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Topological invariants of bifurcation

Jacobo Pejsachowicz

Abstract. I will shortly discuss an approach to bifurcation theory based on elliptic topology. The main goal is a construction of an index of bifurcation points for C^1-families of Fredholm maps derived from the index bundle of the family of linearizations along the trivial branch. As illustration, I will present an application to bifurcation of homoclinic solutions of non-autonomous differential equations from a branch of stationary solutions.

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1. Introduction

The classical topological approach to bifurcation of zeroes of parametrized families of maps is essentially of local nature [13, 14, 2, 3]. Sufficient condition for bifurcation are obtained by analyzing the behavior of the linearized family in a small enough neighborhood of an isolated point of the trivial branch at which the linearization fails to be invertible. Here, instead, I would like to discuss an alternative, non-local approach based on elliptic topology. I will consider families of C^1-Fredholm maps of index 0 parametrized by topologically nontrivial spaces and will use the non-vanishing of a global invariant associated to the family of linearizations in order to find on the trivial branch at least one bifurcation point. This kind of argument, applied to families of linear Fredholm operators, was successfully used in many places (see for example [12], [23] among others). My point here is that, after adding one more tool, essentially the same method works for nonlinear Fredholm maps as well. Although some of the results stated here were already proved for families of Fredholm maps of special type in [20, 19], the general case is an ongoing work and full details will appear in [21].

In what follows I will describe more precisely what I mean by bifurcation from the trivial branch and the topological invariants under consideration.

Let X, Y be Banach spaces and let P be an n-dimensional compact connected smooth manifold. Let $f\colon P \times \mathcal{O} \to Y$ be a continuously differentiable map defined on the product of P with an open neighborhood $\mathcal{O}$ of the origin in X. Assume that $f(p,0) = 0$ for all p in P. Solutions of the equation $f(p,x) = 0$ of the form $(p,0)$ are called *trivial* and the set $P \times \{0\}$ is called the *trivial branch.* In what follows I will identify the parameter space P with the set of trivial solutions and will write the parameter variable as a subscript. Accordingly I will denote by $f_p\colon U \to Y$ the map defined by $f_p(x) = f(p,x)$.

A *bifurcation point* for solutions of the equation $f(p,x) = 0$ is a point p_* in P such that every neighborhood of $(p_*,0)$ contains nontrivial solutions of this equation.

Let $L_p = Df_p(0)$ be the Frechet derivative of the map f_p at 0. The map L sending $p \in P$ to L_p is called the *family of linearizations along the trivial branch.* By the Implicit Function Theorem, bifurcation cannot occur at points where the operator L_p is an isomorphism. However, in general, the set $Bif(f)$ of bifurcation points of f is only a proper closed subset of the set $\Sigma(L) = \{p \in P \mid L_p \text{ is singular}\}$. Assuming that L_p is a Fredholm operator of index 0 for all $p \in P$, sufficient conditions for the existence of bifurcation points can be obtained from homotopy invariants of the family of linearizations along the trivial branch.

Since bifurcation arises only at points of $\Sigma(L)$, the first invariant that comes in mind is the obstruction to the existence of a homotopy deforming the family L into a family of isomorphisms. This obstruction is given by an element $Ind\,L$ of the reduced Grothendieck group of virtual vector bundles $\widetilde{KO}(P)$, called *family index* or *index bundle* [4, 16]. However, in dealing with nonlinear perturbations of the family L one has to consider a stronger invariant and, quite naturally, our bifurcation invariant is not $Ind\,L$ but rather its image $J(Ind\,L)$ under the generalized J-homomorphism which associates to each vector bundle the stable fiberwise homotopy class of its unit sphere bundle.

Our main result asserts that if $J(Ind\,L)$ does not vanish and $\Sigma(L)$ is a proper subset of P then there exists at least one bifurcation point from the trivial branch for solutions of the equation $f(p,x) = 0$. The corresponding theorem together with some consequences and generalizations are stated in Section 2. Section 3 is devoted to a localized version of the basic invariant. To each admissible open subset U of the parameter space is assigned a bifurcation index $\sigma(f,U)$ belonging to the finite group $J(P)$ defined in [5] which gives a measure of the number of bifurcation points of f in U. It is related to the global invariant $J(Ind\,L)$ much in the same way as the local fixed-point index is related to the Lefschetz number and provides an interpolation between the global invariant and the Alexander-Ize invariant at isolated bifurcation points [2]. The precise relation with the Alexander -Ize invariant is discussed in Section 4. In Section 5 the previous theory is used in order to show how the topology of the parameter space leads to the appearance of homoclinic trajectories of non-autonomous differential equations emanating from a stationary solution.

Few comments to related work: one-parameter families of C^k-Fredholm mappings of index 0 were studied in [8, 18] among others. The well known Global Bifurcation Theorem of P.Rabinowitz was extended to one parameter families of C^1 Fredholm maps in [22] using an appropriate degree theory (see also [11]). For a special class of bifurcation problems involving Fredholm maps a different method was developed by Zvyagin in [24] using a device due to Ize. Krasnosel'skij-Rabinowitz theory was carried to the setting of several-parameter families of compact perturbations of identity mainly by the work of Alexander and Ize [2, 3, 13, 14, 10]. The review paper [15] contains a complete reference list for this topic. In [6, 7] a different approach to local bifurcation index in the semilinear case was developed by Bartsch.

2. The Main Result

Let us recall the definition of the index bundle. I will use here a construction slightly different from the one given by Atiyah in [4] but, of course, both approaches give the same element in K-theory.

If P is a compact space, the Grothendieck group $KO(P)$ is the group completion of the abelian semigroup Vect (P) of all isomorphisms classes of vector bundles over P. As a group $KO(P) = \text{Vect}\,(P) \times \text{Vect}\,(P)/\Delta$ where Δ is the diagonal sub-semigroup. The elements of $KO(P)$ are called virtual bundles. Each virtual bundle can be written as a difference $[E]-[F]$, where $[E]$ is the equivalence class of $(E, 0)$.

Let X, Y be real Banach spaces, let $\Phi(X, Y)$ be the space of all Fredholm operators. With $\Phi_k(X, Y)$ I will denote the space of operators of index k. Given a continuous family $L\colon P \to \Phi(X, Y)$ of Fredholm operators parametrized by a compact topological space P, using compactness of P one can find a finite dimensional subspace V of Y transverse to the family L i.e., such that

$$Im\, L_p + V = Y \text{ for any } \; p \in P \tag{2.1}$$

It follows from 2.1 that the finite dimensional spaces $E_p = L_p^{-1}(V)$ are fibers of a vector bundle E over P. By definition the index bundle

$$Ind\, L = [E] - [\Theta(V)] \in KO(P),$$

where $\Theta(V) = P \times V$ denotes the trivial vector bundle over P with fiber V. That the above virtual bundle is independent from the choice of V follows easily from the identity $[E] - [F] = [E \oplus H] - [F \oplus H]$, which holds in $KO(P)$.

It is easy to see that $Ind\, L$ depends only on the homotopy class of L. It vanishes whenever L can be deformed by a homotopy to a family of invertible operators. The index bundle is functorial under the change of parameter space and moreover it has the logarithmic property of the ordinary index. Namely, $Ind\,\big(LM\big) = Ind\, L + Ind\; M$.

I will need also the generalized J-homomorphism. Given a vector bundle E, let $S[E]$ be the sphere bundle with respect to some chosen scalar product on E.

Two vector bundles E, F are said to be *stably fiberwise homotopy equivalent* if for some n the sphere bundle $S[E \oplus \Theta(\mathrm{R}^n)]$ is fiberwise homotopy equivalent to the sphere bundle $S[F \oplus \Theta(\mathrm{R}^n)]$. Let $\widetilde{KO}(P)$ be the kernel of the rank homomorphism $rk\colon KO(P) \to \mathrm{Z}$ and let $T(P)$ be the subgroup of $\widetilde{KO}(P)$ generated by elements $[E] - [F]$ such that $S[E]$ and $S[F]$ are stably fiberwise homotopy equivalent. Put $J(P) = \widetilde{KO}(P)/T(P)$. The projection to the quotient $J\colon \widetilde{KO}(P) \to J(P)$ is the *generalized J-homomorphism*.

The groups $J(P)$ were introduced by Atiyah in [5] who also proved that $J(S^n)$ coincide with the image in π^s of the stable j-homomorphism. It follows from this that $J(P)$ is a finite group for any compact CW-complex P. Since Stiefel-Whitney characteristic classes can be obtained from the Thom class using Steenrod squares they depend only on the stable fiber homotopy type of the associated sphere bundle and hence are well defined on elements of $J(P)$.

Theorem 2.1. [21] *Let P be a compact connected orientable n-dimensional manifold and let $f\colon P \times \mathcal{O} \to Y$ be a C^1-family of Fredholm maps of index 0 parametrized by P such that $f(p, 0) = 0$. Assume that the linearization $L_p = Df_p(0)$ at the points of the trivial branch is nonsingular at some point $p_0 \in P$ and that $J(Ind\, L) \neq 0$ in $J(P)$, then the family f possesses at least one bifurcation point from the trivial branch.*

The next theorem uses Stiefel-Whitney classes in order to estimate the covering dimension of the set of bifurcation points.

Theorem 2.2. [21, 6] *Let $f : P \times X \to Y$ be as in Theorem 2.1 and let $m = min\{k/\ \omega_k(Ind\, L) \neq 0\}$, then the Lebesgue covering dimension of the set $Bif(f)$ of all bifurcation points of f is at least $n - m$.*

The proof uses the previous theorem and Poincare duality.

Remark 2.3. For $P = S^1$ this reduces to the bifurcation theorems proved in [11] and [22] by other means. However, in [11, 22] was proved that the bifurcating branch is global.

In the remaining part of the paper, except when differently stated, Fredholm means Fredholm of index 0.

Theorem 2.1 is a particular case of a slightly more general result. In order to formulate it I will need the degree theory constructed in [22]. The construction of the degree in [22] is based on a homotopy invariant of paths of Fredholm operators called parity. Given a path $L\colon [a, b] \to \Phi_0(X, Y)$ with invertible end points and transverse to the one-codimensional analytic variety Σ of all non-invertible Fredholm operators, its *parity* $\sigma(L) \in \mathrm{Z}_2$ is defined by $\sigma(L) = \#(L \cap \Sigma)$-mod 2. This definition can be extended to general paths with invertible end points using approximation by transversal paths (see [11] for this and for a different construction using parametrices).

Let $f\colon X \to Y$ be a C^1-Fredholm map of index 0 that is proper on closed bounded subsets. In order to assign to each regular point of the map f an orientation $\epsilon(x) = \pm 1$, with properties analogous to the sign of the Jacobian determinant in finite dimensions, we choose a fixed regular point b called *base point* and define "ad arbitrium" $\epsilon(b) = \pm 1$. With this said, the multiplicity $\epsilon(x)$ at any regular point x is uniquely defined by the requirement $\epsilon(x) = (-1)^{\sigma(Df\circ\gamma)}\,\epsilon(b)$, where γ is any path in X joining b to x. The independence from the choice of the path follows from the homotopy invariance of the parity. If Ω is an open bounded subset of X such that $0 \notin f(\partial\Omega)$ and is a regular value of the restriction of f to Ω, then the *base point degree of* f is defined by $\deg_p(f,\Omega,0) = \sum_{x\in f^{-1}(0)} \epsilon(x)$.

It was proved in [22] that when 0 is not a regular value of f the degree can still be defined using approximation (although not by regular values since the Sard -Smale theorem does not extend to C^1-Fredholm maps of index 1). This assignment defines an integral-valued degree theory for C^1-Fredholm maps which are proper on closed bounded sets. The base point degree is invariant under homotopies only up to sign and, as a matter of fact, no degree theory for general Fredholm maps can be homotopy invariant. However, the change in sign can be determined as follows: let $h\colon I \times X \to Y$ be a homotopy and let Ω be an open bounded subset of X such that $0 \notin h([0,1]\times\partial\Omega)$. Assume(for simplicity) that b is a regular point both of h_0 and h_1, then

$$\deg_b(h_0,\Omega,0) = (-1)^{\sigma(H)}\,\deg_b(h_1,\Omega,0), \tag{2.2}$$

where H is the path $t \to Dh_t(b)$.

If f is a C^1-Fredholm map and x_0 is an isolated but necessarily regular zero its *multiplicity* $m(f,x_0)$ is defined by $m(f,x_0) = |\deg_b(f, B(x_0,\delta),0)|$, where $B(x_0,\delta)$ open ball centered at x_0 and small enough radius δ and b is any regular point of f. Notice that properness need not be assumed since all Fredholm maps are locally proper. Finally, let us denote by $\mathrm{Z}[\frac{1}{m}]$ the ring of all rational numbers whose denominator is a power of m.

Theorem 2.4. *Let P be as in* 2.1*, let $\mathcal{O} = B(0,\delta)$ be an open ball in X and let $f : P\times\mathcal{O} \to Y$ be a C^1-family of Fredholm maps parametrized by P. Assume that the only solutions of $f(p,x) = 0$ are those of the form $(p,0)$. Suppose moreover that for some (and hence all) $q \in P$ we have that $m = m(f_q,0) \neq 0$, then*

i) *the index bundle $Ind\,L$ is orientable.*
ii) *$J(Ind\,L) = 0$ in $J(P)\otimes \mathrm{Z}[\frac{1}{m}]$.*

Theorem 2.1 follows the above theorem with $m = 1$.

Sketch of proof: For the first claim one must show that $\omega_1(Ind\,L)$ vanishes. For this it is enough to check that for any closed path with $\gamma(0) = q = \gamma(1)$, $<\omega_1(Ind\,L);\gamma_*[S^1]> = <\omega_1(Ind\,L\circ\gamma);[S^1]> = 0$.

By Proposition 2.7 of [9], $<\omega_1(Ind\,L\circ\gamma),[S^1]> = \sigma(L\circ\gamma)$. Consider the homotopy $h(t,x) = f(\gamma(t),x)$ and choose a regular base point $b \in \mathcal{O}$ for f_q (there must be at least one since $m \neq 0$). The parity of the path $t \to Dh_t(b)$ equals

$\sigma(L\circ\gamma)$. On the other hand, since there are no zeroes of h on $I\times\partial\mathcal{O}$ we can apply the homotopy property (2.2) of the base point degree from which we obtain that $\sigma(L\circ\gamma)=0$ being $m\neq 0$. This proves the first claim.

The proof of the second claim is roughly speaking as follows: using a modified version of the Caccioppoli reduction one shows that the zero-set of the map f coincides with the zero-set of a map $\bar f$ defined on a finite dimensional fiber bundle M over P with values in R^s and such that $\bar f$ has degree $\pm m$ on each fiber. By construction, the bundle E of tangents to the fibers of M at the points of the trivial branch represents the index bundle. Composing $\bar f$ with the fiberwise exponential map produces a map g from the sphere bundle $S(E)$ to S^{s-1} of degree $\pm m$ on each fiber. With this, the second assertion follows from the first and the mod-k Dold's theorem of Adams [1].

Corollary 2.5. *If $J(Ind\,L)\neq 0$ and for some $q\in P$, the multiplicity $m=m(f_q,0)$ is defined and is prime to the order of $J(P)$, then $Bif(f)\neq\emptyset$.*

Proof. Assume that there are no bifurcation points. By $ii)$ of theorem 2.4 for some k, $m^kJ(Ind\,L)=0$. Hence the order of $J(Ind\,L)$, divides both m^k and the order of $J(P)$. □

3. The Local Bifurcation Index

In this section I will assume that the range of the family, Y is a Kuiper space, i.e., that $GL(Y)$ is contractible. Let U be an open subset of a compact connected manifold P and let $f\colon U\times X\to Y$ be a family of C^1-Fredholm maps parametrized by U such that $f(p,0)=0$ for any $p\in U$. The pair (f,U) will be called *admissible* if the singular set $\Sigma(L)$ of the family L of linearizations along the trivial branch is a compact, proper subset of U.

Theorem 3.1. *There exists a function assigning to each admissible pair (f,U) an element $\sigma(f,U)\in J(P)$, called* bifurcation index *, verifying the following properties:*

$\mathcal{P}1)$ Existence- *If $\sigma(f,U)\neq 0$ then the family f has a bifurcation point.*
$\mathcal{P}2)$ Normalization- *If $U=P$ then $\sigma(f,U)=J(Ind\,L)$.*
$\mathcal{P}3)$ Homotopy invariance- *Let $h\colon[0,1]\times U\times X\to Y$ be a C^1-Fredholm map of index 1 such that the set $\{(t,p)/Dh_{(t,p)}(0)$ is singular $\}$ is compact, then $\sigma(h_0,U)=\sigma(h_1,U)$.*
$\mathcal{P}4)$ Additivity- *Let $U\subset\bigcup U_i$. Put $f_i=f|_{U_i}$ and $\Sigma_i=\Sigma(f)\cap U_i$. If (f_i,U_i) are admissible and $\Sigma_i\cap\Sigma_j=\emptyset$, then $\sigma(f,U)=\sum_i\sigma(f_i,U_i)$.*

The construction of the local bifurcation index follows the approach of the previous Section. If Y is a Kuiper space, then $GL(X,Y)$, when nonempty, is an open contractible subset of a Banach space. By a theorem of Borsuk any continuous map with values in $GL(X,Y)$ defined on a closed subset of a metric space can be extended. I will use this fact in order to define for any family $L:U\to\Phi_0(X,Y)$

such that $\Sigma(L)$ is a compact subset of U a localized form of the index bundle $Ind(L,U)$ belonging to $\widetilde{KO}(P)$.

For this, let V be any neighborhood of $\Sigma(L)$ in U such that $\Sigma(L) \subset V \subset \bar{V} \subset U$. The restriction $L|_{\partial V}$ of L to the boundary of Vcan be extended to a family $L' \colon P - V \to GL(X,Y)$. Patching L' with L gives a family $\tilde{L}$ of Fredholm operators parametrized by P which coincides with L in a neighborhood of $\Sigma(L) = \Sigma(\tilde{L})$. It is easy to see that $Ind(\tilde{L})$ is independent of the choice of V and the extension. The *index bundle* of the family L on U is defined by $Ind(L,U) = Ind(\tilde{L}) \in \widetilde{KO}(P)$. Now if (f,U) is admissible we define its bifurcation index $\sigma(f,U) \in J(P)$ by

$$\sigma(f,U) = J\,(Ind(L,U)) \tag{3.1}$$

The verification of $\mathcal{P}2 - \mathcal{P}4$ is quite standard. The existence property $\mathcal{P}1$ follows from Theorem 2.1 applied to an appropriate extension of the map f to $P \times X$. This is the only point where the assumption that Y is a Kuiper space is essential since $Ind(L,U)$ can be alternatively constructed via K-theory of locally compact spaces.

Remark 3.2. It follows easily from the results in [8] that if $P = S^1$, viewed as a one point compactification of the real line R and $U = (a,b)$, then the local index of bifurcation points $\sigma(f,U)$ coincides with the parity of the path L.

4. Comparison with the Alexander-Ize invariant

Now let us discuss the relation of the local bifurcation index with the Alexander-Ize invariant. I will consider here only the stable version defined in [2].

Let $g \colon \mathrm{R}^k \times \mathrm{R}^n \to \mathrm{R}^n$ be a C^1-family of maps, parametrized by R^k, such that $g_p(0) = 0$. Assume that p_0 is an isolated point in the set $\Sigma(L)$. The homotopy class of the restriction of L to the boundary of a small closed disk D centered at p_0 defines an element in the homotopy group $\pi_{k-1}(GL(n))$. Stabilizing this element through the inclusion of $GL(n)$ into $GL(m)$, $n \le m$, one obtains the Alexander-Ize invariant γ_g belonging to the homotopy group $\pi_{k-1}(GL(\infty))$. Let $\pi^s_{k-1}(S^0) = \lim_{m\to\infty} \pi_{m+k-1}(S^m)$ be the $(k-1)-$stable homotopy group of S^0. In [2] it is shown that the point p_0 is a bifurcation point of f provided the image of γ_f by the classical j-homomorphism $j \colon \pi_{k-1}(GL(\infty)) \to \pi^s_{k-1}(S^0)$ does not vanish in $\pi^s_{k-1}(S^0)$.

In the remaining part of this Section I will take as parameter space the sphere S^n, viewed as one point compactification of R^n . The definition of γ_g can be easily extended to parametrized families of C^1-Fredholm maps.

For this, let $f \colon \bar{D} \times X \to Y$ be a C^2-Fredholm family such that $f(p,0) = 0$. Here $D = D(p_0,r) \subset \mathrm{R}^n$ be an open disk centered at p_0. Put $L_p = Df_p(0)$ and assume that $\Sigma(L) = \{p_0\}$. Being D contractible by Theorem 1.6.3 of [8], the family L has a parametrix. In other words, there exists a family of isomorphisms $A \colon \bar{D} \to GL(Y,X)$ such that $A_pL_p = Id + K_p$ for any p in $\bar{D}$, where $K \colon \bar{D} \to K(Y)$

is a family of operators such that the image of K_p is contained in a fixed finite dimensional subspace V of Y. Let T_p be the restriction of $Id + K_p$ to V. By the preceding discussion the family T defines an element γ_f in $\pi_{n-1}(GL(\infty))$ which can be easily shown to be independent from the choice of the parametrix.

Theorem 4.1. *If $f\colon \bar{D} \subset S^n \times X \to Y$, p_0 and D are as above, then, upon identification of the group $J(S^n)$ with $Im\ j$,*

$$\sigma(f, D) = j(\gamma_f). \tag{4.1}$$

From the above theorem and the computation of $j(\gamma_f)$ in terms of the n-th Radon-Hurwitz number c_n obtained in [3] it follows:

Corollary 4.2. *Let $f\colon S^n \times X \to Y$ be a C^1 family of Fredholm maps such that $f(p, 0) = 0$. Assume that there exist $\epsilon > 0$, $\delta > 0$ such that $|L_p x| \geq \epsilon |p||x|$, for $|p| \leq \delta$. If $D = D(0, \delta)$, then, for $n \equiv 1, 2, 4, 8$ mod-8, dim $\ker L_0$ is divisible by c_n. Moreover if dim $\ker L_0 = k\, c_n$ with k an odd integer then $\sigma(f, D) \neq 0$ in $J(S^n)$.*

5. Bifurcation of homoclinic trajectories

This section is devoted to the application of the previous results to bifurcation of homoclinic solutions of systems of time dependent ordinary differential equations from the stationary solution.

Let $g\colon \Lambda \times \mathrm{R} \times \mathrm{R}^n \to \mathrm{R}^n$ be a smooth family of time dependent vector fields on R^n parametrized by a compact connected orientable manifold Λ of dimension m. I will assume that $g(\lambda, t, 0) = 0$, (thus $u(t) \equiv 0$ is a stationary solution of $u'(t) - g(\lambda, t, u(t)) = 0$) and I will look for conditions on the linearization of g_λ at $u \equiv 0$ which entails the appearance of nonvanishing (but close to zero) solutions to the problem:

$$\begin{cases} u'(t) - g(\lambda, t, u(t)) = 0, \\ \lim_{t\to\infty} u(t) = \lim_{t\to-\infty} u(t) = 0. \end{cases} \tag{5.1}$$

Nontrivial solutions of (5.1) are precisely the trajectories homoclinic to 0.

The linearization of (5.1) at 0 is

$$\begin{cases} u'(t) - A(\lambda, t)u(t) = 0, \\ \lim_{t\to\infty} u(t) = 0 = \lim_{t\to-\infty} u(t) \end{cases} \tag{5.2}$$

where $A(\lambda, t) = D_u g(\lambda, t, 0)$.

I will assume that g and $D_u g$ are bounded and that the following asymptotic condition holds true:

(A1) As $t \to \pm\infty$ the family $A(\lambda, t)$ converges, to a family of matrices $A(\lambda, \pm\infty)$, such that $A(\lambda, \pm\infty)$ has no eigenvalues on the imaginary axis.

As a consequence of $(A1)$, the map $\lambda \to A(\lambda, \pm\infty)$ is continuous and by perturbation theory [17] the projectors onto the real part of the spectral subspaces of $A(\lambda, \pm\infty)$ corresponding to the eigenvalues with negative (respectively positive) real part are continuous as well. It follows from this that the generalized eigenspaces $E^s(\lambda, \pm\infty)$ and $E^u(\lambda, \pm\infty)$ corresponding to the part of the spectrum of $A(\lambda, \pm\infty)$ on the left and right half plane respectively, are fibers of a pair of vector bundles $E^s(\pm\infty)$ and $E^u(, \pm\infty)$ over Λ which decompose the trivial bundle $\Theta(\mathrm{R}^n)$ with fiber R^n into a direct sum:

$$E^s(\pm\infty) \oplus E^u(\pm\infty) = \Theta(\mathrm{R}^n) \tag{5.3}$$

The bundles E^s, E^u are called *stable and unstable bundle* at $\pm\infty$. They can be alternatively described by

$$E^s(\lambda, \pm\infty) = \{v \in \mathrm{R}^n \mid \lim_{t\to\infty} e^{tA(\lambda,\pm\infty)} v \to 0\} \tag{5.4}$$

$$E^u(\lambda, \pm\infty) = \{v \in \mathrm{R}^n \mid \lim_{t\to-\infty} e^{tA(\lambda,\pm\infty)} v \to 0\} \tag{5.5}$$

My final assumption is

(A2) For some $\lambda_0 \in \Lambda$ both (5.2) and the adjoint problem

$$\begin{cases} u'(t) + A^*(\lambda_0, t)u(t) = 0, \\ \lim_{t\to\infty} u(t) = 0 = \lim_{t\to-\infty} u(t) \end{cases} \tag{5.6}$$

admit only the trivial solution $u \equiv 0$.

Let $\omega(E) = \omega_1(E) + ... + \omega_n(E)$ be the total Stiefel-Whitney class of E.

Theorem 5.1. *If the system* (5.1) *verifies* $(A1), (A2)$ *and if*

$$\omega(E^s(+\infty)) \neq \omega(E^s(-\infty),) \tag{5.7}$$

then, at some $\lambda_* \in \Lambda$, bifurcation of homoclinic trajectories from the stationary solution occurs. More precisely there is a sequence (λ_n, u_n) where $u_n \neq 0$ is solution of (5.1) *with $\lambda_n \to \lambda_*$ and $u_n \to 0$ in the space $C^1_0(\mathrm{R}; \mathrm{R}^n)$ of all C^1 functions vanishing at infinity together with its derivative.*

Moreover if $k = min\{i | \omega_i(E^s(+\infty)) \neq \omega_i(E^s(-\infty))\}$ then the set of all bifurcation points has dimension not less than $m - k$.

Proof. The space $H^1(\mathrm{R}; \mathrm{R}^{2n})$ of all absolutely continuous functions $u \in L^2(\mathrm{R}; \mathrm{R}^{2n})$ with square integrable derivative is a natural function space for our problem since any function $u \in H^1(\mathrm{R}; \mathrm{R}^{2n})$ has the property that $\lim_{t\to\pm\infty} u(t) = 0$. Let $X = H^1(\mathrm{R}; \mathrm{R}^{2n}), Y = L^2(\mathrm{R}; \mathrm{R}^{2n})$ and let us consider the family of maps $f \colon P \times X \to Y$ defined by

$$[f(\lambda, u)](t) = u'(t) - g(\lambda, t, u(t)) \tag{5.8}$$

Because of the continuous embedding of H^1 into $C(\mathrm{R})$ it follows that upon assumption $(A1)$ the map f is C^1 and such that $f(\lambda, 0) = 0$. Moreover the Frechet derivative $D_u f(\lambda, 0)$ is the operator $L_\lambda \colon X \to Y$ defined by

$$[L_\lambda u](t) = u'(t) - A(\lambda, t)u(t) \tag{5.9}$$

The next proposition shows that f is a C^1 Fredholm map of index 0 and computes the index bundle of the family L defined by (5.9) in terms of the asymptotic bundles.

Proposition 5.2. *The family L defined by* (5.9) *verifies*

i) $L_\lambda \in \Phi_0(X,Y)$ *for all* $\lambda \in \Lambda$

ii) $Ind\, L = [E^s(+\infty)] - [E^s(-\infty)] \in \widetilde{KO}(\Lambda)$

Proof. Let us split R into $\mathrm{R} = \mathrm{R}^+ \cup R^-$ with $R^\pm = [0, \pm\infty]$ and denote with $X^\pm, Y^\pm$ the spaces $H^1(\,R_\pm;\mathrm{R}^n)$ and $L^2(\mathrm{R}_\pm;\mathrm{R}^n)$ respectively. Consider the operators $L^\pm_\lambda\colon X^\pm \to Y^\pm$ defined as in (5.9)by the restrictions of A_λ to $\mathrm{R}_\pm$. I will show that $L^\pm_\lambda$ are Fredholm and compute their index bundles. Notice that, if $M^\pm_\lambda\colon X^\pm \to Y^\pm$ are defined by

$$[M^\pm_\lambda u](t) = u'(t) - A(\lambda, \pm\infty)u(t), \tag{5.10}$$

then $K^\pm_\lambda = M^\pm_\lambda - L^\pm_\lambda$ is a compact operator for each $\lambda \in \Lambda$. Indeed, if ϕ_m is a smooth function in R_+ such that $\phi_m \equiv 1$ on $[0, m-1]$ and $\phi_m \equiv 0$ on $[m, +\infty)$, then K^+_λ is limit of

$$[K^m_\lambda u](t) = \phi_m(t)[A(\lambda, +\infty) - A(\lambda, t)](u(t). \tag{5.11}$$

Moreover the operator K^m_λ is compact because it can be factorized through the inclusion $H^1([0,m]\,\mathrm{R}^n) \subset L^2(\mathrm{R}^+;\mathrm{R}^n)$ which is compact. On the other hand it is well known that M_λ is surjective with $\ker M_\lambda = E^s(\lambda, +\infty)$. Indeed the second assertion is clear while for the first it is enough to observe that a right inverse for M_λ is is given by

$$S_\lambda(v)(t) = \int_0^t P_\lambda e^{(s-t)A_\lambda(s)} v(s)ds + \int_t^\infty (\mathrm{id} - P_\lambda) e^{(t-s)A_\lambda(s)} v(s)ds$$

where P_λ is the projector onto $E^s(\lambda, +\infty)$.

Thus M^+_λ and hence also L^+_λ are Fredholm operators whose numerical index equals $\dim E^s(\lambda, +\infty)$. Moreover by homotopy invariance of the index bundle

$$Ind\ L^+ = Ind\ M^+ = [E^s(+\infty)] \tag{5.12}$$

Similarly we have that L^-_λ is Fredholm of index $\dim E^u(\lambda, -\infty)$ by $(A3)$ and $Ind\ L^- = [E^u(-\infty)]$.

In order to compute the index of L let us observe that the restriction map $I\colon Y \to Y^- \oplus Y^+$ defined by $Iv = (v_{|\,\mathrm{R}^-}, v_{|\,\mathrm{R}^+})$ is an isomorphism, while the analogous map $J\colon H \to X^- \oplus X^+$ is injective with

$$\mathrm{Im}\, J = \{(u^-, u^+)/u^-(0) = u^+(0)\}.$$

Thus $\mathrm{Im}\, J = \ker \psi$ where $\psi(u^-, u^+) = u^-(0) - u^+(0)$ and hence J is a Fredholm operator of index $-n$. Moreover there is a commutative diagram

$$\begin{array}{ccc} X^- \oplus X^+ & \xrightarrow{L_\lambda^- \oplus L_\lambda^+} & Y^- \oplus Y^+ \\ J \Big\uparrow & & I \Big\uparrow \\ X & \xrightarrow[L_\lambda]{} & Y \end{array} \qquad (5.13)$$

Thus L_λ is Fredholm and by assumption $(A3)$, index $L_\lambda = 0$ which proves $i)$. Now $ii)$ follows from (5.13) by the logarithmic property of the index bundle. Indeed, considering I and J as constant families, $Ind\ I = 0, Ind\ J = -\Theta(\mathrm{R}^n)$. Hence, by (5.3),

$$Ind\, L = [E^u(-\infty)] + [E^s(+\infty)] - [\Theta(\mathrm{R}^n)] = [E^s(+\infty)] - [E^s(-\infty)],$$

as claimed. □

Theorem 5.1 follows from Theorem 2.2 and the above Proposition. Indeed, since L takes values in $\Phi_0(X;Y)$ it follows that, for δ small enough, the restriction of f to $\Lambda \times B(0,\delta)$ is a family of C^1-Fredholm maps such that $f(\lambda, 0) = 0$ By hypothesis, the total Stiefel-Whitney class $\omega(Ind\, L) \neq 0$ and hence by Theorem 2.2 the set of bifurcation points of H^1-solutions of (5.1) must be of dimension at least $m \geq 0$. Being $H^1(\mathrm{R};\mathrm{R}^n) \subset C(\mathrm{R};\mathrm{R}^n)$ the regularity and convergence in $C^1_0(\mathrm{R};\mathrm{R}^n)$ are easily obtained by bootstrap. □

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