POLITECNICO DI TORINO
Repository ISTITUZIONALE

Prime numbers between squares

Original
Prime numbers between squares / Bazzanella, D.. - In: RIVISTA DI MATEMATICA DELLA UNIVERSITÀ DI PARMA. - ISSN 0035-6298. - STAMPA. - 3*:(2004), pp. 159-164.

Availability:
This version is available at: 11583/2496963 since:

Publisher:
Università di Parma

Published
DOI:

Terms of use:

This article is made available under terms and conditions as specified in the corresponding bibliographic description in the repository

Publisher copyright

(Article begins on next page)

25 November 2026

PRIME NUMBERS BETWEEN SQUARES

DANILO BAZZANELLA

Abstract

A well known conjecture about the distribution of primes asserts that between two consecutive squares there is always at least one prime number. The proof of this conjecture is out of reach at present, even under the assumption of the Riemann Hypothesis. The aim of this paper is to provide a conditional proof of the conjecture assuming a hypothesis about the behavior of Selberg's integral in short intervals.

Mathematical Subject Classification :
11N5 - Distribution of Primes

This is the authors' post-print version of an article published on
Riv. Mat. Univ. Parma (7) 3* (2004), 159–164.[1]

1. INTRODUCTION.

A well known conjecture about the distribution of primes asserts that

Conjecture 1.
For every integer n, the interval $[n^2, (n+1)^2]$ contains a prime.

The proof of this conjecture is quite out of reach at present, even under the assumption of the Riemann Hypothesis (RH). On the other side it is not difficult to prove unconditionally that Conjecture 1 holds for almost all integers.

More precisely we can prove that, for every $[n^2, (n+1)^2] \subset [1, N]$, we have the expected number of primes with at most $O(N^{1/4+\varepsilon})$ exceptions, and $O((\ln N)^{2+\varepsilon})$ exceptions under the assumption of RH, see Bazzanella [1]. To obtain some results in the direction of Conjecture 1 we need to assume hypotheses stronger than RH. Define

$$J(N,h) = \int_1^N (\vartheta(x+h) - \vartheta(x) - h)^2 dx,$$

[1]This version does not contain journal formatting and may contain minor changes with respect to the published version. The final publication is available at http://rivista.math.unipr.it/. The present version is accessible on PORTO, the Open Access Repository of Politecnico di Torino (`http://porto.polito.it`).

with $\vartheta(x) = \sum_{p \le x} \log p$ and p a prime number, and consider the following strong form of Montgomery's pair correlation conjecture.

Conjecture 2. Let γ Euler's constant. For any $\varepsilon > 0$ we have

$$J(N,h) = hN\log(N/h) - (\gamma + \log 2\pi)hN + o(hN) + O(N),$$

uniformly for $1 \le h \le N^{1-\varepsilon}$.

Goldston [5] deduced the validity of Conjecture 1 assuming Conjecture 2. The basic idea of this paper is to connect the distribution of primes in intervals of the type $[n^2, (n+1)^2]$ to the exceptional set for the asymptotic formula of the distribution of primes in short intervals, and using the properties of this set, see Bazzanella [2] and Bazzanella and Perelli [3], to obtain a new conditional proof of Conjecture 1.

With this in mind we state the following conjecture.

Conjecture 3.

$$J(N+Y,h) - J(N,h) = o(hN),$$

uniformly for $1 \le Y \le N^{1/2}$ and $N^{1/2} \ll h \ll N^{1/2}$.

Assuming Conjecture 3 we can state our main theorem.

Theorem *Assume Conjecture 3. The intervals of type* $[n^2, (n+1)^2]$ *contain the expected number of primes for* $n \to \infty$.

We note that although Conjecture 3 is weaker than Conjecture 2, our Theorem is stronger than the result of Goldston [5], which asserts only the existence of a prime in intervals of type $[n^2, (n+1)^2]$.

2. BASIC LEMMA.

The proof of the Theorem is based on a result about the structure of the exceptional set for the asymptotic formula

$$\psi(x + h(x)) - \psi(x) \sim h(x) \text{ as } x \to \infty. \tag{1}$$

Let X be a large positive number, $\delta > 0$, $h(x)$ an increasing function such that $x^{\varepsilon} \le h(x) \le x$ for some $\varepsilon > 0$,

$$\Delta(x,h) = \psi(x + h(x)) - \psi(x) - h(x),$$

and

$$E_{\delta}(X,h) = \{X \le x \le 2X : |\Delta(x,h)| \ge \delta h(x)\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_{\delta}(X,h) = \emptyset$ for $X \ge X_0(\delta)$. Hence for small $\delta > 0$ and X tending to ∞

the set $E_\delta(X,h)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X,h) \subset E_{\delta'}(X,h) \quad \text{if} \quad 0<\delta'<\delta.$$

The lemma about the structure of the exceptional set is the following.

Lemma. *Let $0<\theta<1$, $h(x)$ increasing fuunction such that $h(x) \asymp x^\theta$, X be sufficiently large depending on the function $h(x)$ and $0<\delta'<\delta$ with $\delta-\delta' \geq exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X,h)$ then $E_{\delta'}(X,h)$ contains the interval $[x_0-ch(X), x_0+ch(X)] \cap [X,2X]$, where $c=(\delta-\delta')\theta/5$. In particular, if $E_\delta(X,h) \neq \emptyset$ then*

$$|E_{\delta'}(X,h)| \gg_\theta (\delta-\delta')h(X).$$

Proof.

We will always assume that x and X are sufficiently large as prescribed by the various statements, and $\varepsilon>0$ is arbitrarily small and not necessarily the same at each occurrence. From the Brun-Titchmarsh theorem, see Montgomery-Vaughan [7], we have that

$$\psi(x+y)-\psi(x) \leq \frac{21}{10} y \frac{\log x}{\log y} \tag{2}$$

for $10 \leq y \leq x$. From (2) we easily see that

$$\psi(x+y)-\psi(x) \leq \frac{9}{4\alpha} cY \tag{3}$$

for $X \leq x \leq 3X$ and $0 \leq y \leq cY$, where $0<\alpha<1$, $X^{\alpha-\varepsilon} \leq Y \leq X$ and

$$\frac{\alpha}{5}\exp(-\sqrt{\log X}) \leq c \leq 1.$$

Let $h(x) \asymp x^\theta$, $x_0 \in E_\delta(X,h)$ and $x \in [x_0-ch(X), x_0+ch(X)] \cap [X,2X]$, where c satisfies the above restrictions. We have

$$|\Delta(x,h)| = |\Delta(x_0,h)+\Delta(x,h)-\Delta(x_0,h)| \geq$$

$$|\Delta(x_0,h)| - |\psi(x+h(x))-\psi(x_0+h(x_0))| - |\psi(x)-\psi(x_0)| - |h(x)-h(x_0)|.$$

Hence from (3) with $\alpha=\theta$ we obtain

$$|\Delta(x,h)| \geq \delta h(x) - \frac{9}{2\theta} ch(X) + O(X^{2\theta-1+\varepsilon}) \geq \delta h(x) - \frac{5}{\theta} ch(X) \geq \delta' h(x),$$

by choosing $c=(\delta-\delta')\theta/5$, since $h(x)$ is increasing. Hence $x \in E_{\delta'}(X,h)$ and the Lemma follows.

3. PROOF OF THE THEOREM.

The prime number theorem implies that

$$\psi(x+h(x))-\psi(x) \sim h(x) \quad \text{as} \quad x \to \infty,$$

for $h(x)$ sufficiently large with respect to x. Hence the expected number of primes in intervals of type $[n^2,(n+1)^2]$ is $(n+1)^2-n^2\sim 2n$ and then the Theorem asserts that

$$\psi((n+1)^2)-\psi(n^2)\sim 2n \quad \text{as} \quad n\to\infty. \tag{4}$$

In order to prove the Theorem we assume that (4) does not hold. Then there exist $\delta_0>0$ and a sequence $x_j\to\infty$ with $|\Delta(x_j,h)|\geq\delta_0 h(x_j)$ and

$$h(x)=2\sqrt{x}+1. \tag{5}$$

For x_j sufficiently large, choose $\delta'=\delta_0/2$ in the Lemma. Hence

$$|\Delta(x,h)|\geq\frac{\delta_0}{2}h(x)\geq\frac{\delta_0}{2}\sqrt{x_j} \quad \text{for} \quad x_j\leq x\leq x_j+\frac{\delta_0}{20}\sqrt{x_j}.$$

From our assumption it follows that

$$x_j^{3/2}\ll\int_{x_j}^{x_j+Y}|\Delta(x,h)|^2dx, \tag{6}$$

where

$$Y=\frac{\delta_0}{20}\sqrt{x_j}. \tag{7}$$

From (5) we see that

$$h(x)=h(x_j)+O(1) \quad \text{uniformly for} \quad x_j\leq x\leq x_j+Y,$$

and therefore

$$\int_{x_j}^{x_j+Y}|\Delta(x,h)|^2dx=\int_{x_j}^{x_j+Y}|\psi(x+h(x_j))-\psi(x)-h(x_j)|^2dx+O(x_j^{1/2}). \tag{8}$$

Recalling the definitions of the functions $\psi(x)$ and $\vartheta(x)$ we find that

$$\int_{x_j}^{x_j+Y}|\psi(x+h(x_j))-\psi(x)-h(x_j)|^2dx=$$

$$\int_{x_j}^{x_j+Y}|\vartheta(x+h(x_j))-\vartheta(x)-h(x_j)|^2dx+O(x_j^{1/2}\log^2 x_j). \tag{9}$$

From (6), (8) and (9) we can conclude that

$$x_j^{3/2}\ll\int_{x_j}^{x_j+Y}|\vartheta(x+h(x_j))-\vartheta(x)-h(x_j)|^2dx=J(x_j+Y,h(x_j))-J(x_j,h(x_j)).$$

Assuming Conjecture 3 we get a contradiction, and then the Theorem follows.

References

[1] D. Bazzanella. *Primes between consecutive square. Arch. Math.*, 75 (2000), 29-34.

[2] D. Bazzanella. *Primes in almost all short intervals II. Boll. UMI*, (8) 3-B (2000), 717-726.

[3] D. Bazzanella and A. Perelli. *The exceptional set for the number of primes in short intervals. J. Number Theory*, 80 (2000), 109-124.

[4] H. Davenport. *Multiplicative Number Theory*, volume GTM 74. Springer - Verlag, 1980. second edition.

[5] D.A. Goldston. *Linnik's theorem on Goldbach numbers in short intervals. Glasgow Math. J.*, 32 (1990), 285–297.

[6] D.A. Goldston and M.L. Montgomery. *Pair correlation of zero and primes in short intervals. Analytic Number Theory and Dioph. Prob., ed. by A.C. Adolphson et al.*, pages 183–203, Birkhäuser Verlag 1987.

[7] H.L.Montgomery and R.C.Vaughan. *The large sieve. Mathematika*, 20 (1973), 119–134.

[8] A. Selberg. *On the normal density of primes in small intervals, and the difference between consecutive primes. Arch. Math. Naturvid.*, 47 (1943), 87–105.

Danilo Bazzanella
Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino, Italy
e-mail: danilo.bazzanella@polito.it