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ON THE DIVISOR FUNCTION IN SHORT INTERVALS

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Abstract. Let $d(n)$ denote the number of positive divisors of the natural number n. The aim of this paper is to investigate the validity of the asymptotic formula

$$\sum_{x<n\le x+h(x)} d(n) \sim h(x)\log x$$

for $x \to +\infty$, assuming a hypothetical estimate on the mean

$$\int_X^{X+Y} (\Delta(x+h)-\Delta(x))^2 \,\mathrm{d}x,$$

which is a weakened form of a conjecture of M. Jutila.

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1. Introduction

As usual, let

$$\Delta(x) = \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1) \tag{1.1}$$

denote the error term in the Dirichlet divisor problem, where $d(n)$ is the number of divisors of n and γ is the Euler-Mascheroni constant. The current best upper bound for the error term $\Delta(x)$ is due to M. N. Huxley [3] who showed that for every $\varepsilon > 0$ we have

$$\Delta(x) \ll x^{131/416+\varepsilon}. \tag{1.2}$$

The above estimate implies that for $h(x) = x^\theta$ and $\theta > 131/416$ we can deduce that

$$\sum_{x<n\le x+h(x)} d(n) \sim h(x)\log x, \tag{1.3}$$

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for $x \to +\infty$. To date, this is the best known result about the above asymptotic formula, as remarked by M. Z. Garaev, F. Luca and W. G. Nowak [1].

The aim of this paper is to investigate the validity of the asymptotic formula (1.3) for smaller values of $h(x)$, under the assumption of an unproved heuristic hypothesis. We note that the upper bound of the correct order for the sum of the divisor function in short intervals it is a simpler problem solved by P. Shiu [9], who proved

$$\sum_{x<n\le x+h(x)} d(n) \ll h(x)\log x,$$

for $x^{\varepsilon} \le h(x) \le x$.

We notice that M. Jutila [6] conjectured that

$$\Delta(x+h) - \Delta(x) \ll \sqrt{h}\, x^{\varepsilon}, \tag{1.4}$$

for $x^{\varepsilon} \ll h \ll x^{1/2-\varepsilon}$, which is close to being best possible in view of the omega result

$$\Delta(x+h) - \Delta(x) = \Omega\left(\sqrt{h}\,\log^{3/2}\left(\frac{\sqrt{x}}{h}\right)\right), \tag{1.5}$$

valid for $T \le x \le 2T$, $T^{\epsilon} \le h = h(T) \le T^{1/2-\epsilon}$, see A. Ivić [4, Corollary 2]. If we assume the conjecture (1.4) of M. Jutila one easily obtains

$$\sum_{x<n\le x+h(x)} d(n) \sim h(x)\log x,$$

for every $h(x) \gg x^{\varepsilon}$, since for the large values of $h(x)$ the validity of the asymptotic formula is insured by the cited result of M. N. Huxley. With the aim to relax our assumption we may request that the upper bound (1.4) holds on average. Then we state the following weaker conjecture.

Conjecture. *Let $h(x) = x^{\theta}$ with $\theta > 0$, $k \ge 1$, $Y = X^{\alpha}$ with $\theta < \alpha \le 1$ and $\varepsilon > 0$ arbitrarily small. There exist $X_0 > 0$ such that*

$$\int_X^{X+Y} (\Delta(x+h(x)) - \Delta(x))^k \,\mathrm{d}x \ll h(X)^{k/2}\, Y X^{\varepsilon} \tag{1.6}$$

uniformly for $X \ge X_0$.

In 1984, M. Jutila [7] proved that

$$\int_X^{X+Y} (\Delta(x+h) - \Delta(x))^2 \,\mathrm{d}x \ll hY \log^3\left(\frac{\sqrt{X}}{h}\right),$$

with $X^{\varepsilon} \ll h \le \sqrt{X}/2$ and $hY \gg X^{1+\varepsilon}$, which implies the Conjecture for $k=2$ and $1-\alpha < \theta < 1/2$. Moreover, he conjectured that

$$\int_X^{2X} (\Delta(x+h) - \Delta(x))^4 \,\mathrm{d}x \ll h^2 X^{1+\varepsilon}, \tag{1.7}$$

which is our Conjecture with $k = 4$ and $\alpha = 1$. In 2009 A. Ivić [4, Theorem 4] proved (1.7) for $X^{3/8} \ll h \ll X^{1/2}$, which implies the Conjecture for $k = 4$, $\alpha = 1$ and $\theta > 3/8$.

Assuming the Conjecture we obtain the following result.

Theorem 1.1. *Let $h(x) = x^\theta$ and assume that Conjecture holds for fixed values of $k \geq 1$ and $0 < \alpha < 1$. Then we have*

$$\sum_{x<n\leq x+h(x)} d(n) \sim h(x)\log x,$$

for $2\alpha/(k + 2) < \theta < \alpha$.

We observe that the Conjecture may be further weakened at least in two ways. The first is to assume that (1.6) holds for $2\alpha/(k + 2) < \theta \leq 131/416$, since the Theorem is not useful for $\theta > 131/416$ and values of θ such that $\theta \leq 2\alpha/(k + 2)$ it is not used in the proof. Note that even if we assume (1.6) for very small values of θ we do not obtain the result for θ smaller than $2\alpha/(k + 2)$, which is indeed the limit of the method. The second way is to substitute in the Conjecture the term

$$\Delta(x + h(x)) - \Delta(x)$$

with

$$\Delta(x + h(x)) - \Delta(x) + \Sigma(x, h),$$

where $\Sigma(x, h)$ is an arbitrarily function negligible respect to $h(x)\log x$, and following the method introduced by D. R. Heath-Brown in [2].

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2. The basic lemma

Let $h(x) = x^\theta$ for some $0 < \theta < 1$ and

$$E_\delta(X, h) = \{X \leq x \leq 2X : |\Delta(x + h) - \Delta(x)| \geq \delta h(x)\log x\}.$$

It is clear that (1.3) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, h) = \emptyset$ for $X \geq X_0(\delta)$. Hence for small $\delta > 0$, X tending to ∞ and $h(x)$ suitably small with respect to x, the set $E_\delta(X, h)$ contains the exceptions, if any, to the expected asymptotic formula (1.3). The first result of the paper is about a property of the set $E_\delta(X, h)$ which is fundamental for our method.

Lemma 2.1. *Let $h(x) = x^\theta$ with $0 < \theta < 1$, $\delta > 0$ and X be sufficiently large. If $x_0 \in E_\delta(X, h)$ then for every $0 < \delta' < \delta$ and $0 < \theta' < \theta$ we have*

$$E_{\delta'}(X, h) \supseteq [x_0 - X^{\theta'}, x_0 + X^{\theta'}] \cap [X, 2X].$$

Proof. Let $h(x) = x^\theta$, $0 < \theta' < \theta < 1$,

$$x_0 \in E_\delta(X, h) \tag{2.1}$$

and

$$x \in [x_0 - X^{\theta'}, x_0 + X^{\theta'}] \cap [X, 2X]. \tag{2.2}$$

Let

$$F(x, h) = \Delta(x + h(x)) - \Delta(x),$$

where $\Delta(x)$ is defined in (1.1). Then

$$|F(x,h)| = \left| \sum_{x < n \le x + h(x)} d(n) - M(x,h) \right|,$$

where

$$M(x,h) = (x + h(x)) \left(\log(x + h(x)) + 2\gamma - 1 \right) - x \left(\log x + 2\gamma - 1 \right).$$

Then we have

$$\begin{aligned} |F(x,h)| =& |F(x_0,h) + F(x,h) - F(x_0,h)| \\ \ge& \, |\Delta(x_0 + h(x_0)) - \Delta(x_0)| \\ &- \left| \sum_{x < n \le x + h(x)} d(n) - \sum_{x_0 < n \le x_0 + h(x_0)} d(n) \right| \\ &- \Big| M(x_0, h) - M(x,h) \Big|. \end{aligned} \tag{2.3}$$

We define

$$R = \sum_{x < n \le x + h(x)} d(n) - \sum_{x_0 < n \le x_0 + h(x_0)} d(n)$$

and we easily see that

$$|R| \le \sum_{x < n \le x_0} d(n) + \sum_{x + h(x) < n \le x_0 + h(x_0)} d(n),$$

for $x < x_0$ and

$$|R| \le \sum_{x_0 < n \le x} d(n) + \sum_{x_0 + h(x_0) < n \le x + h(x)} d(n),$$

for $x > x_0$. In either case we deduce that

$$R \ll X^{\theta' + \varepsilon},$$

since $d(n) \ll n^\varepsilon$ for every $\varepsilon > 0$. For every fixed value of θ we can choose $\varepsilon > 0$ sufficiently small such that

$$R \ll X^{\theta' + \varepsilon} = o(X^\theta).$$

Moreover, by the definition of $M(x,h)$, we have

$$\Big|M(x_0,h) - M(x,h)\Big| \ll X^{\theta+\theta'-1} = o(X^\theta),$$

since $\theta' < 1$. Then from (2.1) and (2.3) we conclude that

$$|\Delta(x+h(x)) - \Delta(x)| \geq \delta' h(x)\log x,$$

for every $0 < \delta' < \delta$ and the Lemma follows. □

3. Proof of the Theorem

We will always assume that X_n are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence. Our theorem asserts that, under the assumption of the Conjecture for fixed values of $k \geq 1$ and $0 < \alpha < 1$, the asymptotic formula

$$\sum_{x<n\leq x+h(x)} d(n) \sim h(x)\log x = x^\theta \log x, \tag{3.1}$$

holds for $2\alpha/(k+2) < \theta < \alpha$. In order to prove the theorem we assume that (3.1) does not hold. Then there exist a constant $\delta > 0$ and a sequence $X_n \to \infty$ such that

$$E_\delta(X_n,h) = \{X_n \leq x \leq 2X_n : |\Delta(x+h(x)) - \Delta(x)| \geq \delta h(x)\log x\} \neq \emptyset.$$

The use of the Lemma with $\delta' < \delta$ and $0 < \theta' < \theta$ implies that there exists a sequence $x_n \to \infty$ such that

$$[x_n, x_n + X^{\theta'}] \subset E_{\delta'}(X_n,h)$$

and then

$$\begin{aligned} X_n^{k\theta+\theta'}\log^k X_n &\ll \int_{x_n}^{x_n+X^{\theta'}} (\Delta(x+h(x)) - \Delta(x))^k \, \mathrm{d}x \\ &\ll \int_{x_n}^{x_n+Y} (\Delta(x+h(x)) - \Delta(x))^k \, \mathrm{d}x, \end{aligned} \tag{3.2}$$

where $Y = X_n^\alpha$, with $\theta < \alpha \leq 1$. Besides, assuming the Conjecture, we have

$$\int_{x_n}^{x_n+Y} (\Delta(x+h(x)) - \Delta(x))^k \, \mathrm{d}x \ll X_n^{\theta k/2+\alpha+\varepsilon}. \tag{3.3}$$

For X_n sufficiently large and $\theta > 2\alpha/(k+2)$ we have a contradiction between (3.2) and (3.3), and this completes the proof of the Theorem.

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