Orientability of Fredholm families and topological degree

Orientability of Fredholm Families and Topological Degree for Orientable Nonlinear Fredholm Mappings

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* Partially supported by the CNR (Italy) and the NATO Research Council
** Partially supported by U.S. Air Force Grant AFOSR-90-0094

Introduction

A C^k map $f: X \to Y$ between real Banach spaces X and Y is called *Fredholm* provided that at each point x in its domain its Fréchet derivative $Df(x)$ is a Fredholm operator, i.e., Im $Df(x)$ is closed and both Ker $Df(x)$ and Coker $Df(x)$ are finite dimensional. The integer $\dim \operatorname{Ker} Df(x) - \dim \operatorname{coker} Df(x)$, which is independent of the choice of x, is called the *index* of the map f. Generalizations of Sard's Theorem can be used in order to show that for proper C^2-Fredholm maps *of index 0* the inverse image under f of a generic point y in Y is a finite subset of X. In 1936, Caccioppoli ([7]) observed this this was so and proved that the cardinality of $f^{-1}(y)$ is independent (mod 2) of the choice of the point y and is invariant under deformations. The resulting invariant is known as the Caccioppoli mod 2 degree.

Actually, the technique used by Caccioppoli was not based on the density of regular values in infinite dimensions. Instead, he used transversality in quite a modern way to show that a C^1-Fredholm map of index 0 is equivalent, in a neighborhood of a compact set, to a map from a finite-dimensional manifold to a Euclidean space of the same dimension.

In 1965, this degree was rediscovered by Smale ([43]) (in the C^2 case), as a consequence of his generalization of Sard's Theorem to Fredholm maps. More generally to any proper Fredholm map of positive index n between Banach manifolds Smale associate a degree, defined as the unoriented cobordism class of $f^{-1}(y)$ where y regular value of the map.

Subsequently, the mod 2 degree of Cacciopoli, for maps of index 0 was improved to an oriented degree (integer-valued) by Elworthy and Tromba([10,11]). Their theory largerly parallels the construction of the degree for maps between oriented finite dimensional manifolds and is based upon the notion of oriented Fredholm structure. Another approach, using infinite dimensional cohomology, was developed more or less at the same time by Murkherjea in [8].

In [10], Elworthy and Tromba constucted an oriented degree for maps of positive index as well. But, in this case, several restrictions has to be imposed on the class of manifolds under consideration. One of the main results of thieir theory is a classification theorem (Theorem 4.3 of [10]) of Hopf type, which describe the set of all homotopy classes of proper C^2 - Fredholm maps of positive index from a Banach manifold into its model space in terms of the degree and Fredholm reductions of its tangent bundle. In the special case of maps of index zero, which is the only case that will be considered in this paper, their result implies that the correspondence $[f] \to \pm \deg f$ classifies all homotopy classes of proper C^2 - Fredholm maps of a Kuiper space X into itself. The indeterminacy $\pm$ occurs for quite fundamental reasons. Indeed, using Kuiper's Theorem on the contractibility of the general linear group of a real Hilbert space one can easily show that no degree in the Fredholm setting can be homotopy invariant.

Since in the applications of degree to the study of multiplicity and bifurcation of solutions of nonlinear differential equations the homotopy invariance property of a degree theory plays an essential role, in the past twenty years many other degree theories have been constructed for particular classes of Fredholm maps of index 0 for which homotopy invariance holds. Recently we considered this problem from a different point of view. In [**15**], we developed a new approach to oriented degree theory for general C^2-Fredholm maps (of index 0) defined on simply connected subsets of Banach spaces (see [**14**]for a concise description) whose novelty lay in a veri simple formula for the changes of degree that can possibly occur in an admissible homotopy. Such a formula provides essential information for the analysis of multiplicity and bifurcation of the zeroes of C^2-Fredholm maps and indeed it was used in [**15**] in order to extend the Rabinowitz Global Bifurcation Theorem ([**38**]) to this setting.

In [**16**], we considered the case of maps equivariant under the action of a compact Lie group and proved some theorems of Borsuk-Ulam type. The purpose of this paper is to generalize our degree theory to the case of C^2-Fredholm maps of index 0 defined on multiply connected domains as well as on Banach manifolds. In this way we obtain a degree theory which is more general and, in our opinion, more natural than the one based on orientable Fredholm structures constructed by Elworthy and Tromba in this case. Moreover, from the general viewpoint presented here it is possible to see clearly the essential ingredients which underlie any degree theory for nonlinear Fredholm maps. In order to illustrate this, we shall relate the present theory to several other degree theories for particular classes of Fredholm maps which have appeared in the literature since the seminal paper of Caccioppoli ([**7**]) and which, at first glance, seem to be unrelated.

While the topic of this paper is a natural continuation of that of [**15**], the presentation will be different, and for this reason we have made this paper self-contained. To be sure, the basic ingredient of our construction is still the assignment of multiplicities to regular points of a map by means of the parity of a path of linear Fredholm operators. However, while previously we adopted a geometric approach using base-points (see Section 2 bellow), here we shall employ the more abstract notion of orientation function for Fredholm maps. This leads to a construction which is more symmetric than the one made by choosing a base point. Moreover, orientation functions are more suitable for describing the relationship between various degree theories.

What follows is a nontechnical description of our construction. We shall begin by motivating the concept of orientation in infinite dimensional Banach spaces which we will use here.

Let U be an open subset of a Banach space X and let $f\colon U \to Y$ be a proper C^2-Fredholm map. In order to define the degree of f by means of regular value approximation (using a generalization of Sard's Theorem) it is necessary to assign to each regular point of the map f a multiplicity ± 1. This must be done in a coherent manner which will make the sum of the multiplicities of points in the preimage of a regular value independent of the choice of the regular value. If X and Y are of the same finite dimension, this is usually done by assigning multiplicities ± 1 to each of the two connected components of the space $GL(X, Y)$. Fixing bases in X and Y respectively, one can distinguish the two connected components by assigning to each linear isomorphism the sign of the determinant of its associated matrix in the given bases. If x is a regular point of f, its multiplicity $\epsilon(x)$ is, by definition, sgndet $Df(x)$. With such a definition, the sum of the multiplicities of points in the inverse image of a regular value of f extends to the Brouwer degree.

In infinite dimensions, $GL(X, Y)$ does not, in general, split into two components and hence a different approach is needed. Plainly, the simplest strategy for overcoming this obstacle is to consider only maps whose derivatives take values in a proper subset $\mathcal{F}$ of the set of all Fredholm operators of index 0. A prominent example of one such choice is when X=Y and $\mathcal{F}$ is the set of all linear compact perturbations of the identity (i.e., compact vector fields). The set $GL_C(X)$ of invertible linear compact vector fields has two components which are distinguished by the function ϵ which is defined for $T \in GL_C(X)$ by $\epsilon(T) = \deg_{L.S.}(T) = (-1)^m$, where m is the sum of the algebraic multiplicities of the negative eigenvalues of T. This induces an assignment of multiplicities to the regular points of a smooth compact vector field which extends to the Leray-Schauder degree.

To understand what, in general, should be required of a choice of $\mathcal{F}$ in order to obtain a coherent assignment of multiplicities at regular points for maps whose derivatives take values in $\mathcal{F}$, we examine a geometric property of the "sgndet" function which is independent of the orientation. The pertinent property is that the sign of the determinant of a path of matrices switches by -1 each time the path crosses transversally the one codimensional stratified subset of singular matrices. While the notion of oriented basis cannot be extended to infinite dimensions, the preceding property has an appropriate correspondent for paths in the space $\Phi_0(X,Y)$ of all linear Fredholm operators of index 0. Indeed, the highest stratum $\mathcal{S}'$ of the set $\mathcal{S}$ of non-invertible elements of $\Phi_0(X,Y)$ is the one codimensional submanifold consisting of the operators having a one-dimensional kernel. All the other strata are of higher codimension. It follows then, by using standard methods in transversality theory, that any continuous path in $\Phi_0(X,Y)$ with invertible end-points may, by means of a small perturbation, be made to be both smooth and transverse to $\mathcal{S}'$. By definition, the *parity* of a path $L\colon I \to \Phi_0(X,Y)$ with invertible end-points is given by $\sigma(L,I) = (-1)^m$, where m is the number of intersection points with $\mathcal{S}'$ of any smooth approximation of L which is transverse to $\mathcal{S}'$. In other words, the parity of a path is the mod-2 intersection index of the path with the stratified set $\mathcal{S}$ (see Section 1 for a better definition). The parity depends only on the homotopy class of the path (relative to ∂I) and it is multiplicative under union of intervals. Clearly, the parity of a path of matrices with invertible end-points is nothing but the product of the sign of the determinant at the end-points.

Motivated by this observation, given a subset $\mathcal{F}$ of $\Phi_0(X,Y)$ we shall say that $\mathcal{F}$ is *orientable* provided that the parity of any path in $\mathcal{F}$ with invertible end-points depends only on its end-points. On the set of isomorphisms belonging to an orientable set, a function having the above described property of the sign of the determinant can be defined. Such a function will induce a coherent assignment of multiplicities to regular points of maps with $Df(x) \in \mathcal{F}$ and hence a degree. We will show that many of the degree theories for restricted classes of Fredholm maps that appear in the literature correspond to particular choices of orientable subsets $\mathcal{F}$.

In infinite dimensions, the whole set $\Phi_0(X,Y)$ of all linear Fredholm operators of index 0 will not, in general, be orientable. It is not orientable if X is a Hilbert space. In fact, Kuiper's Theorem implies that any two linear isomorphisms on an infinite dimensional Hilbert space X may be joined by a path of linear isomorphisms, so that one can easily construct closed paths in $\Phi_0(X)$ which cross transversally $\mathcal{S}$ at only one point and hence have parity -1.

Therefore, in order to construct a degree theory for C^2-Fredholm maps without imposing further restrictions on the values taken by the derivatives, one has to consider a more refined notion of oriented map. The idea is the following: we shall think of Df as a family of linear Fredholm operators parametrized by the points in the domain of the map and then, rather than assigning a multiplicity to the linear operator $Df(x)$, we shall instead assign a multiplicity directly to the parameter value x.

The map f is defined to be *orientable* if given any two regular points of f, the parity of the family Df of derivatives of f along any path between them is independent of the choice of the path. Such a notion of orientation is more sensitive to the topology of the domain of f. For instance, irrespective of the image of the family Df, any map having a simply connected domain is orientable. If the map f is orientable, we can assign multiplicities $\epsilon(x)$ to all regular points of f so that the following rule holds: $\epsilon(x) \cdot \epsilon(x') =$ the parity of Df along any path between x and x'. A function ϵ verifying the above property will be called an *orientation* of the map f. For connected U, once the multiplicity $\epsilon(x_0) = \pm 1$ of some fixed regular point x_0 of f is chosen, the values of ϵ is completely determined by the above rule.

Much like the finite dimensional case, such a distribution of multiplicities produces a properly defined additive degree theory. However, the homotopy property of the resulting degree requires a separate discussion. The degree theories for maps having derivatives in an orientable set are homotopy invariant in the usual sense. However, since the identity operator on an infinite dimensional Hilbert space can be connected by a path of linear isomorphisms to a linear compact vector field of degree -1, it follows that the homotopy invariance property cannot hold in the more general setting described in the preceding paragraph, and needs to be reformulated. The formula describing the behavior of the degree under admissible homotopies, as well as its geometric interpretation, is given in Section 2. It fills a major gap in the Elworthy-Tromba approach.

There is another reason why, in infinite dimensions, it appears to be more natural to orient maps rather than manifolds (or, more precisely, rather than Fredholm structures on Banach manifolds). Orientability of C^1-Fredholm maps depends intrinsically on the differentiable structure of the domain and the range. Indeed, it is invariant by C^1 homotopies of the map. In contrast, orientability of Fredholm structures presents the unpleasant feature that an infinite dimensional manifold can have both orientable and nonorientable Fredholm structures. In our construction, the Fredholm structure on the manifold is not used at all and the resulting degree is defined for any oriented map between two Banach manifolds. In finite dimensions the degree in [**10**] coincides with the classical Brouwer degree for maps between orientable manifolds, while our degree coincides with the more general Olum degree for orientable maps between not necessarily orientable manifolds.

To the best of our knowledge, the idea of orienting maps rather than manifolds was first considered by Olum in [**36**] using homology with coefficients in a local system. In the infinite dimensional setting, a degree theory for oriented maps between smooth, separable Fredholm manifolds was constructed by Mukherjea in his thesis (cf. [**8**]) using Fredholm filtrations and finite codimensional cohomology. This construction, however, does not give a genuine degree theory since only the absolute value of this degree is independent of the choice of filtration. In his study of the Gysin homomorphism, Morava ([**32**]) considered orientation for Fredholm maps of arbitrary index. His definition is less elementary than the one used here and requires the model space to be Kuiper.

In order to place technical matters in the background, we will split our construction into two separate levels of generality. First, we shall consider the case of oriented, C^2-Fredholm maps defined on open subsets of Banach spaces, the restrictions of which to closed, bounded subsets of their domain are proper. These kind of maps arise typically in the applications of degree theory to boundary value problems for nonlinear elliptic equations, and the construction of the degree in this case requires almost no vector bundle formalism. We shall then extend our theory to proper, oriented, C^2-Fredholm maps between Banach manifolds.

The paper is organized as follows. In Section 1, we record a few properties of the parity which we will need and also introduce the notion of oriented family of linear Fredholm operators. In Section 2, we first describe the construction of the degree for C^2-Fredholm maps defined on open subsets of Banach spaces and discuss the homotopy property of this degree. In the final part of the second section, we consider a particular case of the general theory, namely the case of maps whose family of derivatives takes values in an orientable subset of $\Phi_0(X, Y)$, and we analyze several examples of this type that have appeared in the literature. Section 3 is technical and can be skipped at the first reading. It deals with necessary and sufficient conditions for orientability of all maps with a given domain. Section 4 is devoted to comparing the present degree theory with a variant of the Elworthy-Tromba degree.

Let us recall, on this respect, that we consider in this paper only degree theories for maps of index 0. Therefore when we refer to the Elworthy Tromba theory we are speaking about degree introduced in Section 3 of [10] for maps in this class and not the one in Section 4 of the same paper for maps of positive index. In Section 5, we extend our construction to a degree for proper C^2-Fredholm maps between Banach manifolds. As an illustration of the usefulness of this extension, we shall prove a general reduction property of the degree, analogous to the well-known reduction property of the Leray-Schauder degree for compact vector fields. In the absence of any considerations about orientation, such a property was already considered by Caccioppoli in [**7**] as a starting point for the definition of his mod-2 degree. The basic theorem of [**42**] will be obtained as a corollary of the reduction property of the degree. The final section is devoted to the proof of a property of the parity which is needed in Section 5 but which is also of independent interest.

Section 1 Parity and Orientation

Our approach to orientation is based on the notion of parity of a path of linear Fredholm operators, which was introduced in [**17**] (cf. also [**18**,**20**]). We begin by recalling its definition and some of the properties.

Throughout, X and Y will denote real Banach spaces. Let $L(X, Y)$ be the set of all continuous linear operators from X to Y, with the norm topology, and let $\Phi_0(X, Y)$ be the open subset of $L(X, Y)$ consisting of all linear Fredholm operators of index 0. The open set of all invertible operators from X to Y will be denoted by $GL(X, Y)$. By $GL_C(X)$ we will denote the group of all operators in $GL(X)$ which are compact perturbations of the identity map on X, i.e., the linear invertible compact vector fields. Each operator L in $GL_C(X)$ has a properly defined Leray-Schauder degree, $\deg_{L.S.}(L)$, which in terms of the spectrum of L is given by the Leray-Schauder formula, $\deg_{L.S.}(L) = (-1)^m$, where m

is the sum of the algebraic multiplicities of the negative eigenvalues of L. We recall that the group $GL_C(X)$ has two connected components, $GL_C^+(X)$ and $GL_C^-(X)$ on which the Leray-Schauder degree assumes the values $+1$ and -1, respectively.

By a *family of Fredholm operators parametrized by a topological space* Λ we mean a continuous mapping $h\colon \Lambda \to \Phi_0(X,Y)$. A point $\lambda \in \Lambda$ having the property that $h(\lambda)$ is invertible is called a *regular point* of the family. The set of regular points of h is denoted by R_h, and its complement in Λ, namely the set of *singular points* of the family, is denoted by Σ_h.

A parametrix for a family $h\colon \Lambda \to \Phi_0(X,Y)$ is a family of invertible operators $g\colon \Lambda \to GL(Y,X)$ having the property that each $g(\lambda) \circ h(\lambda)$ is a linear compact vector field.

A family of Fredholm operators parametrized by a contractible metric space always has a parametrix (cf. [**20**]). In particular, a path in $\Phi_0(X,Y)$ is a family parametrized by a closed, bounded interval $I = [a,b]$ of real numbers and so it has a parametrix. We will call a path with regular end-points an admissible path.

Definition *Given an admissible path $\alpha\colon I \to \Phi_0(X,Y)$, the parity of the path α on I, $\sigma(\alpha,I)$, is the element of $Z_2 = \{1,-1\}$ given by*

$$\sigma(\alpha,I) = deg_{L.S.}(\beta(a) \circ \alpha(a)) \cdot deg_{L.S.}(\beta(b) \circ \alpha(b)),$$

where $\beta\colon I \to GL(Y,X)$ is any parametrix for α.

As we explained in the introduction, the parity has a simple geometric interpretation as a mod-2 intersection index of the path α with the one codimensional stratified subset S of $\Phi_0(X,Y)$ whose elements are noninvertible operators(cf. [**18**]). Clearly, the parity of a path of isomorphisms is 1 and the parity of an admissible path of compact vector fields is the product of the Leray-Schauder degrees of the end-points of the path. We record the following properties for further reference [**20**]:

(1.1) *Homotopy Invariance*: If $\alpha\colon [0,1] \times I \to \Phi_0(X,Y)$ is a family whose sections are admissible paths, then $\sigma(\alpha_0, I) = \sigma(\alpha_1, I)$.

(1.2) *Multiplicativity under partitions of I*: If $\alpha\colon I = [a,b] \to \Phi_0(X,Y)$ is admissible and if $c \in I$ is a regular point of the path α, then $\sigma(\alpha,I) = \sigma(\alpha,[a,c]) \cdot \sigma(\alpha,[c,b])$.

(1.3) *Multiplicativity under composition*: If $\alpha\colon I \to \Phi_0(X,Y)$ and $\beta\colon I \to \Phi_0(Y,Z)$ are admissible and if $\delta\colon I \to \Phi_0(X,Z)$ is the pointwise composition of α with β, then $\sigma(\delta,I) = \sigma(\beta,I) \cdot \sigma(\alpha,I)$. In particular, the parity of a path of Fredholm operators is invariant under composition with a path of isomorphisms.

(1.4) *Multiplicativity under direct sum*: If $\alpha\colon I \to \Phi_0(X,Y)$ and $\beta\colon I \to \Phi_0(X',Y')$ are admissible and if δ is the pointwise direct sum of α with β, then $\sigma(\delta,I) = \sigma(\beta,I) \cdot \sigma(\alpha,I)$.

In the sequel, we shall also need the following proposition, which reduces the calculation of the parity of a path of Fredholm operators to that of the parity of a morphism of finite dimensional vector bundles over I. This proposition is a particular case of the general reduction property of the parity which we will prove in Section 6.

Proposition 1.5 *Let $\alpha\colon I = [a, b] \to \Phi_0(X, Y)$ be a path of Fredholm operators with regular end -points. Then, given any any finite dimensional subspace V of Y such that $\alpha(\lambda)(X) + V = Y$ for each λ in Λ, the family of subspaces $E_\lambda = \alpha(\lambda)^{-1}(V)$ is a finite dimensional vector bundle E over I (which is trivial since I is contractible). Moreover, if $\psi\colon I \to L(V, X)$ is any path of linear injective maps from V to X such that $(\lambda, x) \in I \times X \mapsto (\lambda, \psi(\lambda)(x))$ is a trivialization of the bundle E, then*

$$\sigma(\alpha, I) = \text{ sgndet}(\alpha(a) \circ \psi(a)) \cdot \text{sgndet}(\alpha(b) \circ \psi(b)).$$

Notice that for any path of Fredholm operators one can find a finite dimensional subspace V and a path ψ as above.

Now we shall return to families of Fredholm operators parametrized by general spaces and introduce the notion of orientation.

Let $h\colon \Lambda \to \Phi_0(X, Y)$ be a family of Fredholm operators and let $\gamma\colon I \to \Lambda$ be a path in the parameter space whose end-points are regular points of the family h. The number $\sigma(h, \gamma) \in Z_2 = \{\pm 1\}$ defined by $\sigma(h, \gamma) = \sigma(h \circ \gamma, I)$ will be called *the parity of the family h along the admissible path γ.*

Definition *A family $h\colon \Lambda \to \Phi_0(X, Y)$ will be called orientable if there exist a function $\epsilon : R_h \longrightarrow Z_2$ having the property that for any path γ in Λ joining two regular points p and q of h, the parity of h along the path γ is given by the product of the values of the function ϵ at those points: i.e.,*

$$\sigma(h, \gamma) = \epsilon(p) \cdot \epsilon(q). \tag{1.6}$$

Any function ϵ verifying (1.6) will be called orientation

Remark Since the parity of a path of isomorphisms is always 1, it follows from (1.6) that any orientation ϵ of h has to be a locally constant function on the open set R_h and hence it is constant on the connected components of this set.

Proposition 1.7 *For a family $h\colon \Lambda \to \Phi_0(X, Y)$ parametrized by a pathwise connected space Λ, the following assertions are equivalent:*

i) The family h is orientable.

ii) The parity of h along any admissible path in Λ depends only on the end-points of the path.

iii) For some (and hence any) regular point p of h, if $\gamma : I \longrightarrow \Lambda$ is any closed path with $\gamma(a) = \gamma(b) = p$, then $\sigma(h, \gamma) = 1$.

Proof. That (i) implies (ii) and (ii) implies (iii) follows immediately from the multiplicativity with respect to partitions and homotopy invariance properties of the parity. Now assume (iii) and choose p to be a regular point of h. Then for each regular point q define $\epsilon(q)$ to be the parity of h along any path joining p and q. This defines the required orientation.

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A family which has no regular points will be called *degenerate*. A degenerate family is trivially orientable and has a unique orientation since $Map(\emptyset; Z_2)$ has a unique element; the function with empty graph. If h is a nondegenerate orientable family and Λ is pathwise connected, then there are precisely two orientations $\pm\epsilon$ for h, where the orientation ϵ is constructed by choosing a fixed regular point p of h and then defining, for any $q \in R_h$,

$$\epsilon(q) = \sigma(h, \gamma), \tag{1.8}$$

where γ is any path in Λ joining p to q.

Remark Notice that if Λ' is any subspace of the parameter space, then the restriction of an orientation to $R_h \bigcap \Lambda'$ gives an orientation for the restricted family. If Λ is not path connected, then clearly any orientation for h is given by orienting the restriction of h to each path component of Λ. This together with (1.8) describes all of the orientation functions of an orientable family of Fredholm operators.

Definition *A subset $\mathcal{F}$ of $\Phi_0(X, Y)$ is called orientable if the inclusion map of $\mathcal{F}$ into $\Phi_0(X, Y)$ is an orientable family.*

It is clear that if $\mathcal{F}$ is orientable, then so is any family $h: \Lambda \to \mathcal{F}$. In particular, any subset of an orientable set of operators is also orientable. Several examples of orientable subsets of $\Phi_0(X, Y)$ will be constructed in Section 2.

We close this section by observing that from the homotopy invariance of the parity, together with assertion iii) of Proposition 1.7 one immediately obtains

Proposition 1.8 *If Λ is simply connected, then every family $h: \Lambda \to \Phi_0(X, Y)$ is orientable. In particular, every simply connected subset of $\Phi_0(X, Y)$ is orientable.*

Section 2 The Degree for Oriented Nonlinear Fredholm Maps

A Fréchet differentiable map $f\colon U \to Y$ from an open subset U of a Banach space X into a Banach space Y is called Fredholm provided that at each point x in the domain its derivative $Df(x)$ is a Fredholm operator. In this section, we will develop a degree theory for orientable C^2-Fredholm maps (of index 0) between Banach spaces which are proper on closed, bounded subsets of their domain. Except for the C^2 assumption, which is a restriction imposed by technical aspects of the construction, the assumptions are typically satisfied by the operators associated with nonlinear elliptic boundary value problems.

Let us recall that a *regular point* of a C^1-map f is a point at which the derivative $Df(x)$ is surjective. The set R_f of all regular points of f is an open subset of the domain of the map. Its complement is the set S_f of *singular points* of the map f. A *regular value* of a C^1-map is a point of the range whose preimage consists of regular points. Since the set of singular points is closed and since proper maps send closed sets into closed sets, it follows that the set of regular values of a proper differentiable map is open. A generalization of the Sard-Smale Theorem due to Quinn and Sard ([**36**]) asserts that if $U \subset X$ is open and $f\colon U \to Y$ is a C^{k+1} map, the derivative of which is a linear Fredholm operator of index k at each point, then the set of regular values of f is of the first category in Y. Our present construction of degree relies on this result, in the cases $k = 0$ and $k = 1$. The case $k = 1$ arises in the consideration of homotopies, since by definition a *homotopy* in our category will be a C^2-Fredholm map $H\colon [0,1] \times U \to Y$ of index one (homotopies of more general type were considered in [**15**]). Notice that every section $H_t : U \to Y$, defined by $H_t(x) = H(t,x)$, of a homotopy is a C^2-Fredholm map of index 0.

Definition *An orientable C^2-Fredholm map $f\colon U \to Y$ is a C^2-Fredholm map such that its family of derivatives $Df\colon U \to \Phi_0(X,Y)$ is orientable. A homotopy H will be called orientable provided that the family $D_xH\colon [0,1] \times U \to \Phi_0(X,Y)$ is orientable. By an orientation of a map or homotopy we mean an orientation for the associated family of derivatives.*

A map having at least one regular point is called *nondegenerate*. Notice that, if ϵ is an orientation of a homotopy $H\colon [0,1] \times U \to Y$, then ϵ_t defined by $\epsilon_t(x) = \epsilon(t,x)$ is an orientation of each section $H_t : U \to Y$. Therefore sections of orientable homotopies are orientable maps. Conversely, we have the following

Proposition 2.1 *Suppose that some section H_t of $H\colon [0,1] \times U \to Y$ is nondegenerate and orientable. Then the homotopy H is orientable.*

Proof. Since the space $[0,1] \times U$ is homotopy equivalent to $\{t\} \times U$, any closed path in $[0,1] \times U$ with base point in $\{t\} \times U$ is homotopic to a closed path contained in $\{t\} \times U$. The result now follows from iii) of Proposition 1.7.

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Let $f\colon U \to Y$ be an orientable C^2-Fredholm map that is proper on closed, bounded subsets of U and let ϵ be an orientation of f. Suppose that Ω is open and bounded, with $\bar{\Omega} \subset U$. If $y \notin f(\partial\Omega)$ and y is a regular value of $f\colon \Omega \to Y$, then $f^{-1}(y) \cap \Omega$ is finite and *the degree of f in Ω with respect to y and ϵ* is defined by

$$\deg_\epsilon(f,\Omega,y) = \sum_{x\in f^{-1}(y)\cap\Omega} \epsilon(x). \tag{2.1}$$

Using the Inverse Function Theorem and the properness of the restriction of f to $\bar{\Omega}$, it can be easily shown that $\deg_\epsilon(f,\Omega,y)$ is a locally constant function on the open set RV_f of all regular values of the restriction of f to $\bar{\Omega}$.

In the rest of this section, unless explicitly stated otherwise, all maps and homotopies will be assumed to be proper on closed, bounded subsets of the domain. When no confusion can arise, we shall not distinguish in the notation between the map $f\colon U \to Y$ and its restriction to Ω.

Using regular value approximation, we will extend the degree function to points $y \in Y$ which are not necessarily regular values of f. In order to justify this extension, the main tool is the following

Proposition 2.2 *Suppose that $H\colon[0,1]\times U \to Y$ is an orientable homotopy and let ϵ be an orientation of H. Let Ω be open and bounded, with $\bar{\Omega}\subset U$ and $y \notin H([0,1]\times\partial\Omega)$. If y is a regular value of H_0 and H_1, then*

$$\deg_{\epsilon_0}(H_0,\Omega,y) = \deg_{\epsilon_1}(H_1,\Omega,y). \tag{2.3}$$

Proof. Since the degree is locally constant on the set of regular values, we can find a neighborhood V_y of y such that $\deg_{\epsilon_0}(H_0,\Omega,y')$ and $\deg_{\epsilon_1}(H_1,\Omega,y')$ are independent of the choice of $y' \in V_y$. By the generalized Sard-Smale Theorem for C^2-Fredholm maps of index 1, this neighborhood contains a point $\bar{y}$ which is a regular value of the homotopy H. Therefore, with no loss of generality we can assume that y is also a regular value of the homotopy H.

If $H^{-1}(y)\cap([0,1]\times\Omega)$ is empty, then, of course, each side of (2.3) is zero. Otherwise, in view of our regularity and properness assumptions, it follows that $M = H^{-1}(y)\cap([0,1]\times\Omega)$ is a compact one dimensional manifold whose boundary ∂M is the disjoint union of the two finite sets $M_0 = H^{-1}(y)\cap(\{0\}\times\Omega)$ and $M_1 = H^{-1}(y)\cap(\{1\}\times\Omega)$. As the boundary of a one dimensional manifold, ∂M has a natural involution τ which sends each point $m \in \partial M$ to the other point of ∂M belonging to the component of m in M.

Let m be a point in M_0. We will show that

$$\epsilon(m) = \epsilon(\tau(m)) \quad \text{if and only if} \quad \tau(m) \in M_1. \tag{2.4}$$

To do so, let $\eta\colon[0,1]\to H^{-1}(y)\cap(I\times\Omega)$ be a regular parametrization of the component of m in M such that $\eta(0) = m$ and $\eta(1) = \tau(m)$. In (t,x) coordinates η has the form $\eta(s) = (\phi(s), x(s))$ for s in $[0,1]$. By the very definition of orientation, $\sigma(D_xH\circ\eta,[0,1]) = \epsilon(m)\epsilon(\tau(m))$, and hence (2.4) is equivalent to the assertion that

$$\sigma(D_xH\circ\eta,[0,1]) = 1 \quad \text{if and only if} \quad \phi(0) \neq \phi(1). \tag{2.5}$$

Let us rewrite (2.4) in a yet another but equivalent form. For this, we apply the chain rule to the identity $H(\phi(s),x(s)) \equiv y$ obtaining:

$$D_tH(\phi(s),x(s))\phi'(s)+D_xH(\phi(s),x(s))x'(s)\equiv 0$$

Recalling that $\eta'(s)$ does not vanish in $[0,1]$, and that y a regular value of both H_0 and H_1 it follows from the above expression that neither $\phi'(0)$ nor $\phi'(1)$ can be zero. On the other hand, since the function ϕ sends both the interval $[0,1]$ and its boundary into itself, it is clear that $\phi(0)\neq\phi(1)$ if and only if the signs of $\phi'(0)$ and $\phi'(1)$ coincide.

Therefore, (2.4) is equivalent to

$$\sigma(D_xH\circ\eta,[0,1])=\mathrm{sgn}(\phi'(0))\cdot\mathrm{sgn}(\phi'(1)). \tag{2.6}$$

Next we will prove that (2.6) holds. For this we define $\alpha\colon[0,1]\to\Phi_0(X,Y)$ by $\alpha(s)=D_xH(\phi(s),x(s))$ and consider an auxiliary homotopy $\bar\alpha\colon[0,1]\times[0,1]\to\Phi_0(\mathbf{R}\times X,\mathbf{R}\times Y)$ defined by

$$\bar\alpha(\lambda,s)(r,x)=(r,\lambda D_tH(\phi(s),x(s))r+\alpha(s)x).$$

From the the multiplicativity of the parity under direct sum and the block-diagonal structure of $\bar\alpha_0(s)$ it follows that

$$\sigma(\bar\alpha_0,[0,1])=\sigma(\alpha,[0,1]).$$

On the other hand, by homotopy invariance of the parity

$$\sigma(\bar\alpha_0,[0,1])=\sigma(\bar\alpha_1,[0,1]).$$

We will use Proposition 1.5 in order to compute $\sigma(\bar\alpha_1,[0,1])$ and hence $\sigma(\alpha,[0,1])$. Since y is a regular value of H, the subspace $V\equiv\mathbf{R}\times\{0\}\subset\mathbf{R}\times Y$ is transverse to the image of each $\bar\alpha_1(s)$. The total space of the corresponding pull-back bundle E of Proposition 1.5 is

$$\{(s,r,x)|DH(\phi(s),x(s))(r,x)=0\}.$$

The bundle E has an explicit trivialization given by $(s,r)\mapsto r(\phi'(s),x'(s))$. By Proposition 1.5 we have

$$\sigma(\bar\alpha_1,[0,1])=\mathrm{sgn}(\phi'(0))\cdot\mathrm{sgn}(\phi'(1)).$$

Thus (2.6) is verified, and hence so is (2.4).

Finally, we will use (2.4) to complete the proof. If $m=(0,x)$, and $\tau(m)=(0,x')$ also lies in M_0, then, according to (2.4), $\epsilon_0(x)=-\epsilon_0(x')$, so that the contributions from x and x' to the degree of H_0 cancel. In other words,

$$\deg_{\epsilon_0}(H_0,\Omega,y)=\sum_{(0,x)\in M_0}\epsilon_0(x)=\sum_{(0,x)\in M_0,\tau(0,x)\in M_1}\epsilon_0(x).$$

The same argument shows that

$$\deg_{\epsilon_1}(H_1,\Omega,y)=\sum_{(1,x)\in M_1}\epsilon_1(x)=\sum_{(1,x)\in M_1,\tau(1,x)\in M_0}\epsilon_1(x).$$

The assertion (2.3) follows from these last two formulas, since τ induces a bijection between the sets of subindices which occur in the right-hand sides of the above two sums, and since, again by (2.4), if m in M_i is such that $\tau(m)$ is not in M_i, then $\epsilon(m) = \epsilon(\tau(m))$.

■

We now can state the main result of this section.

Theorem 2.7. *Let $f\colon U \to Y$ be an orientable C^2-Fredholm map defined on an open connected subset U of a Banach space X which is proper on closed, bounded subsets of U and let ϵ be an orientation for f. There exists an function which assigns to any open and bounded set Ω, with $\bar{\Omega} \subset U$, and any $y \notin f(\partial\Omega)$ an integer $deg_\epsilon(f, \Omega, y)$ having the following properties*

i) **Additivity:** *If Ω_1, Ω_2, are disjoint open subsets of Ω such that $y \notin f(\bar{\Omega} - \Omega_1 \cup \Omega_2)$, then*

$$deg_\epsilon(f, \Omega, y) = deg_\epsilon(f, \Omega_1, y) + deg_\epsilon(f, \Omega_2, y).$$

ii) **Existence:** *If $deg_\epsilon(f, \Omega, y) \neq 0$, then the equation $f(x) = 0$ has a solution in Ω.*

iii) **Homotopy:** *If $H\colon [0, 1] \times U \to Y$ is an orientable homotopy with orientation ϵ and if $y \notin H([0, 1] \times \partial\Omega)$, then*

$$\deg_{\epsilon_0}(H_0, \Omega, y) = \deg_{\epsilon_1}(H_1, \Omega, y). \tag{2.8}$$

Proof. If $y \notin f(\partial\Omega)$ is a singular value of f, we want to define the degree of f in Ω with respect to y by means of regular value approximation. In case f is degenerate, we define $\deg_\epsilon(f, \Omega, y)$ to be 0. From now we assume that f is not degenerate.

Using the Generalized Sard-Smale Theorem we can choose a regular value y' of f whose distance to y is less than the distance of y from $f(\partial\Omega)$, and then define

$$\deg_\epsilon(f, \Omega, y) = \deg_\epsilon(f, \Omega, y') = \sum_{x' \in f^{-1}(y') \cap \Omega} \epsilon(x'). \tag{2.9}$$

To see that the right hand side of (2.9) is independent of the choice of y', let us take two regular values y_0 and y_1 each of which verifies the above condition and consider the homotopy $H : [0, 1] \times U \to Y$ defined by $H(t, x) = f(x) - ty_0 - (1 - t)y_1$. It is clear that the orientation ϵ of f extends to an orientation ϵ of H by setting $\epsilon(t, x) = \epsilon(x)$ if $D_x H(t, x) = Df(x)$ is invertible. Clearly, 0 is a regular value of both H_0 and H_1. Moreover, with the above choice of orientation for H we have

$$\deg_{\epsilon_0}(H_0, \Omega, 0) = \deg_\epsilon(f, \Omega, y_0) \quad \text{and} \quad \deg_{\epsilon_1}(H_1, \Omega, 0) = \deg_\epsilon(f, \Omega, y_1).$$

Notice also that, under our assumptions, the segment joining y_0 with y_1 does not intersect $f(\partial\Omega)$ and hence $0 \notin H([0, 1] \times \partial\Omega)$. The equality between the degrees with respect to y_0 and y_1 now follows directly from Proposition 2.2. This shows that the degree is properly defined. The verification of the properties i) to iii) is a straightforward application of the density of regular values.

∎

We emphasize that the homotopy property iii) is not a genuine homotopy invariance property, since the orientation ϵ_t varies with t. A better geometric understanding of this property can be achieved by introducing base-points. To explain this, we shall briefly describe the base-point degree. By the very definition of the degree, if the map $f\colon U \to Y$ is degenerate, then for any admissible Ω and y, $\deg_\epsilon(f,\Omega,y) = 0$. On the other hand, a nondegenerate orientable map has precisely two orientations which are completely determined by their value at a given point. Let us assume that f is orientable, nondegenerate and let p be any regular point of f. Let ϵ_p be the unique orientation of f such that $\epsilon_p(p) = 1$. The *degree of f in Ω with respect to y and the base point p* is defined by

$$\deg_p(f,\Omega,y) = \deg_{\epsilon_p}(f,\Omega,y). \tag{2.10}$$

In the case that the domain is simply connected, this degree coincides with the base-point degree introduced in [**15**]. If ϵ is any orientation of f, then $\deg_\epsilon(f,\Omega,y)$ and $\deg_p(f,\Omega,y)$ are related by

$$\deg_\epsilon(f,\Omega,y) = \epsilon(p)\deg_p(f,\Omega,y). \tag{2.11}$$

In terms of the base-point degree, the homotopy property iii) can be reinterpreted as follows: Let H be an orientable homotopy with orientation ϵ and let y be such that $y \notin H([0,1]\times\partial\Omega)$. If either H_0 or H_1 are degenerate, then $\deg_{\epsilon_t}(H_t,\Omega,y) = 0$ for any $t \in [0,1]$. If both H_0 and H_1 are nondegenerate and if p_0, p_1 are base-points of H_0 and H_1, respectively, then substituting (2.11) into iii) and using (1.8) we obtain

$$\deg_{p_0}(H_0,\Omega,y) = \sigma(H)\ \deg_{p_1}(H_1,\Omega,y), \tag{2.12}$$

where $\sigma(H)$ is the parity of the family D_xH along any path in $[0,1]\times U$ joining $(0,p_0)$ with $(1,p_1)$. In the simply connected case, this is essentially the homotopy variance property (1.3) of [**15**].

For a generic class of homotopies, the change in sign of the degree along an admissible homotopy has a particularly simple geometric interpretation. Assume, just for the sake of definiteness, that p is a regular point of both H_0 and H_1 and let $\alpha(t) = D_xH(t,p)$. If α crosses the singular set $\mathcal{S}$ only at the maximal stratum $\mathcal{S}' = \{L \mid \dim \operatorname{Ker} L = 1\}$, transversally, then $\deg_{\epsilon_t}(H_t,\Omega,y)$ changes sign at each of the crossing points.

Notice also that, if the map $f\colon U \to Y$ is such that $Df\colon U \to \Phi_0(X,Y)$ is transverse to $\mathcal{S}$, then the distribution of multiplicities given by an orientation of f can be described rather geometrically: for such a map, the inverse image by Df of $\mathcal{S}$ will be a stratified subset S_f of the domain whose top stratum is a submanifold of codimension one. The multiplicity is constant on each connected component of the complement R_f of S_f in U, and once the multiplicity is assigned to the connected component of a chosen point p, the multiplicity of any other connected component will be the same or the opposite depending on whether a generic curve joining the two components crosses S_f transversally an even or an odd number of times.

Remark Our presentation of the degree has an unusual feature in that we keep separate the domain U of the orientable map from the set Ω on which we compute the degree. We choose this approach since, in the case of connected domain U, one has a convenient description of the homotopy property in terms of base points. A formally different approach would be to consider admissible quadruples (Ω, f, y, ϵ) made by an open bounded subset Ω of X, a proper, C^2-Fredholm map $f : \bar{\Omega} \to Y$ which is orientable on Ω, a point y not in the image of the boundary of Ω and an orientation ϵ for the map f. Then our construction will assign a degree to each admissible quadruple. We leave to the reader the reformulation of the properties i) to iii) according to this scheme.

Remark There is a classical example by Whitney [**45**] of a C^1 function from $\mathbf{R^2}$ to $\mathbf{R}$ whose gradient vanishes on a connected set on which the function is not constant. Consequently, the set of regular values of a C^1-Fredholm map of index 1 need not be dense. It follows that the density of regular values argument which we used in Proposition 2.2 is not valid without the assumption that the homotopies are C^2.

We shall conclude this section with the discussion of a special but important case of the degree constructed above: namely, the degree theory for $\mathcal{F}$-maps, where $\mathcal{F}$ is an orientable subset of $\Phi_0(X, Y)$ (see Section 1). As we mentioned briefly in the introduction, in this case the orientation is defined *directly* on the set $\mathcal{F}$ and because of this the resulting degree becomes homotopy invariant in the classical sense. We shall show that this degree theory provides a reasonably complete understanding of various constructions of homotopy invariant degree theories for nonlinear Fredholm mappings which have been made in the past years. In what follows $\mathcal{F}$ will be a given orientable subset of $\Phi_0(X, Y)$ with a fixed orientation ϵ on it.

Definition *A C^2-Fredholm map $f{:}\, U \to Y$ is called an $\mathcal{F}$-map if the family of derivatives $Df{:}\, U \to \Phi_0(X, Y)$ takes values in $\mathcal{F}$. An $\mathcal{F}$-homotopy is a homotopy whose sections are $\mathcal{F}$-maps.*

Clearly any $\mathcal{F}$-map $f{:}\, U \to Y$ inherits a unique orientation induced by the given one on $\mathcal{F}$ by defining $\epsilon(x) = \epsilon(Df(x))$ for each regular point of f. Therefore, from Theorem 2.7 we immediately obtain the following:

Corollary 2.13 *Let $\mathcal{F}$ be an orientable subset of $\Phi_0(X, Y)$ with orientation ϵ. Then the formula*

$$\deg_\epsilon(f, \Omega, y) = \lim_{z \to y, z \in RV_f} \sum_{x \in f^{-1}(z) \cap \Omega} \epsilon(Df(x)) \tag{2.14}$$

defines an integer-valued degree for the class of all $\mathcal{F}$-maps that are proper on closed, bounded sets. This degree has the existence property, it is additive over domains and it has the (usual) homotopy invariance property with respect to $\mathcal{F}$-homotopies.

At least in the C^2-Fredholm setting, many degree theories can be obtained from the above by an appropriate choice of orientable subset $\mathcal{F}$.

(i) The Brouwer, Leray-Schauder and Nussbaum -Sadovskii Degrees.

First, suppose that X and Y are finite dimensional oriented Banach spaces of the same dimension. Then $\epsilon{:}\, GL(X, Y) \to \{1, -1\}$, defined by $\epsilon(T) = $ sgndet T, is an orientation for

$\mathcal{F} = \Phi_0(X, Y) = L(X, Y)$. In the class of C^2 mappings, $\deg_\epsilon$ coincides with the Brouwer degree. In the infinite dimensional case, suppose that $X = Y$ and let $\mathcal{F}$ be the set of linear compact perturbations of the identity, i.e., linear compact vector fields. By the very definition of parity, $\mathcal{F}$ is orientable with orientation $\epsilon\colon GL_C(X, Y) \to \{1, -1\}$ defined by $\epsilon(T) = \deg_{L.S.}(T)$. Clearly, C^2 compact vector fields are $\mathcal{F}$-maps and on this class the degree $\deg_\epsilon$ coincides with the Leray-Schauder degree. Lastly, again suppose that $X = Y$, but now let $\mathcal{F}$ be the set of all linear α-contractive perturbations of the identity, where α is some measure of noncompactness. Each operator in $\mathcal{F}$ has only a finite number of negative eigenvalues, each of which has finite algebraic multiplicity. For each isomorphism $T \in \mathcal{F}$, define $\epsilon(T) = (-1)^m$, where m is the sum of the algebraic multiplicities of the negative eigenvalues of T. The function ϵ defines an orientation of $\mathcal{F}$: this is a special case of Proposition 2.18 below. Moreover, for each isomorphism $T \in \mathcal{F}$, $\epsilon(T)$ coincides with $\deg_{N.S.}(T)$, the degree being that defined by Nussbaum in [**34**] and by Sadovskii in [**41**]. But perturbations of the identity by C^2 nonlinear α-contractive maps are $\mathcal{F}$-maps. Thus, among C^2 maps, the degree of Nussbaum and of Sadovskii for α-contractive perturbations of the identity is a special case of Corollary 2.13.

(ii) The Mawhin Coincidence Degree and Related Degrees for Semilinear Maps.

Let $T \in \Phi_0(X, Y)$ be fixed. A map $N\colon X \to Y$ is called T-α-contractive if there exists a compact linear operator A such that $T + A$ is an isomorphism and $(T + A)^{-1}N$ is α-contractive. Define $\mathcal{F}$ to be the set of all linear T-α-contractive perturbations of T. Choose some linear compact operator A such that $T + A$ is an isomorphism. Then it is easy to see that for any isomorphism S in $\mathcal{F}$ the operator $(T + A)^{-1} \circ S$ is an α-contractive perturbation of the identity and the function $\epsilon\colon GL(X, Y) \cap \mathcal{F} \to \{1, -1\}$ defined by $\epsilon(S) = \deg_{N.S.}((T + A)^{-1} \circ S)$ is an orientation. The degree for T-α-contractive perturbations of a bounded Fredholm operator T was constructed by Pejsachowicz and Vignoli in [**36**]. In the same paper, it is shown that the degree constructed by Gaines and Mawhin in [**21**] for T-α-contractive perturbations of unbounded Fredholm operators is actually equivalent to theirs. Indeed, the general case reduces to that of perturbations of bounded operators by changing the norm of the range (this trick is due to M. Furi). As before, smooth T-α-contractive perturbations of T have derivatives in $\mathcal{F}$ and with the above choice of orientation $\deg_\epsilon$ agrees, up to the sign, with the degree in [**36**]. This also includes the Mawhin coincidence degree (cf. [**30**]) since this degree is a special case of the one in [**21**].

(iii) The Tromba Degree for Röthe Maps and the Isnard Degree.

A C^2 map $f\colon X \to X$ is called a Röthe map if for any $x \in X, Df(x)$ is a compact perturbation of an isomorphism belonging to the star of the identity * in $GL(X)$. The smooth generalized Leray-Schauder vector fields considered in [**40**] are examples of Röthe maps. Let $\mathcal{F}$ be the set of all linear compact perturbations of operators belonging to the star of the identity in $GL(X)$. Since $\mathcal{F}$ is contractible, by Proposition 1.8 it is orientable with a unique orientation ϵ on $\mathcal{F}$ such that $\epsilon(Id) = 1$. Any isomorphism T in $\mathcal{F}$ has the form $T = S + K$, with K compact and S is in the star of the identity in $GL(X)$. We will

* The star of the identity in $\mathcal{F}$, where $\mathcal{F}$ is a subset of $L(X)$, consists of all operators T in $\mathcal{F}$ having the property that the segment between T and the identity is contained in $\mathcal{F}$.

show that $\epsilon(T) = \deg_{L.S.}(S^{-1}T)$. Indeed $\alpha(t) = tId + (1-t)T, 0 \le t \le 1$ is an admissible path in $\mathcal{F}$ which has as a parametrix the path defined by $\beta(t) = (tI+(1-t)S)^{-1}, 0 \le t \le 1$. By the definition of parity, $\epsilon(T)\cdot\epsilon(Id) = \sigma(\alpha,[0,1]) = \deg_{L.S}(S^{-1}T)\cdot\deg_{L.S}(I)$, and hence $\epsilon(T) = \deg_{L.S}(S^{-1}T)$.

In [**42**], Tromba defined a homotopy invariant degree for Röthe maps using the fact that $GL(X) \cap \mathcal{F}$ has two connected components and assigning multiplicity one to the connected component of the identity. From the above formula for the orientation ϵ it follows immediately that his degree is a special case of the degree defined in Corollary 2.13.

In [**23**], Isnard introduced a degree theory for proper C^1-Fredholm maps between oriented C^1-Fredholm manifolds modeled by a Banach space X, whose derivative at each point belongs $\mathcal{G}$, the star of the identity in $\Phi_0(X,X)$. The approach is rather convoluted. The definition is based on regular value approximation but in order to prove the homotopy invariance in the C^1 case he uses the Caccioppoli technique combined with some results of [**10**] (see the remark at the end of Section 5). In order to define multiplicities at regular points he uses the fact that the set of isomorphisms in $\mathcal{G}$ has two connected components. The set $\mathcal{G}$, being contractible, has a unique orientation ϵ which coincides with the one defined by Isnard. This orientation extends to the orientation on $\mathcal{F}$ which we just described for the degree for Röthe maps. Therefore, for C^2 proper maps of the above type which are defined on X, both degrees agree.

A further comment on perturbations of the identity by α-contractions is in order. Nussbaum (cf. [35]) has defined a fixed-point index for mappings which have an iterate which is a k-set -contraction with $k < 1$. Furthermore, he also has shown that an iterate of a C^1 map is a k-set -contraction with $k < 1$ on a neighborhood of a compact fixed-point set if and only if the radius of the essential spectrum of its linearization, at each point in its fixed-point set, is less that unity . The preceding two observations provide an alternative construction of degree for proper C^1 maps f having the property that $(1-z)I + zDf(x)$ is Fredholm of index zero, for each complex number z of modulus less that unity and each point x in the zero set of f. This class is closely related to that considered by Isnard [23].

(iv) Degree theories for Maps with Essentially Positive Derivative.

Let X and Y be Banach spaces such that X is continuously embedded in Y. An operator $T\colon X \to Y$ which is Fredholm of index 0 is called *essentially positive* provided that the spectrum of T lying in the non-positive real axis consists of a finite number of eigenvalues, each of which has finite algebraic multiplicity. Essentially positive operators arise typically as realizations on appropriate function spaces of strongly elliptic linear partial differential operators. If T is an essentially positive isomorphism, define

$$\epsilon(T) = (-1)^m, \tag{2.15}$$

where m is the sum of the algebraic multiplicities of the negative eigenvalues of T. We shall now describe three orientable subsets of the set of all essentially positive operators. Each of these arises in degree theories for fully nonlinear elliptic equations and, on the isomorphisms in each of them, formula (2.15) defines an orientation.

Proposition 2.16 *Let $\mathcal{F}$ be the set of essentially positive operators T for which there* *is some real number $c = c(T)$ such that the family $(T + \lambda I)^{-1}, \lambda \geq c$, is uniformly bounded in $L(Y, X)$. Then formula (2.15) defines an orientation for $\mathcal{F}$.*

Proof. It is easy to see that $\mathcal{F}$ is open, and that ϵ is constant on paths of isomorphisms in $\mathcal{F}$. Thus, since each path in $\Phi_0(X, Y)$ may be approximated arbitrarily closely by a path having only a finite number of singular points ([**18**]), it is sufficient to check that if γ is a path in $\mathcal{F}$ having t_0 as an isolated singular point, and if η is sufficiently small, then

$$\sigma(\gamma, [t_0 - \eta, t_0 + \eta]) = \epsilon(\gamma(t_0 - \eta))\epsilon(\gamma(t_0 + \eta)). \tag{2.17}$$

But 0, as an eigenvalue of $\gamma(t_0)$, is of finite algebraic multiplicity, and so it is isolated in the spectrum of $\gamma(t_0)$. Choose $\delta > 0$ such that the spectrum of $\gamma(t_0)$ in $[-\delta, \delta]$ consists of 0, and then choose $\eta > 0$ small enough so that for $t \in [t_0 - \eta, t_0 + \eta]$ the spectrum of $\gamma(t)$ in $[-\delta, \delta]$ is a finite number of eigenvalues which do not include the end points. For $t \in [t_0 - \eta, t_0 + \eta]$, define $P(t)$ to be the projection onto the sum of the eigenspaces corresponding to the spectrum of $\gamma(t)$ in $(-\infty, \delta]$. Then $t \mapsto (\gamma(t) + P(t))^{-1}$ is a parametrix for γ on $[t_0 - \eta, t_0 + \eta]$. Using this parametrix to compute the parity, we easily obtain (2.17).

■

The degree theory for $\mathcal{F}$-maps, where $\mathcal{F}$ is as in the above proposition was introduced by Fenske in [**12**] (cf. [**9**] and [**25**] for closely related theories). Notice that the derivatives of smooth α-contractive perturbations of the identity are essentially positive operators and have the spectral growth properties prescribed in the above proposition. This relates the degree theories considered in (i) to the Fenske degree.

The next two orientable sets were introduced in Section 9 of [**20**] in connection with the degree for strongly orientable quasilinear Fredholm maps.

We will say that the resolvent of an essentially positive operator T is of *minimal growth* provided that there is some c such that $\lambda(T + \lambda I)^{-1}, \lambda \geq c$, are uniformly bounded in $L(Y, Y)$.

Proposition 2.18 *Assume that X is compactly embedded in Y. Let $\mathcal{F}_1$ be the set of essentially positive operators which have resolvents of minimal growth. Let $\mathcal{F}_2$ be any set of essentially positive operators having the property that for each pair of operators T and S in $\mathcal{F}_2$ there exists λ_* such that $tS + (1-t)T + \lambda I$ is invertible, for all $t \in [0, 1]$ and $\lambda \geq \lambda_*$. Then both $\mathcal{F}_1$ and F_2 are orientable, with orientation ϵ given by 2.15.*

The proof of this proposition follows immediately from the results in Section 9 of [**20**]. In [**29**], Li developed a homotopy invariant degree for maps associated with fully nonlinear elliptic second-order partial differential equations with Dirichlet boundary conditions. For smooth maps of this type, this is a special case of our degree for $\mathcal{F}_2$-maps, where $\mathcal{F}_2$ is as in the preceding proposition.

(v) The Degree for Quasilinear Fredholm Maps.

In [**17**,**20**], a degree is constructed for the class of quasilinear Fredholm mappings. The construction is of a rather different type from that presented here: it uses representations and reduction to the compact vector field case. The degree for the full class of quasilinear Fredholm maps is not homotopy invariant and therefore not of $\mathcal{F}$-type. However, this degree has the same homotopy property as the one in Theorem 2.7. This, of course, is not a coincidence. Indeed, the parity was originally introduced in connection with the homotopy formula in [**17**,**20**]. The notion of orientation used in that paper is weaker than the one used here*. Nevertheless, using the regular value formula in [**20**], the following can easily be proved: Let $o: \Phi_0(X,Y) \to Z_2$ be any orientation in the sense of [**20**]. If $f: X \to Y$ is a C^2 quasilinear Fredholm map which has 0 as a regular point, then the function ϵ defined by $\epsilon(x) = o[Df(0)] \cdot \sigma(Df, \gamma)$, where γ is any path between 0 and the regular point x, is an orientation of the map f. Moreover, for any admissible set Ω

$$\deg_o(f, \Omega, 0) = \deg_\epsilon(f, \Omega, 0).$$

Remark Using finite dimensional approximations in a standard way, our degree theory can easily be improved to a degree theory for compact perturbations of orientable C^2-Fredholm maps. Such a degree theory will contain as particular case several classes of $\mathcal{F}$ maps described in this section.

We consider here only C^2 maps, because in this case the use of parity and the density of regular values makes the construction of the degree almost as simple and transparent as the classical construction of Brouwer's degree based on differential topology (cf. [**31**]. But the smooth case is always conceptually relevant and plays a central role in the construction of a degree for any class of maps.

Remark There are many other constructions of degree theories for maps of Fredholm type, both oriented and unoriented, which we have not mentioned so far. They split essentially into the following three types. The first is based on the use of representations. The origin of these goes back to the Leray-Schauder paper on degree for compact vector fields [**28**]; examples of this approach are found in [**5**], [**4**] and [**26**]. Here, usually, orientation is not taken into account and the degree is either unoriented or it depends on the choice of the representation. A bridge between the above theories and our degree is the paper [**20**]. The second type uses the original Caccioppoli approach ([**7**]) and oriented Fredholm structures: this type of construction can be found in [**3**], [**2**] and [**46**]. Its main advantage is that it work also for C^1 maps. For the relation with our degree see the remark at the end of Section 5. The third type is the multivalued degree theory based on Galerkin approximation (see [**6**] and [**42**]). This one uses a completely different technique, since it is tailored to the case of nonsmooth maps. However, it also can be related to our approach through a stability property of the parity established in [**13**].

* What in [**20**] is called a strong orientation of a subset of $\Phi_0(X,Y)$ is what is here called an orientation.

Section 3 Obstructions to Orientability

The degree is defined for maps which are orientable. In this section, we shall study necessary and sufficient conditions ensuring orientability of all C^2-Fredholm maps with a given domain.

In what follows we shall not distinguish between the additive notation for $Z_2 = \{0, 1\}$ and the multiplicative one which has been used in the definition of the parity.

Let U be a pathwise connected topological space and let $\alpha: U \to \Phi_0(X, Y)$ be a nondegenerate family of linear Fredholm operators parametrized by U. Choose a regular point p of α. Let $\pi_1(U, p)$ denote the first homotopy group of U based at p. The family α induces a homomorphism $\sigma(\alpha): \pi_1(U, p) \to Z_2$ defined as follows: if $[\gamma] \in \pi_1(U, p)$ denotes the homotopy class of the closed path γ based at p, then

$$\sigma(\alpha)([\gamma]) = \sigma(\alpha, \gamma). \tag{3.1}$$

That $\sigma(\alpha)$ is a properly defined homomorphism from $\pi_1(U, p)$ to Z_2 follows from properties (1.1) and (1.2) of the parity. Moreover, it is clear that α is orientable if and only if the homomorphism $\sigma(\alpha)$ is trivial. Recall the standard identification of the group $\mathrm{Hom}(\pi_1(U, p), Z_2)$ with the first singular cohomology group $H^1(U; Z_2)$ of U with Z_2 coefficients [**22**]. Indeed, any homomorphism $\sigma: \pi_1(U, p) \to Z_2$ factors through the quotient of $\pi_1 \equiv \pi_1(U, p)$ by its commutator $[\pi_1, \pi_1]$. This quotient is nothing but the first singular homology group $H_1(U; Z_2)$. Hence, σ induces a homomorphism $\sigma': H_1(U; Z_2) \to Z_2$. By the Universal Coefficient Theorem in cohomology, σ' corresponds to a unique cohomology class $\omega \in H^1(U; Z_2)$. Through this identification, we associate to each family $\alpha: U \to \Phi_0(X, Y)$ its *obstruction class* $\omega(\alpha) \in H^1(U; Z_2)$, which has the property that α is orientable if and only if $\omega(\alpha)$ vanishes.

If $H^1(U; Z_2) = 0$, then all obstructions vanish and any Fredholm family parametrized by U is orientable. Moreover, since U and $U \times [0, 1]$ are of the same homotopy type, $H^1(U; Z_2) = 0$ implies that $H^1(U \times [0, 1]; Z_2) = 0$ also, and hence all Fredholm families parametrized by $U \times [0, 1]$ are also orientable. These observations lead to the following

Theorem 3.2 *Let U be an open subset of a Banach space X such that $H^1(U; Z_2) = 0$. Then all C^2-Fredholm maps defined on U are orientable, as are all C^2-Fredholm homotopies defined on $U \times [0, 1]$). In this case the degree theory given by Theorem 2.7 is defined for all C^2 Fredholm maps with domain U that are proper on closed bounded sets. Moreover it verifies the homotopy property (2.8) with respect to all homotopies of this type.*

The obstruction class $\omega(\alpha)$ has another interpretation which can be used to show that, when X is a Kuiper space, the vanishing of $H^1(U; Z_2)$ is indeed a necessary condition for orientability of all families parametrized by U. For this let us recall a result from [**19**].

According to the Atiyah-Jänich theory ([**1**], [**24**]), each family $\alpha: S^1 \to \Phi_0(X, Y)$ has an associated stable equivalence class of vector bundles over S^1, called the *analytical index* and denoted by $\mathrm{Ind}\ \alpha$. If V is any finite dimensional subspace of Y such that $\alpha(s)(X) + V = Y$ for each s in S^1, then $\mathrm{Ind}\ \alpha$ is the stable equivalence class of the bundle with total space $E = \{(s, x) : \alpha(s)x \in V\}$. The group $KO(S^1)$ of all stable equivalence classes of vector bundles over S^1 is a cyclic group of order 2 generated by the canonical line bundle over the real projective space $RP^1 \cong S^1$. There is a natural isomorphism $w_1: KO(S^1) \to Z_2$ that associates to each stable equivalence class of vector bundles over S^1 its first Steifel-Whitney number, i.e., the evaluation of the first Steifel-Whitney class on the fundamental class of S^1. From Proposition 2.7 of [**19**] it follows that if α is a nondegenerate Fredholm family parametrized by U and $\gamma: S^1 \to U$ is a closed path based at a regular point p of α, then

$$w_1(Ind\ \alpha \circ \gamma) = \sigma(\alpha, \gamma) = \sigma(\alpha)([\gamma]). \tag{3.3}$$

We shall say that U is *totally orientable with respect to X* if any family $\alpha: U \to \Phi_0(X, Y)$ of linear Fredholm operators parametrized by U is orientable.

Theorem 3.4. *Let X be a Kuiper space. A pathwise connected topological space U is totally orientable with respect to X if and only if $H^1(U; Z_2) = 0$.*

Proof. We have already shown that, for general Banach space spaces X and Y, if $H^1(U; Z_2) = 0$, then U is totally orientable with respect to X. To verify the converse, it is enough to show that the obstruction correspondence $\alpha \mapsto \omega(\alpha)$ is surjective. Without any loss of generality, we can suppose that $X = Y$.

Using the identification of $H^1(U; Z_2)$ with $\mathrm{Hom}(\pi_1(U,p), Z_2)$, the problem reduces to that of showing that any homomorphism $\rho: \pi_1(U,p) \to Z_2$, is of the form $\sigma(\alpha)$ for some Fredholm family $\alpha: U \to \Phi_0(X, Y)$.

Let $\rho: \pi_1(U,p) \to Z_2$ be such a homomorphism. Since $Z_2 = \{1, -1\}$ is the orthogonal group $O(1)$ of the real line, we may consider ρ as a one dimensional representation of the group $\pi_1(U,p)$. In a standard way, any such representation induces a (flat) line bundle η on U, characterized by the fact that

$$\rho([\gamma]) = w_1(\gamma^*\eta), \tag{3.5}$$

where γ is any closed path and $\gamma^*\eta$ is the stable equivalence class of the vector bundle induced by γ from η.

Let us denote by $\theta(F)$ the trivial bundle over U with fiber F. Let X_1 be a subspace of X such that $X \cong X_1 \oplus R$, and let $l: \theta(X_1) \oplus \eta \to \theta(X_1) \oplus R \cong \theta(X)$ be the Fredholm bundle morphism defined by $l(x,e) = (x,0)$. Since X is a Kuiper space, all Banach bundles with fiber X are trivial and hence $\theta(X_1) \oplus \eta$ is isomorphic to the trivial bundle $\theta(X)$. Thus, up to isomorphism, l is equivalent to a continuous family $\alpha: U \to \Phi_0(X)$. From the construction of the index bundle described above, it is clear that for any closed path γ based at p

$$\mathrm{Ind}\ \alpha \circ \gamma = \gamma^*\eta. \tag{3.6}$$

This, together with (3.3) and (3.5), shows that $\rho = \sigma(\alpha)$.

■

Corollary 3.7 (cf. Theorem 2.11 of [**11**]) *Suppose that X is a Kuiper space, the norm of which is smooth on the complement of the origin. Let U be an open subset of*

X. Then in order that all C^2-Fredholm maps $f: U \to Y$ be orientable it is necessary and sufficient that $H^1(U; Z_2) = 0$.

Proof. According to Proposition 2.4 of [**11**], any Fredholm family parametrized by an open subset of a smooth Kuiper space is homotopic to the family of derivatives of a C^2-Fredholm map. Orientability being a homotopy invariant, the corollary follows from this observation and the preceding theorem.

■

Section 4 Comparison with the Elworthy-Tromba theory

We will now compare the Elworthy-Tromba construction with ours. While the original construction of the degree in [**10**] was done for proper C^2-Fredholm maps between manifolds, for simplicity and clarity of exposition, in the present section we shall describe a variant of their theory adapted to the context of maps of Banach spaces that are proper on closed, bounded sets. We shall take the opposite point of view in Section 5, where we shall extend our construction to manifolds.

The approach to orientation in [**10**] which we will now describe is based on oriented Fredholm structures. Let U be an open connected subset of X and let $f\colon U \to Y$ be a C^2-Fredholm map. Given $x \in U$, let K be any linear compact operator such that $Df(x) + K$ is an isomorphism. By the Inverse Function Theorem, the restriction ϕ of $f + K$ to a neighborhood V of x is a diffeomorphism. Moreover, the map $f \circ \phi^{-1}\colon \phi(V) \to Y$ is a compact vector field. In particular, we have

$$D(f \circ \phi^{-1})(y) = Id + C_y \quad for \quad y \in \phi(V), \tag{4.1}$$

where each C_y is a linear compact operator.

Consider the family $\mathcal{U}_f = \{(V, \phi) \mid (4.1) \text{ holds}\}$. Clearly $\{V \mid (V, \phi) \in \mathcal{U}_f\}$ is a covering of U. Moreover, it follows from (4.1) that given two members (V, ϕ) and (W, ψ) of $\mathcal{U}_f$, for any $y \in \psi(V \cap W)$

$$D(\phi \circ \psi^{-1})(y) \in GL_C(Y). \tag{4.2}$$

It is immediate that $\mathcal{U}_f$ is maximal with respect to this property. Let us recall that an atlas which is maximal with respect to (4.2) is called a *Fredholm structure*, and that a map f which satisfies (4.1) on any chart of the structure is said to be an *admissible map* with respect to the structure [**8**]. (In [**11**], the names are $C(I)$-structure and $C(I)$-map, respectively.) The above discussion may be summarized in the following proposition which is a special case of Theorem 2.2 in [**11**].

Proposition 4.3 *A C^2-Fredholm map $f\colon U \to Y$ induces a unique Fredholm structure $\mathcal{U}_f$ with respect to which it is admissible.*

We call a Fredholm structure *orientable* provided that there is a subatlas $\mathcal{A}$ of the Fredholm structure, called an *oriented structure*, which is maximal with respect to the following property:

$$D(\phi \circ \psi^{-1}) \text{ takes values in } GL_C^+(Y) \text{ for any charts } (V, \phi) \text{ and } (W, \psi) \text{ in } \mathcal{A}. \tag{4.4}$$

Now suppose that $\mathcal{A} = \{(V, \phi)\}$ is an oriented Fredholm structure for U. Suppose also that $f\colon U \to Y$ is an admissible map which is proper on closed, bounded subsets of U. Then, if Ω is open with $\bar{\Omega} \subset U$, and $y \notin f(\partial\Omega)$ is a regular value of f, the Elworthy-Tromba degree of f is defined by

$$\deg_{E.T.}(f, \Omega, y) = \sum_{x \in f^{-1}(y) \cap \Omega} \epsilon_x, \tag{4.5}$$

where

$$\epsilon_x = \deg_{L.S.}(D(f \circ \phi^{-1})(f(x)) \quad \text{for} \quad (V, \phi) \text{ a chart at } x.$$

Clearly the orientability of the structure ensures that the above definition of multiplicity ϵ_x is independent of the choice of chart.

Theorem 4.6 *Let U be an open, connected subset of X and let $f: U \to Y$ be C^2-Fredholm and nondegenerate. The map f is orientable if and only if the Fredholm structure $\mathcal{U}_f$ induced by f is orientable. Moreover, if f is proper on closed, bounded sets and if Ω is open and bounded, with $\bar{\Omega} \subset U$ and $y \notin f(\partial\Omega)$ is a regular value of f, then*

$$\deg_{E.T.}(f, \Omega, y) = \deg_{\epsilon}(f, \Omega, y), \tag{4.7}$$

where ϵ is the unique orientation of $f: U \to Y$ which agrees with ϵ_x at some regular point of f.

Proof. We first suppose that $f: U \to Y$ is orientable. We shall construct an oriented subatlas of the Fredholm structure induced by f. For this let us consider a fixed regular point p of f. For any point x in U, choose a path $\gamma_x: [0,1] \to U$ in U joining p to x. Also choose a parametrix η_x for the restriction of Df to γ_x, which also has the property that $\eta_x(0)$ is the inverse of $Df(p)$. If we let $K_x(t) = \eta_x(t)^{-1} - Df(\gamma_x(t))$, for $t \in I$, then $K_x: [0,1] \to L(X,Y)$ is a path of compact linear operators such that $K_x(0) = 0$ and such that $Df(\gamma_x(t)) + K_x(t)$ is invertible for each $t \in I$. Since $Df(x) + K_x(1)$ is invertible, it follows that the restriction of $f + K_x(1)$ to a sufficiently small disc $B(x, \delta_x)$ of radius δ_x centered at x is a diffeomorphism. Letting $V_x = B(x, \frac{1}{3}\delta_x)$ and ϕ_x be the restriction of $f + K_x(1)$ to V_x, we obtain a chart (V_x, ϕ_x). We shall show that $\mathcal{A} = \{(V_x, \phi_x)\}_{x \in U}$ extends to an oriented structure. To do so, let x and y be points in U for which $V_x \cap V_y \neq \emptyset$. If γ_x, γ_y, K_x and K_y are as described in the above construction, let γ be the closed path in U obtained by first following γ_x, then the straight line segment from x to y, and then γ_y^{-1} from y back to p. Using the interval $[0,3]$ as parameter space, γ is defined by

$$\gamma(t) = \begin{cases} \gamma_x(t), & 0 \leq t \leq 1 \\ (2-t)x + (t-1)y, & 1 \leq t \leq 2 \\ \gamma_y(3-t), & 2 \leq t \leq 3. \end{cases} \tag{4.8}$$

Let $L(t) = Df(\gamma(t))$ and let $M(t) = L(t) + K(t)$, where $K: [0,3] \to L(X,Y)$ is defined by

$$K(t) = \begin{cases} K_x(t), & 0 \leq t \leq 1 \\ (2-t)K_x(1) + (t-1)K_y(1), & 1 \leq t \leq 2 \\ K_y(3-t), & 2 \leq t \leq 3. \end{cases} \tag{4.9}$$

By construction, $M(0) = L(0) = L(3) = M(3) = Df(p)$ and since M is a compact perturbation of L it follows that the closed path L is homotopic to M by a base point preserving homotopy. From the assumption that f is orientable and the homotopy invariance of the parity, it follows that $\sigma(L,[0,3]) = \sigma(M,[0,3]) = 1$. But the restriction of M to $[0,1] \cup [2,3]$ takes values in $GL_C(X,Y)$, so that $\sigma(M,[0,1]) = \sigma(M,[2,3]) = 1$. Thus, by the multiplicativity with respect to partitions property of the parity,

$$\sigma(M,[1,2]) = 1. \tag{4.10}$$

Now, since $V_x \cap V_y \neq \emptyset$ either $V_x \subset B(y,\delta_y)$ or $V_y \subset B(x,\delta_x)$. Assuming the former it follows that $L(t) + K_y(1)$ is invertible for $t \in [1,2]$. Hence $N(t) = [L(t) + K_y(1)]^{-1}$ is a parametrix for the restriction of M to $[1,2]$. The definition of parity, (4.10) and the fact that $N(2) = M(2)^{-1}$ imply that

$$1 = \deg_{L.S.}(N(1)M(1)) \cdot \deg_{L.S.}(N(2)M(2)) = \deg_{L.S.}(N(1)M(1)),$$

i.e.,

$$\deg_{L.S.}((D\phi_y(x))^{-1} D\phi_x(x)) = 1. \tag{4.11}$$

The homotopy invariance of the Leray-Schauder degree, the connectedness of $V_x \cap V_y$, and (4.11) show that if $z \in V_x \cap V_y$ and $w = \phi_y(z)$, then

$$\begin{aligned} \deg_{L.S.}(D(\phi_x \circ \phi_y^{-1})(w)) &= \deg_{L.S.}(D\phi_x(z) \circ (D\phi_y(z))^{-1}) \\ &= \deg_{L.S.}((D\phi_y(z))^{-1} \circ D\phi_x(z)) \\ &= \deg_{L.S.}((D\phi_y(x))^{-1} \circ D\phi_x(x)) \\ &= 1. \end{aligned}$$

Thus $D(\phi_x \circ \phi_y^{-1})(w) \in GL_C^+(Y)$, and so the atlas $\mathcal{A}$ is oriented. It can be extended to a maximal oriented subatlas, which completes the proof that $\mathcal{U}_f$ is orientable.

To prove the converse, suppose that the Fredholm structure induced by f has an oriented subatlas $\mathcal{A} = \{(V,\phi)\}$. Define $\epsilon: R_f \to Z_2$ by

$$\epsilon(p) = \deg_{L.S.}(D(f \circ \phi^{-1})(z)), \tag{4.12}$$

where (V,ϕ) is a chart at p and $z = \phi(p)$. Observe that, in view of the orientability of $\mathcal{A}$, the right-hand side of (4.12) is independent of the choice of chart. We shall show that ϵ is an orientation for f and, at the same time, verify the last assertion of the proposition.

Indeed, let p and q be regular points of f and let $\gamma: [0,1] \to V$ be a path joining p to q. Now choose $\{(V_k,\phi_k)\}_{1 \le k \le n}$, a finite collection of charts in $\mathcal{A}$, together with a partition $0 = t_0 \le t_1 \le \quad \le t_n = 1$ of I such that, if $I_k = [t_{k-1}, t_k]$, then $\gamma(I_k) \subset V_k$, for $1 \le k \le n$. Define $L = Df \circ \gamma: I \to \Phi_0(X,Y)$ and then, letting L_k be the restriction of L to I_k, observe that $M_k: I_k \to GL(Y,X)$, defined by $M_k(t) = (D\phi_k(\gamma(t)))^{-1}$, is a parametrix for L_k, if $1 \le k \le n$.

The orientability of $\mathcal{A}$ implies that

$$M_k^{-1}(t_k) M_{k+1}(t_k) \in GL_C^+(Y) \quad \text{for} \quad 1 \le k \le n-1. \tag{4.13}$$

Since $GL^+_C(Y)$ is connected, one can choose for each k, a path $\tau_k\colon I_k \to GL^+_C(Y)$ such that $\tau_k(t_{k-1}) = Id$ and $\tau_k(t_k) = M_k^{-1}(t_k)M_{k+1}(t_k)$. Observe that if we define $N_k(t) = M_k(t)\tau_k(t)$ for $1 \leq k \leq n-1$, and $N_n(t) = M_n(t)$, then each N_k is again a parametrix for L_k. But now the parametrices $N_k, 1 \leq k \leq n$, have the property that N_k and N_{k-1} agree at t_k, and hence they can be patched together to obtain a parametrix $N\colon I \to GL(Y,X)$ for L. By the very definition of the parity,

$$\begin{aligned}\sigma(Df\circ\gamma, I) &= \deg_{L.S.}(N(0)L(0))\cdot\deg_{L.S.}(N(1)L(1))\\ &= \deg_{L.S.}(Df(p)(D\phi_0(\gamma(0)))^{-1})\cdot\deg_{L.S.}(Df(q)(D\phi_n(\gamma(1)))^{-1})\\ &= \deg_{L.S.}(D(f\circ\phi_0^{-1})(p))\cdot\deg_{L.S.}(D(f\circ\phi_n^{-1})(q)).\end{aligned}$$

This formula is just the assertion that

$$\sigma(Df\circ\gamma, I) = \deg_{E.T.}(Df(\gamma(0))\cdot\deg_{E.T.}(Df(\gamma(1)),$$

from which it is clear that $f\colon U \to Y$ is orientable and also that if ϵ is an orientation for $f\colon U \to Y$ which agrees with the Elworthy-Tromba degree at some regular point, then (4.7) holds.

∎

Remark (cf. [**11**]) Given a Fredholm structure $\mathcal{A} = \{(V,\phi)\}$ on U, one can define a 1-Čech cocycle σ of the covering $\{V\}$ with values in Z_2 by taking

$$\sigma(V,W) = \deg_{L.S.}(D(\phi\circ\psi^{-1})(y)$$

for $y \in \phi(V\cap W)$ (in general, a refinement needs to be taken since $V\cap W$ might not be connected). It is easy to see that the structure is orientable if and only if the cocycle σ is a coboundary, i.e., there exists $\epsilon\colon\{V\}\to Z_2$ such that

$$\sigma(V,W) = \epsilon(V)\cdot\epsilon(W).$$

If the first Čech cohomology group of U, $\check{H}^1(U;Z_2)$ with Z_2 coefficients, is trivial, then (after refinement) any cycle is a coboundary and hence any Fredholm structure on U is orientable. This is a corollary of Theorem 2.10 in [**11**]. In general, the cocycle σ will define a cohomology class $\omega(\mathcal{A})$ belonging to $\check{H}^1(U;Z_2)$ which is an obstruction to the orientability of the structure. If $\mathcal{A} = \mathcal{U}_f$, then the proof of the preceding theorem can be modified to show that, after appropriate identification, $\omega(\mathcal{A}) = \omega(Df)$.

Section 5 Degree for Maps Between Banach Manifolds

We shall extend our construction of the degree to the case of proper C^2-Fredholm maps between Banach manifolds. A general reference for Banach manifolds and bundles will be the book of Lang [**27**], although we will use slightly different notation.

Given a morphism $\phi\colon\zeta \to \eta$ between the Banach bundles $\zeta = (E, \pi, \Lambda)$ and $\eta = (F, \pi, \Lambda)$ over the base Λ, its restriction to the fiber $E_\lambda = \pi^{-1}(\lambda)$ at λ of ζ with values in the fiber $F_\lambda = \pi^{-1}(\lambda)$ at λ of η will be denoted by $\phi(\lambda)$. The pullback of a bundle ζ or a morphism ϕ, by a map $g\colon\Lambda' \to \Lambda$ will be denoted by $g^*(\zeta)$ and $g^*(\phi)$, respectively. The trivial bundle $\pi : \Lambda \times X \to \Lambda$ will be denoted by $\theta_\Lambda(X)$ or by $\theta(X)$, when the base space is clear from the context.

A morphism $\phi\colon\zeta \to \eta$ will be called a *Fredholm morphism (of index 0)* if each $\phi(\lambda)\colon E_\lambda \to F_\lambda$ is Fredholm (of index 0). As before regular points of a morphism are points λ of the base such that $\phi(\lambda)$ is an isomorphism.

Given a Fredholm morphism $\phi\colon\zeta \to \eta$ of Banach bundles over $I = [0, 1]$ such that $\phi(0)$ and $\phi(1)$ are isomorphisms, its parity will be defined as follows:

By the contractibility of I, all bundles over I are trivializable and hence there exist trivializations $\psi\colon\theta(E_0) \to \zeta$ and $\mu\colon\theta(F_0) \to \eta$. Since $\mu^{-1} \circ \phi \circ \psi\colon\theta(E_0) \to \theta(F_0)$ is a morphism of trivial bundles, it is necessarily of the form $(t, x) \mapsto (t, \alpha(t)x)$, where $\alpha\colon I \to \Phi_0(X, Y)$ is a path of Fredholm maps. By definition, the *parity of the morphism* ϕ on the interval I is given by

$$\sigma(\phi, I) = \sigma(\alpha, I). \tag{5.1}$$

That the right-hand side of (5.1) is independent of the choice of trivialization follows immediately from the invariance of the parity under composition with a path of isomorphisms. All of the properties of the parity extend to this more general context. In particular, it is multiplicative with respect to the direct sum of bundles.

More generally, if $\phi\colon\zeta \to \eta$ is a morphism of bundles over the base space Λ, and $\gamma\colon I \to \Lambda$ is a path joining two regular points of ϕ, then, exactly as before, one can define the *parity* $\sigma(\phi, \gamma)$ *of* ϕ *along* γ by

$$\sigma(\phi, \gamma) = \sigma(\gamma^*(\phi), I). \tag{5.2}$$

Definition *A Fredholm morphism of Banach bundles $\phi\colon\zeta \to \eta$ is said to be orientable if there exists a function $\epsilon\colon R_\phi \to \{-1, +1\}$ which has the property that*

$$\sigma(\phi, \gamma) = \epsilon(\gamma(0)) \cdot \epsilon(\gamma(1)), \tag{5.3}$$

for each path $\gamma\colon [0, 1] \to \Lambda$ joining two regular points of ϕ.

We now turn to the degree for proper C^2-Fredholm maps. First, notice that if U is an open subset of a Banach space X and if $f\colon U \to Y$ is a proper C^2-Fredholm map with values in a Banach space Y having an orientation ϵ, then, by the homotopy invariance and the additivity properties of the degree, the quantity $\deg_\epsilon(f, \Omega, y)$ is independent of the choice of $y \in Y$ and the open bounded subset Ω of U containing $f^{-1}(y)$. This common value will be called the *degree* of the proper map f and will be denoted by $\deg_\epsilon(f)$. In particular, if y is any regular value of f, then

$$\deg_\epsilon(f) = \sum_{x \in f^{-1}(y)} \epsilon(x). \tag{5.4}$$

We shall use this formula in order to extend the above notion of degree to proper maps between Banach manifolds.

Let M and N be smooth Banach manifolds with N connected, and let $f: M \to N$ be a C^2-Fredholm map. The family of derivatives of f is now the Fredholm morphism

$$Df: T_M \to f^*(T_N),$$

where T_M and T_N are the tangent bundles of M and N, respectively. As before, we shall say that f is orientable if Df possesses an orientation ϵ, which will be called an orientation for f.

Definition *Let $f: M \to N$ be a proper C^2-Fredholm map with orientation ϵ. The degree of f with respect to ϵ is defined by*

$$deg_\epsilon(f) = \sum_{x \in f^{-1}(n)} \epsilon(x), \tag{5.5}$$

where $n \in N$ is any regular value of f.

To have the degree properly defined it is necessary to show that the right-hand side of (5.5) is independent of the choice of regular value n. By properness of f and the Inverse Function Theorem, the right-hand side of (5.5) is locally constant on the set of regular values of the map f. We will now show that it is actually constant and that the resulting degree verifies the homotopy property (2.8) with respect to proper, orientable homotopies, by following closely the type of argument that we used to establish the corresponding properties in Section 2.

It is easy to see that Proposition 1.5 remains true if we substitute Fredholm families by Fredholm morphisms and transversal finite dimensional subspaces of Y by finite dimensional subbundles of the range transversal to the morphism. Using this, the geometric argument in the proof of Proposition 2.2 extends word by word to maps between manifolds. Hence one obtains

Proposition 5.6 *Suppose that $H: [0,1] \times M \to N$ is a proper, orientable homotopy of C^2-Fredholm maps and that n is a regular value of both H_0 and H_1. If ϵ is an orientation for H, then*

$$\sum_{x \in H_0^{-1}(n)} \epsilon_0(x) = \sum_{x \in H_1^{-1}(n)} \epsilon_1(x)$$

The independence of the right hand side of (5.5) from the choice of the regular value n follows from this proposition and an argument from [**31**] which we shall sketch below for the sake of completeness.

A homotopy $G\colon [0,1] \times N \to N$ is called an isotopy if each section G_t is a diffeomorphism. Two points n and n' belonging to N are called isotopic if there exists an isotopy between the identity map and a diffeomorphism sending one point to the other. It is easy to see that two points in a Banach space are always isotopic by an isotopy whose sections coincide with the identity map outside of a bounded set. This implies that each point $n \in N$ has a neighborhood of points isotopic to n. Hence the property of being isotopic is an open equivalence relation. But, since N is connected, there is only one equivalence class and any two points in N are isotopic.

Now let n_0 and n_1 be two regular values of f and let $G\colon [0,1] \times N \to N$ be an isotopy such that $G_0 = Id$ and $G_1(n_1) = n_0$. Define $H(t,x) = G_t[f(x)]$. Since $DG_t(y)$ is always an isomorphism it follows easily from the properties of the parity that if ϵ is an orientation of f, then $\bar{\epsilon}(t,x) = \epsilon(x)$ defines an orientation for H. Using this orientation and Proposition 5.6, it follows that

$$\sum_{x \in f^{-1}(n_0)} \epsilon(x) = \sum_{x \in H_0^{-1}(n_0)} \bar{\epsilon}_0(x) = \sum_{x \in H_1^{-1}(n_0)} \bar{\epsilon}_1(x) = \sum_{x \in f^{-1}(n_1)} \epsilon(x).$$

This shows that the degree is properly defined. That it verifies the homotopy property (2.8) with respect to proper, orientable homotopies follows immediately from Proposition 5.6 and the generalized Sard-Smale theorem. All of the other properties listed in Theorem 2.7 are possessed by this degree. Moreover, much as we did in [**15**], it can be shown that our degree is multiplicative with respect to the composition of maps. If $H^1(M; Z_2) = 0$, then the above construction provides a degree for all proper C^2-Fredholm maps defined on M that is homotopy invariant (in the above sense) with respect to proper homotopies.

Given a Fredholm structure on a Banach manifold N a C^2-Fredholm map $f : M \to N$ induces a pull-back Fredholm structure on M such that f becomes admissible with respect to these two structures (see Theorem 2.2 in [**11**]). In our construction of degree, Fredholm structures are not used at all. Nevertheless, it is of some interest to have the following proposition which asserts that the orientability of a map can be characterized in terms of the pull-back operation, at least in some special case.

Proposition 5.7 *Assume that the target manifold N admits Fredholm structures and smooth partition of unity. If a C^2-Fredholm map $f\colon M \to N$ is nondegenerate and orientable, then the pull-back of any orientable structure on N is an orientable structure on M. If N admits an orientable Fredholm structure and $f\colon M \to N$ is such that the pull-back by f of this structure is orientable, then f is orientable.*

Proof. By Theorem 2.2 in [**11**], any Fredholm structure on N is induced by a Fredholm map $g\colon N \to Y$, where Y is the model space for N. With the obvious modifications, the proof of Theorem 4.6 works also for maps from manifolds into Banach spaces. Therefore. the Fredholm structure induced by g will be orientable if and only if $g\colon N \to Y$ is an orientable map . From the chain rule it follows that the composition of orientable maps is orientable. The first assertion follows from this, since the pull-back by f of the Fredholm structure induced by g on N coincides with the one induced by $g \circ f$. To prove the second assertion, we again use the fact that the given orientable structure is induced by an orientable map $g : N \to Y$ and that, by the chain rule, if γ is any homotopy class of closed paths in M based at a regular point of $h = g \circ f$, then $\sigma(Dh, \gamma) = \sigma(Dg, f \circ \gamma) \cdot \sigma(Df, \gamma)$. Therefore, if $\sigma(Dh, \gamma) = 1$, then $\sigma(Df, \gamma) = 1$ also.

■

In [**10**], the degree was defined for proper C^2-Fredholm maps between completely orientable, separable Banach manifolds (completely orientable means that all the Fredholm structures on the manifold are orientable). Using the above discussion, together with the same arguments as were used in the proof of Theorem 4.6, one can easily show that in this particular setting our degree coincides, up to the choice of orientable structure on N, with the Elworthy-Tromba degree. But, of course, our degree is defined even if N fails to have any orientable Fredholm structure. Also, because we are using Quinn and Sard's generalization of Sard's Theorem in [**38**], we do not require separability. Although we shall not show this here, it is not difficult to prove that in finite dimensions our degree coincides with Olum's degree [**36**] for orientable maps between not necessarily orientable manifolds.

Remark In terms of the obstruction classes introduced in Section 3, the orientability of a nondegenerate map $f: M \to N$ can be characterized as follows: f is orientable if and only if for any $g: N \to Y$, $\omega(D[g \circ f]) = f^*[\omega(Dg)]$.

In the final part of this section we shall prove a reduction property of our degree similar to the well-known reduction property of the Leray-Schauder degree. Then we will apply our result to give a short and simple proof of the Basic Theorem 3.1 of [**44**].

The following situation occurs frequently in nonlinear analysis and differential topology. There is a map $f: M \to N$ and a closed submanifold Z of N transversal to f. In this case, $W = f^{-1}(Z)$ is a submanifold of M and f induces in an obvious way a map $g: W \to Z$. Under reasonable assumptions, the calculation of the degree of f reduces to the calculation of the degree of g. We shall prove such a reduction property for C^2-Fredholm maps. Solely for the sake of simplicity, we shall consider only the case of maps between Banach spaces and we will take Z to be a closed, complemented subspace of the range.

Theorem 5.8 *Let X and Y be Banach spaces and let $f: X \to Y$ be a C^2-Fredholm map of index 0. Let Z be a closed, complemented subspace of Y which is transverse to f, i.e., $ImDf(x) + Z = Y$ for any $x \in f^{-1}(Z)$. Then:*

(i) $M = f^{-1}(Z)$ is a Banach submanifold of X.

(ii) The map $g: M \to Z$ which is the restriction of f to M is a C^2-Fredholm map of index 0.

(iii) The set R_g of regular points of g coincides with $R_f \cap M$ and the restriction ϵ' to R_g of any orientation ϵ for f is an orientation of g.

Moreover, if f is proper, then so is g and

$$\deg_{\epsilon}(f) = \deg_{\epsilon'}(g). \tag{5.7}$$

Proof. By the Implicit Function Theorem, M is a Banach submanifold of X. Identifying the tangent space T_x of $x \in X$ with X, the tangent space of each $m \in M$ is given by $T_m M = Df(m)^{-1}(Z)$. Denote the tangent bundle of M by T_M.

We will apply Theorem 6.1, with $\Lambda = M$ and α the restriction of Df to M. The bundle ζ of that theorem is precisely T_M. Also, under the identification of $g^*(T_Z)$ with $\theta(Z)$ we see that Dg coincides with the morphism β of Theorem 6.1. From this Theorem it follows that g is Fredholm of index 0, and also that the regular values of f and g coincide. Finally, to verify assertion (iii), let ϵ be an orientation for f and let $\gamma: I \to M$ be a path joining regular points of g. According to the last assertion in Theorem 6.1, $\epsilon(\gamma(0)) \cdot \epsilon(\gamma(1)) = \sigma(Df, \gamma) = \sigma(Dg, \gamma)$, from which it follows that the restriction of ϵ to the regular values of g is an orientation of g.

■

Remark If $f = Id + C: X \to X$ with C compact and $C(X) \subset Z$, then$f^{-1}(Z) \subset Z$ and hence, except for the C^2 assumption, Theorem 5.6 contains as a very special case the standard reduction property of the Leray-Schauder degree.

Remark We observe that a reduction of the type described in the preceding theorem was already used by Caccioppoli [**7**]. Indeed, he introduced such a reduction with Z of finite dimension and regardless orientability of the manifold M used the identity (5.7) mod-2 as the very definition of his degree.

Remark As we already mentioned, the general Sard-Smale Theorem for a nonlinear Fredholm map of index 1 requires that the map be C^2. However, the assertions (i)-(iii) of Theorem 5.6 also hold in the C^1-setting. This can be used in order to extend the present degree theory to orientable, C^1-Fredholm maps much like in [**3**].

Corollary 5.9 (cf. Th. 3.1 of [**42**]) *Let X and Y be Banach spaces and suppose that $F: X \times Y \to X$ is a C^2-Fredholm map which has 0 as a regular value. Assume, moreover, that for each $y \in Y$, $F_y: X \to X$ is a compact perturbation of the identity. Let M be the manifold $F^{-1}(0)$ and let g be the restriction to M of the projection map of $X \times Y$ onto Y. Then $g: M \to Y$ is an orientable C^2-Fredholm map of index 0. A point $m = (x, y)$ is a regular point of g if and only if $DF_y(x) = D_xF(x, y) \in GL_C(X)$, and the function $\epsilon: R_g \to \{-1, +1\}$ defined by*

$$\epsilon(m) = \deg_{L.S.}(DF_y(x)) \quad \text{for} \quad m = (x, y) \in R_g \tag{5.8}$$

is an orientation for g.

Proof. Define $f(x, y) = (F(x, y), y)$. It is clear that $f: X \times Y \to X \times Y$ is a C^2-Fredholm map having the property that at each point $(x, y) \in X \times Y$, $DF(x, y)$ is a compact perturbation of the identity. Thus $f: X \times Y \to X \times Y$ is an $\mathcal{F}$-map, where $\mathcal{F} \subset \Phi_0(X \times Y, X \times Y)$ is the class of compact perturbations of the identity. According to the discussion in Section 2, it is orientable with orientation given by $\epsilon(x, y) = \deg_{L.S.}(Df(x, y))$. On the other hand, the reduction property of the Leray-Schauder degree implies that $\deg_{L.S.}(Df(x, y)) = \deg_{L.S.}(DF_y(x))$, and hence the corollary follows from Theorem 5.6 applied to f with $Z = \{0\} \times Y \subset X \times Y$.

■

Corollary 5.10 *Let $f: X \times Y \to X$ be a C^2 map of the form $f(x, y) = x - C(x, y)$ where $C: X \times Y \to X$ is compact. Assume that 0 is a regular value of f and that there exists* *a continuous function* $\rho: Y \to R^+$ *having the property that* $f_y(x) \neq 0$ *if* $\|x\| \geq \rho(y)$. *Then the map* $g: f^{-1}(0) \to Y$, *defined by restriction to this manifold of the projection map* $\pi: X \times Y \to Y$, *is a proper, orientable,* C^2*-Fredholm map and*

$$\deg_\epsilon(g) = \deg_{L.S.}(f_y, B(0, \rho(y)), 0), \tag{5.10}$$

where y *is any point in* Y *and* $B(0, \rho(y))$ *is the ball centered at the origin of radius* $\rho(y)$.

Section 6 A General Reduction Property of the Parity

We will devote this last section to a proof of the reduction property of the parity which was used in the proof of Theorem 5.8 (cf. [**20**] and [**15**] for special cases).

Theorem 6.1 *Let Λ be a paracompact topological space and let $\alpha\colon \Lambda \to \Phi_0(X,Y)$ be a parametrized family of Fredholm operators of index 0. Let V be a closed, complemented subspace of Y having the property that*

$$\alpha(\lambda)X + V = Y \quad \text{for all} \quad \lambda \in \Lambda. \tag{6.2}$$

Then:
i) The set $E = \{(\lambda, x) \in \Lambda \times X : \alpha(\lambda)x \in V\}$ is the total space of a subbundle ζ of $\theta(X)$ with fibers $E_\lambda = \alpha(\lambda)^{-1}(V)$.
ii) The morphism $\beta\colon \zeta \to \theta(V)$ defined on E by $\beta(\lambda, x) = (\lambda, \alpha(\lambda)x)$ is a Fredholm morphism of index 0.
iii) The set of regular points of α and β coincide. Furthermore, if $\gamma\colon I \to \Lambda$ is any path joining regular points of α, then

$$\sigma(\alpha, \gamma) = \sigma(\beta, \gamma).$$

Proof. We begin by showing that each E_λ has a complement in X. Let $\lambda \in \Lambda$, and set $K_\lambda = \text{Ker } \alpha(\lambda)$, $R_\lambda = \text{Im } \alpha(\lambda)$ and $V_\lambda \equiv R_\lambda \cap V$. In view of the canonical isomorphism $V/(V \cap R_\lambda) \cong (V + R_\lambda)/R_\lambda$, and the assumption that $V + R_\lambda = Y$, it follows that

$$\dim V/(V \cap R_\lambda) = \dim (V + R_\lambda)/R_\lambda = \dim Y/R_\lambda = \text{codim } R_\lambda. \tag{6.3}$$

In particular, (6.3) implies that V_λ is of finite codimension in V. Consequently, V_λ is complemented in V. Since, by assumption, V itself is complemented in Y, we conclude that V_λ has a complement Y_λ in Y. From this it follows immediately that R_λ splits into a direct sum

$$R_\lambda = V_\lambda \oplus (Y_\lambda \cap R_\lambda). \tag{6.4}$$

On the other hand, K_λ, being finite dimensional, has a complement X_λ in X. Moreover, in view of the Open Mapping Theorem, the restriction of $\alpha(\lambda)$ to X_λ is an isomorphism of X_λ with R_λ. Using the decomposition (6.4), we see that the inverse of this isomorphism maps $Y_\lambda \cap R_\lambda$ onto a complement of $E_\lambda \cap X_\lambda$ in X_λ. But this subspace is also a complement of E_λ in X.

Let us consider now the vector bundle morphism $\gamma\colon \theta(X) \to \theta(Y/V)$ given by the composition

$$\theta(X) \xrightarrow{\bar\alpha} \theta(Y) \xrightarrow{p} \theta(Y/V), \tag{6.5}$$

where $\bar\alpha$ is the morphism induced by α (i.e., $\bar\alpha(\lambda, x) = (\lambda, \alpha(\lambda)x),$) and p is the quotient projection. It follows from (6.2) that each $\gamma(\lambda)$ is surjective. Since for any λ, $E_\lambda =$ Ker $\bar\alpha(\lambda)$ is complemented in X, by Proposition 6 of Chap.3, Sect.3 of [**27**],the sequence

$$\theta(X) \xrightarrow{\gamma} \theta(Y/X) \to 0$$

is exact, which means that $\zeta \equiv \text{Ker } \gamma$ is a subbundle of the trivial bundle $\theta(X)$. This proves i).

To prove ii), first we note that β is a morphism, since it is the restriction of the morphism $\bar{\alpha}$ to a subbundle. Moreover, for any $\lambda \in \Lambda$, one has $\text{Ker } \beta(\lambda) = \text{Ker } \alpha(\lambda)$ and also $\text{Im } \beta(\lambda) = \text{Im } \alpha(\lambda) \cap V = V_{\lambda}$. Thus, from (6.3) it follows that each $\beta(\lambda)$ is Fredholm of index 0.

The first assertion in iii) is a consequence of the fact that $\text{Ker } \alpha(\lambda) = \text{Ker } \beta(\lambda)$. To verify the second assertion, recall that, according to Proposition 8 of Chap.3, Sect.5 of [**27**], there is a subbundle η of $\theta(X)$ having the property that $\zeta \oplus \eta = \theta(X)$. Define $\delta\colon \eta \to \theta(Y)$ to be the restriction of $\bar{\alpha}$ to η. Then each $\delta(\alpha)$ is injective and $\text{Im } \delta(\alpha)$ is a complement, in Y, of Z. Hence, by Prop.5 of Chap.3, Sect.3 of [**27**], $\psi = \text{Im } \delta$ is a subbundle of $\theta(Y)$ and the morphism $\bar{\alpha}$ splits into a direct sum

$$\bar{\alpha} = \delta \oplus \beta\colon \eta \oplus \zeta \to \psi \oplus \theta(V). \tag{6.6}$$

Now let $\gamma\colon I \to \Lambda$ be a path which joins regular points of α. Then (6.6) and γ^* induce the following splitting of morphisms and bundles over I:

$$\bar{\alpha}' = \delta' \oplus \beta'\colon \eta' \oplus \zeta' \to \psi' \oplus \theta(V),$$

where for notational convenience we have denoted by $'$ the action of γ^* on morphisms and bundles. From this, using the fact that $\delta'\colon \eta' \to \psi'$ is a path of isomorphisms together with the multiplicative property of the parity under direct sums, it follows that

$$\sigma(\alpha, \gamma) = \sigma(\alpha', I) = \sigma(\delta', I) \cdot \sigma(\beta', I) = \sigma(\beta', I) = \sigma(\beta, \gamma).$$

■

Proposition 1.5 follows immediately from the above theorem by taking V to be a finite dimensional subspace of Y.

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