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THE WRONSKIAN AND ITS DERIVATIVES

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(Communication presented by Prof. Gaetana Restuccia)

ABSTRACT. This note reports on some joint work with I. Scherbak, aiming to overview a connection between generalized wronskians (of fundamental systems of solutions of linear ordinary differential equations with constant coefficients) and the intersection theory of complex Grassmann varieties. Detailed computations are performed in the easiest case of the intersection theory of the grassmannian of lines in projective spaces.

Introduction

This paper aims to show the existence, according to [5] and [6], of a bridge between a classical and fairly elementary analytic device, the *wronskian* associated to a basis of solutions of a linear ordinary differential equation with constant coefficients, and some classical enumerative algebraic geometry known as *Schubert Calculus*. The latter is the set of formal rules governing the intersection theory of Grassmann varieties, which permits to solve a class of enumerative problems like, e.g., the very popular *how many lines do intersect four others in general position in the three dimensional complex projective space?*

Informally, for $0 \leq r \leq d - 1$, the grassmannian $G(r, \mathbb{P}^d)$ parameterizes all closed linear subvarieties of dimension r of the d -dimensional complex projective space \mathbb{P}^d . It is a projective algebraic variety of dimension $(r + 1)(d - r)$ and, in fact, it can be embedded in the projective space $\mathbb{P}(\bigwedge^{r+1} \mathbb{C}^{d+1})$ via the so called *Plücker morphism*.

The easiest example of a grassmannian which is not a projective space is $G(1, \mathbb{P}^3)$, the variety that parameterizes all the projective lines in \mathbb{P}^3 . It can be embedded *à la Plücker* in \mathbb{P}^5 as a quadric hypersurface, the celebrated *Klein's quadric*. An amusing exercise shows that the set of lines of \mathbb{P}^3 intersecting another one at some point, determines a hyperplane section of the Klein's quadric. Such an observation solves the aforementioned enumerative problem: there are exactly two lines intersecting four others in general position in \mathbb{P}^3 , because a line of \mathbb{P}^5 (the intersection of four hyperplanes) intersects a quadric hypersurface precisely in two points. It is a particular case of a more general fact: the number of r -dimensional projective linear subvarieties of \mathbb{P}^d intersecting $(r + 1)(d - r)$ linear

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subspaces of codimension $r + 1$ in \mathbb{P}^d coincides with the Plücker degree of the grassmannian $G(r, \mathbb{P}^d)$ embedded in $\mathbb{P}(\bigwedge^{r+1} \mathbb{C}^{d+1})$. Unlike the special case of the grassmannian $G(1, \mathbb{P}^3)$, it is hard to find the degree of $G(r, \mathbb{P}^d)$ via direct geometrical methods. This is why H. C. H. Schubert (1848–1911) developed a formalism especially intended to automatize the solution of such a kind of enumerative problems. Using a modern terminology, the Calculus by Schubert amounts to a precise knowledge of the product structure of the singular cohomology ring of grassmannians, responsible for their intersection theory. The *cap product* between the singular cohomology and the singular homology of $G(r, \mathbb{P}^d)$, which via Poincaré duality are isomorphic as \mathbb{Z} -modules, can be interpreted as intersecting subvarieties of the grassmannian in general position. It is nice to recall, at this point, that the product structure of the cohomology of Grassmann varieties is completely determined thanks to formulas due to Mario Pieri (1860–1913), who was professor of “Geometria Proiettiva e Descrittiva” at the University of Catania, and Giovanni Zeno Giambelli (1879–1953), who was professor of “Analisi Algebrica” at the University of Messina.

The main purpose of this paper is to indicate what J. M. Hoene–Wronski (1776–1853), and wronskians, have to do with the story told above. Recall that the wronskian $W(\mathbf{f})$ of a basis $\mathbf{f} = (f_0, f_1, \dots, f_r)$ of solutions of a linear ODE (with constant coefficients) is the determinant whose j th row ($0 \leq i \leq r$) is the j th derivative of the row \mathbf{f} . A *generalized wronskian* of type $0 \leq i_0 < i_1 < \dots < i_r$, instead, is the determinant whose j th row is the i_j th derivative of \mathbf{f} . It is easily seen that the derivative of order $h \geq 0$ of a wronskian is a linear combination of generalized wronskians with non negative integral coefficients. Then an amazing fact occurs: the constant multiplying the generalized wronskian of type $(d - r, d - r + 1, \dots, d)$ occurring in the derivative of order $(r + 1)(d - r)$ of $W(\mathbf{f})$ is precisely the *Plücker degree of the grassmannian* $G(r + 1, \mathbb{P}^d)$! The reason of this fact, announced in [4], is explained in the paper [5], whose main result can be informally phrased by saying that Pieri’s formula can be reduced to the primordial operation of taking derivatives of generalized wronskian determinants. In addition, since each generalized wronskian is obviously a multiple of the wronskian (see Section 2.4 below), the formula computing the coefficient of multiplicity is nothing else than the famous Giambelli’s formula.

Indeed we consider generalized wronskians associated to a basis of solutions of a *universal differential equation*, whose coefficients are indeterminates in a polynomial ring. To speak in a slightly more technical way, the *wronskian module* $\mathcal{W}(\mathbf{f})$, i.e. the free \mathbb{Z} -module generated by the generalized wronskians of \mathbf{f} , can be identified with the singular homology of the infinite grassmannian. Therefore $\mathcal{W}(\mathbf{f})$ comes equipped with two additional module structures: one over the polynomial ring $\mathbb{Z}[e_1, \dots, e_{r+1}]$, whose indeterminates are the coefficients of the *universal differential equation*, and another one over the cohomology of the infinite grassmannian. The result is that these two module structures are isomorphic.

The paper is organized as follows. Section 1 is a review of basic notions regarding wronskians and generalized wronskians. Pieces of notation used in the rest of the paper are introduced there. Section 2 gives a look to the magic behind the combinatorics of wronskians. Some general results got in [5] are stated in full generality and are explained without proofs. The last two sections, instead, are devoted to study the product structure of the *wronskian module* in the particular case of solutions of second order linear ODEs. Pieri’s and Giambelli’s formulas for generalized wronskians will be proven not like in [5],

but via *ad hoc* methods which rely exclusively on elementary techniques of differential equations with constant coefficients and basic properties of determinants.

1. The Wronskian

1.1. Let \mathbb{K} be either the real field \mathbb{R} or the complex field \mathbb{C} , equipped with the standard topology induced by the norm function $z \mapsto |z|$. Let

$$\mathcal{O}(U) := \{\text{regular functions } f : U \rightarrow \mathbb{K}\}.$$

Regular means, here, holomorphic if $\mathbb{K} = \mathbb{C}$ or real analytic if $\mathbb{K} = \mathbb{R}$. For each open set U and each $j \geq 0$, denote by $D^j : \mathcal{O}(U) \rightarrow \mathcal{O}(U)$ the j -th iterated of the usual derivative with respect to a coordinate z of \mathbb{K} , i.e. $D^j f = d^j f / dz^j$ for each $f \in \mathcal{O}(U)$. The wronskian of an $(r + 1)$ -tuple

$$\mathbf{f} := (f_0, f_1, \dots, f_r) \tag{1}$$

of regular functions defined on some open subset U of \mathbb{K} is:

$$W_0(\mathbf{f}) = \mathbf{f} \wedge D\mathbf{f} \wedge \dots \wedge D^r \mathbf{f}, \tag{2}$$

where the right hand side of (2) stands for the determinant

$$\begin{vmatrix} f_0 & f_1 & \dots & f_r \\ Df_0 & Df_1 & \dots & Df_r \\ \vdots & \vdots & \ddots & \vdots \\ D^r f_0 & D^r f_1 & \dots & D^r f_r \end{vmatrix} \in \mathcal{O}(U).$$

Wronskians have been extensively studied in the literature. They enjoy very nice properties. For instance regular functions (f_0, f_1, \dots, f_r) defined on a connected open set U are linearly independent if and only if the wronskian $W(\mathbf{f})$ is not identically zero on U . The very first example is the wronskian of a pair of functions. It naturally arises in the formula expressing the derivative of the ratio of two functions. In fact

$$D\left(\frac{f}{g}\right) = \frac{Df \cdot g - f \cdot Dg}{g^2} = \frac{W_0(g, f)}{g^2},$$

provided that $g(z) \neq 0$ for all $z \in U$. The reader can easily show, as a pleasant exercise, that f, g are linearly independent over \mathbb{K} if and only if $W_0(f, g) = -W_0(g, f)$ is not the zero function.

1.2. It is natural to embed the wronskian (2) into a full family of *generalized wronskians*, as follows. If

$$\boldsymbol{\lambda} := (\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0) \tag{3}$$

is a non increasing sequence of $r + 1$ non negative integers (some among the λ_j s may possibly be zero), the *generalized wronskian* associated to $\boldsymbol{\lambda}$ is:

$$W_{\boldsymbol{\lambda}}(\mathbf{f}) := D^{\lambda_r} \mathbf{f} \wedge D^{1+\lambda_{r-1}} \mathbf{f} \wedge \dots \wedge D^{r+\lambda_0} \mathbf{f} \tag{4}$$

where the right hand side of (4) is a short for the determinant

$$\begin{vmatrix} D^{\lambda_r} f_0 & D^{\lambda_r} f_1 & \dots & D^{\lambda_r} f_r \\ D^{1+\lambda_{r-1}} f_0 & D^{1+\lambda_{r-1}} f_1 & \dots & D^{1+\lambda_{r-1}} f_r \\ \vdots & \vdots & \ddots & \vdots \\ D^{r+\lambda_0} f_0 & D^{r+\lambda_0} f_1 & \dots & D^{r+\lambda_0} f_r \end{vmatrix} \in \mathcal{O}(U) \tag{5}$$

of the square matrix whose i -th row, for $0 \leq i \leq r$, is the $(i + \lambda_{r-i})$ -th derivative $D^{i+\lambda_{r-i}} \mathbf{f} := (D^{i+\lambda_{r-i}} f_0, D^{i+\lambda_{r-i}} f_1, \dots, D^{i+\lambda_{r-i}} f_r)$ of the row \mathbf{f} . In particular

$$W_{\underbrace{(0, \dots, 0)}_{r+1 \text{ times}}}(\mathbf{f}) = W_0(\mathbf{f}),$$

explaining the subscript 0 in the notation (2). We like expression (4) more than its determinantal expression (5), because the derivation operator D satisfies a Leibniz's like rule with respect to the wedge product \wedge , namely:

$$\begin{aligned} DW_{\lambda} &= D(D^{\lambda_r} \mathbf{f} \wedge D^{1+\lambda_{r-1}} \mathbf{f} \wedge \dots \wedge D^{r+\lambda_0} \mathbf{f}) = \\ &= \sum_{\substack{i_0 + i_1 + \dots + i_r = 1 \\ i_j \geq 0}} D^{i_0+\lambda_r} \mathbf{f} \wedge D^{1+i_1+\lambda_{r-1}} \mathbf{f} \wedge \dots \wedge D^{r+i_r+\lambda_0} \mathbf{f}. \end{aligned} \tag{6}$$

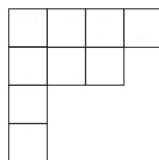
Generalized wronskians provide a further materialization of an important classical combinatorial framework, recalled below. For applications of derivatives of wronskians to Algebraic Geometry see e.g. [9].

2. (Generalized) Wronskians and Combinatorics.

2.1. Partitions. Non increasing sequences (3) of $r + 1$ non negative integers are called *partitions* of length $\leq r + 1$. Each λ_i is said to be a *part*. The *length* of a partition is the number of its non zero parts. Let ℓ_{r+1} be the set of all partitions of length $r + 1$ and $\ell_{\leq r+1}$ be the set of all partitions of length $\leq r + 1$. If the last $r - h$ parts of λ are zero, one simply writes $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_h)$, omitting the zero parts, viewing it as a partition of length less or equal than $r + 1$, for each $r \geq h$. The following chain of inclusions holds:

$$\ell_{\leq 0} \subset \ell_{\leq 1} \subset \ell_{\leq 2} \subset \ell_{\leq 3} \subset \dots$$

and $\ell_{r+1} = \ell_{\leq r+1} \setminus \ell_{\leq r}$. The *weight* of a partition λ is the non negative integer $|\lambda| = \sum_{i=0}^r \lambda_i$. Each partition λ can be represented via a *Young diagram*, an array of left justified rows, with λ_0 boxes in the first row, λ_1 boxes in the second row, \dots , λ_r -boxes in the $(r + 1)$ th row.



The Young diagram of the partition (4, 3, 1, 1) of the integer 9.

Each box of a Young diagram determines a *hook*, consisting of that box and of all the boxes in its row to the right of the box and its column below the box. The *hook length* of the box is the number of boxes in its hook. So, for instance, filling the Young diagram of the partition (3, 2, 1, 1) with the hook length of each box gives the following *Young tableau*:

7	4	3	1
6	2	1	
1			
1			

The hook length filling of (3, 2, 1, 1)

We denote by \mathcal{P} the set of all partitions and by $\mathcal{P}^{(r+1) \times (d-r)}$ the set of all partitions whose Young diagram is contained in a $(r + 1) \times (d - r)$ rectangle, i.e. the set of all partitions λ such that:

$$d - r \geq \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_r \geq 0.$$

One denotes by 0 the unique partition of weight 0. For more on partitions, see [2, 8].

2.2. Schur Polynomials and Littlewood Richardson coefficients. Let $\mathbf{a} := \sum_{n \geq 0} a_n t^n$ be a formal power series with coefficients in some commutative ring A with unit, such that $a_0 = 1$, and let $\lambda \in \ell_{\leq r+1}$ for some $r \geq 0$. The determinant

$$\Delta_\lambda(\mathbf{a}) = \begin{vmatrix} a_{\lambda_r} & a_{\lambda_{r-1}+1} & \dots & a_{\lambda_0+r} \\ a_{\lambda_{r-1}} & a_{\lambda_{r-1}} & \dots & a_{\lambda_0+r-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\lambda_r-r} & a_{\lambda_{r-1}-r+1} & \dots & a_{\lambda_0} \end{vmatrix}$$

is said to be the *Schur polynomial* associated to the formal power series \mathbf{a} and to the partition λ , where we set $a_j = 0$ if $j < 0$. The reader can easily check that adding a string of zeros to a partition λ does not change the Schur polynomial. It is well known that the multiplication in A of two Schur polynomials is an integral linear combination of Schur polynomials

$$\Delta_\lambda(\mathbf{a}) \cdot \Delta_\mu(\mathbf{a}) = \sum_{|\nu|=|\lambda|+|\mu|} L_{\lambda\mu}^\nu \cdot \Delta_\nu(\mathbf{a}).$$

The numbers $L_{\lambda\mu}^\nu \in \mathbb{Z}$ are known as *Littlewood-Richardson Coefficients*. They are very important in our story as suggested by the following proposition.

Theorem 2.1. *The product of two generalized wronskians $W_\lambda(\mathbf{f})$ in the \mathbb{K} -algebra $\mathcal{O}(U)$ obeys the following relation:*

$$W_\lambda(\mathbf{f}) \cdot W_\mu(\mathbf{f}) = \left(\sum_{|\nu|=|\lambda|+|\mu|} L_{\lambda\mu}^\nu \cdot W_\nu(\mathbf{f}) \right) \cdot W_0(\mathbf{f}). \tag{7}$$

See [5, 6].

2.3. Generalized wronskians are related with derivatives of wronskians. For instance

$$DW_0(\mathbf{f}) = W_{(1)}(\mathbf{f}).$$

In fact, by applying Leibniz’s rule (6) and the fact that a determinant vanishes whenever two rows are equals, one gets:

$$DW_0(\mathbf{f}) = D(\mathbf{f} \wedge D\mathbf{f} \wedge \dots \wedge D^r \mathbf{f}) = \mathbf{f} \wedge D\mathbf{f} \wedge \dots \wedge D^{r-1} \mathbf{f} \wedge D^{r+1} \mathbf{f} = W_{(1)}(\mathbf{f}).$$

In general, the j th derivative of a wronskian is a linear combination of generalized wronskians:

$$D^j W_0(\mathbf{f}) = \sum_{|\lambda|=j} g_\lambda W_\lambda(\mathbf{f}). \tag{8}$$

This is a consequence of (6) and of an easy induction. Then an amazing fact occurs:

Theorem 2.2. (see [5]) *The coefficient g_λ in (8) can be computed via the hook length formula:*

$$g_\lambda = \frac{|\lambda|!}{k_1 \cdot \dots \cdot k_j} = \frac{j!}{k_1 \cdot \dots \cdot k_j} \tag{9}$$

where k_i is the hook length of each box of the Young diagram of the partition λ .

A consequence of Theorem 2.2 is the following remarkable fact, already observed in [4]:

Corollary 2.1. *Let $(d - r)^{r+1}$ be the partition with $r + 1$ parts equal to $d - r$. Then the coefficient $g_{(d-r)^{r+1}}$ multiplying $W_{(d-r)^{r+1}}(\mathbf{f})$ in the expansion of $D^{(r+1)(d-r)}W_0(\mathbf{f})$ is precisely the Plücker degree of the Grassmannian $G(r, \mathbb{P}^d)$, parameterizing r dimensional linear subvarieties in the complex projective space, i.e. (cf. [1, p. 274]):*

$$g_{(d-r)^{r+1}} = \frac{1!2! \cdot \dots \cdot r! \cdot (r + 1)(d - r)!}{(d - r)!(d - r + 1)! \cdot \dots \cdot d!}.$$

Example 2.1. Let $\mathbf{f} = (f_0, f_1)$. Then:

$$\begin{aligned} D^4 W_0(\mathbf{f}) &:= D^4(\mathbf{f} \wedge D\mathbf{f}) = D^3 \circ D(\mathbf{f} \wedge D\mathbf{f}) = D^3(\mathbf{f} \wedge D^2 \mathbf{f}) = \\ &D^2(D\mathbf{f} \wedge D^2 \mathbf{f} + \mathbf{f} \wedge D^3 \mathbf{f}) = D(2 \cdot D\mathbf{f} \wedge D^3 \mathbf{f} + \mathbf{f} \wedge D^4 \mathbf{f}) = \\ &= 2 \cdot D^2 \mathbf{f} \wedge D^3 \mathbf{f} + 3 \cdot D\mathbf{f} \wedge D^4 \mathbf{f} + \mathbf{f} \wedge D^5 \mathbf{f} = \\ &= 2 \cdot D^{0+2} \mathbf{f} \wedge D^{1+2} \mathbf{f} + 3 \cdot D^{0+1} \mathbf{f} \wedge D^{1+3} \mathbf{f} + \mathbf{f} \wedge D^{1+4} \mathbf{f} = \\ &= 2 \cdot W_{(2,2)}(\mathbf{f}) + 3 \cdot W_{(3,1)}(\mathbf{f}) + W_{(4)}(\mathbf{f}), \end{aligned}$$

and the coefficient 2 multiplying $W_{(2,2)}(\mathbf{f})$ is the Plücker degree of the grassmannian $G(1, \mathbb{P}^3)$ in its Plücker embedding, i.e the *number of lines meeting 4 others in general position* – see [3, 4] for details. All the other coefficients have enumerative interpretation in either the classical or the quantum cohomology of the grassmannian. See e.g. [4].

2.4. Theorem 2.2 and Corollary 2.1 hold in general, but in [5] they are proven under the additional hypothesis that \mathbf{f} be a basis of solutions of an ordinary differential equation of order $r + 1$ with constant coefficient. Such a proof has two advantages: on one hand it requires less prerequisites and is extremely concrete. On the other hand, it gives the opportunity to give an additional look to the theory of differential equations with constant coefficients. In the paper [5] (see also the survey [6]) one considers ordinary differential equations of order $r + 1$

$$y^{(r+1)} - e_1 y^r + \dots + (-1)^{r+1} e_{r+1} y = 0, \tag{10}$$

with coefficients taken in a commutative associative \mathbb{Q} -algebra A . The most economical one is $A := \mathbb{Q}[e_1, \dots, e_{r+1}]$, where e_1, \dots, e_{r+1} are transcendental and algebraically independent over \mathbb{Q} . One looks for solutions in $A[[t]]$, the A -algebra of formal power series in one indeterminate t and proves that there is an A -basis of solutions $\mathbf{u} := (u_0, \dots, u_1)$, where u_i are elements of $A[[t]]$. In other words, each $u \in A[[t]]$ satisfying (10) is a (unique) A -linear combination of (u_0, \dots, u_r) . If $\mathbf{h} := (h_n)_{n \in \mathbb{Z}}$ is the sequence in A defined by:

$$\frac{1}{1 - e_1 t + \dots + (-1)^{r+1} e_{r+1} t^{r+1}} = \sum_{n \in \mathbb{Z}} h_n t^n,$$

with the equality taken in $A[[t]]$, then one proves that *Giambelli's formula* for wronskians holds, i.e.:

$$W_\lambda(\mathbf{u}) = \Delta_\lambda(\mathbf{h}) W_0(\mathbf{u}). \tag{11}$$

Equation (11) implies (7) for $\mathbf{f} = \mathbf{u}$:

$$\begin{aligned} W_\lambda(\mathbf{u}) W_\mu(\mathbf{u}) &= \Delta_\lambda(\mathbf{u}) \Delta_\mu(\mathbf{u}) W_0(\mathbf{u})^2 = \\ &= \left(\sum_{|\nu|=|\lambda|+|\mu|} L_{\lambda\mu}^\nu \Delta_\nu(\mathbf{h}) W_0(\mathbf{u}) \right) W_0(\mathbf{u}) = \\ &= \left(\sum_{|\nu|=|\lambda|+|\mu|} L_{\lambda\mu}^\nu \cdot W_\nu(\mathbf{u}) \right) \cdot W_0(\mathbf{u}). \end{aligned}$$

Using Giambelli's formula and Lemma A.9.4 in [1] one proves Pieri's formula for wronskians:

$$h_n W_\lambda(\mathbf{u}) = \sum_{\mu} W_\mu(\mathbf{u}), \tag{12}$$

where the sum is over all partitions $\mu := (\mu_0, \mu_1, \dots, \mu_r)$ such that $|\mu| = |\lambda| + n$ and

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \dots \geq \mu_r \geq \lambda_r \geq 0.$$

Pieri's formula (12) finally implies the *hook length formula* (9) (see [2]). In the next section we shall prove Giambelli's and Pieri's formulas for wronskians associated to a fundamental system of solutions of a second order differential equation, offering a concrete proof based on the properties of differential equations and of determinants only.

3. Second order differential equations and Wronskians

3.1. Let $E := \mathbb{Q}[e_1, e_2]$ be the algebra of polynomials in two indeterminates e_1, e_2 and $E[t]$ the polynomial E -algebra in a further indeterminate t (algebraically independent from e_1, e_2) with E -coefficients. The *universal monic polynomial* of degree 2 is by definition:

$$U_2(t) := t^2 - e_1 t + e_2 \in E[t]. \quad (13)$$

The ring $E[t]$ is an E -subalgebra of $E[[t]]$, the ring of formal power series in t with E -coefficients. For each $v \in E[[t]]$ written in the form

$$v = \sum_{n \geq 0} a_n \frac{t^n}{n!}, \quad (a_n \in E) \quad (14)$$

and for each $j \geq 0$, let

$$D^j v = \sum_{n \geq 0} a_{n+j} \frac{t^n}{n!}. \quad (15)$$

For $j > 0$, the map $D^j : E[[t]] \rightarrow E[[t]]$ determined by (15) is the j th iterated of the usual derivation, $D := D^1 = d/dt$, defined on rings of formal power series, while D^0 is just the identity map. To fix another piece of notation, let $D^j v(0)$ be the class of $D^j v$ modulo the principal prime ideal (t) . More concretely $(D^j v)(0) = a_j$ if v is as in (14). The *universal differential operator* (with constant coefficients) is the evaluation of $U_2(t)$ at D :

$$U_2(D) := D^2 - e_1 D + e_2 : E[[t]] \rightarrow E[[t]].$$

3.2. A formal power series $u \in A[[t]]$ is a solution of the *universal* second order ODE

$$U_2(D)y = 0 \quad (16)$$

if and only if $u \in \ker U_2(D)$. By [5, 6] one knows that $\ker U_2(D)$ is a free E -submodule of $E[[t]]$ of rank 2 generated by:

$$u_0 = \sum_{n \geq 0} h_n \frac{t^n}{n!} \quad \text{and} \quad u_1 = \sum_{n \geq 0} h_{n-1} \frac{t^n}{n!},$$

where the coefficients h_i , for all $i \in \mathbb{Z}$, are defined via the equality:

$$\sum_{n \in \mathbb{Z}} h_n t^n = \frac{1}{1 - e_1 t + e_2 t^2} = 1 + \sum_{n \geq 0} (e_1 t - e_2 t^2)^n, \quad (17)$$

holding in $E[[t]]$. Notice that $h_0 = 1$ and $h_1 = e_1$, while $h_j = 0$ if $j < 0$.

3.3. The pair (u_0, u_1) is the *universal fundamental system* of solution of (16). The motivation for the terminology is that if A is any \mathbb{Q} -algebra (e.g. the real or complex field) and $P(t) = t^2 - e_1(P)t + e_2(P) \in A[t]$ is any polynomial, then the unique \mathbb{Q} -algebra homomorphism $E[t] \rightarrow A[t]$ mapping $e_1 \mapsto e_1(P)$ and $e_2 \mapsto e_2(P)$, maps the universal fundamental system to an A -basis of the solutions of the differential equation $P(D)y = 0$ or, put otherwise, $A[[t]] \supseteq \ker P(D) = \ker U_2(D) \otimes_E A$.

3.4. If $\mathbf{f} = (f_0, f_1)$ is any E -basis of $\ker U_2(D) \subset E[[t]]$, let $D^j \mathbf{f} := (D^j f_0, D^j f_1)$. The pair \mathbf{f} itself satisfies the differential equation (16)

$$D^2 \mathbf{f} - e_1 D \mathbf{f} + e_2 \mathbf{f} = \mathbf{0} = (0, 0), \tag{18}$$

because each component f_i does. If $\lambda : \lambda_0 \geq \lambda_1$ is a partition of length at most 2, let

$$W_\lambda(\mathbf{u}) = D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f} = \begin{vmatrix} D^{\lambda_1} f_0 & D^{\lambda_1} f_1 \\ D^{1+\lambda_0} f_0 & D^{1+\lambda_0} f_1 \end{vmatrix} \in E[[t]] \tag{19}$$

be the corresponding *generalized wronskian* and let

$$\mathcal{W}(\mathbf{f}) = \bigoplus_{\lambda \in \ell_{\leq 2}} \mathbb{Z} \cdot W_\lambda(\mathbf{f})$$

be the *wronskian module*, namely the free \mathbb{Z} -module generated by all the generalized wronskians. By construction, the abelian group $\mathcal{W}(\mathbf{f})$ is a module over the polynomial ring $\mathbb{Z}[e_1, e_2]$ as well, a subring of $\mathbb{Q}[e_1, e_2]$. In addition, the natural evaluation map

$$\mathbb{Z}[e_1, e_2] \longrightarrow \mathcal{W}(\mathbf{f}) \tag{20}$$

defined by $P(e_1, e_2) \mapsto P(e_1, e_2) \cdot W_0(\mathbf{f})$ is a \mathbb{Z} -module isomorphism. By construction $\mathcal{W}(\mathbf{f})$ has no torsion over E . It then suffices to prove that (20) is an epimorphism. Indeed each entry of the determinant (19) can be written as an E -linear combination of \mathbf{f} and $D\mathbf{f}$ using the differential equation (18). Therefore $W_\lambda(\mathbf{f})$ itself can be written as a $\mathbb{Z}[e_1, e_2]$ -multiple of $W_0(\mathbf{f}) = \mathbf{f} \wedge D\mathbf{f}$ and then the wronskian module $\mathcal{W}(\mathbf{f})$ is a free $\mathbb{Z}[e_1, e_2]$ -module of rank 1 generated by $W_0(\mathbf{f})$, as claimed. The key result of this section is:

Theorem 3.1. *The equality*

$$e_2 W_{(\lambda_0, \lambda_1)}(\mathbf{f}) = W_{(1+\lambda_0, 1+\lambda_1)}(\mathbf{f}) \tag{21}$$

holds in $\mathcal{W}(\mathbf{f})$.

3.5. We shall assume Theorem 3.1, postponing its proof to the deduction of some consequences. The importance of (21) is that it fully determines the module structure of $\mathcal{W}(\mathbf{f})$ over $\mathbb{Z}[e_1, e_2]$. In principle one would also need a formula expressing the product $e_1 W_{(\lambda_0, \lambda_1)}(\mathbf{f})$ as a linear combination of generalized wronskians, but that is a consequence of (21). In fact, by applying D^{λ_1-1} to equation (18), one may substitute $e_1 D^{\lambda_1} \mathbf{f}$ with $D^{1+\lambda_1} \mathbf{f} + e_2 D^{\lambda_1-1} \mathbf{f}$ in the last member of the equalities below

$$e_1 W_{(\lambda_0, \lambda_1)}(\mathbf{f}) = e_1 (D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}) = (e_1 D^{\lambda_1} \mathbf{f}) \wedge D^{1+\lambda_0} \mathbf{f}, \tag{22}$$

obtaining:

$$\begin{aligned} e_1 W_{(\lambda_0, \lambda_1)}(\mathbf{f}) &= (D^{1+\lambda_1} \mathbf{f} + e_2 D^{\lambda_1-1} \mathbf{f}) \wedge D^{1+\lambda_0} \mathbf{f} = \\ &= D^{1+\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f} + e_2 D^{\lambda_1-1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f} \\ &= W_{(1+\lambda_0, 1+\lambda_1)}(\mathbf{f}) + e_2 W_{(1+\lambda_0, \lambda_1-1)}(\mathbf{f}) = \\ &= W_{(1+\lambda_0, 1+\lambda_1)}(\mathbf{f}) + W_{(2+\lambda_0, \lambda_1)}(\mathbf{f}), \end{aligned} \tag{23}$$

where we used (21) to write the last equality.

Corollary 3.1. *For each $n \geq 0$, the coefficient h_n defined through (17) satisfies the following Leibniz's-like rule:*

$$h_n(D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}) = \sum_{i=0}^n D^{\lambda_1+i} \mathbf{f} \wedge D^{1+\lambda_0+n-i} \mathbf{f}. \quad (24)$$

Proof. By induction on $n \geq 1$. As $h_1 = e_1$, the property holds for $n = 1$ because of 3.5. For $n \geq 2$, formula (17) shows that $h_n = e_1 h_{n-1} - e_2 h_{n-2}$. Therefore:

$$\begin{aligned} h_n(D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}) &= (e_1 h_{n-1} - e_2 h_{n-2})(D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}) = \\ &= e_1 h_{n-1}(D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}) - e_2 h_{n-2}(D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}). \end{aligned}$$

By induction, the last member is equal to:

$$e_1 \sum_{i=0}^{n-1} D^{\lambda_1+i} \mathbf{f} \wedge D^{1+\lambda_0+n-i} \mathbf{f} - e_2 \sum_{i=0}^{n-2} D^{\lambda_1+i} \mathbf{f} \wedge D^{1+\lambda_0+n-2-i} \mathbf{f} =$$

i.e, using (21) and (23):

$$\begin{aligned} &= \sum_{i=0}^{n-1} D^{\lambda_1+1+i} \mathbf{f} \wedge D^{1+\lambda_0+n-i} \mathbf{f} + \sum_{i=0}^{n-1} D^{\lambda_1+i} \mathbf{f} \wedge D^{1+\lambda_0+n+1-i} \mathbf{f} + \\ &- \sum_{i=0}^{n-2} D^{\lambda_1+i+1} \mathbf{f} \wedge D^{1+\lambda_0+n-1-i} \mathbf{f} = \\ &= D^{\lambda_1+n} \mathbf{f} \wedge D^{2+\lambda_0} \mathbf{f} + \sum_{i=0}^{n-1} D^{\lambda_1+i} \mathbf{f} \wedge D^{1+\lambda_0+n+1-i} \mathbf{f} = \\ &= \sum_{i=0}^n D^{\lambda_1+i} \mathbf{f} \wedge D^{1+\lambda_0+n+1-i} \mathbf{f}, \end{aligned}$$

as desired. \square

Notice that in the sum at the second member of (24) there may be cancelations due to the antisymmetry of the wronskian, i.e. each term $D^i \mathbf{f} \wedge D^j \mathbf{f}$ would cancel with a term like $D^j \mathbf{f} \wedge D^i \mathbf{f}$. The result predicting the terms surviving such cancelations is:

Corollary 3.2. *Pieri's formula for generalized 2×2 wronskians holds*

$$h_n W_{\lambda}(\mathbf{f}) = \sum_{\mu} W_{\mu}(\mathbf{f}), \quad (25)$$

where the sum is over all the partition $\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq 0$ such that $|\mu| = |\lambda| + n$.

Proof. First one notices that there is a 1-1-correspondence between partitions $\lambda_1 \geq \lambda_0$ and strictly increasing sequences $0 \leq i_0 < i_1$. It is given by $(\lambda_0, \lambda_1) \mapsto (\lambda_1, 1 + \lambda_0)$. Since h_n satisfies the same formal rules as the operator D_n defined on the exterior algebra of a free \mathbb{Z} -module, introduced in [3], the proof consists in copying the first part of the proof of [3, Theorem 2.4]. \square

A quite immediate consequence of Pieri's formula is:

Corollary 3.3. Giambelli’s formula for 2×2 wronskians hold, i.e. for each $\lambda = (\lambda_0, \lambda_1)$

$$W_\lambda(\mathbf{f}) = \Delta_\lambda(\mathbf{h})W_0(\mathbf{f}) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_0+1} \\ h_{\lambda_1-1} & h_{\lambda_0} \end{vmatrix} \cdot W_0(\mathbf{f}). \tag{26}$$

Proof. In fact, using (24), one can easily check the last of the equalities below (just expand and cancel summands with opposite signs):

$$\begin{aligned} W_\lambda(\mathbf{f}) &= D^{\lambda_1} \mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f} = \\ &= h_{\lambda_1}(\mathbf{f} \wedge D^{1+\lambda_0} \mathbf{f}) + h_{1+\lambda_0}(D^{\lambda_1} \mathbf{f} \wedge \mathbf{f}). \end{aligned} \tag{27}$$

Using (24) again:

$$W_\lambda(\mathbf{f}) = h_{\lambda_1} h_{\lambda_0}(\mathbf{f} \wedge D\mathbf{f}) + h_{\lambda_1-1} h_{\lambda_0+1}(D\mathbf{f} \wedge \mathbf{f}),$$

and because $\mathbf{f} \wedge D\mathbf{f} = -D\mathbf{f} \wedge \mathbf{f}$ one finally gets:

$$W_\lambda(\mathbf{f}) = (h_{\lambda_1} h_{\lambda_0} - h_{\lambda_1-1} h_{\lambda_0+1})W_0(\mathbf{f}) = \begin{vmatrix} h_{\lambda_1} & h_{\lambda_0+1} \\ h_{\lambda_1-1} & h_{\lambda_0} \end{vmatrix} \cdot W_0(\mathbf{f})$$

as desired. □

Remark 3.1. If $\lambda = (1, 0)$ formula (26) gives (see also 3.2):

$$W_{(1)}(\mathbf{f}) = h_1 W_0(\mathbf{f}) = e_1 W_0(\mathbf{f}),$$

which is a classical theorem due to Liouville and Abel, saying that the derivative of the wronskian of a fundamental system of solutions of a (second order with constant coefficients) differential equation is proportional to the wronskian itself. In fact:

$$W_{(1)}(\mathbf{f}) = \begin{vmatrix} h_0 & h_2 \\ h_{-1} & h_1 \end{vmatrix} W_0(\mathbf{f}) = \begin{vmatrix} h_0 & h_2 \\ 0 & h_1 \end{vmatrix} = h_1 W_0(\mathbf{f}) = e_1 W_0(\mathbf{f}).$$

Thus Giambelli’s formula for wronskians can be viewed as a generalization of the aforementioned classical Abel-Liouville’s theorem, which is then the first historical example of Schubert Calculus. See below.

3.6. Let $G_{1,d} := G(1, \mathbb{P}^d)$ be the complex grassmann variety parameterizing projective lines in the d -dimensional projective space. Over $G_{1,d}$ sits an universal exact sequence:

$$0 \longrightarrow \mathcal{S} \longrightarrow G_{1,d} \times \mathbb{C}^{d+1} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{S} = \{([u], v) \in G_{1,d} \times \mathbb{C}^{d+1} \mid v \in [u]\}$ is the tautological bundle and $\mathcal{Q} := G_{1,d} \times \mathbb{C}^{d+1} / \mathcal{S}$ is the universal quotient bundle. Classical results regarding the cohomology of the grassmannian say that the singular cohomology $H^*(G_{1,d}, \mathbb{Q})$ is a \mathbb{Q} -algebra generated by $\mathbb{Q}[\varepsilon_1, \varepsilon_2]$, where ε_i denotes the i th Chern class of \mathcal{S} . In addition the topological space $G_{1,d}$ admits a cellular decomposition: there are precisely $2(d-1)$ cells, parameterized by all the partitions $\lambda := \lambda_0 \geq \lambda_1$ whose Young diagram (Cf. Section 2.1) is contained in a $2 \times (d-1)$ rectangle. Each cell B_λ has codimension $|\lambda|$ in $G_{1,d}$. Standard facts in algebraic topology ensure that the singular homology $H_*(G_{1,d}, \mathbb{Q})$ is a $2(d-1)$ -dimensional vector space generated by the homology classes Ω_λ of the closure of B_λ in $G_{1,d}$. Let $c_i = c_i(\mathcal{Q})$ be the i th Chern class of the universal quotient bundle. The relation $c_t(\mathcal{S})c_t(\mathcal{Q}) = c_t(G_{1,d} \times \mathbb{P}^d)$,

where $c_t(F) = 1 + c_1(F)t + \dots$ denotes the Chern polynomial of a vector bundle F , implies that

$$\sum_{n \geq 0} c_n t^n = \frac{1}{1 - \varepsilon_1 t + \varepsilon_2 t^2} \in H^*(G_{1,d}, \mathbb{Q})[[t]].$$

Classical Giambelli's formula for Schubert Calculus (see e.g. [7]) says that

$$\Omega_\lambda = \Delta_\lambda(\mathbf{c}) \cdot \Omega_0 = \begin{vmatrix} c_{\lambda_1} & c_{\lambda_0+1} \\ c_{\lambda_1-1} & c_{\lambda_0} \end{vmatrix} \Omega_0.$$

Consider the differential equation

$$y'' - \varepsilon_1 y' + \varepsilon_2 y = 0, \quad (28)$$

with coefficients in $H^*(G_{1,d}, \mathbb{Q})$, looking for solutions in $H^*(G_{1,d}, \mathbb{Q})[[t]]$. By the universal property seen in **3.3**, the universal solution $\mathbf{u} = (u_0, u_1)$ of $U_2(D)y = 0$ is mapped to a basis $\mathbf{v} := (v_0, v_1)$ of solutions of (28) via the unique \mathbb{Q} -algebra homomorphism sending $e_i \mapsto \varepsilon_i$. In particular $h_i \mapsto c_i$ and $W_\lambda(\mathbf{u}) \mapsto W_\lambda(\mathbf{v})$. Therefore

$$W_\lambda(\mathbf{v}) = \Delta_\lambda(\mathbf{c})W_0(\mathbf{v}).$$

Since the last equation are defined over the integers, it turns out that the natural map

$$\mathcal{W}(\mathbf{v}) := \bigoplus_{0 \leq |\lambda| \leq 2(d-1)} \mathbb{Z} \cdot W_\lambda(\mathbf{c}) \longrightarrow H_*(G_{1,d}, \mathbb{Z}),$$

sending $W_\lambda(\mathbf{v}) \mapsto \Omega_\lambda$ is an isomorphism of $H^*(G_{1,d}, \mathbb{P}^d)$ -modules. From the practical point of view, doing computation in the cohomology ring of the Grassmannian is the same as multiplying wronskians with polynomials in the coefficients of a differential equation.

4. The proof of Theorem 3.1.

To prove Theorem **3.1** we use two reductions. First we show that is the same to prove the theorem for the universal basis \mathbf{u} of solutions introduced in Section **3.2**. Then we prove the result in a larger \mathbb{Q} -algebra where the polynomial $U_2(t)$ splits.

4.1. A first reduction. If \mathbf{f} is any E -basis of $\ker U_2(D)$, there is an invertible matrix M with entries in E such that $\mathbf{f} = M\mathbf{u}$. If M is invertible, its determinant must be invertible in E , i.e. $\det(M) \in \mathbb{Q}^*$. Therefore $W_\lambda(\mathbf{f})$ coincides with $W_\lambda(\mathbf{u})$ up to a non zero rational multiplicative constant. Hence we may assume $\mathbf{f} = \mathbf{u}$.

4.2. A second reduction. Let $E[\alpha_1]$ be the quotient ring $E[t]/(U_2(t))$, where $\alpha_1 = t + (U_2(t))$. The ring $E[\alpha_1]$ is the *universal splitting algebra* of the polynomial $U_2(t)$ as the product of two linear factors. In the ring $E[\alpha_1]$ define $\alpha_2 = e_1 - \alpha_1$. It is obvious that $E[\alpha_1][t] = E[\alpha_2][t]$ is isomorphic to the integral polynomial algebra $\mathbb{Q}[\alpha_1, \alpha_2]$. The polynomial $U_2(t)$ is defined over $E[\alpha_1]$ as well, and it admits the splitting

$$U_2(t) = (t - \alpha_1)(t - \alpha_2)$$

in $E[\alpha_1][t]$. Setting

$$\exp(\alpha_j t) = \sum_{n \geq 0} \alpha_j^n t^n / n!$$

and thinking of $U_2(D)$ as an endomorphism of $E[\alpha_1][[t]]$, it is easy to check that

$$\exp(\vec{\alpha}t) := (\exp(\alpha_1t), \exp(\alpha_2t)),$$

are linearly independent elements of $\ker U_2(D)$. Indeed $D \exp(\alpha_jt) = \alpha_j \exp(\alpha_jt)$ and thus

$$U_2(D) \exp(\alpha_jt) = (D - (e_1 - \alpha_j))(D - \alpha_j) \exp(\alpha_jt) = 0 \tag{29}$$

which implies:

$$D^{1+\lambda} \exp(\alpha_jt) = \alpha_j D^\lambda \exp(\alpha_jt) \tag{30}$$

for each non negative integer λ . The *universal Euler formula*

$$\exp(\alpha_jt) = u_0 + (\alpha_j - e_1)u_1 \tag{31}$$

expresses the exponential solutions of $U_2(D)y = 0$ as a linear combination of the universal fundamental system u_0 and u_1 . It can be easily checked via a direct verification. In particular

$$\exp(\vec{\alpha}t) = \mathbf{u} \cdot \begin{pmatrix} 1 & 1 \\ \alpha_1 & \alpha_2 \end{pmatrix}, \tag{32}$$

and one sees that in spite $(\exp(\alpha_1t), \exp(\alpha_2t))$ are linearly independent over $E[\alpha_1]$, they are not a fundamental system, because $\alpha_2 - \alpha_1$ is not invertible in $E[\alpha_1]$!

4.3. Equation (32) implies that

$$W_\lambda(\exp(\vec{\alpha}t)) = (\alpha_2 - \alpha_1)W_\lambda(\mathbf{u})$$

for each partition λ of length at most 2, while the equality

$$e_2 W_{(\lambda_0, \lambda_1)}(\exp(\vec{\alpha}t)) = W_{(2+\lambda_0, 1+\lambda_1)}(\mathbf{u}) \tag{33}$$

implies

$$(\alpha_2 - \alpha_1)e_2 W_{(1+\lambda_0, \lambda_1)}(\mathbf{u}) = (\alpha_2 - \alpha_1)W_{(1+\lambda_0, 1+\lambda_1)}(\mathbf{u}),$$

which is equivalent to (21), for $\mathbf{f} = \mathbf{u}$, because $E[\alpha_1][[t]] \cong \mathbb{Q}[\alpha_1, \alpha_2][[t]]$ is an integral domain. Our task is then reduced to prove formula (33). This is easy, because working in $E[\alpha_1][[t]]$, one has:

$$\begin{aligned} e_2 W_{(\lambda_0, \lambda_1)}(\exp(\vec{\alpha}t)) &= \alpha_1 \alpha_2 \begin{vmatrix} D^{\lambda_1} \exp(\alpha_1t) & D^{\lambda_1} \exp(\alpha_2t) \\ D^{1+\lambda_0} \exp(\alpha_1t) & D^{1+\lambda_0} \exp(\alpha_2t) \end{vmatrix} = \\ &= \begin{vmatrix} \alpha_1 D^{\lambda_1} \exp(\alpha_1t) & \alpha_2 D^{\lambda_1} \exp(\alpha_2t) \\ \alpha_1 D^{1+\lambda_0} \exp(\alpha_1t) & \alpha_2 D^{1+\lambda_0} \exp(\alpha_2t) \end{vmatrix} \end{aligned}$$

that is, invoking formula (30):

$$= \begin{vmatrix} D^{1+\lambda_1} \exp(\alpha_1t) & D^{1+\lambda_1} \exp(\alpha_2t) \\ D^{2+\lambda_0} \exp(\alpha_1t) & D^{2+\lambda_0} \exp(\alpha_2t) \end{vmatrix} = W_{(1+\lambda_0, 1+\lambda_1)}(\exp(\vec{\alpha}t))$$

which completes the proof of Theorem 3.1. □

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References

- [1] W. Fulton, *Intersection Theory* (Springer-Verlag, 1984).
- [2] W. Fulton, *Young Tableaux With Application to Representation Theory and Geometry* (Cambridge University Press, 1997).
- [3] L. Gatto, “Schubert Calculus via Hasse–Schmidt Derivations”, *Asian J. Math.* **9**, No. 3, 315–322, (2005).
- [4] L. Gatto, “Schubert Calculus: An Algebraic Introduction”, *Instituto de Matemática Pura e Aplicada*, Rio de Janeiro, 2005.
- [5] L. Gatto and I. Scherbak, “Linear ODEs, Wronskians and Schubert Calculus”, *Moscow Math. J.* (to appear).
- [6] L. Gatto and I. Scherbak, “On Generalized Wronskians”, in *Contributions to Algebraic Geometry* (P. Pragacz ed.), Impanga Lecture Notes (to appear).
- [7] Ph. Griffiths and J. Harris, *Principles of Algebraic Geometry* (Addison Wesley, 1994).
- [8] I. G. Macdonald, *Symmetric functions and Hall polynomials* (Clarendon Press, Oxford, 1979).
- [9] L. Gatto and F. Ponza, “Derivatives of Wronskians with applications to families of special Weierstrass points”, *Trans. Amer. Math. Soc.* **351**, No. 6, 2233–2255 (1999).
- [10] P. Pragacz, “La vita e l’opera di Józef Maria Hoene-Wronski”, *Atti Accad. Pelorit. Pericol. Cl. Sci. Fis. Mat. Nat.* **89**, No. 1, C1C8901001 (2011) [19 pages].

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