Surfaces in R4 with constant principal angles with respect to a plane

SURFACES IN $\mathbb{R}^4$ WITH CONSTANT PRINCIPAL ANGLES WITH RESPECT TO A PLANE

PIERRE BAYARD, ANTONIO J. DI SCALA, OSVALDO OSUNA CASTRO, AND GABRIEL RUIZ-HERNÁNDEZ

Abstract. We study surfaces in $\mathbb{R}^4$ whose tangent spaces have constant principal angles with respect to a plane. Using a PDE we prove the existence of surfaces with arbitrary constant principal angles. The existence of such surfaces turns out to be equivalent to the existence of a special local symplectomorphism of $\mathbb{R}^2$. We classify all surfaces with one principal angle equal to 0 and observe that they can be constructed as the union of normal holonomy tubes. We also classify the complete constant angles surfaces in $\mathbb{R}^4$ with respect to a plane. They turn out to be extrinsic products. We characterize which surfaces with constant principal angles are compositions in the sense of Dajczer-Do Carmo. Finally, we classify surfaces with constant principal angles contained in a sphere and those with parallel mean curvature vector field.

1. Introduction

In [16] Camille Jordan defined the concept of principal angles between two linear subspaces of the Euclidean space. The principal angles are real numbers between 0 and $\frac{\pi}{2}$ which describe the mutual position of the two subspaces. If one subspace has dimension one, the principal angle is just the usual angle between a straight line and a subspace. When both subspaces have dimension two, their principal angles are two real numbers θ_1, θ_2 such that $0 \leq \theta_1 \leq \theta_2 \leq \frac{\pi}{2}$.

In this work we consider the principal angles between the tangent planes of an immersed surface in $\mathbb{R}^4$ and a fixed plane Π in $\mathbb{R}^4$. Every tangent plane $T_p\Sigma$ of the surface Σ, considered as a vector subspace of $\mathbb{R}^4$, has two principal angles $\theta_1(p), \theta_2(p)$ with the fixed plane Π, which depend on the point $p \in \Sigma$. The aim of this article is to investigate local and global geometric properties of those surfaces in which $\theta_1(p), \theta_2(p)$ are constant functions. For simplicity,

1991 *Mathematics Subject Classification.* Primary 53C40, 53C42.
Key words and phrases. Helix surfaces, constant angle surfaces, principal angles.
The fourth author was partially supported by Conacyt.

we will call them helix surfaces or constant angles surfaces with respect to a plane.

In the case where the constant angles surface is contained in some hyperplane $\mathbb{R}^3$ of $\mathbb{R}^4$, we show in Proposition 2.12, that the surface has a constant angle with respect to a direction in the hyperplane $\mathbb{R}^3$; these surfaces are classified in [9] and [17]. The case of constant angle submanifolds in $\mathbb{R}^n$ with respect to some direction in $\mathbb{R}^n$ was investigated by the second and the last authors in [10]. Constant angle surfaces with respect to a direction have been investigated very recently also in other Riemannian manifolds, as in the works [11] and [12].

Here is a theorem collecting some of our main results.

Theorem 1.1. *Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles with respect to a plane $\Pi \subset \mathbb{R}^4$. Then Σ has zero Gauss curvature and has flat normal bundle. If Σ is complete then Σ is an extrinsic product. Moreover, if Σ is compact then Σ is a torus embedded in $\mathbb{R}^4$ as a product of two closed plane curves.*

In Section 2 we recall basic definitions and properties of principal angles between linear subspaces in $\mathbb{R}^4$, and we introduce the notion of helix (or constant angles) surfaces. In Section 3 we study the Gauss map of a helix surface; we show that a surface Σ has constant principal angles if and only if its Gauss map image belongs to a product of circles in $S^2(\sqrt{2}/2)\times S^2(\sqrt{2}/2)$. If Σ is moreover compact then the circles are equators in $S^2(\sqrt{2}/2)$.

In Section 4 we write down the structure equations of a constant angles surface in an adapted frame, and in Section 5 we classify the complete helix surfaces such that $0 < \theta_1 < \theta_2 < \pi/2$ (*the generic case*) and such that $0 = \theta_1 < \theta_2 < \pi/2$. In Section 6, we characterize the constant angles surfaces which are compositions, a concept studied in [4],[5] and [6] by Do Carmo, Dajczer and Tojeiro in the context of local isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ with zero normal curvature. We prove that a generic helix surface is a composition if and only if its first normal space has rank one (Proposition 6.1).

In Section 7 we study constant angles surfaces in spheres of $\mathbb{R}^4$ and in Section 8 we describe the local structure of constant angles surfaces whose lower principal angle vanishes; we show that these surfaces can be constructed as the union of holonomy tubes along a curve in the normal space of a given curve of $\mathbb{R}^4$. A similar construction was used in [7].

In Section 9 and 10 we show the existence of surfaces with constant principal angles; we use the Cauchy-Kowalewski existence theorem for partial differential equations. In Theorem 10.3, we show the existence of non trivial helix surfaces with generic principal angles and whose first normal spaces have rank two; in particular these helix surfaces are not compositions. In order to use the Cauchy-Kowalewski theorem we consider the surface Σ as the graph of a local diffeomorphism $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$. Then we observe that Σ is a helix surface if and only if F is a symplectomorphism whose

jacobian matrix has constant length. It is interesting to remark that, by Theorem 1.1, a global symplectomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ whose Jacobian matrix has constant length is necessarily an affine map.

In the last section, we consider constant angle surfaces with parallel mean curvature vector. We prove in Theorem 11.1, that the latter condition is equivalent to the fact that Σ is a product.

2. Preliminaries

2.1. **Principal angles.** Here we recall the notion of principal angles between two planes in $\mathbb{R}^4$. We refer to [16] or [14] for more details.

Definition 2.1. Let V and W be two-dimensional subspaces of $\mathbb{R}^4$. The principal angles between V and W, $0 \leq \theta_1 \leq \theta_2 \leq \pi/2$, are defined by

$$\cos\theta_1 := \langle v_1, w_1\rangle := \max\{\langle v, w\rangle | v \in V, w \in W, |v| = |w| = 1\},$$

$$\cos\theta_2 := \langle v_2, w_2\rangle := \max\{\langle v, w\rangle | v \in V, w \in W, v \perp v_1, w \perp w_1, |v| = |w| = 1\}.$$

If $p_W : \mathbb{R}^4 \to W$ stands for the orthogonal projection on W, the expression $\mathrm{Q}_{WV}(v) := \langle \mathrm{p}_W(v), \mathrm{p}_W(v)\rangle$ defines a quadratic (positive semidefinite) form on the subspace V. Let us denote by $\mathcal{S}_{WV} \in Sym(V)$ the symmetric endomorphism such that

$$\langle \mathrm{p}_W(v), \mathrm{p}_W(v')\rangle = \langle \mathcal{S}_{WV}(v), v'\rangle, \ \ \forall v, v' \in V.$$

The following is well-known.

Proposition 2.2. *The eigenvalues of $\mathcal{S}_{WV}$ are $\cos^2(\theta_1)$ and $\cos^2(\theta_2)$. In particular, there exists an orthonormal basis (v_1, v_2) of V such that, for all $v = X_1 v_1 + X_2 v_2$ belonging to V,*

$$Q_{WV}(v) = \cos^2\theta_1 X_1^2 + \cos^2\theta_2 X_2^2. \tag{1}$$

The next lemma links the principal angles between V and $W^\perp$ to the principal angles between V and W :

Lemma 2.3. *Let V and W be two-dimensional subspaces of $\mathbb{R}^4$. If the principal angles between V and W are θ_1 and θ_2, then the principal angles between V and $W^\perp$ are $\theta_1^\perp = \pi/2 - \theta_2 \leq \theta_2^\perp = \pi/2 - \theta_1$.*

Proof. Since, for all $v \in V$, $v = p_W(v) + p_{W^\perp}(v)$ with $p_W(v) \perp p_{W^\perp}(v)$, we readily get

$$|v|^2 = Q_{WV}(v) + Q_{W^\perp V}(v).$$

Thus, by (1),

$$Q_{W^\perp V}(v) = \sin^2\theta_1 X_1^2 + \sin^2\theta_2 X_2^2.$$

Using Proposition 2.2 again (with $W^\perp$ instead of W), we deduce that $\cos^2(\theta_1^\perp) = \sin^2\theta_2$ and $\cos^2(\theta_2^\perp) = \sin^2\theta_1$, and the result follows. □

Proposition 2.4. *Given any two angles $0 \leq \theta_1 \leq \theta_2 \leq \pi/2$, there exist two planes V and W in $\mathbb{R}^4$ with these two principal angles.*

Proof. Let (w_1, w_2, w_3, w_4) be an orthonormal basis of $\mathbb{R}^4$, and consider the plane $W = span\{w_1,\ w_2\}$. Let v_1 and v_2 be the orthonormal vectors given by

$$v_1 := \cos(\theta_1)w_2 + \sin(\theta_1)w_4 \quad v_2 := \cos(\theta_2)w_1 + \sin(\theta_2)w_3.$$

Let us define $V = span\{v_1,\ v_2\}$. Since $p_W(v_1) = \cos(\theta_1)w_2$ and $p_W(v_2) = \cos(\theta_2)w_1$, writing $v = X_1v_1 + X_2v_2$ we readily get

$$Q_{WV}(v) = |p_W(v)|^2 = \cos^2(\theta_1)X_1^2 + \cos^2(\theta_2)X_2^2,$$

and the result follows from (1). □

2.2. **Principal angles and bivectors.** We consider the vector space $\Lambda^2\mathbb{R}^4$ endowed with its natural scalar product, defined on decomposable bivectors by

$$\langle v_1 \wedge v_2, w_1 \wedge w_2\rangle := \langle v_1, w_1\rangle\langle v_2, w_2\rangle - \langle v_2, w_1\rangle\langle v_1, w_2\rangle.$$

Let V and W be two oriented planes of $\mathbb{R}^4$. If (v_1, v_2) and (w_1, w_2) are positively oriented and orthonormal basis of V and W, we define the angle $\theta \in [0, \pi]$ between V and W by the formula

$$\cos\theta = \langle v_1 \wedge v_2, w_1 \wedge w_2\rangle.$$

Let us denote by $\theta^\perp \in [0, \pi]$ the angle between V and $W^\perp$, where the orientation of $W^\perp$ is such that the union of two positively oriented basis of W and $W^\perp$ is a positively oriented basis of $\mathbb{R}^4$. Note that $\theta^\perp$ is also the angle between $V^\perp$ and W. The following result may be find in [14], Theorem 5.

Lemma 2.5. *The angles θ and $\theta^\perp$ are linked to the principal angles θ_1 and θ_2 between V and W by the formulae*

$$|\cos\theta| = \cos\theta_1 . \cos\theta_2 \qquad \textit{and} \qquad |\cos\theta^\perp| = \sin\theta_1 . \sin\theta_2. \tag{2}$$

2.3. **Surfaces with constant principal angles.** Recall that a surface $\Sigma \subset \mathbb{R}^4$ is called *full* if it is not contained in an affine hyperplane.

Definition 2.6. Let Σ be an immersed surface in $\mathbb{R}^4$ and let $\Pi \subset \mathbb{R}^4$ be a two-dimensional plane. We say that Σ is a *helix surface* or a *constant angles surface* with respect to Π, if the principal angles between T_pM and Π do not depend on $p \in \Sigma$. We will also say that Σ has *constant principal angles* with respect to the plane Π.

Example 2.7. *(The Clifford torus is a helix surface)*
Let us consider the torus $T^2 = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{R}^4$ with the metric induced by the metric of $\mathbb{R}^4$. Let us see that T^2 is a helix surface with respect to the plane $\Pi_{12} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_3 = x_4 = 0\}$. The tangent space of T^2 at the point $p = (x_1, x_2, x_3, x_4) \in T^2$ has an orthonormal basis given by $(v_1 = (-x_2, x_1, 0, 0),\ v_2 = (0, 0, -x_4, x_3))$. Since $w_1 := v_1$ belongs to $T_pT^2 \cap \Pi_{12}$, we readily get $\cos\theta_1 = \langle v_1, w_1\rangle = 1$, i.e. $\theta_1 = 0$. Moreover, we get that $v_2 \perp v_1$, $w_2 := (x_1, x_2, 0, 0)$ belongs to Π_{12} and that $w_2 \perp w_1$. Thus $\cos\theta_2 = \langle v_2, w_2\rangle = 0$, i.e. $\theta_2 = \pi/2$. So, the flat torus has constant

principal angles $\theta_1 = 0, \theta_2 = \pi/2$ with respect to the plane Π_{12}, i.e. T^2 is a helix with respect to Π_{12}. Analogously T^2 is a helix with respect to the plane $\Pi_{34} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 | x_1 = x_2 = 0\}$, with the same constant principal angles.

Example 2.8. We construct a helix surface in $\mathbb{R}^4$ with respect to a plane by using a nonplanar curve in $\mathbb{R}^3$. The surface will be full and will be a Riemannian product of $\mathbb{R}$ with a curve in $\mathbb{R}^3$. Let γ be a classical regular helix curve in $\mathbb{R}^3$ with respect to a fixed direction d, i.e. such that the tangent vectors of γ make a constant angle θ with d. We define Σ as the Riemannian product $\gamma \times \mathbb{R}$ which is an immersed surface in $\mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. Then Σ has constant principal angles with respect to the plane generated by d and $e_4 = (0, 0, 0, 1)$. The constant principal angles are θ and 0.

Lemma 2.9. *Let $\Sigma^2 \subset \mathbb{R}^4$ be a compact immersed surface. Let Π be any two-dimensional plane in $\mathbb{R}^4$. Then there exists $p \in \Sigma$, depending on Π, such that $T_p\Sigma$ and Π have a principal angle equal to zero.*

Proof. Let H be any hyperplane containing the plane Π. Since Σ is compact there exists $p \in \Sigma$ such that $T_p\Sigma \subset H$. So, the two planes $T_p\Sigma$ and Π belong to the hyperplane H. Therefore, they have a common straight line and thus a principal angle has to be zero. □

Proposition 2.10. *If Σ is a compact immersed helix surface in $\mathbb{R}^4$ with respect to a plane, then it has constant principal angles equal to zero and $\pi/2$.*

Proof. By Lemma 2.9, there exists $p \in \Sigma$ with a principal angle at p equal to zero. Because Σ is a helix, Σ has a zero principal angle at every point. Now, the same argument applied to $\Pi^\perp$ shows that the other principal angle is equal to $\pi/2$ (using also Lemma 2.3). □

Example 2.11. We construct a noncompact helix surface in $\mathbb{R}^4$ with respect to a plane, with one principal angle equal to zero. In this example the helix surface is not full. Let Σ be an immersed surface in the Euclidean space $\mathbb{R}^3$ with its standard Riemannian metric. Let us assume that there is a unit vector $d \in \mathbb{R}^3$ such that every tangent space of Σ makes a constant angle $0 < \theta < \pi/2$ with the direction d. The authors Di Scala and Ruiz-Hernández investigated this class of submanifolds in [9] and [10]; they are called helix surfaces with respect to the direction d, or constant angle surfaces. They are never compact. For example a cone of revolution is a helix with respect to a direction of its axis of revolution. Now, let us consider Σ as an immersed surface in $\mathbb{R}^4$, using the inclusion $\mathbb{R}^3 \subset \mathbb{R}^3 \times \mathbb{R} = \mathbb{R}^4$. Let Π be the plane generated by d and $e_4 = (0, 0, 0, 1) \in \mathbb{R}^4$. Then Σ is a helix surface with respect to Π and its constant principal angles are $\pi/2$ and θ. Let $\Pi^\perp$ be the orthogonal complement of Π in $\mathbb{R}^4$. By Lemma 2.3, Σ is a helix surface with respect to $\Pi^\perp$ and its constant principal angles are $\pi/2 - \theta$ and 0.

There is a natural relation between non full helix surfaces in $\mathbb{R}^4$ with respect to a plane and helix surfaces in $\mathbb{R}^3$ with respect to a direction:

Proposition 2.12. Classification of helix surfaces in $\mathbb{R}^4$ which are not full. *Let Σ be a non full immersed surface in $\mathbb{R}^4$ which is a helix with respect to a plane $\Pi \subset \mathbb{R}^4$. Assume that Σ is contained in $\mathbb{R}^3 \subset \mathbb{R}^4$. Then Σ is a helix surface in $\mathbb{R}^3$ with respect to some direction in $\mathbb{R}^3$.*

Proof. First case: Π is contained in the hyperplane $\mathbb{R}^3$. Let d be a unit vector in $\mathbb{R}^3$ normal to Π. Then Σ is a helix surface with respect to the direction d. *Second case:* Π is transversal to $\mathbb{R}^3$. Thus $\Pi \cap \mathbb{R}^3$ is a line l in $\mathbb{R}^3$. Let us denote by d a fixed unit direction in $\mathbb{R}^3$ parallel to l. We prove that Σ is a helix surface with respect to the direction d. For this, consider e_4 a unit vector normal to $\mathbb{R}^3$ in $\mathbb{R}^4$ and ξ a local unit vector field orthogonal to Σ in $\mathbb{R}^3$. Since (ξ, e_4) is an orthonormal basis of $T\Sigma^\perp$, the bivector $\xi \wedge e_4$ represents the normal plane $T\Sigma^\perp$. By hypothesis, for every $p \in \Sigma$, $T_p\Sigma$ and Π have constant principal angles, and by Lemma 2.3 $T_pM^\perp$ and Π also have constant principal angles. Let $w \in \mathbb{R}^4$ be a fixed direction such that (d, w) is an orthonormal basis of Π. We conclude that $\langle \xi \wedge e_4, d \wedge w\rangle := \langle \xi, d\rangle\langle e_4, w\rangle - \langle \xi, w\rangle\langle e_4, d\rangle = \langle \xi, d\rangle\langle e_4, w\rangle$ is constant (see Lemma 2.5). Taking the derivative along a direction T tangent to Σ we get

$$\begin{aligned} 0 = T.(\langle \xi \wedge e_4, d \wedge w\rangle) &= (T.\langle \xi, d\rangle)\langle e_4, w\rangle + \langle \xi, d\rangle(T.\langle e_4, w\rangle) \\ &= (T.\langle \xi, d\rangle)\langle e_4, w\rangle \end{aligned}$$

since $\langle e_4, w\rangle$ is constant (e_4 and w are fixed directions). Finally, let us observe that $\langle e_4, w\rangle \neq 0$: otherwise w would be in $\mathbb{R}^3$, which is not possible since Π, transversal to $\mathbb{R}^3$, is generated by the pair $(d,\ w)$, with d belonging to $\mathbb{R}^3$. Therefore $T.\langle \xi, d\rangle = 0$, which means that $\langle \xi, d\rangle$ is constant along Σ. This is equivalent to say that Σ is a helix surface with respect to the direction d. □

Example 2.13. A helix in $\mathbb{R}^4$ which is full and is not a Riemannian product of two curves. *Let Π be a two-dimensional subspace of $\mathbb{R}^4$ and let G be the group of all isometries of $\mathbb{R}^4$ that fix pointwise Π. So, G is isomorphic to the group $SO(2)$. Let γ be a connected regular curve in $\mathbb{R}^4$, whose tangent lines make a constant angle with the plane Π. We define an immersed surface Σ in $\mathbb{R}^4$ by taking $\Sigma := G \cdot \gamma$, the orbit of γ under the action of G. Let us observe that Σ is foliated by its geodesics $g \cdot \gamma$, for every $g \in G$. The other curves $G \cdot p$ for every $p \in \gamma$ (these curves are planar circles in $\mathbb{R}^4$) are orthogonal to such family of geodesics in Σ. Let us observe that the geodesics on Σ given by $g \cdot \gamma$ have the same property as the original γ: their tangent lines make the same constant angle with respect to the plane Π, since G consists of isometries in $\mathbb{R}^4$ that fix pointwise Π.*

3. Characterization of helix surfaces using the Gauss map

The Grassmannian of the oriented 2-planes in $\mathbb{R}^4$ identifies with the set

$$Q = \{\eta \in \Lambda^2\mathbb{R}^4 : \ \langle \eta, \eta \rangle = 1, \ \eta \wedge \eta = 0\}$$

of unit and decomposable bivectors of $\mathbb{R}^4$. Recall that the Hodge operator is the symmetric map $* : \Lambda^2\mathbb{R}^4 \to \Lambda^2\mathbb{R}^4$ such that $\langle \eta, *\eta' \rangle = \eta \wedge \eta'$ for all $\eta, \eta' \in \Lambda^2\mathbb{R}^4$, where $\Lambda^4\mathbb{R}^4$ is identified with $\mathbb{R}$ using the canonical volume form on $\mathbb{R}^4$. Since $** = id_{\Lambda^2\mathbb{R}^4}$, $\Lambda^2\mathbb{R}^4$ splits into the orthogonal sum

$$\Lambda^2\mathbb{R}^4 = E^+ \oplus E^-$$

where $E^+ = \{\eta : *\eta = \eta\}$ and $E^- = \{\eta : *\eta = -\eta\}$, and the natural map

$$\begin{array}{rcl} \Lambda^2\mathbb{R}^4 & \to & E^+ \oplus E^- \\ \eta & \mapsto & (\eta^+, \eta^-), \end{array}$$

induces an isometry between Q and the product of spheres $S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$. Consider the Gauss map

$$\begin{array}{rcl} G : \Sigma & \to & S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2) \\ x & \mapsto & \big((e_1 \wedge e_2)^+, (e_1 \wedge e_2)^-\big) \end{array}$$

where (e_1, e_2) is a positively oriented and orthonormal basis of $T_x\Sigma$. We first give a characterization of an helix surface in terms of its Gauss map image:

Proposition 3.1. *Σ is an immersed helix surface in $\mathbb{R}^4$ with respect to a plane if and only if its Gauss map image belongs to a product of circles in $S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$.*

Proof. Fix Π an oriented plane of $\mathbb{R}^4$, represented by $(\eta_o^+, \eta_o^-) \in S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$. For $x \in \Sigma$, $G(x) = (\eta^+, \eta^-)$ represents the plane $T_x\Sigma$, with its orientation. We define the two angles α^+, α^- by the formulae

$$\cos(\alpha^+) = 2\langle \eta_o^+, \eta^+ \rangle, \qquad \cos(\alpha^-) = 2\langle \eta_o^-, \eta^- \rangle.$$

The formulae

$$\langle \eta_o, \eta \rangle = \langle \eta_o^+, \eta^+ \rangle + \langle \eta_o^-, \eta^- \rangle, \qquad \langle \eta_o, *\eta \rangle = \langle \eta_o^+, \eta^+ \rangle - \langle \eta_o^-, \eta^- \rangle$$

read

$$\cos\theta = \frac{1}{2}(\cos\alpha^+ + \cos\alpha^-), \qquad \cos\theta^\perp = \frac{1}{2}(\cos\alpha^+ - \cos\alpha^-),$$

where θ and $\theta^\perp$ are the angles between Π and $T_x\Sigma$, and Π and $T_x\Sigma^\perp$ defined Section 2.2. We thus have

$$\cos\alpha^+ = \cos\theta + \cos\theta^\perp \qquad \text{and} \qquad \cos\alpha^- = \cos\theta - \cos\theta^\perp. \tag{3}$$

By Lemma 2.5 we deduce that the angles α^+ and α^- are constant if and only if the principal angles θ_1, θ_2 are, and thus that Σ is an helix surface with respect to Π if and only if its Gauss map image belongs to a product of circles centered at η_0^+ and η_0^- in $S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$. $\square$

Proposition 3.2. *If Σ is a compact immersed helix surface in $\mathbb{R}^4$ with respect to a plane Π, then it has constant principal angles equal to zero and $\pi/2$. That means that its Gauss map image is a product of two equators in $S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$.*

Proof. The first part was proved in Proposition 2.10. Lemma 2.5 and formulae (3) imply that the two angles α^+ and α^- between Π and $T_p\Sigma$ are equal to $\pi/2$, and thus that the Gauss map image is a product of two equators in $S^2(\sqrt{2}/2) \times S^2(\sqrt{2}/2)$. □

4. Structure equations

In this section we compute the structure equations (see [1, p.10]) of a helix surface in a frame adapted to the helix structure.

Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles with respect to the plane $\Pi \subset \mathbb{R}^4$. Let $T_1, T_2 \in \Gamma(\mathrm{T}\Sigma)$ be a local frame such that $T_1(p)$ and $T_2(p)$ are unit eigenvectors of $S_{\Pi T_p\Sigma}$ at every point $p \in \Sigma$, and let e_1, e_2 be the corresponding frame of Π, defined by

$$\begin{aligned} e_1 &= \cos(\theta_1)T_1 + \sin(\theta_1)\xi_1\,, \\ e_2 &= \cos(\theta_2)T_2 + \sin(\theta_2)\xi_2 \end{aligned}$$

where ξ_1, ξ_2 are normal vector fields (θ_1 and θ_2 still denote the constant principal angles). Note that T_1 and T_2 (and thus e_1, e_2, ξ_1 and ξ_2) do exist since the eigenvalues of $S_{\Pi T_p\Sigma}$ are constant. Let X be a vector field of Σ. Taking derivatives in both hands we get

$$\begin{aligned} D_X e_1 &= \cos(\theta_1)D_X T_1 + \sin(\theta_1)D_X\xi_1 \\ &= \cos(\theta_1)(\nabla_X T_1 + \alpha(X,T_1)) + \sin(\theta_1)(\nabla^\perp_X\xi_1 - A_{\xi_1}(X)) \\ &= \cos(\theta_1)\nabla_X T_1 - \sin(\theta_1)A_{\xi_1}(X) + \cos(\theta_1)\alpha(X,T_1) + \sin(\theta_1)\nabla^\perp_X\xi_1 \end{aligned}$$

and

$$\begin{aligned} D_X e_2 &= \cos(\theta_2)D_X T_2 + \sin(\theta_2)D_X\xi_2 \\ &= \cos(\theta_2)(\nabla_X T_2 + \alpha(X,T_2)) + \sin(\theta_2)(\nabla^\perp_X\xi_2 - A_{\xi_2}(X)) \\ &= \cos(\theta_2)\nabla_X T_2 - \sin(\theta_2)A_{\xi_2}(X) + \cos(\theta_2)\alpha(X,T_2) + \sin(\theta_2)\nabla^\perp_X\xi_2. \end{aligned}$$

We can regard $\Sigma \times \Pi \to \Sigma$ as a trivial bundle endowed with a flat connection D. Then, there exists a function $f : \Sigma \to \mathbb{R}$ such that

$$D_X e_1 = df(X)e_2 \qquad \text{and} \qquad D_X e_2 = -df(X)e_1.$$

Then from the above equations we get

$$\begin{aligned} \cos(\theta_2)df(X)T_2 + \sin(\theta_2)df(X)\xi_2 &= \cos(\theta_1)\nabla_X T_1 - \sin(\theta_1)A_{\xi_1}(X) \\ &\quad + \cos(\theta_1)\alpha(X,T_1) + \sin(\theta_1)\nabla^\perp_X\xi_1\,, \\ -\cos(\theta_1)df(X)T_1 - \sin(\theta_1)df(X)\xi_1 &= \cos(\theta_2)\nabla_X T_2 - \sin(\theta_2)A_{\xi_2}(X) \\ &\quad + \cos(\theta_2)\alpha(X,T_2) + \sin(\theta_2)\nabla^\perp_X\xi_2. \end{aligned}$$

Taking the normal and the tangent components we get

$$\begin{aligned} \cos(\theta_2)df(X)T_2 &= \cos(\theta_1)\nabla_X T_1 - \sin(\theta_1)A_{\xi_1}(X)\,, \\ -\cos(\theta_1)df(X)T_1 &= \cos(\theta_2)\nabla_X T_2 - \sin(\theta_2)A_{\xi_2}(X) \end{aligned} \tag{4}$$

$$\begin{array}{rcl} \sin(\theta_2)df(X)\xi_2 & = & \cos(\theta_1)\alpha(X,T_1)+\sin(\theta_1)\nabla^\perp_X\xi_1\ , \\ -\sin(\theta_1)df(X)\xi_1 & = & \cos(\theta_2)\alpha(X,T_2)+\sin(\theta_2)\nabla^\perp_X\xi_2. \end{array} \tag{5}$$

Here is the first consequence of the above equations.

Lemma 4.1. *The Levi-Civita connection and the normal connection are flat.*

Proof. Indeed, from equations (5) it follows that

$$\alpha(T_1,T_2)=0 \qquad \text{and} \qquad \alpha(T_1,T_1)\perp\alpha(T_2,T_2)\,.$$

Now the first claim follows from Gauss equation and the second from Ricci equation. □

In the frame T_1, T_2 we have

$$A_{\xi_1}=\begin{pmatrix} 0 & 0 \\ 0 & m_1 \end{pmatrix}, \qquad A_{\xi_2}=\begin{pmatrix} m_2 & 0 \\ 0 & 0 \end{pmatrix}$$

where $m_1, m_2 : \Sigma \to \mathbb{R}$ are two smooth functions, and

$$\alpha(T_1,T_1)=m_2\xi_2, \qquad \alpha(T_2,T_2)=m_1\xi_1.$$

Define $t, n : \Sigma \to \mathbb{R}$ two smooth functions such that

$$\nabla T_1 = dt\ T_2,\ \nabla T_2=-dt\ T_1,\ \nabla^\perp\xi_1=dn\ \xi_2 \text{ and } \nabla^\perp\xi_2=-dn\ \xi_1. \tag{6}$$

We may thus re-write the structure equations (4)-(5) as follows:

$$\begin{array}{rcl} \cos(\theta_2)df(X) & = & \cos(\theta_1)dt(X)-\sin(\theta_1)\langle X,T_2\rangle m_1\,, \\ -\cos(\theta_1)df(X) & = & -\cos(\theta_2)dt(X)-\sin(\theta_2)\langle X,T_1\rangle m_2 \end{array} \tag{7}$$

and

$$\begin{array}{rcl} \sin(\theta_2)df(X) & = & \cos(\theta_1)\langle X,T_1\rangle m_2+\sin(\theta_1)dn(X)\ , \\ -\sin(\theta_1)df(X) & = & \cos(\theta_2)\langle X,T_2\rangle m_1-\sin(\theta_2)dn(X). \end{array} \tag{8}$$

Observe that equations (8) imply the existence of two functions λ_1, λ_2 such that

$$\nabla\lambda_1=m_1T_2 \qquad \text{and} \qquad \nabla\lambda_2=m_2T_1.$$

Remark 4.2. The fact that m_1T_2 and m_2T_1 are gradients also follows from Codazzi equations (C1) and (C2) below.

4.1. **The case $\theta_1=0$ and $0<\theta_2<\frac{\pi}{2}$.** Under these assumptions equations (7) and (8) are equivalent to

$$\begin{array}{rcl} \cos(\theta_2)df(X) & = & dt(X)\,, \\ -df(X) & = & -\cos(\theta_2)dt(X)-\sin(\theta_2)\langle X,T_1\rangle m_2 \end{array} \tag{9}$$

and

$$\begin{array}{rcl} \sin(\theta_2)df(X) & = & \langle X,T_1\rangle m_2\ , \\ 0 & = & \cos(\theta_2)\langle X,T_2\rangle m_1-\sin(\theta_2)dn(X), \end{array} \tag{10}$$

and are thus equivalent to

$$d\lambda_1=\tan(\theta_2)dn, \qquad d\lambda_2=\tan(\theta_2)dt \qquad \text{and} \qquad df=\frac{dt}{\cos(\theta_2)}. \tag{11}$$

As a consequence we get the following result.

Proposition 4.3. *Under the above assumptions the vector field T_2 is geodesic.*

Proof. Indeed, from the second equation of the system (11) it follows that $dt(T_2) \equiv 0$, which implies $\nabla_{T_2} T_2 = -dt(T_2)T_1 \equiv 0$. □

4.2. **The generic case.** By the *generic case* we mean the case in which the principal angles $\theta_1, \theta_2 \notin \{0, \frac{\pi}{2}\}$. Under this hypothesis, we may eliminate df in the first equation of (7) using successively the second, the third, and the last equation in (7)-(8), to get the system

$$\left(\frac{\cos(\theta_1)}{\cos(\theta_2)} - \frac{\cos(\theta_2)}{\cos(\theta_1)}\right) dt = \frac{\sin(\theta_1)}{\cos(\theta_2)} d\lambda_1 + \frac{\sin(\theta_2)}{\cos(\theta_1)} d\lambda_2 \tag{12}$$

$$-\frac{\sin(\theta_1)}{\sin(\theta_2)} dn + \frac{\cos(\theta_1)}{\cos(\theta_2)} dt = \frac{\sin(\theta_1)}{\cos(\theta_2)} d\lambda_1 + \frac{\cos(\theta_1)}{\sin(\theta_2)} d\lambda_2 \tag{13}$$

$$-\frac{\sin(\theta_2)}{\sin(\theta_1)} dn + \frac{\cos(\theta_1)}{\cos(\theta_2)} dt = \left(\frac{\sin(\theta_1)}{\cos(\theta_2)} - \frac{\cos(\theta_2)}{\sin(\theta_1)}\right) d\lambda_1. \tag{14}$$

4.3. **Codazzi equations.** Here we compute the Codazzi equations. Since

$$\begin{aligned}
(\nabla_{T_1} A_{\xi_1})(T_2) &= \nabla_{T_1}\left(A_{\xi_1}(T_2)\right) - A_{\nabla^{\perp}_{T_1}\xi_1}(T_2) - A_{\xi_1}(\nabla_{T_1} T_2) \\
&= \nabla_{T_1}\left(A_{\xi_1}(T_2)\right) \\
&= \nabla_{T_1}(m_1 T_2) \\
&= dm_1(T_1)T_2 - m_1 dt(T_1)T_1,
\end{aligned}$$

$$\begin{aligned}
(\nabla_{T_2} A_{\xi_1})(T_1) &= \nabla_{T_2}\left(A_{\xi_1}(T_1)\right) - A_{\nabla^{\perp}_{T_2}\xi_1}(T_1) - A_{\xi_1}(\nabla_{T_2} T_1) \\
&= -A_{\nabla^{\perp}_{T_2}\xi_1}(T_1) - A_{\xi_1}(\nabla_{T_2} T_1) \\
&= -A_{dn(T_2)\xi_2}(T_1) - A_{\xi_1}(dt(T_2)T_2) \\
&= -dn(T_2)m_2 T_1 - dt(T_2)m_1 T_2,
\end{aligned}$$

$$\begin{aligned}
(\nabla_{T_1} A_{\xi_2})(T_2) &= \nabla_{T_1}\left(A_{\xi_2}(T_2)\right) - A_{\nabla^{\perp}_{T_1}\xi_2}(T_2) - A_{\xi_2}(\nabla_{T_1} T_2) \\
&= -A_{\nabla^{\perp}_{T_1}\xi_2}(T_2) - A_{\xi_2}(\nabla_{T_1} T_2) \\
&= A_{dn(T_1)\xi_1}(T_2) + A_{\xi_2}(dt(T_1)T_1) \\
&= dn(T_1)A_{\xi_1}(T_2) + dt(T_1)A_{\xi_2}(T_1) \\
&= dn(T_1)m_1 T_2 + dt(T_1)m_2 T_1
\end{aligned}$$

and

$$\begin{aligned}
(\nabla_{T_2} A_{\xi_2})(T_1) &= \nabla_{T_2}\left(A_{\xi_2}(T_1)\right) - A_{\nabla^{\perp}_{T_2}\xi_2}(T_1) - A_{\xi_2}(\nabla_{T_2} T_1) \\
&= \nabla_{T_2}\left(A_{\xi_2}(T_1)\right) \\
&= \nabla_{T_2}(m_2 T_1) \\
&= dm_2(T_2)T_1 + m_2 dt(T_2)T_2,
\end{aligned}$$

the Codazzi equations are satisfied if and only if

$$\begin{cases} m_1 dt(T_2) &= -dm_1(T_1) & (C1) \\ m_1 dt(T_1) &= m_2 dn(T_2) & (C2) \\ m_2 dt(T_1) &= dm_2(T_2) & (C3) \\ m_2 dt(T_2) &= m_1 dn(T_1). & (C4) \end{cases}$$

Observe that $(C1)$ (resp. $(C3)$) is equivalent to $m_1 T_2$ (resp. $m_2 T_1$) be a gradient.

5. Complete surfaces with constant principal angles.

5.1. **The generic case.** Here we show that a complete surface $\Sigma \subset \mathbb{R}^4$ with constant principal angles $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ is totally geodesic.

From equation (12) the differential dt is a linear combination of $d\lambda_1, d\lambda_2$. Namely,

$$dt = A d\lambda_1 + B d\lambda_2$$

where A, B are constants which depend on θ_1, θ_2. The following lemma is crucial.

Lemma 5.1. *The following formulae hold:*

$$\nabla_{T_2} \nabla \lambda_2 = \|\nabla \lambda_2\| (A \nabla \lambda_1 + B \nabla \lambda_2)$$

$$\nabla_{T_1} \nabla \lambda_1 = -\|\nabla \lambda_1\| (A \nabla \lambda_1 + B \nabla \lambda_2).$$

Proof. Keep in mind that

$$\nabla \lambda_1 = m_1 T_2 \qquad \text{and} \qquad \nabla \lambda_2 = m_2 T_1 \,.$$

For the first equality we have to show that

$$\langle \nabla_{T_2} \nabla \lambda_2, T_1 \rangle = \|\nabla \lambda_2\| B \langle \nabla \lambda_2, T_1 \rangle \qquad (*)$$

and

$$\langle \nabla_{T_2} \nabla \lambda_2, T_2 \rangle = \|\nabla \lambda_2\| A \langle \nabla \lambda_1, T_2 \rangle. \qquad (**)$$

We compute

$$\begin{aligned} \langle \nabla_{T_2} \nabla \lambda_2, T_2 \rangle &= \langle \nabla_{T_2} m_2 T_1, T_2 \rangle \\ &= m_2 \langle \nabla_{T_2} T_1, T_2 \rangle \\ &= m_2 \langle dt(T_2) T_2, T_2 \rangle \\ &= m_2 \langle (A d\lambda_1(T_2) + B d\lambda_2(T_2)) T_2, T_2 \rangle \\ &= m_2 \langle (A d\lambda_1(T_2)) T_2, T_2 \rangle \\ &= m_2 A d\lambda_1(T_2) \\ &= \|\nabla \lambda_2\| A \langle \nabla \lambda_1, T_2 \rangle, \end{aligned}$$

which proves $(**)$. The proof of $(*)$ is analogous:

$$\begin{aligned}
\langle \nabla_{T_2}\nabla\lambda_2, T_1\rangle &= \langle \nabla_{T_1}\nabla\lambda_2, T_2\rangle \\
&= m_2\langle \nabla_{T_1}T_1, T_2\rangle \\
&= m_2\langle dt(T_1)T_2, T_2\rangle \\
&= m_2\langle (Bd\lambda_2(T_1))T_2, T_2\rangle \\
&= m_2\langle (Bd\lambda_2(T_1))T_2, T_2\rangle \\
&= m_2 Bd\lambda_2(T_1) \\
&= \|\nabla\lambda_2\| B\langle \nabla\lambda_2, T_1\rangle.
\end{aligned}$$

The proof of the second equality is analogous and is therefore omitted. This proves the lemma. □

Theorem 5.2. *Assume that $\Sigma \subset \mathbb{R}^4$ with constant principal angles θ_1, θ_2 such that $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ is a complete surface. Then Σ is totally geodesic.*

Proof. Indeed, taking inner product with $\nabla\lambda_2$ in both sides of the first equation of the above lemma we get

$$\langle \nabla_{T_2}\nabla\lambda_2, \nabla\lambda_2\rangle = \|\nabla\lambda_2\| B\langle \nabla\lambda_2, \nabla\lambda_2\rangle.$$

Then along the flow $F_t^{T_2}$ of the vector field T_2 the function $f(t) = \|\nabla\lambda_2\|$ satisfies

$$\left(\frac{f^2}{2}\right)' = Bf^3.$$

Observe that $B \neq 0$ because $\theta_2 \neq 0$. Since the flow of T_2 is complete it is not difficult to see that $f(t) \equiv 0$. This shows $m_2 \equiv 0$.

Analogously, taking the inner product with $\nabla\lambda_1$ in both sides of the second equation of the above lemma we get

$$\langle \nabla_{T_1}\nabla\lambda_1, \nabla\lambda_1\rangle = -\|\nabla\lambda_1\| A\langle \nabla\lambda_1, \nabla\lambda_1\rangle.$$

Then along the flow $F_t^{T_1}$ of the vector field T_1 the function $g(t) = \|\nabla\lambda_1\|$ satisfies

$$\left(\frac{g^2}{2}\right)' = -Ag^3.$$

Note that $A \neq 0$ because $\theta_1 \neq 0$. Since the flow of T_1 is complete we get that $g(t) \equiv 0$. This shows $m_1 \equiv 0$. Thus $\alpha \equiv 0$ and Σ is totally geodesic. □

Corollary 5.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ by a symplectomorphism and let Jf be its Jacobian matrix. If $\|Jf\|$ is a constant function then f is affine.*

Proof. As explained in Proposition 9.4 below the graph of such a symplectomorphism can be used to construct a surface with constant principal angles. Since it is an entire graph it is complete, and the above theorem implies that the graph is totally geodesic. Thus, f must be an affine map. □

5.2. **Complete surfaces with $\theta_1 = 0$.** Here we prove that a complete not totally geodesic surface with constant principal angles is a product.

Theorem 5.4. *Assume $\Sigma \subset \mathbb{R}^4$ with constant principal angles such that $\theta_1 = 0$ to be complete and not totally geodesic. Then T_1, T_2 are parallel vector fields and Σ is an extrinsic product.*

Proof. If $\theta_2 = \frac{\pi}{2}$, from the first equation in (7) we get $dt = 0$, and thus that T_1 and T_2 are parallel vector fields. So assume $0 < \theta_2 < \frac{\pi}{2}$. As in the proof of Theorem 5.2 (using Lemma 5.1 and $B \neq 0$) we get that λ_2 is a constant function. The second equation in (11) then implies $dt = 0$, and thus that T_1 and T_2 are parallel vector fields. That Σ is an extrinsic product follows from the well-known Moore's Lemma [1, p. 28] since $\alpha(T_1, T_2) \equiv 0$. □

5.3. **Proof of Theorem 1.1.** The first claim was already proved in Lemma 4.1. The second claim is a consequence of Theorem 5.2 and Theorem 5.4. The third part is a consequence of the previous theorem since a compact surface is complete.

6. Do Carmo-Dajczer-Tojeiro compositions

In [4],[5] and [6] local isometric immersions of $\mathbb{R}^2$ into $\mathbb{R}^4$ were studied. The authors introduced the concept of *compositions*: an isometric immersion $i : U \subset \mathbb{R}^2 \to \mathbb{R}^4$ is called a composition (or is said to be trivial) if there exist isometric immersions $g : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ and $F : W \subset \mathbb{R}^3 \to \mathbb{R}^4$ such that

$$i = F \circ g.$$

Since the surfaces $\Sigma \subset \mathbb{R}^4$ with constant principal angles are flat it is natural to understand when they are compositions.

Proposition 6.1. *Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$. The following facts are equivalent:*

(i) *Σ is a composition,*
(ii) *the first normal space N_1 has rank one,*
(iii) *either T_1 or T_2 is a totally geodesic vector field.*

Proof. We first prove $(i) \Rightarrow (ii)$ by contradiction. Assume that $\Sigma \subset \mathbb{R}^4$ is a composition such that $rank(N_1) = 2$. It follows from [6, p. 209, Proposition 2.2] that either $dn(T_1) \equiv 0$ or $dn(T_2) \equiv 0$. Then Codazzi equation $(C2)$ or $(C4)$ implies that either T_1 or T_2 is totally geodesic. Assume first that T_1 is totally geodesic. Then equation (12) implies $d\lambda_2(T_1) \equiv 0$, which in turn implies $m_2 \equiv 0$ and contradicts the hypothesis $rank(N_1) = 2$. A similar argument shows that if T_2 is geodesic then $m_1 \equiv 0$ and so $rank(N_1) = 1$, a contradiction. Thus (i) implies (ii). $(ii) \Rightarrow (i)$ is proved in [4]. $(iii) \Rightarrow (ii)$ follows from equation (12) and $(ii) \Rightarrow (iii)$ from Codazzi equations. □

Remark 6.2. A surface $\Sigma \subset \mathbb{R}^4$ with constant principal angles $\{\theta_1, \theta_2\} = \{0, \frac{\pi}{2}\}$ is a composition since it is a product. In general, a product has first normal space N_1 of rank 2.

Proposition 6.3. *Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles such that $\theta_1 = 0$. Then Σ is a composition.*

Proof. If $rank(N_1) = 1$ then the claim follows from [4]. Assume $rank(N_1) = 2$. Equations (11) imply $dn(T_1) = 0$ and so by [6, p. 209, Proposition 2.2] we get that Σ is a composition. □

7. Constant angles surfaces in spheres

A product of two circles is a surface of constant principal angles contained in a sphere. The aim of this section is to classify the surfaces with constant principal angles contained in some sphere. Along this section we assume $\Sigma \subset \mathbb{R}^4$ to be a surface of constant principal angles with respect to the plane $\Pi \subset \mathbb{R}^4$. The following lemma is well-known:

Lemma 7.1. *The surface Σ is contained in some sphere if and only if there exists a normal parallel vector field ξ whose shape operator is a multiple of the identity, i.e. is of the form $A_\xi = \delta \mathrm{Id}$ with $\delta \neq 0$.*

Now assume that Σ is contained in some sphere. Then there exists a function θ such that the normal vector field ξ defined as

$$\xi := \cos(\theta)\xi_1 + \sin(\theta)\xi_2$$

is normal, parallel and such that $A_\xi = \delta \mathrm{Id}$, $\delta \neq 0$.

Lemma 7.2. *The surface $\Sigma \subset \mathbb{R}^4$ with constant principal angles is contained in some sphere if and only if there exist a function θ and a number $\delta \neq 0$ such that*

$$m_1 \cos(\theta) = \delta, \quad m_2 \sin(\theta) = \delta \quad \text{and} \quad d\theta = -dn. \tag{15}$$

Proof. If $\xi = \cos(\theta)\xi_1 + \sin(\theta)\xi_2$, we have

$$A_\xi = \cos(\theta)A_{\xi_1} + \sin(\theta)A_{\xi_2} = \begin{pmatrix} \sin(\theta)m_2 & 0 \\ 0 & \cos(\theta)m_1 \end{pmatrix}.$$

Thus $A_\xi = \delta Id$ if and only if the first two equations in (15) hold. Moreover, recalling (6), we get

$$\begin{aligned} \nabla^\perp_X \xi &= \nabla^\perp_X (\cos(\theta)\xi_1 + \sin(\theta)\xi_2) \\ &= -\sin(\theta)\left(d\theta(X) + dn(X)\right)\xi_1 + \cos(\theta)\left(dn(X) + d\theta(X)\right)\xi_2, \end{aligned}$$

and the vector fielf ξ is parallel if and only if the last equation in (15) holds. □

We now prove by contradiction that $\theta_1 = 0$ or $\theta_2 = \pi/2$. Equations (12)-(14) together with (15) and Codazzi equations (C1)-(C2) give the following

system for $d\theta(T_1), d\theta(T_2), dt(T_1)$ and $dt(T_2)$:

$$\begin{cases} (\frac{\cos(\theta_1)}{\cos(\theta_2)} - \frac{\cos(\theta_2)}{\cos(\theta_1)})dt(T_1) & = & \frac{\sin(\theta_2)\delta}{\cos(\theta_1)\sin(\theta)} & (1-T_1) \\ (\frac{\cos(\theta_1)}{\cos(\theta_2)} - \frac{\cos(\theta_2)}{\cos(\theta_1)})dt(T_2) & = & \frac{\sin(\theta_1)\delta}{\cos(\theta_2)\cos(\theta)} & (1-T_2) \\ \frac{\sin(\theta_1)}{\sin(\theta_2)}d\theta(T_1) + \frac{\cos(\theta_1)}{\cos(\theta_2)}dt(T_1) & = & \frac{\cos(\theta_1)\delta}{\sin(\theta_2)\sin(\theta)} & (2-T_1) \\ \frac{\sin(\theta_1)}{\sin(\theta_2)}d\theta(T_2) + \frac{\cos(\theta_1)}{\cos(\theta_2)}dt(T_2) & = & \frac{\sin(\theta_1)\delta}{\cos(\theta_2)\cos(\theta)} & (2-T_2) \\ \frac{\sin(\theta_2)}{\sin(\theta_1)}d\theta(T_1) + \frac{\cos(\theta_1)}{\cos(\theta_2)}dt(T_1) & = & 0 & (3-T_1) \\ \frac{\sin(\theta_2)}{\sin(\theta_1)}d\theta(T_2) + \frac{\cos(\theta_1)}{\cos(\theta_2)}dt(T_2) & = & (\frac{\sin(\theta_1)}{\cos(\theta_2)} - \frac{\cos(\theta_2)}{\sin(\theta_1)})\frac{\delta}{\cos(\theta)} & (3-T_2) \\ dt(T_2) & = & -\tan(\theta)d\theta(T_1) & (C_1) \\ dt(T_1) & = & -\cot(\theta)d\theta(T_2) & (C_2) \end{cases}.$$

Observe that the values of $d\theta(T_1), d\theta(T_2), dt(T_1)$ and $dt(T_2)$ are given by the first two and the last two equations. Then using equation $(3-T_2)$ we easily get

$$-2\delta\cos^2(\theta_1)\csc(\theta_1-\theta_2)\csc(\theta_1+\theta_2)\sec(\theta_2)\sec(\theta)\sin(\theta_1) = 0.$$

So we have a contradiction unless either $\theta_1 = 0$ or $\theta_2 = \frac{\pi}{2}$.

Assume that $\theta_1 = 0$ (the case $\theta_2 = \frac{\pi}{2}$ is reduced to that case, switching the role of Π and $\Pi^\perp$). Since Σ is flat we can take local coordinates (x, y) where the metric is given by $dx^2 + dy^2$. Using the structure equations (11), we see that the flow lines of T_2 are level curves of the functions λ_2 and t, whereas the flow lines of T_1 are the level curves of λ_1, n and θ. Thus the vector field T_2 is geodesic and its flow in (x, y) consists of straight lines, and we get a nice intrinsic description of the surface using Fermi coordinates. Namely, let $\gamma(s)$ be the arc length parametrization of a level curve of the function θ and let (s, r) be the Fermi coordinates around a point of γ. The flat metric $dx^2 + dy^2$ is expressed in coordinates (s, r) as

$$dr^2 + (1 - r\kappa(s))^2 ds^2,$$

where $\kappa(s)$ is the curvature of $\gamma(s)$. As explained above the function θ only depends on r, and the function t only on s. We have

$$\nabla t = \frac{t'(s)}{(1-r\kappa(s))^2}\frac{\partial}{\partial s} \qquad \text{and} \qquad \nabla\theta = \theta'(r)\frac{\partial}{\partial r}.$$

Moreover, using the second equation in (11), the very definition of m_2 and the second equation in (15), we get

$$\|\nabla t\| = \frac{\|\nabla\lambda_2\|}{\tan(\theta_2)} = \frac{|m_2|}{\tan(\theta_2)} = \frac{\delta}{\tan(\theta_2)|\sin(\theta(r))|}.$$

Thus

$$\frac{|t'(s)|}{|1-r\kappa(s)|} = \frac{\delta}{\tan(\theta_2)|\sin(\theta(r))|}.$$

Taking $r = 0$ we see that $t'(s)$ is a constant, t'_0, which is such that

$$\frac{t'_0}{|1-r\kappa(s)|} = \frac{\delta}{\tan(\theta_2)|\sin(\theta(r))|};$$

thus $\kappa(s)$ is a constant, κ_0. This shows that the flow lines of T_1 are arcs of circles in coordinates x, y. Actually, the flow lines of T_1 are arcs of circles in the space $\mathbb{R}^4$. To see this, observe that the flow lines of T_1 are contained in a plane parallel to Π because T_1 is always tangent to Π. The absolute value of the curvature of a given flow line of T_1 is given by

$$\|D_{T_1}T_1\|^2 = \|\nabla_{T_1}T_1\|^2 + \|\alpha(T_1,T_1)\|^2 = \kappa_0^2 + \frac{\delta^2}{\sin^2(\theta(r))},$$

which does not depend on the parameter s. So the flow lines of T_1 are arcs of circles contained in parallel planes. We have the following result.

Theorem 7.3. *Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles with respect to the plane $\Pi \subset \mathbb{R}^4$. If Σ is contained in a sphere $S^3 \subset \mathbb{R}^4$ then either $\theta_1 = 0$ or $\theta_2 = \frac{\pi}{2}$ and Σ is a surface of revolution around a fixed plane $\Pi_0 \subset \mathbb{R}^3 \subset \mathbb{R}^4$ obtained by revolving a spherical helix curve of $\mathbb{R}^3$.*

Proof. Let $\mathcal{C}$ be a flow line of T_1. As explained above $\mathcal{C}$ is an arc of circle contained in a plane parallel to Π. So we can assume $\mathcal{C} \subset \Pi$. Let $p \in \mathcal{C}$ and let $\gamma(r) \subset \Sigma \subset \mathbb{R}^4$ be the flow line of T_2 starting from p. Notice that the vectors $\xi(r) := \gamma(r) - p$ are always perpendicular to $\mathcal{C}$ at p. Indeed,

$$D_{T_2}T_1 = \nabla_{T_2}T_1 + \alpha(T_2,T_1) = 0 + 0$$

since $dt(T_2) = 0$ (T_2 is geodesic). This shows that T_1 is constant in $\mathbb{R}^4$ along $\gamma(r)$ and thus that the vectors $\xi(r) := \gamma(r) - p$ are always perpendicular to $\mathcal{C}$ at p. In other words, the curve $\gamma(r)$ is contained in the normal space $\nu_p(\mathcal{C})$ of the curve $\mathcal{C} \subset \mathbb{R}^4$. The above discussion is independent of the point $p \in \mathcal{C}$. Denote by $\gamma_p(r)$ the corresponding curve in $\nu_p(\mathcal{C})$. Now fix $r = r_0$, and let $\xi_{r_0}(p) := p - \gamma_p(r_0)$ be a field of normal vectors along $\mathcal{C}$ (assuming that p varies in $\mathcal{C}$). Observe that the derivative of $\gamma_p(r_0)$ in the direction of T_1 is a multiple of T_1 because T_1 is parallel along γ_p. This shows that the normal vector field $\xi_{r_0}(p)$ is parallel with respect to the normal connection of $\mathcal{C}$. Since $\mathcal{C}$ is a circle contained in Π the parallel transport with respect to the normal connection is given by the rotations around Π which fix $\Pi^\perp$.

Now fix $p_0 \in \mathcal{C}$. Identifying the normal space $\nu_{p_0}(\mathcal{C})$ to $\mathbb{R}^3$ we see that the flow line $\gamma(r)$ of T_2 starting at p_0 is a classical helix curve of $\mathbb{R}^3$ with respect to $N_{p_0} \in \nu_{p_0}(\mathcal{C}) \cong \mathbb{R}^3$, where N_{p_0} is the unit normal at p_0 of $\mathcal{C} \subset \Pi$. Indeed, $\nu_{p_0}(\mathcal{C}) = \mathbb{R}N_{p_0} \oplus \Pi^\perp$ and, as we explained above, $\gamma(r) \subset \nu_{p_0}(\mathcal{C})$. Since $\gamma(r)$ is a helix curve and since it is contained in Σ it follows that $\gamma(r)$ is a *spherical helix*; see [13, p. 248, Lemma 8.22] for a complete discussion about such curves. □

Remark 7.4. It is interesting to notice that the above proof shows that Σ is the union of the so called *holonomy tubes* (see [1, p. 124]) over $\mathcal{C}$ starting with the normal vectors of the curve $\gamma(r)$. A similar construction was used in [7, Section 5] to give local examples of non isoparametric immersions of spheres with curvature normals of constant length.

8. Structure of constant angle surfaces when $\theta_1 = 0$

Here we show that any surface Σ with constant principal angles $\theta_1 = 0$ and arbitrary θ_2 with respect to a plane Π is (up to rigid motion) a union of holonomy tubes over a plane curve $\mathcal{C} \subset \Pi$. More precisely, there exists a curve $\gamma \subset \nu_p(\mathcal{C})$ such that Σ is locally the union of the holonomy tubes of $\mathcal{C}$ through the points of γ.

Theorem 8.1. *Any surface Σ with constant principal angles $\theta_1 = 0$ and arbitrary θ_2 with respect to a plane Π is (up to rigid motion) a union of holonomy tubes over a plane curve $\mathcal{C} \subset \Pi$. Moreover, if $\gamma \subset \nu_p(\mathcal{C})$ is the curve of the starting points of the tubes, then γ is a helix curve of $\mathbb{R}^3 \cong \nu_p(\mathcal{C})$.*

Proof. Observe that the structure equations (11) imply that T_2 is a geodesic vector field. Now, as in the proof of Theorem 7.3 we introduce flat coordinates x, y and Fermi coordinates r, s, where s is the arc length parameter of a level curve $\mathcal{C}$ of the function n, i.e. a flow line of the vector field T_1. Since T_1 is always tangent to Π we can assume that $\mathcal{C} \subset \Pi$. Fix $p_0 \in \mathcal{C} \subset \Pi$ and let $\gamma(r) \subset \mathbb{R}^4$ be the flow line of T_2 starting from p_0. So $\gamma(r) \subset \nu_{p_0}(\mathcal{C})$ since T_1 is constant in $\mathbb{R}^4$ along $\gamma(r)$. Then the same argument as in the proof of Theorem 7.3 shows that the holonomy tubes through $\gamma(r)$ are contained in Σ. Thus, Σ is locally the union of such holonomy tubes. Finally, notice that $\gamma \subset \mathbb{R}^3 \cong \nu_{p_0}(\mathcal{C})$ is a helix curve with respect to $\Pi^\perp \subset \nu_{p_0}(\mathcal{C})$, i.e. is a curve of constant slope with respect to the normal vector N_{p_0} of $\mathcal{C}$ at p_0. □

Remark 8.2. Actually any surface Σ with constant principal angles with either T_1 or T_2 geodesic can be described as a union of holonomy tubes over a curve $\mathcal{C}$ by using a helix curve $\gamma_p \subset \nu_p(\mathcal{C})$ as starting points of the holonomy tubes; moreover, the point p may be arbitrarily chosen. Notice that the helices $(\gamma_p)_{p \in C}$ are all congruent since the parallel transport with respect to the normal connection is an isometry between the normal spaces. So, surfaces with a geodesic vector field T_1 or T_2 are union of congruent helix curves.

9. Existence

In this section we discuss the existence of non-trivial surfaces $\Sigma \subset \mathbb{R}^4$ with prescribed constant principal angles with respect to a plane Π. More precisely, we seek a surface Σ such that θ_1 and θ_2 belong to $(0, \pi/2)$; Σ is thus locally the graph of a function $F : \Pi \to \Pi^\perp$. Taking orthonormal bases of Π and $\Pi^\perp$, and writing $F(x, y) = (f(x, y), g(x, y))$ in these bases, the surface $\Sigma \subset \mathbb{R}^4$ is locally parametrized by

$$(x, y) \to (x, y, f(x, y), g(x, y)).$$

Let $\partial_x := (1, 0, f_x(x, y), g_x(x, y)), \partial_y := (0, 1, f_y(x, y), g_y(x, y))$ be a basis of the tangent plane $\mathrm{T}_{(x,y)}\Sigma$. As usual, let $\mathrm{E}, \mathrm{F}, \mathrm{G}$ denote the coefficients of

the metric tensor of Σ in the coordinates (x, y), i.e.

$$\begin{array}{rcccl} \mathrm{E} & := & \langle \partial_x, \partial_x \rangle & = & 1 + f_x^2 + g_x^2 \\ \mathrm{F} & := & \langle \partial_x, \partial_y \rangle & = & f_x f_y + g_x g_y \\ \mathrm{G} & := & \langle \partial_y, \partial_y \rangle & = & 1 + f_y^2 + g_y^2. \end{array}$$

Set $W := \mathrm{T}_{(x,y)}\Sigma$. Then the symmetric operator $\mathcal{S}_{\Pi W}$ defined Section 2.1 satisfies

$$\langle S_{\Pi W}(\partial_x), \partial_x \rangle = \langle S_{\Pi W}(\partial_y), \partial_y \rangle = 1 \quad \text{and} \quad \langle S_{\Pi W}(\partial_x), \partial_y \rangle = 0.$$

So the matrix of $\mathcal{S}_{\Pi W}$ in (∂_x, ∂_y) is the inverse of the matrix of the metric, i.e.

$$\mathcal{S}_{\Pi W} = \begin{pmatrix} \mathrm{E} & \mathrm{F} \\ \mathrm{F} & \mathrm{G} \end{pmatrix}^{-1}.$$

We deduce the following proposition.

Proposition 9.1. *With the notation above, Σ has constant principal angles with respect to the plane Π if and only if the matrix tensor $\begin{pmatrix} \mathrm{E} & \mathrm{F} \\ \mathrm{F} & \mathrm{G} \end{pmatrix}$ has constant eigenvalues.*

Corollary 9.2. *Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles with respect to a plane Π. If the principal angle θ_1 has multiplicity 2 then Σ is totally geodesic, i.e. is an open subset of a 2-plane.*

Proof. If the principal angle θ_1 has multiplicity 2 then the immersion is orthogonal, i.e. $\mathrm{F} \equiv 0$. Thus E and G are constants. This implies that there exist constants c_1, c_2 such that the map $(x, y) \to (c_1 f(x, y), c_2 g(x, y))$ is a local isometry of $\mathbb{R}^2$. Thus f, g are linear functions and Σ is totally geodesic. □

Since $\mathcal{S}_{\Pi W}$ has constant eigenvalues if and only if its characteristic polynomial does not depend on the point (x, y), we get:

Proposition 9.3. *With the notation above, Σ has constant principal angles $\theta_1, \theta_2 \in (0, \pi/2)$ with respect to the plane Π if and only if*

$$\begin{array}{lcl} \mathrm{E} + \mathrm{G} & = & \sec^2(\theta_1) + \sec^2(\theta_2) \\ \mathrm{EG} - \mathrm{F}^2 & = & \sec^2(\theta_1)\sec^2(\theta_2). \end{array}$$

Since

$$\mathrm{EG} - \mathrm{F}^2 = 1 + f_x^2 + g_x^2 + f_y^2 + g_y^2 + (f_x g_y - f_y g_x)^2,$$

in order to show the existence of surfaces with constant principal angles we have to solve the following PDE system:

$$\left\{ \begin{array}{lcl} 2 + f_x^2 + g_x^2 + f_y^2 + g_y^2 & = & \sec^2(\theta_1) + \sec^2(\theta_2) \\ (f_x g_y - f_y g_x)^2 & = & \sec^2(\theta_1)\sec^2(\theta_2) + 1 - \sec^2(\theta_1) - \sec^2(\theta_2) \\ & = & \tan^2(\theta_1)\tan^2(\theta_2). \end{array} \right.$$

Taking the square root we get the equivalent system

$$\begin{cases} f_x^2 + g_x^2 + f_y^2 + g_y^2 &= c_1 := \sec^2(\theta_1) + \sec^2(\theta_2) - 2, \\ f_x g_y - f_y g_x &= c_2 := \tan(\theta_1)\tan(\theta_2). \end{cases}$$

Since $c_2 \neq 0$ we can divide both hands by c_2 and obtain the following proposition.

Proposition 9.4. *There exists a non totally geodesic surface with constant principal angles if and only if there exists a non linear local symplectomorphism of $\mathbb{R}^2$ whose Jacobian matrix has constant length.*

Proof. Assume that such a surface exists, and set $\psi(x,y) := (\frac{f}{\sqrt{c_2}}, \frac{g}{\sqrt{c_2}})$. Then the system above implies that ψ is a non linear local symplectomorphism of $\mathbb{R}^2$ whose Jacobian matrix has constant length. The converse follows from the equivalences above. □

9.1. **Construction of local symplectomorphisms.** We prove the following existence result:

Theorem 9.5. *There exist non linear local symplectomorphisms of $\mathbb{R}^2$ whose Jacobian matrix has constant length. Thus, there exist non totally geodesic surfaces with different constant principal angles.*

As explained above we have to show the existence of non linear solutions of the system

$$\begin{cases} f_x^2 + g_x^2 + f_y^2 + g_y^2 &= c_1, \\ f_x g_y - f_y g_x &= 1 \end{cases} \tag{16}$$

where $c_1 \neq 2$ (if $c_1 = 2$, by Corollary 9.2 the solutions are linear functions).

Assume that f is a (non constant) known function and set $A = g_x, B = g_y$. Then from the second equation in (16) there exists a function λ such that

$$A = \frac{-f_y + \lambda f_x}{f_x^2 + f_y^2} \qquad \text{and} \qquad B = \frac{f_x + \lambda f_y}{f_x^2 + f_y^2}.$$

Using the first equation in (16), λ satisfies

$$\frac{1 + (f_x^2 + f_y^2)^2 + \lambda^2}{f_x^2 + f_y^2} = c_1,$$

and setting $\Delta := f_x^2 + f_y^2$, this last equation reads

$$\lambda^2 = c_1\Delta - 1 - \Delta^2 .$$

Remark 9.6. Notice that $c_1 \geq 2$, and that $c_1 = 2$ implies $\lambda \equiv 0$ and $\Delta \equiv 1$. So if $\lambda \equiv 0$ and $\Delta \equiv 1$ then f is harmonic with a gradient of constant length. Then it is not difficult to show (using the theory of complex functions) that f must be linear. This gives another proof of Corollary 9.2.

Remark 9.7. The norm of the gradient Δ is necessarily bounded by

$$\frac{c_1 - \sqrt{c_1^2 - 4}}{2} \leq \Delta \leq \frac{c_1 + \sqrt{c_1^2 - 4}}{2}.$$

Assume that $c_1 > 2$. Then f is a solution of the following second order quasi-linear PDE:

$$\left(\frac{-f_y + \sqrt{c_1\Delta - 1 - \Delta^2} f_x}{\Delta}\right)_y = \left(\frac{f_x + \sqrt{c_1\Delta - 1 - \Delta^2} f_y}{\Delta}\right)_x . \tag{17}$$

Reciprocally, if f is a non linear solution of the above equation then it is possible to construct g such that $F(x, y) = (f(x, y), g(x, y))$ is a (non linear) symplectomorphism whose Jacobian matrix has constant length. The existence of such symplectomorphism, i.e. the proof of Theorem 9.5, follows from the following theorem.

Theorem 9.8. *Let*

$$\mathrm{L}f = A(f_x, f_y)f_{xx} + B(f_x, f_y)f_{xy} + C(f_x, f_y)f_{yy} + E(f_x, f_y)$$

be a second order quasi-linear operator with analytic coefficients A, B, C, E defined in some open subset $\Omega \subset \mathbb{R}^2$. Then there exists a non linear function $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ such that $\mathrm{L}f = 0$.

Proof. We can apply the Cauchy-Kowalewski theorem to solve the equation $\mathrm{L}f = 0$ as soon as we can find non characteristic real analytic initial data (see [15, p. 56] for details). Let us set, as it is standard, $p = f_x, q = f_y$. Let g be the quadratic differential form defined on $\Omega \subset \mathbb{R}^2$ by

$$g = Adq^2 + Bdpdq + Cdp^2 .$$

Notice that g is not identically zero since L is a differential operator of second order. So we can find an analytic vector field V such that

$$g(V, V) \neq 0$$

around a point $I_0 = (p_0, q_0) \in \Omega$. We can also consider (p_0, q_0) as a point in the (x, y) plane. So $V(x, y)$ is also a vector field around (p_0, q_0) in the plane (x, y). It is not difficult to see that there exists a non constant analytic curve $\gamma(t)$ such that $\gamma(0) = (p_0, q_0)$ and $\langle \gamma'(t), V(\gamma(t))\rangle = 0$, where $\langle , \rangle$ is the standard scalar product of $\mathbb{R}^2$, i.e. such that V is normal to γ. Consider the following initial conditions on $\gamma(t)$ for the Cauchy problem for the quasi-linear PDE:

$$f(t) = \frac{\langle \gamma(t), \gamma(t)\rangle}{2} \qquad \text{and} \qquad \frac{\partial f}{\partial V}(t) = \langle \gamma(t), V(\gamma(t))\rangle.$$

Then these initial conditions are analytic and non characteristic. Indeed, the condition $g(V, V) \neq 0$ holds for the initial data $(f(t), \frac{\partial f}{\partial V}(t))$ (since $(\nabla f)|_{\gamma(t)} = \gamma(t)$) and this is exactly the condition on the initial data to be non characteristic. Thus we can apply the Cauchy-Kowalewski theorem to get a solution f around (p_0, q_0). Moreover ∇f is not constant since $(\nabla f)|_{\gamma(t)} = \gamma(t)$. □

10. Existence of surfaces without geodesic principal directions

Here we show that there exist surfaces with constant principal angles such that both vector fields T_1 or T_2 are not geodesic.

10.1. **The rank of the first normal space of a graph.** Assume that Σ is parametrized by $F(x,y) = (x, y, f(x,y), g(x,y))$. The tangent space at the point $F(x,y)$ is generated by the vectors

$$F_x = (1, 0, f_x, g_x) \qquad \text{and} \qquad F_y = (0, 1, f_y, g_y),$$

and the normal space by the vectors

$$N_1 = (-f_x, -f_y, 1, 0) \qquad \text{and} \qquad N_2 = (-g_x, -g_y, 0, 1).$$

The first normal space of the graph is generated by the normal vectors

$$\alpha(\partial_x, \partial_x) = (F_{xx})^\perp, \qquad \alpha(\partial_x, \partial_y) = (F_{xy})^\perp \quad \text{and} \quad \alpha(\partial_y, \partial_y) = (F_{yy})^\perp,$$

where $(\cdot)^\perp$ means the normal component.

Lemma 10.1. *The first normal space of the graph $F(x,y)$ has rank one if and only if the matrix*

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{yy} \\ g_{xx} & g_{xy} & g_{yy} \end{pmatrix}$$

has rank one.

Proof. Writing

$$(F_{xx})^\perp = A_{xx}N_1 + B_{xx}N_2, \qquad (F_{xy})^\perp = A_{xy}N_1 + B_{xy}N_2$$

and

$$(F_{yy})^\perp = A_{yy}N_1 + B_{yy}N_2,$$

by straightforward computations we get

$$\begin{pmatrix} \langle F_{xx}, N_1\rangle & \langle F_{xy}, N_1\rangle & \langle F_{yy}, N_1\rangle \\ \langle F_{xx}, N_2\rangle & \langle F_{xy}, N_2\rangle & \langle F_{yy}, N_2\rangle \end{pmatrix} = G. \begin{pmatrix} A_{xx} & A_{xy} & A_{yy} \\ B_{xx} & B_{xy} & B_{yy} \end{pmatrix}$$

where

$$G = \begin{pmatrix} \langle N_1, N_1\rangle & \langle N_1, N_2\rangle \\ \langle N_1, N_2\rangle & \langle N_2, N_2\rangle \end{pmatrix}$$

is the Gram matrix of N_1, N_2. Since the first normal space of the graph $F(x,y)$ has rank one if and only if the matrix

$$\begin{pmatrix} A_{xx} & A_{xy} & A_{yy} \\ B_{xx} & B_{xy} & B_{yy} \end{pmatrix}$$

has rank one, and since the Gram matrix G is invertible, the first normal space of the graph $F(x,y)$ has rank one if and only if the matrix

$$\begin{pmatrix} \langle F_{xx}, N_1\rangle & \langle F_{xy}, N_1\rangle & \langle F_{yy}, N_1\rangle \\ \langle F_{xx}, N_2\rangle & \langle F_{xy}, N_2\rangle & \langle F_{yy}, N_2\rangle \end{pmatrix} = \begin{pmatrix} f_{xx} & f_{xy} & f_{yy} \\ g_{xx} & g_{xy} & g_{yy} \end{pmatrix}$$

has rank one. □

Lemma 10.2. *Let $f, g : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ be two analytic functions, where Ω is open and connected, such that the matrix*

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{yy} \\ g_{xx} & g_{xy} & g_{yy} \end{pmatrix}$$

has rank one for all point in Ω. Assume that f is not an affine function. Then one of the following conditions holds:

- *The functions $f, g, 1$ are linearly dependent over $\mathbb{R}$;*
- *The determinant of the Hessian of f vanishes identically.*

Proof. Notice that the matrix

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{yy} \\ g_{xx} & g_{xy} & g_{yy} \end{pmatrix}$$

has rank one if and only if there exists a function $M : \Omega \to \mathbb{R}$ such that

$$\begin{cases} M df_x &= dg_x \\ M df_y &= dg_y. \end{cases}$$

If M is constant in some open subset of Ω then $f, g, 1$ are linearly independent over $\mathbb{R}$. So assume that M is not constant. Taking the exterior derivative we get

$$\begin{cases} dM \wedge df_x &= 0 \\ dM \wedge df_y &= 0. \end{cases}$$

Since $dM \neq 0$ we get $df_x \wedge df_y \equiv 0$, which implies that the determinant of the Hessian of f vanishes identically. □

10.2. **The existence revisited.** Recall from Section 9 that to construct a graph with constant principal angles we need a solution of the partial differential equation

$$\left(\frac{-f_y + \sqrt{c_1 \Delta - 1 - \Delta^2} f_x}{\Delta} \right)_y = \left(\frac{f_x + \sqrt{c\Delta - 1 - \Delta^2} f_y}{\Delta} \right)_x \tag{18}$$

where $\Delta := f_x^2 + f_y^2$. We are going to show that there exist analytical solutions f with non vanishing determinant $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$. According to Lemmas 10.1 and 10.2, this will imply that the first normal spaces of the graph of (f, g) have rank 2 (the function g is arbitrary). Finally, Proposition 6.1 yields that in this example neither of the two vector fields T_1 or T_2 is geodesic.

So we are going to prove the following theorem and its corollary.

Theorem 10.3. *There exist surfaces $\Sigma \subset \mathbb{R}^4$ with constant principal angles $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ such that their first normal spaces have rank 2. For such a surface the vector fields T_1 and T_2 are not geodesic.*

Corollary 10.4. *There exist surfaces $\Sigma \subset \mathbb{R}^4$ with constant principal angles $0 < \theta_1 < \theta_2 < \frac{\pi}{2}$ which are not compositions in the sense of Do Carmo-Dajczer.*

Proof. As explained above it is enough to show the existence of a solution f of the PDE (18) with non vanishing determinant $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}$. Let $E(u,v)$ be the analytic function defined by

$$E(u,v) := \frac{u + \sqrt{c_1\Delta - 1 - \Delta^2}v}{\Delta}$$

where $\Delta := u^2 + v^2$; equation (18) thus reads

$$E(-f_y, f_x)_y = E(f_x, f_y)_x,$$

or

$$E_u(f_x, f_y)f_{xx} + (E_v(f_x, f_y) - E_v(-f_y, f_x))f_{xy} + E_u(-f_y, f_x)f_{yy} = 0.$$

We look for a solution of this equation with the initial conditions

$$f(x,0) = \phi(x) \qquad \text{and} \qquad f_y(x,0) = \psi(x),$$

where ψ and ϕ are two functions such that

$$E_u(\phi'(x), \psi(x)) \neq 0, \quad E_v(-\psi(x), \phi'(x)) \neq 0 \quad \text{and} \quad \phi''(x) \neq 0. \tag{19}$$

This means that the initial conditions are not characteristics; thus an analytic solution f of equation (18) does exist by the Cauchy-Kowalewski theorem. Moreover, since $f_{xy}(x,0) \equiv 0$ and $f_{xx}(x,0).f_{yy}(x,0) \neq 0$ we see that the Hessian of f does not vanish and thus that the rank of the first normal space is equal to 2. In particular the graph is not totally geodesic.

We finally prove the existence of ψ and ϕ such that conditions (19) hold. Notice that the function $E(u,v)$ is defined in the annulus $\mathcal{A} = \{(u,v) : \frac{c_1 - \sqrt{c_1^2 - 4}}{2} \leq \Delta \leq \frac{c_1 + \sqrt{c_1^2 - 4}}{2}\}$, where it is moreover analytic. Also notice that E_u and E_v do no vanish identically. Let $\mathcal{B} \subset \mathcal{A}$ be the subset such that either $E_u(u,v) = 0$ or $E_v(-v,u) = 0$. Then $\mathcal{B} \neq \mathcal{A}$ and since $\mathcal{B}$ is closed there exists an open disc $D \subset \mathcal{A}$ such that $\mathcal{B} \bigcap D = \emptyset$. If u_0, v_0 are the coordinates of the center of D, the functions $\phi(x) = x^2 + xu_0$ and $\psi(x) = x + v_0$ for small values of x satisfy the system of initial conditions (19). □

10.3. **Deformations.** Here we explain how to use a solution f of the PDE (18) in order to locally construct a surface with constant principal angles θ_1, θ_2. Actually, we will show that a solution f produces a one parameter *deformation* $\Sigma_m \subset \mathbb{R}^4$ of flat surfaces with constant principal angles.

Let f be a solution of the PDE (18). Then there exists g such that

$$g_x = \frac{-f_y + \sqrt{c_1\Delta - 1 - \Delta^2}f_x}{\Delta} \qquad \text{and} \qquad g_y = \frac{f_x + \sqrt{c\Delta - 1 - \Delta^2}f_y}{\Delta};$$

so we have

$$\begin{cases} f_x^2 + g_x^2 + f_y^2 + g_y^2 & = & c\ , \\ f_x g_y - f_y g_x & = & 1. \end{cases}$$

Let $F_m(x,y) := (mf(x,y), mg(x,y))$, with $m \in \mathbb{R}$. As explained in Proposition 9.3, the graph of F_m is a surface with constant principal angles θ_1, θ_2 if and only if

$$\left\{ \begin{array}{rcl} 2 + m^2(f_x^2 + g_x^2 + f_y^2 + g_y^2) & = & \sec^2(\theta_1) + \sec^2(\theta_2) \\ 1 + m^2(f_x^2 + g_x^2 + f_y^2 + g_y^2) + m^4(f_x g_y - f_y g_x)^2 & = & \sec^2(\theta_1)\sec^2(\theta_2), \end{array} \right.$$

i.e. if and only if

$$\left\{ \begin{array}{rcl} 2 + m^2 c & = & \sec^2(\theta_1) + \sec^2(\theta_2) \\ 1 + m^2 c + m^4 & = & \sec^2(\theta_1)\sec^2(\theta_2). \end{array} \right.$$

Thus the graph of F_m is a surface Σ_m with constant principal angles

$$\left\{ \begin{array}{rcl} \theta_1 & = & \operatorname{arcsec}\sqrt{1 + \frac{m^2}{2}(c - \sqrt{c^2 - 4})} \\ \theta_2 & = & \operatorname{arcsec}\sqrt{1 + \frac{m^2}{2}(c + \sqrt{c^2 - 4})}\,. \end{array} \right.$$

It is not difficult to see that the map $(m,c) \to (\theta_1, \theta_2)$ from $(0, +\infty) \times (2, +\infty)$ into $\Delta = \{(\theta_1, \theta_2) : 0 < \theta_1 < \theta_2 < \frac{\pi}{2}\}$ is surjective. This shows how to construct the surface Σ_m with constant principal angles θ_1, θ_2 by starting with a solution f of the PDE (18).

11. Surfaces with parallel mean curvature

Here we classify surfaces $\Sigma \subset \mathbb{R}^4$ having parallel mean curvature vector field H. Recall that the mean curvature vector H is the normal vector field given by

$$\mathrm{H} := \frac{\alpha(T_1, T_1) + \alpha(T_2, T_2)}{2}.$$

We have the following result.

Theorem 11.1. *Let $\Sigma \subset \mathbb{R}^4$ be a surface with constant principal angles. The mean curvature vector* H *is parallel if and only if Σ is a product. Hence, up to a rigid motion, Σ is either an open subset of a torus $S_1(r_1) \times S_2(r_2) \subset \mathbb{R}^2 \times \mathbb{R}^2 \subset \mathbb{R}^4$ or an open subset of the cylinder $\mathbb{R} \times \gamma \subset \mathbb{R} \times \mathbb{R}^3 \subset \mathbb{R}^4$, where $\gamma \subset \mathbb{R}^3$ is an helix, i.e. a curve with constant curvature and torsion.*

Proof. It is enough to show that T_1 and T_2 are geodesic vector fields. Indeed, since $\alpha(T_1, T_2) \equiv 0$, Moore's Lemma implies that Σ is a local product. The vector field H is parallel if and only if

$$dm_2 = -m_1 dn \qquad \text{and} \qquad dm_1 = m_2 dn. \tag{20}$$

If m_1 and m_2 are constant functions then the Codazzi equations imply that both T_1, T_2 are geodesic vector fields. So we can assume that m_1 or m_2 is not constant. Observe that the above equations imply that if one is not constant so is the other. So we have that both functions m_1 and m_2 are not

constant. Now we have

$$\begin{aligned} dt(T_1) &= \frac{dm_2(T_2)}{m_2} && \text{by Codazzi equation (C3)} \\ &= \frac{-m_1 dn(T_2)}{m_2} && \text{by the first equation in (20)} \\ &= \frac{-m_1 m_1 dt(T_1)}{m_2^2} && \text{by Codazzi equation (C2)}\,. \end{aligned}$$

Thus $dt(T_1) \equiv 0$ i.e. T_1 is a geodesic vector field. In a similar way we have

$$\begin{aligned} dt(T_2) &= \frac{-dm_1(T_1)}{m_1} && \text{by Codazzi equation (C1)} \\ &= \frac{-m_2 dn(T_1)}{m_1} && \text{by the second equation in (20)} \\ &= \frac{-m_2 m_2 dt(T_2)}{m_1^2} && \text{by Codazzi equation (C4)}\,. \end{aligned}$$

Thus $dt(T_2) \equiv 0$ i.e. T_2 is a geodesic vector field. □

References

[1] J. Berndt, S. Console and C. Olmos, *Submanifolds and holonomy,* Chapman & Hall/CRC , Research Notes in Mathematics 434 (2003).

[2] B. Y. Chen, *Geometry of submanifolds*, Pure and Applied Mathematics, **22** (1973). MR0353212

[3] B. Y. Chen, *On the surface with parallel mean curvature vector*, Indiana Univ. Math. J. **2** (1973), 655-666.

[4] M. Dajczer and R. Tojeiro, *Submanifolds with nonparallel first normal bundle,* Canad. Math. Bull. **37:3** (1994), 330-337.

[5] M. Dajczer and R. Tojeiro, *On flat surfaces in space forms,* Houston J. Math. **21:2** (1995), 319-338.

[6] M. Dajczer and M. do Carmo, *Local isometric immersions of* $\mathbb{R}^2$ *into* $\mathbb{R}^4$*,* J. Reine Angew. Math. **442** (1993), 205-219.

[7] A. J. Di Scala and C. Olmos, *Submanifolds with curvature normals of constant length and the Gauss map,* J. Reine Angew. Math. **574** (2004), 79-102.

[8] A. Di Scala, *Weak helix submanifolds of euclidean spaces*, Abh. Math. Semin. Univ. Hambg. **79** (2009), 37-46.

[9] A. Di Scala and G. Ruiz-Hernández, *Helix submanifolds of euclidean space*, Monatsh. Math. **157** (2009), 205-215.

[10] A. Di Scala and G. Ruiz-Hernández, *Higher codimensional Euclidean helix submanifolds*, Kodai Math. J. **33:2** (2010), 192-210.

[11] F. Dillen and M. I. Munteanu, *Constant Angle Surfaces in* $\mathbb{H}^2 \times \mathbb{R}$*,* Bull. Braz. Math. Soc. **40:1** (2009), 85-97.

[12] F. Dillen, J. Fastenakels, J. Van der Veken and L. Vrancken, *Constant Angle Surfaces in* $\mathbb{S}^2 \times \mathbb{R}$*,* Monatsh. Math. **152:2** (2007), 89-96. MR2346426

[13] A. Gray, E. Abbena, S. Salamon, *Modern Differential Geometry of CURVES and SURFACES with Mathematica ,* CRC Press, (2006) .

[14] S. Jiang, R. *Angles between Euclidean subspaces*, Geometriae Dedicata **63:2** (1996), 113-121.

[15] F. John, *Partial Differential Equations,* Fourth Edition, Applied Mathematical Sciences, Springer-Verlag (1982).

[16] C. Jordan, *Essais sur la Géométrie à n Dimensions* , Bull. Soc. Math. France **3**, 103-174 (1875).

[17] M. I. Munteanu and A. I. Nistor, *A new approach on constant angle surfaces in* $\mathbb{R}^3$, Turkish J. Math. **33:2** (2009), 169-178.

[18] J. L. Weiner, *The Gauss map for surfaces in 4-Space*, Math. Ann. **269** (1984), 541-560.

Authors' Addresses:
P. Bayard,
Instituto de Física y Matemáticas, Universidad Michoacana,
Edif. C-3, Cd. Universitaria, C.P. 58040 Morelia, Mich. México
email: bayard@ifm.umich.mx

A. J. Di Scala,
Dipartimento di Matematica, Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
email: antonio.discala@polito.it

O. Osuna Castro,
Instituto de Física y Matemáticas, Universidad Michoacana,
Edif. C-3, Cd. Universitaria, C.P. 58040 Morelia, Mich. México
email: osvaldo@ifm.umich.mx

G. Ruiz-Hernández,
Instituto de Matemáticas, Universidad Nacional Autonoma de México,
Ciudad Universitaria, C.P. 04510 D.F. México
email: gruiz@matem.unam.mx