Approximation of Fourier Integral Operators by Gabor Multipliers

APPROXIMATION OF FOURIER INTEGRAL OPERATORS BY GABOR MULTIPLIERS

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ABSTRACT. A general principle says that the matrix of a Fourier integral operator with respect to wave packets is concentrated near the curve of propagation. We prove a precise version of this principle for Fourier integral operators with a smooth phase and a symbol in the Sjöstrand class and use Gabor frames as wave packets. The almost diagonalization of such Fourier integral operators suggests a specific approximation by (a sum of) elementary operators, namely modified Gabor multipliers. We derive error estimates for such approximations. The methods are taken from time-frequency analysis.

1. INTRODUCTION

A fundamental principle expressed by Cordoba and Fefferman [10] says that Fourier integral operators map wave packets to wave packets. Furthermore each wave packet is transported according to the canonical flow in phase space that is associated to the operator.

Rigorous versions of this principle were proved for various types of Fourier integral operators and wave packets. Cordoba and Fefferman used generalized Gaussians as wave packets and considered Fourier integral operators with constant symbol. In the last decade the transport of wave packets by Fourier integral operators was investigated for curvelet-like frames by Smith [29], for curvelets by Candes and Demanet [3], for shearlets by Guo and Labate [15], and more recently for Gabor frames by Rodino and two us [8].

In this paper we investigate Fourier integral operators with a "tame" phase and non-smooth symbols with respect to time-frequency shifts and Gabor frames. Following [1], we will assume that the phase function Φ on $\mathbb{R}^{2d}$ satisfies the conditions (i) $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, (ii) $\partial_z^\alpha \Phi \in L^\infty(\mathbb{R}^{2d})$ for $\alpha \geq 2$, and (iii) $\inf_{x,\eta\in\mathbb{R}^{2d}} |\det\, \partial^2_{x,\eta}\Phi(x,\eta)| \geq \delta > 0$. For brevity, we call Φ a tame phase function. Then the Fourier integral operator T with phase function Φ and the symbol σ is formally defined to be

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta)\, d\eta. \tag{1}$$

Fourier integral operators with a tame phase were investigated by many authors, both in hard analysis [2] and in time-frequency analysis [5, 8, 9]. They arise the

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study of Schrödinger operators, for instance, for the description of the resolvent of the Cauchy problem for the Schrödinger equation with a quadratic type Hamiltonian [1, 22, 23]. In our preceeding work [8, 9, 7] we have adopted the point of view that Fourier integral operators with a tame phase are best studied with those time-frequency methods that correspond to a constant geometry of the wavepackets. Precisely, let $z = (x, \eta) \in \mathbb{R}^{2d}$ be a point in phase space and

$$\pi(z)f(t) = e^{2\pi i \eta \cdot t} f(t - x), \qquad t \in \mathbb{R}^d$$

be the corresponding phase space shift (or time-frequency shift). Roughly speaking, these wave packets correspond to a uniform partition of the phase space.

In this paper we continue the investigation of the matrix of a Fourier integral operator with respect to (frames of) phase space shifts. A first version of a time-frequency analysis of Fourier integral operators was proved in [8]. The formulation is in the spirit of Candes and Demanet and is intriguing in its simplicity.

Theorem 1.1. *Assume that Φ is a tame phase function and χ is the corresponding canonical transformation of $\mathbb{R}^{2d}$. If $g \in \mathcal{S}(\mathbb{R}^d)$ and if σ belongs to the Sobolev space $W^\infty_{2N}(\mathbb{R}^{2d})$, then there exists a constant $C_N > 0$ such that*

$$|\langle T\pi(\mu)g, \pi(\lambda)g\rangle| \leq C_N \langle \chi(\mu) - \lambda\rangle^{-2N}, \qquad \forall \lambda, \mu \in \mathbb{R}^{2d}. \tag{2}$$

Our first goal is to sharpen Theorem 1.1 in several respects.

(a) We keep the assumptions on the phase function Φ (tame phase), but allow non-smooth symbols taken in a generalized Sjöstrand class.

(b) We enlarge the class of "basis" functions or "windows" g and extend Theorem 1.1 to hold for windows in certain modulation spaces $M^1_v(\mathbb{R}^d)$.

The appearance of modulation spaces is not longer a surprise. It is well established that modulation spaces are the appropriate function spaces for phase space analysis and time-frequency analysis with constant geometry. In particular the Sjöstrand class and its generalizations are a special case of the modulation spaces and they arise as suitable symbol classes for pseudodifferential operators. The use of the Sjöstrand class for Fourier integral operators goes back to Boulkhemair [2] and features prominently in recent work on pseudodifferential operators and Fourier integral operators (see [19] for a survey). Furthermore, the theory of time-frequency analysis and modulation spaces is now well developed and offers new tools for the investigation of Fourier integral operators, in addition to the classical methods.

The arguments and proofs of the extension of Theorem 1.1 are in the spirit of [8, 18], but require a substantial amount of technicalities. A main technical result is a new characterization of the Sjöstrand class in Proposition 3.10. This characterization is perfectly adapted to the investigation of Fourier integral operators and measures modulation space norms with a variable family of windows rather than a fixed window.

Our second goal is to derive a discretized version of Theorem 1.1. We will show that a Fourier integral operator can be approximated by (sums of) elementary operators of the form

$$M_{\mathbf{a}} f = M_{\mathbf{a}}^{\chi',g,\Lambda} f = \sum_{\lambda\in\Lambda} a_\lambda \langle f, \pi(\lambda)g\rangle \pi(\chi'(\lambda))g\,, \tag{3}$$

where Λ is a lattice in $\mathbb{R}^{2d}$, (a_λ) a symbol sequence on Λ, and $\chi' : \Lambda \to \Lambda$ is a discrete version of the canonical transformation associated to a phase function Φ. Such operators are suggested by a recent approximation theory for pseudodifferential operators through Gabor multipliers [20]. The modified Gabor multipliers in (3) are adapted precisely to the canonical transformation. We will approximate a Fourier integral operator with tame phase by modified Gabor multipliers and prove an approximation theorem for Fourier integral operators (Theorem 5.4).

The approximation theory also yields an alternative proof for the boundedness of Fourier integral operators on modulation spaces. The boundedness and Schatten class properties of Fourier integral operators were studied under various assumptions on the phase and the symbol. We refer to the book [30] and the articles [2, 10, 5, 7, 27] for a sample of contributions.

Gabor multipliers have many applications in signal processing and acoustics [11, 31] and are especially useful for the numerical realization of pseudodifferential operators. For the study of Fourier integral operators we introduce a new type of Gabor multipliers. We hope that the modified Gabor multipliers will also prove useful for the numerical realization and approximation of Fourier integral operators.

Our paper is organized as follows: In Section 2 we collect some concepts and the necessary facts from time-frequency analysis. Section 3 is devoted to the almost diagonalization of Fourier integral operators with a tame phase. We will prove a substantial generalization of Theorem 1.1. In Section 4 we define and study the modified Gabor multipliers mentioned in (3) and prove their boundedness on modulation spaces. In Section 5 we study the representation and approximation of Fourier integral operators by sums of modified Gabor multipliers and derive a quantitative approximation theorem. Further remarks are contained in Section 6.

2. Notation and preliminary results

We recall some notation and tools from time-frequency analysis. For an exposition and details we refer to the book [17].

(i) First we define the translation and modulation operators

$$T_x f(t) = f(t-x) \quad \text{and} \quad M_\eta f(t) = e^{2\pi i \eta t} f(t),$$

for $\lambda = (x,\eta) \in \mathbb{R}^{2d}$, $f \in \mathcal{S}'(\mathbb{R}^d)$. Their composition is the time-frequency shift

$$\pi(\lambda) = M_\eta T_x.$$

(ii) Now consider a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ (which will be called *window*). The short time Fourier transform of f with respect to g is given by

$$V_g f(x,\eta) = \langle f, M_\eta T_x g\rangle = \langle f, \pi(\lambda)g\rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i t\cdot\eta}\, dt, \quad \text{for } x,\eta \in \mathbb{R}^d\,.$$

The last integral makes sense for dual pairs of function spaces, e.g., for $f, g \in L^2(\mathbb{R}^d)$. The bracket $\langle \cdot, \cdot \rangle$ makes sense for dual pairs of function or distribution spaces, in particular for $f \in \mathcal{S}'(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$.

(iii) In the sequel, we set $v(x,\eta) = v_s(x,\eta) = \langle (x,\eta) \rangle^s = (1+|x|^2+|\eta|^2)^{s/2}$ and we denote by $\mathcal{M}_v(\mathbb{R}^{2d})$ the space of v-moderate weights on $\mathbb{R}^{2d}$; these are measurable functions $m > 0$ satisfying $m(z+\zeta) \leq Cv(z)m(\zeta)$ for every $z, \zeta \in \mathbb{R}^d$.

(iv) Then, for $1 \leq p \leq \infty$ and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ we denote by $M^p_m(\mathbb{R}^d)$ the space of distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^p_m(\mathbb{R}^d)} := \|V_g f\|_{L^p_m(\mathbb{R}^{2d})} < \infty.$$

This definition is meaningful and does not depend on the choice of the window $g \in \mathcal{S}(\mathbb{R}^d), g \neq 0$. If fact, different $g \in M^1_v(\mathbb{R}^d)$ yield equivalent norms on $M^p_m(\mathbb{R}^d)$ whenever $m \in \mathcal{M}_v(\mathbb{R}^{2d})$.

(v) Modulation spaces possess a discrete description as well. Consider a lattice of the form $\Lambda = A\mathbb{Z}^{2d}$ with $A \in GL(2d, \mathbb{R})$. The collection of time-frequency shifts $\mathcal{G}(g,\Lambda) = \{\pi(\lambda)g : \ \lambda \in \Lambda\}$ for a non-zero $g \in L^2(\mathbb{R}^d)$ is called a Gabor system. The set $\mathcal{G}(g,\Lambda)$ is a Gabor frame, if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^2 \leq B\|f\|_2^2 \qquad \forall f \in L^2(\mathbb{R}^d). \tag{4}$$

Gabor frames give the following characterization of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ and of the modulation spaces $M^p_m(\mathbb{R}^d)$. Fix $g \in \mathcal{S}(\mathbb{R}^d), g \neq 0$, then

$$f \in \mathcal{S}(\mathbb{R}^d) \Leftrightarrow \sup_{\lambda\in\Lambda} (1+|\lambda|)^N \langle f, \pi(\lambda)g\rangle < \infty \quad \forall N \in \mathbb{N}, \tag{5}$$

$$f \in M^p_m(\mathbb{R}^d) \Leftrightarrow \Big(\sum_{\lambda\in\Lambda} |\langle f, \pi(\lambda)g\rangle|^p m(\lambda)^p\Big)^{1/p} < \infty, \tag{6}$$

and the latter expression gives an equivalent norm for $M^p_m(\mathbb{R}^d)$.

Finally we recall that, if $A = B = 1$ in (4), then $\mathcal{G}(g,\Lambda)$ is called a Parseval frame, and (4) implies the expansion

$$f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle \pi(\lambda)g \qquad \forall f \in L^2(\mathbb{R}^d) \tag{7}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$. If $f \in M^p_m(\mathbb{R}^d)$ and $g \in M^1_v(\mathbb{R}^d)$ ($m \in \mathcal{M}_v(\mathbb{R}^{2d})$), then this expansion converges unconditionally in $M^p_m(\mathbb{R}^d)$ for $1 \leq p < \infty$.

(vi) Amalgam spaces: Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. The amalgam space $W(L^p)(\mathbb{R}^d)$ consists of all essentially bounded functions such that the norm

$$\|f\|_{W(L^p_m)} = \sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |f(x+k)| m(k) < \infty$$

is finite. If $f \in W(L^p_m)$ is also continuous and Λ is a lattice in $\mathbb{R}^d$, then the sequence $(f(\lambda))_{\lambda\in\Lambda}$ is in the sequence space $\ell^p_m(\Lambda)$. We will use several times the usual convolution relations for amalgam spaces

$$L^p_m * W(L^1_v) \subseteq W(L^p_m)\,, \tag{8}$$

which hold for $1 \le p \le \infty$ and all $m \in \mathcal{M}_v(\mathbb{R}^{2d})$.

3. Almost diagonalization of FIOs

For a given function f on $\mathbb{R}^d$, the Fourier integral operator (FIO in short) with symbol σ and phase Φ can be formally defined by

$$Tf(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x,\eta)} \sigma(x,\eta) \hat{f}(\eta) d\eta. \tag{9}$$

To avoid technicalities we take $f \in \mathcal{S}(\mathbb{R}^d)$ or, more generally, $f \in M^1(\mathbb{R}^d)$. If $\sigma \in L^\infty(\mathbb{R}^{2d})$ and if the phase Φ is real, the integral converges absolutely and defines a function in $L^\infty(\mathbb{R}^d)$.

We will consider a class of Fourier integral operators which arises in the study of Schrödinger type equations. This class of operators was already studied in the spirit of time-frequency analysis in [8].

Definition 3.1. *A phase function $\Phi(x,\eta)$ is called* tame*, if Φ satisfies the following properties:*

(i) $\Phi \in \mathcal{C}^\infty(\mathbb{R}^{2d})$*;*
(ii) *for $z = (x,\eta)$,*

$$|\partial_z^\alpha \Phi(z)| \le C_\alpha, \quad |\alpha| \ge 2; \tag{10}$$

(iii) *there exists $\delta > 0$ such that*

$$|\det \partial^2_{x,\eta} \Phi(x,\eta)| \ge \delta. \tag{11}$$

If we set

$$\begin{cases} y = \nabla_\eta \Phi(x,\eta) \\ \xi = \nabla_x \Phi(x,\eta), \end{cases} \tag{12}$$

we can solve with respect to (x,ξ) by the global inverse function theorem (see e.g. [24]), obtaining a mapping χ defined by $(x,\xi) = \chi(y,\eta)$. As observed in [8], χ is a smooth bi-Lipschitz canonical transformation. This means that

– χ is a smooth diffeomorphism on $\mathbb{R}^{2d}$;
– both χ and χ^{-1} are Lipschitz continuous;
– χ preserves the symplectic form. That is, if $(x,\xi) = \chi(y,\eta)$,

$$\sum_{j=1}^d dx_j \wedge d\xi_j = \sum_{j=1}^d dy_j \wedge d\eta_j. \tag{13}$$

Observe that condition (13) is equivalent to saying that the differential $d\chi(y,\eta)$, at every point $(y,\eta) \in \mathbb{R}^{2d}$, as a linear map $\mathbb{R}^{2d} \to \mathbb{R}^{2d}$, is represented by a symplectic matrix, i.e. belonging to the group

$$Sp(d,\mathbb{R}) := \left\{ A \in GL(2d,\mathbb{R}) : \, {}^tAJA = J \right\},$$

where

$$J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}.$$

In particular, when χ is linear, this means that χ itself is represented by a symplectic matrix.

We now address to the problem of the almost diagonalization of Fourier integral operators with a tame phase function. It turns out that, when the symbol is regular enough, the matrix representation of such an operator is almost diagonal with respect to a Gabor system.

Theorem 1.1 in the introduction was the first precise result about the almost diagonalization of Fourier integral operators in time-frequency analysis: *Assume that Φ is a tame phase function and $g \in \mathcal{S}(\mathbb{R}^d)$. If σ belongs to the Sobolev space $W^\infty_{2N}(\mathbb{R}^{2d})$ (consisting of all distributions with essentially bounded derivatives up to order $2N$), then there exists a constant $C_N > 0$ such that*

$$|\langle T\pi(\mu)g, \pi(\lambda)g\rangle| \leq C_N \langle \chi(\mu) - \lambda\rangle^{-2N}, \qquad \forall \lambda, \mu \in \mathbb{R}^{2d}, \tag{14}$$

where χ is the canonical transformation generated by Φ.

This statement proved in [8, Theorem 1]. For the extension of Theorem 1.1 we next introduce the appropriate symbol class. If $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$, with $v_s = v_s(x,\eta)$, $(x,\eta) \in \mathbb{R}^{2d}$, $s \in \mathbb{R}$, we consider the symbol class $M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})$ of $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ satisfying

$$\|\sigma\|_{M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})} = \|V_\Psi \sigma\|_{L^{\infty,1}_{1\otimes m}(\mathbb{R}^{4d})} = \int_{\mathbb{R}^{2d}} \sup_{z\in\mathbb{R}^{2d}} |V_\Psi(z,\zeta)| m(\zeta)\, d\zeta < \infty.$$

In time-frequency analysis this symbol class is just a special case of a modulation space [12], in theory of pseudodifferential operators $M^{\infty,1}_{1\otimes m}$ is often referred to as a generalized Sjöstrand class after [28].

For FIOs with symbols in $M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})$ there is an almost diagonalization result similar to Theorem 1.1.

The window g will be chosen sufficiently regular, as follows. Given $s \geq 0$, let

$$N \in \mathbb{N}, \quad N > \frac{s}{2} + d, \quad \text{and} \quad g \in M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^d), \tag{15}$$

here v_{4N} is a weight function defined on $\mathbb{R}^d$.

Theorem 3.2. *Let $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$, $s \geq 0$, N, g satisfying* (15) *and such that $\mathcal{G}(g,\Lambda)$ is a Parseval frame for $L^2(\mathbb{R}^d)$. If the phase Φ satisfies conditions* **(i)**, **(ii)** *and $\sigma \in M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})$ then there exists $H \in W(L^1_m)(\mathbb{R}^{2d})$ such that*

$$|\langle T\pi(x,\eta)g, \pi(x',\eta')g\rangle| \leq H(\eta' - \nabla_x\Phi(x',\eta), x - \nabla_\eta\Phi(x',\eta)), \tag{16}$$

for every $(x,\eta),(x',\eta') \in \mathbb{R}^{2d}$, with $\|H\|_{W(L^1_m)} \lesssim \|\sigma\|_{M^{\infty,1}_{1\otimes m}}$

Observe that Theorem 3.2 says we can control the coefficient matrix $|\langle T\pi(x,\eta)g, \pi(x',\eta')g\rangle|$ by a function H of the difference $(\eta' - \nabla_x\Phi(x',\eta), x - \nabla_\eta\Phi(x',\eta))$. It will be useful in the sequel to have a control of the matrix which depends on the difference $(\eta' - \chi_2(x,\eta), x' - \chi_1(x,\eta))$. This is achieved by adding the assumption (**iii**) of [1] on the phase and by choosing another, even larger modulation space as the symbol class.

Theorem 3.3. *Let $s \geq 0$, N, g satisfying* (15) *and assume that $\mathcal{G}(g,\Lambda)$ is a Parseval frame for $L^2(\mathbb{R}^d)$. If the phase Φ is tame and $\sigma \in M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$, then there exists $C>0$ such that*

$$|\langle T\pi(\mu)g, \pi(\lambda)g\rangle| \leq C\frac{\|\sigma\|_{M^\infty_{1\otimes v_s}}}{\langle \chi(\mu)-\lambda\rangle^s}, \quad \forall \lambda,\mu \in \mathbb{R}^{2d}. \tag{17}$$

Theorem 3.3 improves Theorem 1.1 in several respects: If $s=2N$, then $W^\infty_{2N}(\mathbb{R}^{2d}) \subseteq M^\infty_{1\otimes v_{2N}}(\mathbb{R}^{2d})$ [12, Prop. 6.7] and $\mathcal{S}(\mathbb{R}^d) \subseteq M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^d)$. Thus we obtain the same quality of off-diagonal decay for significantly larger classes of symbols and windows.

The remainder of this section is devoted to set up the tools for the proof of these theorems.

For $z,w \in \mathbb{R}^{2d}$, let $\Phi_{2,z}$ the remainder in the second order Taylor expansion of the phase Φ, i.e.,

$$\Phi_{2,z}(w) = 2\sum_{|\alpha|=2}\int_0^1 (1-t)\partial^\alpha \Phi(z+tw)dt\frac{w^\alpha}{\alpha!}\,.$$

For a given window g, we set

$$\Psi_z(w) = e^{2\pi i \Phi_{2,z}(w)}\overline{g}\otimes\widehat{g}(w). \tag{18}$$

The main technical work is to show that the set of windows Ψ_z possesses a joint time-frequency envelope. This property will allow us to replace the z-dependent family of windows Ψ_z by a single window in many estimates.

Before proving the existence of a time-frequency envelope in Lemma 3.9 below, we first look at the phase factor $e^{2\pi i \Psi_{2,z}}$ occurring in (18).

Lemma 3.4. *For every $s\in\mathbb{R}$, $N\in\mathbb{N}$, $N>\frac{s}{2}+d$, and $\Psi\in\mathcal{S}(\mathbb{R}^{2d})$, we have*

$$\sup_{z\in\mathbb{R}^{2d}} |V_\Psi e^{2\pi i\Phi_{2,z}}| \in L^{\infty,1}_{v_{-4N}\otimes v_s}(\mathbb{R}^{4d}), \tag{19}$$

with v_{-4N} and v_s being weight functions on $\mathbb{R}^{2d}$.

Proof. We proceed as in [8]. Let $\Psi\in\mathcal{S}(\mathbb{R}^{2d})$, then

$$|V_\Psi e^{2\pi i\Phi_{2,z}}(u,w)| = \left|\int_{\mathbb{R}^{2d}} e^{2\pi i \Phi_{2,z}(\zeta)}T_u\bar{\Psi}(\zeta)e^{-2\pi i\zeta\cdot w}d\zeta\right|.$$

Using the identity

$$(1-\Delta_\zeta)^N e^{-2\pi i\zeta w} = \langle 2\pi w\rangle^{2N}e^{-2\pi i \zeta w},$$

we integrate by parts and obtain

$$|V_\Psi e^{2\pi i\Phi_{2,z}}(u,w)| = \frac{1}{\langle 2\pi w\rangle^{2N}}\left|\int_{\mathbb{R}^{2d}}(1-\Delta_\zeta)^N\big(e^{2\pi i\Phi_{2,z}(\zeta)}T_u\bar{\Psi}(\zeta)\big)e^{-2\pi i\zeta\cdot w}d\zeta\right|.$$

By means of Leibniz's formula the factor $(1-\Delta_\zeta)^N(e^{2\pi i\Phi_{2,z}(\zeta)}T_u\bar\Psi(\zeta))$ can be expressed further as

$$e^{2\pi i\Phi_{2,z}(\zeta)}\sum_{|\alpha|+|\beta|\leq 2N} p_\alpha(\partial\Phi_{2,z}(\zeta))(T_u\partial_\zeta^\beta\bar\Psi)(\zeta),$$

where $p_\alpha(\partial\Phi_{2,z}(\zeta))$ is a polynomial of derivatives of $\Phi_{2,z}$ of degree at most $|\alpha|$.

As a consequence of (10) we have $|p_\alpha(\partial\Phi_{2,z}(\zeta))|\leq C_\alpha\langle\zeta\rangle^{2|\alpha|}$ for every $z\in\mathbb{R}^{2d}$ with a constant independent of z. Moreover, the assumption $\Psi\in\mathcal{S}(\mathbb{R}^{2d})$ yields that

$$\sup_{|\beta|\leq 2N}|T_u\partial_\zeta^\beta\bar\Psi|\leq C_{N,s}\langle\zeta-u\rangle^{-\ell}$$

for every $\ell\geq 0$. Consequently,

$$\begin{aligned}|V_\Psi e^{2\pi i\Phi_{2,z}}(u,w)| &\lesssim \frac{1}{\langle 2\pi w\rangle^{2N}}\int_{\mathbb{R}^{2d}}\sum_{|\alpha|+|\beta|\leq 2N}\langle\zeta\rangle^{2|\alpha|}\langle\zeta-u\rangle^{-\ell}d\zeta\\ &\lesssim \frac{1}{\langle 2\pi w\rangle^{2N}}\int_{\mathbb{R}^{2d}}\langle\zeta\rangle^{4N}\langle\zeta-u\rangle^{-\ell}d\zeta\\ &\lesssim \frac{1}{\langle 2\pi w\rangle^{2N}}\langle u\rangle^{4N},\end{aligned}$$

whenever $\ell>4N+2d$. Since $-2N+s<-2d$ by assumption, we obtain

$$\int_{\mathbb{R}^{2d}}\sup_{u\in\mathbb{R}^{2d}}\sup_{z\in\mathbb{R}^{2d}}|V_\Psi e^{2\pi i\Phi_{2,z}}|(u,w)\langle u\rangle^{-4N}\langle w\rangle^s\,dw<\infty,$$

whence (19) follows. □

Remark 3.5. *Notice that the two weights v_{-4N} and v_s compensate each other in* (19)*. It is easy to see that some form of compensation is necessary. For example, if $\Phi(\zeta)=|\zeta|^2/2$, then $\Phi_{2,z}(\zeta)=|\zeta|^2/2$ independent of z. A direct computation shows that the STFT of $e^{2\pi i\Phi_{2,z}(\zeta)}$ with Gaussian window $\Psi(\zeta)=e^{-\pi|\zeta|^2}$ has modulus, up to constants, $e^{-\pi|u_1-u_2|^2/2}$, $u_1,u_2\in\mathbb{R}^{2d}$. This function belongs to $L^{\infty,1}_{v_{-4N}\otimes v_s}(\mathbb{R}^{4d})$ if and only if $-4N+s<-2d$. This is in fact better than the condition $-2N+s<-2d$ in the assumptions, due to the fact that here the derivatives of order ≥ 1 of $\Phi_{2,z}(\zeta)$ are bounded from above by $\langle\zeta\rangle$, instead of by $\langle\zeta\rangle^2$, as in the general case. This explains the presence of $2N$ instead of $4N$. In the general case the best upper bound for the derivatives of $\Phi_{2,z}(\zeta)$ is $\langle\zeta\rangle^2$, so that the condition $-2N+s<-2d$ should be sharp.*

Lemma 3.6. *Let $m\in\mathcal{M}_v(\mathbb{R}^d)$, v, ν,w be weight functions on $\mathbb{R}^d$, and $\{f_z : z\in\mathbb{R}^{2d}\}\subseteq\mathcal{S}'(\mathbb{R}^d)$ be a set of distributions in $\mathcal{S}'$. If*

$$\sup_{z\in\mathbb{R}^d}|V_\varphi f_z|\in L^{\infty,1}_{\nu^{-1}\otimes m}(\mathbb{R}^{2d}),\tag{20}$$

for given $\varphi\in\mathcal{S}(\mathbb{R}^d)$, then for every $h\in M^1_{w\nu\otimes v}(\mathbb{R}^d)$

$$\sup_{z\in\mathbb{R}^d}|V_{\varphi^2}(f_z h)|\in L^1_{w\otimes m}(\mathbb{R}^{2d})\,.\tag{21}$$

Proof. The proof of the multiplier property is similar to the convolution property of modulation spaces in [6]. Since $V_g f(x,\eta) = (\widehat{f \cdot T_x\bar{g}})(\eta)$ [17, Lemma 3.1.1], we obtain the identity

$$V_{\varphi^2}(f_z h)(x,\eta) = \big((f_z\overline{T_x\varphi})(hT_x\overline{T_x\varphi})\big)\widehat{\ }(\eta) = (f_z\overline{T_x\varphi})\widehat{\ } *_\eta (h\overline{T_x\varphi})\widehat{\ }(\eta)\,,$$

and consequently

$$|V_{\varphi^2}(f_z h)(x,\eta)| \le |(\widehat{f_z \cdot T_x\bar{\varphi}})| *_\eta |(\widehat{h \cdot T_x\bar{\varphi}})|(\eta)\,,$$

where the convolution is in the second variable η. Now set $F(x,\eta) = \sup_{z\in\mathbb{R}^{2d}} |V_\varphi f_z(x,\eta)| = \sup_{z\in\mathbb{R}^{2d}} |(f_z \cdot T_x\bar{\varphi})\widehat{\ }(\eta)|$ and $H(x,\eta) = |V_\varphi h(x,\eta)|$. Then

$$\sup_{z\in\mathbb{R}^{2d}} |V_{\varphi^2}(f_z\, h)(x,\eta)| \le (F *_\eta H)(x,\eta)\,.$$

Finally,

$$\begin{aligned}\|F *_\eta H\|_{L^1_{w\otimes m}} &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x,\eta-\zeta)H(x,\zeta)d\zeta\, w(x)m(\eta)\,dxd\eta \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} F(x,\eta-\zeta)\nu(x)^{-1}m(\eta-\zeta)H(x,\zeta)\,w(x)\nu(x)v(\zeta)\,dxd\eta d\zeta \\ &\le \|F\|_{L^{\infty,1}_{\nu^{-1}\otimes m}}\,\|H\|_{L^1_{\nu w\otimes v}}\,.\end{aligned}$$

In the last expression both norms are finite by assumption. ∎

Corollary 3.7. *Let $s \ge 0$, $N \in \mathbb{N}$, $N > \frac{s}{2} + d$ and $g \in M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^d)$ and $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$. Then the function $\Psi_z(w) = e^{2\pi i\Phi_{2,z}(w)}\overline{g}\otimes\widehat{g}(w)$ defined as in (18) satisfies*

$$\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi(u_1,u_2)| \in L^1_{1\otimes v_s}(\mathbb{R}^{4d}).$$

Proof. The symmetry property of the weight $\langle x\rangle^{4N}\langle\eta\rangle^{4N}$ implies that $M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^d)$ is invariant under the Fourier transform, and thus $\hat{g} \in M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^d)$. A tensor product argument then shows that $\bar{g}\otimes\hat{g} \in M^1_W(\mathbb{R}^{2d})$ with $W(x_1,x_2,\eta_1,\eta_2) = \langle x_1\rangle^{4N}\langle x_2\rangle^{4N}\langle\eta_1\rangle^{4N}\langle\eta_2\rangle^{4N}$. Since $W(x_1,x_2,\eta_1,\eta_2) \ge \langle(x_1,x_2)\rangle^{4N}\langle(\eta_1,\eta_2)\rangle^{4N}$, we also obtain that $\bar{g}\otimes\hat{g} \in M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^{2d})$.

By Lemma 3.4 we have $\sup_{z\in\mathbb{R}^{2d}} |V_\Psi e^{2\pi i\Phi_{2,z}}| \in L^{\infty,1}_{v_{-4N}\otimes v_s}(\mathbb{R}^{4d})$.

We now apply Lemma 3.6 with $f_z = e^{2\pi i\Phi_{2,z}}$ and $h = \bar{g}\otimes\hat{g}$, $\nu = v_{4N}, m = v_s$, $w \equiv 1$, and $v = v_{4N}$ (observe that v_s is v_{4N}-moderate, since $s < 4N$). As a conclusion we obtain that $\sup_{z\in\mathbb{R}^{2d}} |V_\Psi\Psi_z| \subseteq L^1_{1\otimes v_s}(\mathbb{R}^{4d})$. ∎

We recall the following pointwise inequality of the short-time Fourier transform [17, Lemma 11.3.3]. It is often useful when one needs to change window functions.

Lemma 3.8. *If $g_0, g_1, \gamma \in \mathcal{S}(\mathbb{R}^d)$ such that $\langle\gamma, g_1\rangle \ne 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$, then the inequality*

$$|V_{g_0} f(x,\eta)| \le \frac{1}{|\langle\gamma,g_1\rangle|}(|V_{g_1}f| * |V_{g_0}\gamma|)(x,\eta),$$

holds pointwise for all $(x,\eta)\in\mathbb{R}^{2d}$.

Lemma 3.9. *Let $s \geq 0$, $N \in \mathbb{N}$, $\quad N > \frac{s}{2} + d$ and $g \in M^1_{v_{4N}\otimes v_{4N}}(\mathbb{R}^d)$. Then*

$$\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi| \in W(L^1_{1\otimes v_s})(\mathbb{R}^{4d}). \tag{22}$$

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$ such that $\|\varphi\|_2 = 1$. Using Lemma 3.8,

$$\begin{aligned} |V_{\Psi_z}\Psi|(u_1,u_2) &\leq |V_\varphi \Psi| * |V_{\Psi_z}\varphi|(u_1,u_2) \\ &\leq |V_\varphi\Psi| * (\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\varphi|)(u_1,u_2). \end{aligned}$$

Since $V_\varphi\Psi \in \mathcal{S}(\mathbb{R}^{4d}) \subset W(L^1_{1\otimes v_s})$ and $\|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\varphi|\,\|_{L^1_{1\otimes v_s}} < \infty$ by Corollary 3.7, the convolution relation $L^1_{1\otimes v_s} * W(L^1_{1\otimes v_s}) \subseteq W(L^1_{1\otimes v_s})$ of [17, Theorem 11.1.5] implies that

$$\|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi|\,\|_{W(L^1_{1\otimes v_s})} \leq \|V_\varphi\Psi\|_{W(L^1_{1\otimes v_s})} \|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\varphi|\,\|_{L^1_{1\otimes v_s}} < \infty\,,$$

and so $\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi| \in W(L^1_{1\otimes v_s})$. □

We next formulate an new characterization of the symbol classes $M^{\infty,1}_{1\otimes m}$ and $M^\infty_{1\otimes m}$ that is perfectly adapted to the investigation of Fourier integral operators with a tame phase Φ.

Proposition 3.10. *Let $s \geq 0$, $m \in \mathcal{M}_{v_s}(\mathbb{R}^{2d})$, N, g be as in* (15) *and $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$.*
(i) Then the symbol σ is in $M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})$, if and only if

$$\|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma|\|_{L^{\infty,1}_{1\otimes m}(\mathbb{R}^{4d})} = \int_{\mathbb{R}^{2d}} \sup_{u_1\in\mathbb{R}^d} \sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma(u_1,u_2)| m(u_2)\, du_2 < \infty, \tag{23}$$

with $\|\sigma\|_{M^{\infty,1}_{1\otimes m}} \asymp \|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma|\|_{L^{\infty,1}_{1\otimes m}}$.

In this case the function $H(u_2) = \sup_{u_1\in\mathbb{R}^d} \sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma(u_1,u_2)|$ is in $W(L^1_m)(\mathbb{R}^{2d})$.

(ii) Likewise, $\sigma \in M^\infty_{1\otimes m}(\mathbb{R}^{2d})$ if and only if $\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma| \in L^\infty_{1\otimes m}(\mathbb{R}^{4d})$ with $\|\sigma\|_{M^\infty_{1\otimes m}} \asymp \|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma|\|_{L^\infty_{1\otimes m}}$.

Proof. We detail the proof of case (i). Case (ii) is obtained similarly. Let $\Psi \in \mathcal{S}(\mathbb{R}^{2d})$ with $\|\Psi\|_2 = 1$.

Assume first that $\sigma \in M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})$. Then by Lemma 3.8 we have

$$|V_{\Psi_z}\sigma(u_1,u_2)| \leq |V_\Psi\sigma| * |V_{\Psi_z}\Psi|(u_1,u_2) \leq |V_\Psi\sigma| * \sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi|(u_1,u_2).$$

Set $F(u_1,u_2) = \sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi(u_1,u_2)|$, so that

$$|V_{\Psi_z}\sigma(u_1,u_2)| \leq \big(|V_\Psi\sigma| * F\big)(u_1,u_2).$$

Finally

$$\begin{aligned} \|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma|\,\|_{L^{\infty,1}_{1\otimes m}} &= \int_{\mathbb{R}^{2d}} \sup_{u_1\in\mathbb{R}^d} \sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma(u_1,u_2)| m(u_2)\, du_2 \\ &\leq \|\,|V_\Psi\sigma| * F\|_{L^{\infty,1}_{1\otimes m}} \\ &\leq \|V_\Psi\sigma\|_{L^{\infty,1}_{1\otimes m}} \|F\|_{L^1_{1\otimes v_s}} \lesssim \|\sigma\|_{M^{\infty,1}_{1\otimes m}} \|F\|_{L^1_{1\otimes v_s}} < \infty, \end{aligned}$$

where in the last step we used the independence of the weighted $M^{\infty,1}$-norm of the window [17, Thm. 11.3.7] and Corollary 3.7.

To obtain the sharper estimate, set $H(u_2) = \sup_{u_1\in\mathbb{R}^d}\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma(u_1,u_2)|$, $F_1(u_2) = \int_{\mathbb{R}^{2d}} F(u_1,u_2)\,du_1$, and $G_1(u_2) = \sup_{u_1\in\mathbb{R}^{2d}} |V_\Psi\sigma(u_1,u_2)|$. Then the definition of $M^{\infty,1}_{1\otimes m}(\mathbb{R}^{2d})$ implies that $G_1 \in L^1_m(\mathbb{R}^{2d})$, and Lemma 3.9 implies that $F_1 \in W(L^1_{v_s})(\mathbb{R}^{2d})$. With these definitions we obtain a pointwise estimate for H, namely

$$\begin{aligned} H(w) &= \sup_{u_1\in\mathbb{R}^d}\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma(u_1,w)| \\ &\le \int_{\mathbb{R}^{2d}} \sup_{u_1} |V_\Psi\sigma(u_1,u_2)| F(z-u_1,w-u_2)du_1du_2 \\ &= (G_1 * F_1)(w)\,. \end{aligned}$$

The convolution estimate (8) now yields that $H \le G_1 * F_1 \in L^1_m * W(L^1_{v_s}) \subseteq W(L^1_{v_s})(\mathbb{R}^{2d})$, as claimed.

Conversely, assume (23). Using Lemma 3.8 again, we deduce that

$$|V_\Psi\sigma(u_1,u_2)| \le \frac{1}{\langle\Psi_z,\Psi_z\rangle}|V_{\Psi_z}\sigma| * |V_\Psi\Psi_z|(u_1,u_2),$$

and

$$\begin{aligned} \langle\Psi_z,\Psi_z\rangle &= \int_{\mathbb{R}^{2d}} |e^{2\pi i\Phi_{2,z}(w)}(\overline{g}\otimes\widehat{g})(w)|^2\,dw \\ &= \int_{\mathbb{R}^{2d}} |(\overline{g}\otimes\widehat{g})(w)|^2\,dw = \|\overline{g}\|_2^2\|\widehat{g}\|_2^2 = \|g\|_2^4 \end{aligned}$$

is in fact a constant (depending on g). Using the involution $f^*(z) = \overline{f(-z)}$, we continue with

$$\begin{aligned} |V_\Psi\sigma|(u_1,u_2) &\lesssim |V_{\Psi_z}\sigma| * |V_\Psi\Psi_z|(u_1,u_2) \\ &= |V_{\Psi_z}\sigma| * |(V_{\Psi_z}\Psi)^*|(u_1,u_2) \\ &\le \left(\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma|\right) * \left(\sup_{z\in\mathbb{R}^{2d}} |(V_{\Psi_z}\Psi)^*|\right)(u_1,u_2). \end{aligned}$$

Since by Corollary 3.7

$$\sup_{z\in\mathbb{R}^{2d}} |(V_{\Psi_z}\Psi)^*| = \sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi| \in L^1_{1\otimes v_s}(\mathbb{R}^{4d}),$$

we get

$$\|V_\Psi\sigma\|_{L^{\infty,1}_{1\otimes m}} \lesssim \|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\sigma|\|_{L^{\infty,1}_{1\otimes m}} \|\sup_{z\in\mathbb{R}^{2d}} |V_{\Psi_z}\Psi|\|_{L^1_{1\otimes v_s}} < \infty\,,$$

and the proof is complete. ∎

We now prove Theorems 3.2 and 3.3.

Proof of Theorem 3.2. We know from [8, Equ. (39)] or [7, Equ. (3.3)] that

$$\begin{aligned}|\langle T\pi(x,\eta)g,\pi(x',\eta')g\rangle| &= |V_{\Psi_{(x',\eta)}}\sigma\big((x',\eta),(\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta))\big)| \\ &\leq \sup_{z\in\mathbb{R}^{2d}}|V_{\Psi_z}\sigma\big((x',\eta),(\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta))\big)|. \end{aligned} \tag{24}$$

We claim that the function that controls the off-diagonal decay of $|\langle T\pi(x,\eta)g,\pi(x',\eta')g\rangle|$ is exactly the function $H(w) := \sup_{u_1\in\mathbb{R}^{2d}}\sup_{z\in\mathbb{R}^{2d}}|V_{\Psi_z}\sigma(u_1,w)|$ introduced in Proposition 3.10. There we have already proved that H is in $W(L^1_m)(\mathbb{R}^{2d})$, and now (24) implies that

$$|\langle T\pi(x,\eta)g,\pi(x',\eta')g\rangle| \leq H(\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)).$$

□

A decay estimate in terms of the canonical transformation χ can be obtained by imposing a stronger condition on the symbol class.

Proof of Theorem 3.3. We follow the pattern of the proof of Theorem 3.2. If Φ is a tame phase function, then the argument of $V_{\Psi_z}\sigma$ in (24) can be estimated further as

$$|\eta'-\nabla_x\Phi(x',\eta)+|x-\nabla_\eta\Phi(x',\eta)| \geq |\eta'-\chi_2(x,\eta)|+|x'-\chi_1(x,\eta)| \tag{25}$$

by [8, Lemma 3.1].

Set $H(u_1,u_2) := \sup_{z\in\mathbb{R}^{2d}}|V_{\Psi_z}\sigma(u_1,u_2)|$. By Proposition 3.10 (ii), $H\in L^\infty_{1\otimes v_s}(\mathbb{R}^{4d})$ and $\|H\|_{L^\infty_{1\otimes v_s}} \lesssim \|\sigma\|_{M^\infty_{1\otimes v_s}}$. Now, using (24),

$$\begin{aligned}|\langle T\pi(x,\eta)g,\pi(x',\eta')g\rangle| &\leq \sup_{u_1\in\mathbb{R}^{2d}} H(u_1,\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)) \\ &\lesssim \sup_{u_1\in\mathbb{R}^{2d}} H(u_1,\eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)) \\ &\qquad\times \frac{\langle \eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)\rangle^s}{\langle \eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)\rangle^s} \\ &\lesssim \frac{\sup_{u_2}\sup_{u_1\in\mathbb{R}^{2d}} H(u_1,u_2)v_s(u_2)}{\langle \eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)\rangle^s} \\ &= \frac{\|H\|_{L^\infty_{1\otimes v_s}}}{\langle \eta'-\nabla_x\Phi(x',\eta),x-\nabla_\eta\Phi(x',\eta)\rangle^s} \\ &\lesssim \frac{\|\sigma\|_{M^\infty_{1\otimes v_s}}}{\langle \eta'-\chi_2(x,\eta),x'-\chi_1(x,\eta)\rangle^s},\end{aligned}$$

where in the last inequality we have used (25). □

4. Gabor multipliers

We now introduce a type of Gabor multipliers that are tailored to the canonical transformation χ of a FIO. In the case of the identity map $\chi = \mathrm{id}$ we get the standard Gabor multipliers studied, e.g., in [14, 20].

Since in general the map χ does not preserve a given lattice Λ, we first replace χ by a mapping χ' that enjoys $\chi'(\Lambda)\subset\Lambda$.

We define the integer part of a vector $y = (y_1, ..., y_{2d})$ as

$$\lfloor y \rfloor = (\lfloor y_1 \rfloor, ..., \lfloor y_{2d} \rfloor),$$

i.e., by taking the integer parts of its components.

Now consider a lattice of the form $\Lambda = A\mathbb{Z}^{2d}$ with $A \in GL(2d, \mathbb{R})$. Given a map $\chi : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, we approximate χ by a map $\chi' : \Lambda \to \Lambda$ defined by

$$\chi'(\lambda) = A\lfloor A^{-1}\chi(\lambda)\rfloor.$$

Observe that $|y - \lfloor y \rfloor| < \sqrt{2d}$ and therefore

$$|\chi(\lambda) - \chi'(\lambda)| < \sqrt{2d}\|A\|\,.$$

The almost diagonalization of Theorem 3.3 can now be formulated in terms of χ' as follows: using the inequality $\langle x \rangle \leq \sqrt{2}\langle x + y\rangle\langle y\rangle$ we have

$$|\langle T\pi(\mu)g, \pi(\lambda)g\rangle| \leq C_N 2^s \langle \sqrt{2d}\,\|A\|\rangle^{2s}\,\langle \chi'(\mu) - \lambda\rangle^{-2s}, \qquad \forall \lambda, \mu \in \Lambda. \tag{26}$$

We collect here some properties which will be used in the sequel.

Lemma 4.1. *Let $v(z) = v_s(z) = \langle z\rangle^s$, $z \in \mathbb{R}^{2d}$, $s \geq 0$, and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$. Then*

$$v \circ \chi \asymp v, \qquad v \circ \chi' \asymp v, \tag{27}$$

and

$$m \circ \chi \in \mathcal{M}_v(\mathbb{R}^{2d})\,, \qquad m \circ \chi' \in \mathcal{M}_v(\mathbb{R}^{2d}). \tag{28}$$

Proof. Since χ is Lipschitz continuous, we have

$$|\chi(z)| \leq |\chi(0)| + |\chi(z) - \chi(0)| \leq |\chi(0)| + C|z| \leq C'(1 + |z|) \qquad \forall z \in \mathbb{R}^{2d}\,.$$

By applying the same argument to χ^{-1}, which is also Lipschitz continuous, we obtain $1 + |\chi(z)| \asymp 1 + |z|$. Since we also have $\chi'(z) = \chi(z) + \mathcal{O}(1)$, both the estimates in (27) follow.

Concerning (28), observe that

$$m(\chi(z+\zeta)) = m(\chi(z+\zeta)-\chi(z)+\chi(z)) \leq v(\chi(z+\zeta)-\chi(z))m(\chi(z)) \lesssim v(\zeta)m(\chi(z)),$$

where in the last step we used the fact that $|\chi(z + \zeta) - \chi(z)| \lesssim |\zeta|$. This proves the first formula in (28). The second formula is obtained similarly, because $\chi'(z) = \chi(z) + \mathcal{O}(1)$, so that $\chi'(z + \zeta) - \chi'(z) = \chi(z + \zeta) - \chi(z) + \mathcal{O}(1)$, and we still have $1 + |\chi'(z + \zeta) - \chi'(z)| \lesssim 1 + |\zeta|$. ∎

Remark 4.2. Notice that the above proposition holds only for weights with polynomial growth, but not for weights with super-polynomial growth, e.g., $v(z) = e^{a|z|^b}$, $0 < b < 1$, $a > 0$. This is related to the fact that the weights v_s satisfy the doubling condition $v_s(2z) \asymp v_s(z)$, whereas weights with super-exponential growth do not.

Now, let $\mathcal{G}(g, \Lambda)$ be a Gabor system, and χ, χ' be the canonical transformations of an FIO. Given a sequence $\mathbf{a} = (a_\nu)_{\nu\in\Lambda}$, we define (formally) the Gabor multiplier

$$M_{\mathbf{a}}f = M_{\mathbf{a}}^{\chi',g,\Lambda}f = \sum_{\lambda\in\Lambda} a_\lambda\,\langle f, \pi(\lambda)g\rangle\pi(\chi'(\lambda))g.$$

In order to give a precise meaning to this definition, we need to study the convergence of the above series.

Lemma 4.3. *Let $g \in M_v^1(\mathbb{R}^d)$, $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, and $1 \le p \le \infty$. If $h = (h_\lambda)_{\lambda\in\Lambda} \in \ell^p_{m\circ\chi'}(\Lambda)$, then*

$$\|\sum_{\lambda\in\Lambda} h_\lambda \pi(\chi'(\lambda))g\|_{M^p_m} \le C\Big(\sum_{\lambda\in\Lambda}|h_\lambda|^p m(\chi'(\lambda))^p\Big)^{1/p}. \tag{29}$$

The series $\sum_{\lambda\in\Lambda} h_\lambda\pi(\chi'(\lambda))g$ converges unconditionally in $M^p_m(\mathbb{R}^d)$ for $p<\infty$ and weak unconditionally for $p=\infty$.*

Proof. We assume $p<\infty$ and leave the modification for the case $p=\infty$ to the reader.

Since the finite sequences are norm-dense in $\ell^p_m(\Lambda)$ for $p<\infty$ (and weak*-dense for $p=\infty$), it suffices to show (29) for finite sequences. For a finite set $F\subset\Lambda$ we set $\tilde F = \chi'(F)$. Since $g\in M^1_v(\mathbb{R}^d)$, the synthesis operator $\{c_\alpha\}\mapsto\sum_{\alpha\in\Lambda}c_\alpha\pi(\alpha)g$ is bounded from $\ell^p_m(\Lambda)$ to $M^p_m(\mathbb{R}^d)$ [17, Thm. 12.2.4], so that

$$\begin{aligned}\|\sum_{\lambda\in F} h_\lambda\pi(\chi'(\lambda))g\|_{M^p_m} &= \|\sum_{\alpha\in\tilde F}\Big(\sum_{\lambda\in{\chi'}^{-1}(\alpha)} h_\lambda\Big)\pi(\alpha)g\|_{M^p_m}\\ &\lesssim \Big(\sum_{\alpha\in\tilde F}\Big|\sum_{\lambda\in{\chi'}^{-1}(\alpha)} h_\lambda\Big|^p m(\alpha)^p\Big)^{1/p}.\end{aligned} \tag{30}$$

Since χ^{-1} is Lipschitz continuous, χ' is "uniformly almost injective", in the sense that $\sup_{\alpha\in\Lambda}\#{\chi'}^{-1}(\{\alpha\})<\infty$. Hence the last expression in (30) is bounded by

$$\lesssim\Big(\sum_{\alpha\in\tilde F}\sum_{\lambda\in{\chi'}^{-1}(\alpha)}|h_\lambda|^p m(\alpha)^p\Big)^{1/p} = \Big(\sum_{\lambda\in F}|h_\lambda|^p m(\chi'(\lambda))^p\Big)^{1/p}.$$

By density (29) holds for all $(h_\lambda)\in\ell^p_{m\circ\chi'}(\Lambda)$. The unconditional convergence follows from the norm estimate. □

Proposition 4.4. *Let $g\in M^1_{v^2}(\mathbb{R}^d)$, $m,\tilde m\in\mathcal{M}_v(\mathbb{R}^{2d})$, and $1\le p\le\infty$. If $\mathbf{a} = (a_\lambda)_{\lambda\in\Lambda}$ is a sequence in $\ell^\infty_{\tilde m}(\Lambda)$, then the Gabor multiplier $M_{\mathbf a} = M_{\mathbf a}^{\chi',g,\Lambda}$ is bounded from $M^p_{\frac{m\circ\chi'}{\tilde m}}(\mathbb{R}^d)$ to $M^p_m(\mathbb{R}^d)$. Its operator norm is bounded by $\|M_{\mathbf a}\|_{M^p_{\frac{m\circ\chi'}{\tilde m}}\to M^p_m}\lesssim\|\mathbf a\|_{\ell^\infty_{\tilde m}}$.*

Proof. Since $\frac{m\circ\chi'}{\tilde m}\in\mathcal{M}_{v^2}(\mathbb{R}^d)$ and $g\in M^1_{v^2}(\mathbb{R}^d)$, the coefficient operator $f\mapsto\langle f,\pi(\lambda)g\rangle$ is bounded from $M^p_{\frac{m\circ\chi'}{\tilde m}}(\mathbb{R}^d)$ to $\ell^p_{\frac{m\circ\chi'}{\tilde m}}(\Lambda)$, see [17, Thm. 12.2.3.]. Hence, if $\mathbf a = (a_\lambda)_{\lambda\in\Lambda}\in\ell^\infty_{\tilde m}(\Lambda)$ and $f\in M^p_{\frac{m\circ\chi'}{\tilde m}}(\mathbb{R}^d)$, then the sequence $(a_\lambda\langle f,\pi(\lambda)g\rangle)_{\lambda\in\Lambda}$ belongs to $\ell^p_{m\circ\chi'}(\Lambda)$. Therefore, by Lemma 4.3 the series $\sum_{\lambda\in\Lambda}a_\lambda\langle f,\pi(\lambda)g\rangle\pi(\chi'(\lambda))g$ converges unconditionally in $M^p_m(\mathbb{R}^d)$. The estimate for the operator norm follows from

$$\|\sum_{\lambda\in\Lambda} a_\lambda \langle f, \pi(\lambda)g\rangle \pi(\chi'(\lambda))g\|_{M^p_m} \lesssim \Big(\sum_{\lambda\in\Lambda} |a_\lambda \langle f,\pi(\lambda)g\rangle|^p m(\chi'(\lambda))^p\Big)^{1/p}$$
$$\leq \|\tilde{m}a\|_{\ell^\infty}\Big(\sum_{\lambda\in\Lambda} |\langle f,\pi(\lambda)g\rangle|^p \Big(\frac{m(\chi'(\lambda))}{\tilde{m}(\lambda)}\Big)^p\Big)^{1/p}$$
$$\lesssim \|a\|_{\ell^\infty_{\tilde{m}}} \|f\|_{M^p_{\frac{m\circ\chi'}{\tilde{m}}}}.$$

∎

Corollary 4.5. *If* $\mathbf{a} \in \ell^\infty_{v_s}(\Lambda)$ *for* $s \in \mathbb{R}$ *and* $g \in \mathcal{S}(\mathbb{R}^d)$*, then* $M_{\mathbf{a}}^{\chi',g,\Lambda}$ *maps* $\mathcal{S}(\mathbb{R}^d)$ *into* $\mathcal{S}(\mathbb{R}^d)$*.*

Proof. Since $v_N \circ \chi' \asymp v_N$ by Lemma 4.1, $M^p_{v_N\circ\chi'/v_s}(\mathbb{R}^d) = M^p_{v_{N-s}}(\mathbb{R}^d)$. By Proposition 4.4 $M_{\mathbf{a}}$ maps $M^p_{v_{N-s}}(\mathbb{R}^d)$ to $M^p_N(\mathbb{R}^d)$. Since $\mathcal{S}(\mathbb{R}^d) = \bigcap_{N\geq 0} M^\infty_{v_N}(\mathbb{R}^d)$ by (5), $M_{\mathbf{a}}$ maps $\mathcal{S}(\mathbb{R}^d) = \bigcap_{N\geq 0} M^\infty_{v_{N-s}}(\mathbb{R}^d)$ into $\mathcal{S}(\mathbb{R}^d)$. ∎

5. Approximation of FIOs by Gabor Multipliers

Our next aim is to approximate a FIO T with a tame phase by Gabor multipliers associated to a Parseval frame $\mathcal{G}(g,\Lambda)$. First we will derive a representation

$$T = \sum_{\nu\in\Lambda} \pi(\nu) M_{\mathbf{a}_\nu}, \tag{31}$$

with convergence in several operator norms. To find the candidate symbols $\mathbf{a}_\nu$, we argue as in [20].

Using the commutation relations $M_\eta T_x = e^{2\pi i x\eta} T_x M_\eta$, we can write

$$\pi(\chi'(\mu)+\nu) = c_{\nu,\mu}\pi(\nu)\pi(\chi'(\mu)), \quad \text{for } \nu,\mu \in \Lambda\,, \tag{32}$$

with $|c_{\nu,\mu}| = 1$. We will show that the choice of

$$\mathbf{a}_\nu(\mu) = c_{\nu,\mu}\langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle \tag{33}$$

leads to the formal representation (31).

Since $\mathcal{G}(g,\Lambda)$ is a Parseval frame, f and Tf possess the Gabor expansions

$$f = \sum_\mu \langle f,\pi(\mu)g\rangle \pi(\mu)g, \quad \text{and} \quad Tf = \sum_\lambda \langle Tf,\pi(\lambda)g\rangle\pi(\lambda)g.$$

Setting $\nu = \lambda - \chi'(\mu) \in \Lambda$, we can write

$$\begin{aligned} Tf &= \sum_{\mu} \langle f, \pi(\mu)g\rangle \sum_{\lambda} \Big\langle T\pi(\mu)g, \pi(\lambda)g\Big\rangle \pi(\lambda)g \\ &= \sum_{\mu}\sum_{\nu} \langle f, \pi(\mu)g\rangle \Big\langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\Big\rangle \pi(\chi'(\mu)+\nu)g \\ &= \sum_{\mu}\sum_{\nu} c_{\nu,\mu}\langle f, \pi(\mu)g\rangle \Big\langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\Big\rangle \pi(\nu)\pi(\chi'(\mu))g \\ &= \sum_{\nu} \pi(\nu) \sum_{\mu} c_{\nu,\mu}\langle f, \pi(\mu)g\rangle\langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle \pi(\chi'(\mu))g \\ &= \sum_{\nu} \pi(\nu) M_{\mathbf{a}_\nu} f. \end{aligned} \tag{34}$$

The following result gives a precise meaning to the above computation for test functions in the Schwartz class $\mathcal{S}(\mathbb{R}^d)$.

Proposition 5.1. *Let $\mathcal{G}(g,\Lambda)$ be a Parseval frame and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$.*

(i) If $|\mathbf{a}_\nu(\mu)| \le C(1+|\mu|+|\nu|)^N$ for some $N \ge 0$, then the series $\sum_{\nu\in\Lambda} \pi(\nu) M_{a_\nu}$ converges unconditionally in the strong topology of $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$ and defines a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ ($\mathcal{S}'(\mathbb{R}^d)$ is always endowed with the weak topology).*

(ii) Let A be a continuous operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$, and choose $\mathbf{a}_\nu(\mu)$, $\nu, \mu \in \Lambda$ as

$$\mathbf{a}_\nu(\mu) = c_{\nu,\mu}\langle A\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle$$

with the constant $c_{\nu,\mu}$ as in (32). Then the grow estimates

$$|\mathbf{a}_\nu(\mu)| \le C(1+|\mu|+|\nu|)^N, \tag{35}$$

are satisfied for some constants $C, N \ge 0$ depending on A and g.

Furthermore, A can be represented as the sum $\sum_{\nu\in\Lambda} \pi(\nu) M_{a_\nu}$ of shifted Gabor multipliers in the strong operator topology of $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$.

Proof. (i) By Corollary 4.5 the Gabor multiplier

$$M_{\mathbf{a}_\nu} f = \sum_{\mu\in\Lambda} \mathbf{a}_\nu(\mu)\langle f, \pi(\mu)g\rangle \pi(\chi'(\mu))g \in \mathcal{S}(\mathbb{R}^d)$$

maps $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$. We need to show that the series $\sum_{\nu\in\Lambda} \pi(\nu) M_{\mathbf{a}_\nu} f$, $f \in \mathcal{S}(\mathbb{R}^d)$, converges unconditionally in $\mathcal{S}'(\mathbb{R}^d)$.

Let $h \in \mathcal{S}(\mathbb{R}^d)$ and choose s and l large enough, precisely $s > N + 2d$ and $l > N + s + 2d$. Then

$$
\begin{aligned}
|\langle \sum_\nu \pi(\nu) M_{\mathbf{a}_\nu} f, h\rangle| &= |\sum_\nu \pi(\nu) \sum_\mu \mathbf{a}_\nu(\mu)\langle f, \pi(\mu)g\rangle \pi(\chi'(\mu))g, h\rangle| \\
&\leq \sum_\nu \sum_\mu |\mathbf{a}_\nu(\mu)||\langle f, \pi(\mu)g\rangle||\langle \pi(\nu+\chi'(\mu))g, h\rangle| \\
&\lesssim \sum_\nu \sum_\mu (1+|\mu|+|\nu|)^N (1+|\mu|))^{-l}(1+|\chi'(\mu)|)^s(1+|\nu|)^{-s} < \infty. \qquad (36)
\end{aligned}
$$

In the last step we used the assumption $|a_\nu(\mu)| \leq C(1+|\mu|+|\nu|)^N$ and the fact that $1+|\chi'(\mu)| \asymp 1+|\mu|$ from Lemma 4.1. Then the double series converges because of our choice $s > N + 2d$ and $l > N + s + 2d$.

(ii) The proof is similar to that of Proposition 5 (ii) of [20]. Let $K \in \mathcal{S}'(\mathbb{R}^{2d})$ be the Schwartz kernel of the operator A. Writing $\mu = (\mu_1, \mu_2)$, $\nu = (\nu_1, \nu_2) \in \mathbb{R}^{4d}$ and $\chi'(\mu) = (\chi_1'(\mu), \chi_2'(\mu)) \in \mathbb{R}^{4d}$, we have

$$
\begin{aligned}
|\mathbf{a}_\nu(\mu)| &= |\langle A\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle| \\
&= |\langle K, \pi(\chi'(\mu)+\nu)g \otimes \overline{\pi(\mu)g}\rangle| \\
&= |V_{g\otimes\overline{g}}K(\chi_1'(\mu)+\nu_1, \mu_1, \chi_2'(\mu)+\nu_2, -\mu_2)| \\
&\leq C(1+|\chi'(\mu)|+|\mu|+|\nu|)^N
\end{aligned}
$$

for some $C, N > 0$. In the last step we used the fact that the STFT of a tempered distribution grows at most polynomially. Since $1+|\chi'(\mu)| \lesssim 1+|\mu|$ we see that the sequence $\mathbf{a}_\nu(\mu)$ satisfies the polynomial growth condition (35). It follows then from part (i) that the series $\sum_{\nu\in\Lambda} \pi(\nu) M_{\mathbf{a}_\nu}$ converges in $\mathcal{L}(\mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d))$. Furthermore, its sum must coincide with A, because the formal computation (34) was justified in part (i) and therefore

$$\langle \sum_\nu \pi(\nu) M_{\mathbf{a}_\nu} f, h\rangle = \langle Af, h\rangle, \qquad \forall f, h \in \mathcal{S}(\mathbb{R}^d).$$

(One needs to interchange two summations, which is possible by (36)). □

Remark 5.2. Proposition 5.1 extends to more general windows provided that the sequences $\mathbf{a}_\nu$ satisfy stronger estimates.

For example, if $\sup_{\mu,\nu\in\Lambda} |\mathbf{a}_\nu(\mu)| = C < \infty$, then one may use windows $g \in M^1(\mathbb{R}^d)$. In fact, for $f, h \in M^1(\mathbb{R}^d)$, (36) can be modified to yield

$$
\begin{aligned}
|\langle \sum_\nu \pi(\nu) M_{\mathbf{a}_\nu} f, h\rangle| &\leq \sum_\nu \sum_\mu |\mathbf{a}_\nu(\mu)||\langle f, \pi(\mu)g\rangle||\langle \pi(\nu+\chi'(\mu))g, h\rangle| \\
&\leq C \sum_\mu |\langle f, \pi(\mu)g\rangle| \sum_\nu |\langle \pi(\nu+\chi'(\mu))g, h\rangle| \\
&\leq C'\|f\|_{M^1}\|h\|_{M^1} < \infty\,.
\end{aligned}
$$

where in the last step we have used the characterization (5) of $M^1(\mathbb{R}^d)$ (with $m \equiv 1$ and $p = 1$).

We now consider the convergence of series of Gabor multipliers on modulation spaces. In what follows the space $\mathcal{M}^\infty_m(\mathbb{R}^d)$ denotes the closure of the Schwartz class with respect to the M^∞_m-norm. Whereas Gabor expansions converge only weak* on $M^\infty_m(\mathbb{R}^d)$, they are norm convergent of $\mathcal{M}^\infty_m(\mathbb{R}^d)$.

Proposition 5.3. *Let $\mathcal{G}(g,\Lambda)$ be a Parseval frame and $g \in M^1_v(\mathbb{R}^d)$. If the sequence of symbols $\mathbf{a}_\nu$ satisfies*

$$\sum_{\nu\in\Lambda} \|\mathbf{a}_\nu\|_{\ell^\infty} v(\nu) < \infty\,,$$

then the series $\sum_\nu \pi(\nu) M_{\mathbf{a}_\nu}$ converges in $\mathcal{L}(M^p_{m\circ\chi'}, M^p_m)$ for every $p \in [1,\infty)$ and v-moderate weight m. If $p=\infty$, the series $\sum_\nu \pi(\nu) M_{\mathbf{a}_\nu}$ converges in $\mathcal{L}(\mathcal{M}^\infty_{m\circ\chi'}, \mathcal{M}^\infty_m)$.

Proof. By Proposition 4.4 each operator $M_{\mathbf{a}_\nu}$ is bounded from $M^p_{m\circ\chi'}(\mathbb{R}^d)$ to $M^p_m(\mathbb{R}^d)$ with the norm being dominated by $\|\mathbf{a}_\nu\|_{\ell^\infty}$. Furthermore, $\|\pi(\nu)\|_{M^p_m\to M^p_m} \lesssim v(\nu)$. Hence

$$\begin{aligned}\sum_\nu \|\pi(\nu) M_{\mathbf{a}_\nu}\|_{M^p_{m\circ\chi'}\to M^p_m} &\leq \sum_\nu \|\pi(\nu)\|_{M^p_m\to M^p_m} \|M_{\mathbf{a}_\nu}\|_{M^p_{m\circ\chi'}\to M^p_m}\\ &\lesssim \sum_\nu \|\mathbf{a}_\nu\|_{\ell^\infty} v(\nu) < \infty.\end{aligned}$$

This gives the desired conclusion. □

We return to the study of Fourier integral operators. The following approximation theorem is the main result of our work.

Theorem 5.4. *Let $\mathcal{G}(g,\Lambda)$ be a Parseval frame and $g \in \mathcal{S}(\mathbb{R}^d)$. Let T be a Fourier integral operator with a tame phase and with the associated lattice transformation χ', and define the multiplier symbol as*

$$\mathbf{a}_\nu(\mu) = c_{\nu,\mu} \langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle\,.$$

(i) If $\sigma \in W^\infty_{2N}(\mathbb{R}^{2d})$ for $N > d$ and if $0 \leq r < 2N-2d$, then the series $\sum_{\nu\in\Lambda} \pi(\nu) M_{\mathbf{a}_\nu}$ converges to T in $\mathcal{L}(M^p_{m\circ\chi'}, M^p_m)$ and in $\mathcal{L}(\mathcal{M}^\infty_{m\circ\chi'}, \mathcal{M}^\infty_m)$, for every $p \in [1,\infty)$, and v_r-moderate weight m ($v_r(z) = \langle z\rangle^r$, $z \in \mathbb{R}^{2d}$), and

$$\|T - \sum_{|\nu|\leq L} \pi(\nu) M_{\mathbf{a}_\nu}\|_{M^p_{m\circ\chi'}\to M^p_m} \lesssim L^{r+2d-2N}.$$

(ii) If the symbol σ is in $M^\infty_{1\otimes v_s}(\mathbb{R}^{2d})$ with $s > 2d$, then $T = \sum_{\nu\in\Lambda} \pi(\nu) M_{\mathbf{a}_\nu}$ with the error estimate

$$\|T - \sum_{|\nu|\leq L} \pi(\nu) M_{\mathbf{a}_\nu}\|_{M^p_{m\circ\chi'}\to M^p_m} \lesssim L^{r+2d-s}\,.$$

Proof. (i) We first estimate the magnitude of the multiplier symbols $\mathbf{a}_\nu$: Theorem 1.1, written with χ' as in (26), implies that

$$|\mathbf{a}_\nu(\mu)| = |\langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle| \lesssim \langle \chi'(\mu) - (\chi'(\mu)+\nu)\rangle^{-2N} = \langle\nu\rangle^{-2N},$$

for every $\mu \in \Lambda$. Hence

$$\|\mathbf{a}_\nu\|_\infty \lesssim \langle\nu\rangle^{-2N}$$

and

$$\sum_{\nu\in\Lambda}\langle\nu\rangle^r\|\mathbf{a}_\nu\|_{\ell^\infty}\le\sum_{\nu\in\Lambda}\langle\nu\rangle^{r-2N}<\infty\,.$$

By Proposition 5.3 the series $\sum_\nu \pi(\nu)M_{\mathbf{a}_\nu}$ converges in the operator norm from $M^p_{m\circ\chi'}(\mathbb{R}^d)\to M^p_m(\mathbb{R}^d)$ for every $1\le p\le\infty$. Since the series converges to T in $\mathcal{L}(\mathcal{S}(\mathbb{R}^d),\mathcal{S}'(\mathbb{R}^d))$ by Proposition 5.1, the sum must be identical to T.

For the error estimate we observe that

$$\begin{aligned}\|T-\sum_{|\nu|\le L}\pi(\nu)M_{\mathbf{a}_\nu}\|_{M^p_{m\circ\chi'}\to M^p_m} &\le \sum_{|\nu|>L}\|\pi(\nu)\|_{M^p_m\to M^p_m}\|M_{\mathbf{a}_\nu}\|_{M^p_{m\circ\chi'}\to M^p_m}\\ &\lesssim \sum_{|\nu|>L}\langle\nu\rangle^r\|\mathbf{a}_\nu\|_{\ell^\infty}\\ &\lesssim \sum_{|\nu|>L}\langle\nu\rangle^{r-2N}\lesssim L^{r+2d-2N}.\end{aligned}$$

(ii) is proved similarly by using the decay estimate of Theorem 3.3 instead of Theorem 1.1. ∎

As a byproduct of Theorem 5.4 we obtain an alternative proof of [8, Theorem 4.1]:

Corollary 5.5. *Under the assumptions of Theorem 5.4, the Fourier integral operator T is a bounded operator from $M^p_{m\circ\chi'}(\mathbb{R}^d)$ to $M^p_m(\mathbb{R}^d)$, simultaneously for every $1\le p<\infty$, and from $\mathcal{M}^\infty_{m\circ\chi'}(\mathbb{R}^d)$ to $\mathcal{M}^\infty_m(\mathbb{R}^d)$.*

Remark 5.6. *So far we have used without loss of generality that $\mathcal{G}(g,\Lambda)$ is a tight Gabor frame. If $\mathcal{G}(g,\Lambda)$ is an arbitrary frame with $g\in M^1_v(\mathbb{R}^d)$, then there exists a dual window $\gamma\in M^1_v(\mathbb{R}^d)$ such that every $f\in M^p_m(\mathbb{R}^d)$, for $1\le p\le\infty$ and $m\in\mathcal{M}_v(\mathbb{R}^{2d})$, possesses the Gabor expansion $f=\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)g\rangle\pi(\lambda)\gamma$ with convergence in $M^p_m(\mathbb{R}^d)$. For the existence of a dual window in $M^1_v(\mathbb{R}^d)$ see* [21], *for the result on Gabor expansions see* [17]. *All results of Sections 2 – 5 carry over to general Gabor frames by polarization.*

6. Further Remarks

We conclude with an example and some remarks on how Gabor frames are transformed under FIO.

We first compute explicitly the multiplier symbol (33) for the dilation operator in dimension $d=1$

$$D_sf(x)=f(sx)=\int_{\mathbb{R}}e^{2\pi isx\eta}\hat{f}(\eta)\,d\eta\qquad s>0\,.$$

This is a Fourier integral operator with symbol $\sigma\equiv1$ and phase $\Phi(x,\eta)=sx\eta$. Using (12), the canonical transformation χ is calculated to be

$$\chi(y,\eta)=(y/s,s\eta)\,.$$

For the lattice $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ with $\alpha\beta < 1$ and $\mu = (\alpha k, \beta\ell)$, $k, \ell \in \mathbb{Z}$, we then have $\chi'(\mu) = (\alpha\lfloor k/s \rfloor, \beta\lfloor s\ell \rfloor)$. Consequently, the symbols are

$$\begin{aligned}\mathbf{a}_\nu(\mu) &= \langle T\pi(\mu)g, \pi(\chi'(\mu)+\nu)g\rangle \\ &= e^{2\pi i\alpha\beta\lfloor \frac{k}{s}\rfloor \ell'} \int_{\mathbb{R}} e^{2\pi i\beta(s\ell - \lfloor s\ell\rfloor - \ell')t} g(st-\alpha k)\bar{g}(t-\alpha\lfloor \frac{k}{s}\rfloor - \alpha k')\,dt.\end{aligned}$$

Now, we consider the case of the window function $g(t) = e^{-\pi t^2}$. A straightforward, but lengthy computation (as in [10]) gives

$$\mathbf{a}_{(\alpha k', \beta\ell')}(\alpha k, \beta\ell) = \frac{e^{2\pi i\alpha\beta(\lfloor \frac{k}{s}\rfloor\ell' + (s\ell - \lfloor s\ell\rfloor - \ell')(sk + \lfloor \frac{k}{s}\rfloor + k'))}}{\sqrt{s^2+1}} e^{-\pi\beta^2(s\ell - \lfloor sl\rfloor - \ell')^2} e^{-\pi\alpha^2 s^2(\frac{k}{s} - \lfloor\frac{k}{s}\rfloor + k')^2}.$$

The representation of the dilation operator D_s by a (sum of) modified Gabor multipliers might be of interest for the numerical approximation of D_s within the context of time-frequency analysis, when one is forced to use Gabor frames. Similar formulas can be worked out for general metaplectic operators as well.

More on Gabor frames: Assume that $\mathcal{G}(g,\Lambda)$ is a frame for $L^2(\mathbb{R}^d)$ and that T is a FIO with canonical transformation χ satisfying the standard conditions. Then the transformed system $T\mathcal{G}(g,\Lambda) = \{T\pi(\lambda)g : \lambda \in \Lambda\}$ is a frame, if and only if T is invertible. However, if we fix the window g and only warp the time-frequency space with χ, then we obtain the set $\mathcal{G}(g,\chi(\Lambda)) = \{\pi(\chi(\lambda))g : \lambda \in \Lambda\}$. It is an interesting problem to determine when the transformed Gabor system is still a frame.

If $g(t) = e^{-\pi t^2}$ in dimension $d = 1$, then $\mathcal{G}(g,\chi(\Lambda))$ is a frame, if and only if the lower Beurling density $d^-(\chi(\Lambda)) > 1$. This follows from the characterization of (non-uniform) Gabor frames by Lyubarskii and Seip [25, 26].

On the other hand, if $T = D_s$ is the dilation operator, $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$, and g is a window with compact support, $\operatorname{supp} g \subseteq [-A, A]$, say, then $\chi(\Lambda) = \frac{\alpha}{s}\mathbb{Z} \times s\beta\mathbb{Z}$ is again a lattice. Choosing $\frac{\alpha}{s} > 2A$, then different translates $g(t - s^{-1}\alpha k), k \in \mathbb{Z}$ have disjoint support, and $\mathcal{G}(g,\chi(\Lambda))$ cannot be a frame. Thus the frame property of $\mathcal{G}(g,\chi(\Lambda))$ is subtle.

Based on coorbit theory [13] one can formulate a qualitative result. Recall that a (non-uniform) set $\Lambda \subset \mathbb{R}^{2d}$ is relatively separated if $\max_{k\in\mathbb{Z}^{2d}} \operatorname{card} \Lambda \cap (k + [0,1]^{2d}) < \infty$, and Λ is called δ-dense for $\delta > 0$, if $\bigcup_{\lambda\in\Lambda} \overline{B_\delta(\lambda)} = \mathbb{R}^{2d}$.

The results in [13] assert that for every $g \in M^1(\mathbb{R}^d)$ there exists a $\delta > 0$ depending only on g, such that *every* relatively separated δ-dense set Λ generates a frame $\mathcal{G}(g,\Lambda)$. We fix $g \in M^1(\mathbb{R}^d)$ and $\delta = \delta(g) > 0$.

Now let T be an FIO with canonical transformation χ with Lipschitz constant L. *If $\Lambda \subseteq \mathbb{R}^{2d}$ is δ/L-dense, then $\mathcal{G}(g,\chi(\Lambda))$ is a frame.*

To see this, observe that

$$\begin{aligned}\inf_{\lambda\in\Lambda} |z - \chi(\lambda)| &= \inf_{\lambda\in\Lambda} |\chi(\chi^{-1}(z)) - \chi(\lambda)| \\ &\le L \inf_{\lambda\in\Lambda} |\chi^{-1}(z) - \lambda| \le L\,\frac{\delta}{L} = \delta\,.\end{aligned}$$

Consequently $\chi(\Lambda)$ is δ-dense, and so $\mathcal{G}(g,\chi(\Lambda))$ is a frame.

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