \times S^1 \to GL(2)$. We compute the asymptotic stable bundles of $\mathbf{a}$ at $\pm\infty$:

$$E^s(+\infty) = \{(\lambda, u) \in S^1 \times \mathbb{R}^2 \mid u = t(\cos(\theta/2), \sin(\theta/2)), \lambda = \exp(i\theta), t \in \mathbb{R}\},$$
$$E^s(-\infty) = \{(\lambda, u) \in S^1 \times \mathbb{R}^2 \mid u = (t, 0), t \in \mathbb{R}\}.$$

Thus $E^s(-\infty)$ is a trivial bundle and hence $w_1(E^s(-\infty)) = 1$. In order to compute $w_1(E^s(+\infty))$ we notice that $v_\theta = (\cos(\theta/2), \sin(\theta/2))$ is a basis for $E^s_\theta(+\infty)$ which is the fiber of the pullback E' of $E^s(+\infty)$ by the map $p\colon [0, 2\pi] \to S^1$ defined by $p(\theta) = \exp(i\theta)$. Since $v_0 = (1,0)$ and $v_{2\pi} = (-1, 0)$, the determinant of the matrix C arising in (10) is -1. Hence $w_1(E^s(+\infty)) = -1 \neq w_1(E^s(-\infty))$.

Let us consider the family L of operators L_λ defined by

$$L_\lambda(\mathbf{x})(n) = \begin{cases} x_{n+1} - a(\lambda)x_n & \text{if} \quad n \ge 0, \\ x_{n+1} - a(1)x_n & \text{if} \quad n < 0. \end{cases}$$

Then Remark 3.1 implies that $\operatorname{Ker} L_\lambda$ is isomorphic to $E^s(\lambda, +\infty) \cap E^u(\lambda, -\infty)$. But $E^u(-\infty) = \{(\lambda, u) \in S^1 \times \mathbb{R}^2 \mid u = (0, t), t \in \mathbb{R}\}$ and hence a nontrivial intersection arises only for $\theta = \pi$, i.e., $\lambda = -1$. Thus $\operatorname{Ker} L_\lambda \neq \{\mathbf{0}\}$ only if $\lambda = -1$. Theorem 3.1 implies that L_λ is a Fredholm operator of index: $\operatorname{ind}(L_\lambda) = \dim E^s(\lambda, +\infty) -$

$\dim E^s(\lambda,-\infty) = 1-1 = 0$. Hence we infer that L_λ is an isomorphism for $\lambda \neq -1$, since $\operatorname{Ker} L_\lambda = \{\mathbf{0}\}$, for $\lambda \neq -1$.

Let $\mathbf{h}\colon \mathbb{Z}\times S^1\times\mathbb{R}^2 \to \mathbb{R}^2$ be a continuous family of C^1-dynamical systems satisfying Assumption $(A1)$ and

$(A2')$ $D_x h_n(\lambda,0) \xrightarrow[n\to\pm\infty]{} 0$ uniformly with respect to $\lambda \in S^1$;

$(A3')$ for any $x\in\mathbb{R}^2$ and $\lambda\in S^1$, $h_n(\lambda,x) \xrightarrow[n\to\pm\infty]{} h^\infty_\pm(\lambda,x)$ (uniformly with respect to any bounded set $B\subset\mathbb{R}^2$), and the following two difference equations $x_{n+1} = a(\lambda)x_n + h^\infty_\pm(\lambda,x_n)$ admit only the trivial solution $(x_n = 0)_{n\in\mathbb{Z}}$, for all $\lambda\in S^1$;

$(A4')$ for some $\lambda_0 \in S^1\setminus\{-1\}$ we have that $D_x h_n(\lambda_0,0) = 0$, for all $n\in\mathbb{Z}$, and the difference equation $x_{n+1} = a(\lambda_0)x_n + h_n(\lambda_0,x_n)$ admits only the trivial solution $(x_n=0)_{n\in\mathbb{Z}}$.

Now it is easily seen that whenever the nonlinear perturbation $\mathbf{h}$ verifies $(A1)$ and $(A2')$–$(A4')$ then the family $\mathbf{f} = \mathbf{a}+\mathbf{h}$ satisfies all the assumptions of Theorem 2.1. Therefore it must have a connected branch of nontrivial homoclinic solutions joining $\mathcal{T}_0$ with $\mathcal{T}_\infty$.

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