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The Transshipment Location Problem under Uncertainty with Lower and Upper Capacity Constraints

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Abstract. The problem consists in finding a transshipment facility location which minimizes the expected total cost when the generalized transportation costs are random variables. The optimal location must satisfy supply, demand, and lower and upper capacity constraints. Each generalized transportation cost is given by a deterministic transportation cost from an origin to a destination via a transshipment facility in addition to a random handling cost of the facility, whose probability distribution is not known. In this paper we give the stochastic model, a deterministic approximation of it and an efficient heuristic for solving the deterministic approximation when real-life instances are considered. Computational results show a mean gap of 0; 87% between the stochastic model and its deterministic approximation and 0; 82% between the latter and the heuristic.

Keywords. Double-capacitated transshipment location, random generalized transportation costs, stochastic model, asymptotic approximation, heuristic.

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1 Introduction

Let us consider a set of origins with a given supply, a set of destinations with a given demand, a set of potential transshipment locations with a deterministic fixed cost of location, lower and upper capacity constraints for the facilities, and random generalized transportation costs from origins to destinations via transshipment facilities.

Each generalized transportation cost is a random variable given by the sum of a deterministic transportation cost from an origin to a destination via a transshipment facility plus a random term, with unknown probability distribution, which represents the handling cost of the transshipment facility. The freight, when enters into a transshipment facility, is subject to handling operations which are typically organized in alternative operating paths, where an operating path is a set of options for how the freight may be routed and processed within the transshipment facility. Each operating path has its unit handling cost and a finite capacity. Given the finite capacity of the operating path, congestion effects make the handling cost a random variable, whose distribution is usually not known.

The Transshipment Location Problem under Uncertainty with Lower and Upper Capacity Constraints consists in finding a transshipment location which minimizes the total cost, given by the sum of the total fixed cost and the expected minimum total flow cost, subject to supply, demand, and lower and upper facility capacity constraints. In other words, in this paper we try to integrate two of the main levels of a transshipment network, i.e. the network design (higher level), which leads to a network flow formulation with origins, transshipment points and destinations as nodes of the network, and the transshipment facility management (lower level), where the management variables we consider are represented by the random handling costs of the freight at the facilities.

Only a few papers concerning location problems with stochastic costs are currently available. Among them, Ricciardi et al. [10] develop a heuristic for solving a p-median problem where the throughput costs are random variables with a given probability distribution. Snyder et al. [11] consider a scenario-based stochastic version of a joint location-inventory model that minimizes the expected cost of locating facilities, allowing costs, lead times, demand, and some other parameters to be stochastic. Tadei et al. [12] consider a stochastic p-median problem where the costs for using the facilities are random variables, with unknown probability distribution.

In this paper we address the Transshipment Location Problem under Uncertainty where both upper and lower capacity constraints of the facilities are considered (C^2TLP_u). We give the stochastic model, its deterministic approximation, and an efficient heuristic for solving the deterministic approximation when large real-life instances are considered. We can note that taking into account both upper and lower bounds for the facility size is particularly important from an economic point of view. In fact, this avoids locating facilities whose size is either too large or too small, where both situations are economically inefficient.

Computational results show that the mean gap between the stochastic model and its deterministic approximation is $0,87\%$. Moreover, the performance of the proposed heuristic, designed for solving larger instances of the deterministic model, is very good, both in terms of running time and solution quality, showing a mean gap of $0,82\%$

between the optimum values.
The remainder of the paper is organized as follows. Section 2 introduces the Transshipment Location Problem under Uncertainty with Lower and Upper Capacity Constraints as a stochastic programming model. In Section 3, its deterministic approximation is given. Section 4 presents the heuristic for solving real-life instances of the deterministic model. In Section 5, the computational results of both the deterministic approximation and the proposed heuristic are given. Finally, the conclusions of our work are reported in Section 6.

2 The Transshipment Location Problem under Uncertainty with Lower and Upper Capacity Constraints

We consider the following parameters and data

- I: set of origins
- J: set of destinations
- K: set of potential transshipment locations
- H_k: set of operating paths at transshipment facility $k \in K$
- P_i: supply at origin $i \in I$
- Q_j: demand at destination $j \in J$
- L_k: lower capacity of transshipment facility $k \in K$
- U_k: upper capacity of transshipment facility $k \in K$
- f_k: fixed cost of locating a transshipment facility $k \in K$
- c_{ij}^k: unit deterministic transportation cost from origin $i \in I$ to destination $j \in J$ via transshipment facility $k \in K$

and the variables

- y_k: binary variable which takes value 1 if transshipment facility $k \in K$ is located, 0 otherwise
- θ^{kl}: random variable with unknown probability distribution which represents the unit handling cost of operating path $l \in H_k$ at transshipment facility $k \in K$
- s_{ij}^k: deterministic variable which represents the flow from origin $i \in I$ to destination $j \in J$ via transshipment facility $k \in K$.

Let us assume the system is balanced, i.e. $\sum_{i \in I} P_i = \sum_{j \in J} Q_j = T$. This is a standard assumption and is straightforward to balance the system, if necessary.

Let $r_{ij}^{kl}(\theta^{kl})$ be the random generalized unit transportation cost from origin i to destination j via transshipment facility k in operating path l given by

$$r_{ij}^{kl}(\theta^{kl}) = c_{ij}^k + \theta^{kl}, \quad i \in I, j \in J, k \in K, l \in H_k \tag{1}$$

Let us define with $\tilde{\theta}^k$ the minimum of the random handling costs within the alternative operating paths $\{l\}$ at the transshipment facility k

$$\tilde{\theta}^k = \min_{l \in H_k} \theta^{kl}, \quad k \in K \tag{2}$$

As far as the management level is considered, we assume that the facility management policies are efficiency-based, so that, among the alternative operating paths $\{l\}$ at the transshipment facility k, the one which minimizes the random costs $\{r_{ij}^{k}(\theta^{kl})\}$ will be selected, then

$$\tilde{r}_{ij}^{k}(\tilde{\theta}^{k}) = \min_{l \in H_k} r_{ij}^{kl}(\theta^{kl}) = c_{ij}^{k} + \min_{l \in H_k} \theta^{kl} = c_{ij}^{k} + \tilde{\theta}^{k}, \quad i \in I, j \in J, k \in K \tag{3}$$

The C^2TLP_u is formulated as follows

$$\min_{y} \sum_{k \in K} f_k y_k + \mathbb{E}_{\tilde{\theta}^k} \left[\min_{s} \sum_{i \in I} \sum_{k \in K} \sum_{j \in J} \tilde{r}_{ij}^{k}(\tilde{\theta}^{k}) s_{ij}^{k} \right] \tag{4}$$

subject to

$$\sum_{k \in K} \sum_{j \in J} s_{ij}^{k} = P_i, \quad i \in I \tag{5}$$

$$\sum_{i \in I} \sum_{k \in K} s_{ij}^{k} = Q_j, \quad j \in J \tag{6}$$

$$\sum_{i \in I} \sum_{j \in J} s_{ij}^{k} \leq U_k y_k, \quad k \in K \tag{7}$$

$$\sum_{i \in I} \sum_{j \in J} s_{ij}^{k} \geq L_k y_k, \quad k \in K \tag{8}$$

$$s_{ij}^{k} \geq 0, \quad i \in I, j \in J, k \in K \tag{9}$$

$$y_k \in \{0,1\}, \quad k \in K \tag{10}$$

where $\mathbb{E}_{\tilde{\theta}^k}$ denotes the expected value with respect to $\{\tilde{\theta}^k\}$; the objective function (4) expresses the minimization of the total cost given by the sum of the total fixed cost and the expected total flow cost; constraints (5) and (6) ensure that supply at each origin i and demand at each destination j are satisfied; constraints (7) and (8) ensure the upper and lower capacity restrictions at each transshipment facility k; (9) are the non-negativity constraints, and (10) are the integrality constraints.

In order to make numerical computation easier, instead of (4)-(10) we consider the random-utility counterpart of C^2TLP_u, i.e. the utility maximizing approach rather than the cost minimizing one is used and the *min* operator is replaced by the *max* operator.

Let us define

- $v_{ij}^{k} = -c_{ij}^{k}$: unit deterministic utility from origin $i \in I$ to destination $j \in J$ via transshipment facility $k \in K$
- $u^{kl} = -\theta^{kl}$: unit random utility of operating path $l \in H_k$ at transshipment facility $k \in K$.

Let us assume $\{u^{kl}\}$ are independent and identically distributed (i.i.d.) random variables with a common and *unknown* probability distribution

$$Pr\{u^{kl} \leq x\} = F(x) \tag{11}$$

This assumption is necessary for deriving our deterministic approximation of the stochastic model, but it can be also justified in practice. In fact, the random utility due to the handling operations at any facility k is extremely difficult to measure, so its probability distribution is generally not known and it would be rather arbitrary to assume a particular shape for it. For this reason, we take a common unknown probability distribution for the random utilities, which justifies the identically-distributed assumption. Moreover, for the sake of simplicity, we assume that the alternative operating paths interact only slightly with each other, allowing us to consider their random utilities as independent variables.

Let $\tilde{u}^k$ be the maximum of the random utilities within the alternative operating paths $\{l\}$ at the transshipment facility k

$$\tilde{u}^k = \max_{l \in H_k} u^{kl}, \quad k \in K \tag{12}$$

which is still of course a random variable with unknown probability distribution given by

$$B_k(x) = \Pr\left\{\tilde{u}^k \leq x\right\} \tag{13}$$

As $\tilde{u}^k \leq x \Longleftrightarrow u^{kl} \leq x,\ l \in H_k$ and u^{kl} are independent, using (11) one gets

$$B_k(x) = \Pr\left\{\tilde{u}^k \leq x\right\} = \prod_{l \in H_k} \Pr\left\{u^{kl} \leq x\right\} = \prod_{l \in H_k} F(x) = [F(x)]^{n_k} \tag{14}$$

where $n_k = |H_k|$ is the total number of alternative operating paths at the transshipment facility k.

Let us consider the unit random utility from origin i to destination j via transshipment facility k

$$\tilde{v}_{ij}^k(\tilde{u}^k) = v_{ij}^k + \tilde{u}^k, \quad i \in I, j \in J, k \in K, l \in H_k \tag{15}$$

C^2TLP_u as a random-utility model then becomes

$$\max_y \sum_{k \in K} -f_k y_k + \mathbb{E}_{\tilde{u}^k}\left[\max_s \sum_{i \in I} \sum_{k \in K} \sum_{j \in J} \tilde{v}_{ij}^k(\tilde{u}^k) s_{ij}^k\right] \tag{16}$$

subject to

$$\sum_{k \in K} \sum_{j \in J} s_{ij}^k = P_i, \quad i \in I \tag{17}$$

$$\sum_{i\in I}\sum_{k\in K} s_{ij}^k = Q_j, \quad j \in J \tag{18}$$

$$\sum_{i\in I}\sum_{j\in J} s_{ij}^k \leq U_k y_k, \quad k \in K \tag{19}$$

$$\sum_{i\in I}\sum_{j\in J} s_{ij}^k \geq L_k y_k, \quad k \in K \tag{20}$$

$$s_{ij}^k \geq 0, \quad i \in I, j \in J, k \in K \tag{21}$$

$$y_k \in \{0,1\}, \quad k \in K \tag{22}$$

Let us take the Lagrangian of problem (16)-(22) obtained by relaxing the constraints (18), (19), and (20) by means of the multipliers $\mu_j, j \in J$, and $\lambda_k \geq 0, k \in K$, and $\eta_k \leq 0, k \in K$, respectively

$$\begin{aligned} L = \quad & \sum_{k\in K} -f_k y_k + \mathbb{E}_{\tilde{u}^k}\left[\max_s \sum_{i\in I}\sum_{k\in K}\sum_{j\in J} \tilde{v}_{ij}^k(\tilde{u}^k) s_{ij}^k\right] + \\ & + \sum_{j\in J}\mu_j\left(\sum_{i\in I}\sum_{k\in K} s_{ij}^k - Q_j\right) + \sum_{k\in K}\lambda_k\left(U_k y_k - \sum_{i\in I}\sum_{j\in J} s_{ij}^k\right) + \\ & \sum_{k\in K}\eta_k\left(L_k y_k - \sum_{i\in I}\sum_{j\in J} s_{ij}^k\right) \end{aligned} \tag{23}$$

When the strong duality conditions are satisfied, we know that problem (16)-(22) is equivalent to

$$\begin{aligned} \min_{\mu,\lambda\geq 0,\eta\leq 0}\max_y \quad & \sum_{k\in K}\left[-f_k y_k + \lambda_k U_k y_k + \eta_k L_k y_k\right] - \sum_{j\in J}\mu_j Q_j + \\ & + \mathbb{E}_{\tilde{u}^k}\left[\max_s \sum_{i\in I}\sum_{k\in K}\sum_{j\in J} y_k\left(\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k\right) s_{ij}^k\right] \end{aligned} \tag{24}$$

subject to

$$\sum_{k\in K}\sum_{j\in J} y_k s_{ij}^k = P_i, \quad i \in I \tag{25}$$

$$s_{ij}^k \geq 0, \quad i \in I, j \in J, k \in K \tag{26}$$

$$y_k \in \{0,1\}, \quad k \in K \tag{27}$$

We can note that, because of the relaxation of the capacity constraints (19) and (20), in order to prevent sending flows via closed transshipment facilities (i.e. those facilities for which $y_k = 0$), the term y_k must be introduced into the objective function

(24) and constraints (25).

The term $(\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k)$ in (24) is called the unit "shadow" random utility from i to j via transshipment facility k, due to the fact that the term contains the shadow prices μ_j, λ_k, and η_k.
For any value of the multipliers $\{\mu_j\}$, $\{\lambda_k \geq 0\}$ and $\{\eta_k \leq 0\}$, a freight unit in i will select the alternative (s,t) given by the transshipment facility s and the destination t (for the sake of simplicity, we assume this alternative is unique), whose shadow random utility $(\tilde{v}_{it}^s(\tilde{u}^s) + \mu_t - \lambda_s - \eta_s)$ is the maximum within those of the open transshipment facilities and the destinations. So, the unit shadow random utility from i becomes

$$\tilde{v}_i(\tilde{u}^s) = \tilde{v}_{it}^s(\tilde{u}^s) + \mu_t - \lambda_s - \eta_s = max_{k:y_k=1,j}\left(\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k\right) \tag{28}$$

Problem (24)-(27), for any value of the multipliers $\{\mu_j\}$, $\{\lambda_k \geq 0\}$, and $\{\eta_k \leq 0\}$ and any transshipment facility location $\{y_k\}$, gives the following trivial optimal flows

$s_{ij}^k = P_i$ if $\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k = \tilde{v}_{it}^s(\tilde{u}^s) + \mu_t - \lambda_s - \eta_s$
$s_{ij}^k = 0$ otherwise

and the objective function (24) becomes

$$\min_{\mu,\lambda\geq 0,\eta\leq 0} \max_y \sum_{k\in K} [-f_k y_k + \lambda_k U_k y_k + \eta_k L_k y_k] - \sum_{j\in J} \mu_j Q_j + \mathbb{E}_{\tilde{u}^s}\left[\sum_{i\in I} P_i \tilde{v}_i(\tilde{u}^s)\right] =$$
$$\min_{\mu,\lambda\geq 0,\eta\leq 0} \max_y \sum_{k\in K} [-f_k y_k + \lambda_k U_k y_k + \eta_k L_k y_k] - \sum_{j\in J} \mu_j Q_j + \sum_{i\in I} P_i \mathbb{E}_{\tilde{u}^s}\left[\tilde{v}_i(\tilde{u}^s)\right] \tag{29}$$

To calculate $\mathbb{E}_{\tilde{u}^s}[\tilde{v}_i(\tilde{u}^s)]$ in (29) we first need to know the probability distribution of $\tilde{v}_i(\tilde{u}^s)$, named $G_i(x)$

$$G_i(x) = Pr\{\tilde{v}_i(\tilde{u}^s) \leq x\} = Pr\left\{\max_{k:y_k=1,j}\left(\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k\right) \leq x\right\} \tag{30}$$

As

$$\max_{k:y_k=1,j}\left(\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k\right) \leq x \Longleftrightarrow$$
$$\tilde{v}_{ij}^k(\tilde{u}^k) + \mu_j - \lambda_k - \eta_k \leq x, \quad k \in K : y_k = 1, j \in J \tag{31}$$

and the random variables $\tilde{u}^k$ are independent (because u^{kl} are independent), (30) becomes

$$
\begin{aligned}
G_i(x) \quad = \quad & \Pr\left\{max_{k:y_k=1,j}\left(\tilde{v}_{ij}^k(\tilde{u}^k)+\mu_j-\lambda_k-\eta_k\right)\le x\right\} = \\
& \Pr\left\{\tilde{v}_{ij}^k(\tilde{u}^k)+\mu_j-\lambda_k-\eta_k\le x,\ \ k\in K: y_k=1, j\in J\right\} = \\
& \prod_{k\in K:y_k=1}\prod_{j\in J}\Pr\left\{\tilde{v}_{ij}^k(\tilde{u}^k)+\mu_j-\lambda_k-\eta_k\le x\right\} = \\
& \prod_{k\in K:y_k=1}\prod_{j\in J}\Pr\left\{v_{ij}^k+\tilde{u}^k+\mu_j-\lambda_k-\eta_k\le x\right\} = \\
& \prod_{k\in K:y_k=1}\prod_{j\in J}\Pr\left\{\tilde{u}^k\le x-v_{ij}^k-\mu_j+\lambda_k+\eta_k\right\} = \\
& \prod_{k\in K:y_k=1}\prod_{j\in J}B_k(x-v_{ij}^k-\mu_j+\lambda_k+\eta_k) = \\
& \prod_{k\in K:y_k=1}\prod_{j\in J}\left[F(x-v_{ij}^k-\mu_j+\lambda_k+\eta_k)\right]^{n_k}
\end{aligned} \tag{32}
$$

3 The deterministic approximation of C^2TLP_u

The unknown probability distribution $F(x)$ prevents the calculation of $G_i(x)$ in (32). A possible way to solve this problem and get an explicit form for $G_i(x)$ is to consider an asymptotic approximation for it.
The method we use to derive an asymptotic approximation of $G_i(x)$ is based on the following observation. Under mild conditions on the unknown probability distribution $F(x)$, the probability distribution $G_i(x)$ tends towards a specific functional form as the total number of alternative handling operating paths at transshipment facility k, n_k, becomes large.
Following Galambos [6], we will prove that the only condition requested for $F(x)$ is that is asymptotically exponential in its right tail, i.e. there is a constant $\beta > 0$ such that

$$\lim_{y\to\infty}\frac{1-F(x+y)}{1-F(y)} = e^{-\beta x} \tag{33}$$

This is a very mild condition, as we observe that many probability distributions show such behavior.

Firstly, let us consider the following aspect: the solution of problem (24)-(27) does not change if an arbitrary constant is added to the random variables $\tilde{u}^k$.

Let us choose this constant as the root a_{n_k} of the equation

$$1-F(a_{n_k}) = 1/n_k \tag{34}$$

Replacing $\tilde{u}^k$ with $\tilde{u}^k - a_{n_k}$ in (32)

$$G_i(x\mid n_k) = \prod_{k\in K:y_k=1}\prod_{j\in J}\left[F(x-v_{ij}^k-\mu_j+\lambda_k+\eta_k+a_{n_k})\right]^{n_k}. \tag{35}$$

Let us assume that $n_k, k \in K : y_k = 1$ are large enough to use $\lim_{n_k \to \infty} G_i(x \mid n_k)$ as an approximation of $G_i(x)$.

The following theorem holds

Theorem 1. *Under condition (33), the unknown probability distribution $G_i(x)$ becomes*

$$G_i(x) = \lim_{n_k \to \infty} G(x \mid n_k) = \exp\left(-A_i e^{-\beta x}\right) \tag{36}$$

where

$$A_i = \sum_{k \in K : y_k = 1} \sum_{j \in J} e^{\beta(v_{ij}^k + \mu_j - \lambda_k - \eta_k)} = \sum_{k \in K} \sum_{j \in J} y_k e^{\beta(v_{ij}^k + \mu_j - \lambda_k - \eta_k)}, \quad i \in I \tag{37}$$

is the accessibility, in the sense of Hansen [8], of a freight unit in i to the overall system of located transshipment facilities and destinations.

Proof. By (35) one has

$$\begin{aligned} G_i(x) \quad = \quad & \lim_{n_k \to \infty} G_i(x \mid n_k) = \\ & \lim_{n_k \to \infty} \prod_{k \in K : y_k = 1} \prod_{j \in J} \left[F(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k + a_{n_k})\right]^{n_k} = \\ & \prod_{k \in K : y_k = 1} \prod_{j \in J} \lim_{n_k \to \infty} \left[F(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k + a_{n_k})\right]^{n_k} \end{aligned} \tag{38}$$

As $\lim_{n_k \to \infty} a_{n_k} = \infty$, from (33) one obtains

$$\lim_{n_k \to \infty} \frac{1 - F(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k + a_{n_k})}{1 - F(a_{n_k})} = e^{-\beta(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k)} \tag{39}$$

By (39) and (34) one has

$$\begin{aligned} & \lim_{n_k \to \infty} F(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k + a_{n_k}) = \\ & \lim_{n_k \to \infty} \left(1 - [1 - F(a_{n_k})] e^{-\beta(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k)}\right) = \\ & \lim_{n_k \to \infty} \left(1 - \frac{e^{-\beta(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k)}}{n_k}\right) \end{aligned} \tag{40}$$

and, by reminding that $\lim_{n \to \infty} (1 + \frac{x}{n})^n = e^x$

$$\begin{aligned} & \lim_{n_k \to \infty} \left[F(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k + a_{n_k})\right]^{n_k} = \\ & \lim_{n_k \to \infty} \left[1 - \frac{e^{-\beta(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k)}}{n_k}\right]^{n_k} = \\ & \exp\left(-e^{-\beta(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k)}\right) \end{aligned} \tag{41}$$

Substituting (41) into (38) and using (37) one finally gets

$$G_i(x) = \prod_{k \in K: y_k = 1} \prod_{j \in J} \exp\left(-e^{-\beta(x - v_{ij}^k - \mu_j + \lambda_k + \eta_k)}\right) = \exp\left(-A_i e^{-\beta x}\right) \square \tag{42}$$

$\square$

It is interesting to observe that $G_i(x)$ in (36) becomes a Gumbel (or double exponential) distribution [7].

Having now an explicit form for $G_i(x)$, we can calculate $\mathbb{E}_{\tilde{u}^s}\left[\tilde{v}_i(\tilde{u}^s)\right]$ in (29) as follows

$$\overline{v}_i = \mathbb{E}_{\tilde{u}^s}\left[\tilde{v}_i(\tilde{u}^s)\right] = \int_{-\infty}^{+\infty} x dG_i(x) = \int_{-\infty}^{+\infty} x \exp\left(-A_i e^{-\beta x}\right) A_i e^{-\beta x} \beta dx, \quad i \in I \tag{43}$$

Substituting for $t = A_i e^{-\beta x}$ one gets

$$\begin{aligned}
\overline{v}_i &= -1/\beta \int_0^{+\infty} \ln(t/A_i) e^{-t} dt = \\
&= -1/\beta \int_0^{+\infty} e^{-t} \ln t dt + 1/\beta \ln A_i \int_0^{+\infty} e^{-t} dt = \\
&= \gamma/\beta + 1/\beta \ln A_i = \\
&= 1/\beta(\ln A_i + \gamma)
\end{aligned} \tag{44}$$

where $\gamma = -\int_0^{+\infty} e^{-t} \ln t \, dt \simeq 0.5772$ is the Euler constant.

By substituting (44) in (29), the deterministic approximation of C^2TLP_u, named C^2TLP_d, becomes the following nonlinear mixed-integer problem

$$\min_{\mu, \lambda \geq 0, \eta \leq 0} \max_y \sum_{k \in K} \left[-f_k y_k + \lambda_k U_k y_k + \eta_k L_k y_k\right] - \sum_{j \in J} \mu_j Q_j + 1/\beta \sum_{i \in I} P_i \ln A_i \tag{45}$$

subject to

$$\sum_{k \in K} \sum_{j \in J} y_k s_{ij}^k = P_i, \quad i \in I \tag{46}$$

$$s_{ij}^k \geq 0, \quad i \in I, j \in J, k \in K \tag{47}$$

$$y_k \in \{0, 1\}, \quad k \in K \tag{48}$$

where the constant $\frac{\gamma}{\beta} \sum_{i \in I} P_i$ has been dropped in the objective function.

If we denote by

$$x_{ij}^k = s_{ij}^k / P_i \quad i \in I, j \in J, k \in K \tag{49}$$

the probability that a freight unit in i is delivered towards the alternative (k, j), given by the transshipment facility k and the destination j, then x_{ij}^k is equal to the probability that the pair (k, j) is the alternative of maximum utility.

The following theorem holds

Theorem 2. *At optimality, the probability x_{ij}^k is given by*

$$x_{ij}^k = \frac{e^{\beta(v_{ij}^k+\mu_j-\lambda_k-\eta_k)}}{\sum_{k'\in K:y_{k'}=1}\sum_{j'\in J} e^{\beta(v_{ij'}^{k'}+\mu_{j'}-\lambda_{k'}-\eta_{k'})}}, \quad i \in I, j \in J, k \in K \tag{50}$$

Proof. At optimality, the probability that a freight unit in i chooses the alternative (k, j) is equal to the probability that (k, j) is the alternative of maximum utility. Then, from the Total Probability Theorem [3], condition (33) and eq. (37), one obtains

$$\begin{aligned} x_{ij}^k &= \int_{-\infty}^{+\infty} \prod_{u\neq k}\prod_{v\neq j} \exp\left[-e^{-\beta(x-v_{iv}^u-\mu_v+\lambda_u+\eta_u)}\right] d\left[\exp\left(-e^{-\beta(x-v_{ij}^k-\mu_j+\lambda_k+\eta_k)}\right)\right] = \\ &= e^{\beta(v_{ij}^k+\mu_j-\lambda_k-\eta_k)} \int_{-\infty}^{+\infty} \beta e^{-\beta x} exp(-A_i e^{-\beta x})dx = \\ &= e^{\beta(v_{ij}^k+\mu_j-\lambda_k-\eta_k)} \int_{0}^{+\infty} e^{-A_i t}dt = \frac{e^{\beta(v_{ij}^k+\mu_j-\lambda_k-\eta_k)}}{A_i} = \\ &= \frac{e^{\beta(v_{ij}^k+\mu_j-\lambda_k-\eta_k)}}{\sum_{k'\in K:y_{k'}=1}\sum_{j'\in J} e^{\beta(v_{ij'}^{k'}+\mu_{j'}-\lambda_{k'}-\eta_{k'})}} \quad i \in I, j \in J, k \in K \end{aligned} \tag{51}$$

where $t = e^{-\beta x}$. $\square$

$\square$

The optimal flows s_{ij}^k then become

$$s_{ij}^k = P_i x_{ij}^k = P_i \frac{e^{\beta(v_{ij}^k+\mu_j-\lambda_k-\eta_k)}}{\sum_{k'\in K} y_{k'} \sum_{j'\in J} e^{\beta(v_{ij'}^{k'}+\mu_{j'}-\lambda_{k'}-\eta_{k'})}}, \quad i \in I, j \in J, k \in K \tag{52}$$

and it is trivial to check the satisfaction of constraints (46).

Eq. (50) represents a multinomial Logit model, which is widely used in choice theory [4]. In our case, it describes how the freight delivered from i is split among the pairs (k, j), due to the stochasticity of the random utilities of the transshipment facilities which the freight passes through.

It is interesting to note that eq. (52), although it has been derived for open transshipment facilities, holds for all k's. So, for an open transshipment facility k, s_{ij}^k represents the actual flow from i to j via that open facility. Vice versa, for a closed transshipment facility k, it represents the "potential" flow from i to j via that closed facility. The potential flow will be used in our heuristic of Section 4.

Solving the deterministic approximation C^2TLP_d is much faster than solving the stochastic model C^2TLP_u and it also has a very good performance in terms of solution quality, showing a mean gap of $0,87\%$ between the optimum values (see Section 5). By using one of the best stochastic commercial solvers, C^2TLP_u is just able to solve instances with up to a few origins and some decades of potential facility locations and destinations in one hour of computing time, whereas in the same computing time C^2TLP_d can solve instances which are roughly three times larger than those solved by C^2TLP_u. Unfortunately, if one wanted to consider even larger instances (hundreds of nodes for the total number of origins, destinations and potential facility locations) which may appear in real-life applications, also C^2TLP_d becomes inefficient and some heuristics must be used.
One of these heuristics is given in the next section.

4 A heuristic for solving C^2TLP_d

The heuristic for solving C^2TLP_d is based on three procedures which interact with each other: the first is a procedure for calculating the Lagrangian multipliers $\{\mu_j\}$, $\{\lambda_k \geq 0\}$, and $\{\eta_k \leq 0\}$, when a transshipment facility location $\{y_k\}$ is already given, while the second and the third are, respectively, for opening and closing down facilities in order to improve the given facility location.

Firstly, we are going to present the three procedures, while the overall heuristic is given at the end of this section.

4.1 Lagrangian multipliers calculation

As assumed previously, a transshipment location $\{y_k\}$ is already given.
We calculate the Lagrangian multipliers $\{\mu_j\}$, $\{\lambda_k\}$, and $\{\eta_k\}$ by an efficient iterative method as follows.

Let us start with $\lambda_k = \eta_k = 0, k \in K$.

Calculate $\{\mu_j\}$ such that the demand satisfaction constraints (18), where s^k_{ij} are given by (52), are satisfied.
To do that, solve the following system of equations iteratively in $e^{\beta\mu_j}$, starting with any value for $\{e^{\beta\mu_j}\}$ (e.g., $e^{\beta\mu_j} = 1$ then $\mu_j = 0, j \in J$)

$$e^{\beta\mu_j} = Q_j / \sum_{i\in I} \sum_{k\in K:y_k=1} P_i \frac{e^{\beta v^k_{ij}} e^{-\beta\lambda_k} e^{-\beta\eta_k}}{\sum_{k'\in K} \sum_{j'\in J} y_{k'} e^{\beta v^{k'}_{ij'}} e^{\beta\mu_{j'}} e^{-\beta\lambda_{k'}} e^{-\beta\eta_{k'}}}, \quad j \in J \qquad (53)$$

Once $\{e^{\beta\mu_j}\}$ are calculated, the multipliers $\{\lambda_k\}$ and $\{\eta_k\}$ are updated as follows (note that these multipliers are also calculated for closed facilities).

Let $D_k(\lambda, \eta)$ be the throughput of facility k

$$D_k(\lambda, \eta) = \sum_{i \in I} \sum_{j \in J} s_{ij}^k(\lambda, \eta), \quad k \in K \tag{54}$$

$D_k(\lambda, \eta)$ is expressed only in the unknowns $\{\lambda_k\}$ and $\{\eta_k\}$, because $\{\mu_j\}$ are known yet.

Like flows $\{s_{ij}^k\}$, $\{D_k\}$ are calculated also for closed facilities. When k is open, D_k represents the actual throughput of the facility, whereas it represents its "potential" throughput when k is closed.

By (52), eq. (54) becomes

$$\begin{aligned} D_k(\lambda, \eta) =& e^{-\beta\lambda_k} e^{-\beta\eta_k} \sum_{i \in I} P_i \frac{\sum_{j \in J} e^{\beta v_{ij}^k} e^{\beta\mu_j}}{\sum_{k' \in K} \sum_{j \in J} y_{k'} e^{\beta v_{ij}^{k'}} e^{\beta\mu_j} e^{-\beta\lambda_{k'}} e^{-\beta\eta_{k'}}} = \\ &= e^{-\beta\lambda_k} e^{-\beta\eta_k} \rho_k, \quad k \in K \end{aligned} \tag{55}$$

where

$$\rho_k = \sum_{i \in I} P_i \frac{\sum_{j \in J} e^{\beta v_{ij}^k} e^{\beta\mu_j}}{\sum_{k' \in K} \sum_{j \in J} y_{k'} e^{\beta v_{ij}^{k'}} e^{\beta\mu_j} e^{-\beta\lambda_{k'}} e^{-\beta\eta_{k'}}}, \quad k \in K \tag{56}$$

is the current size of facility k (actual if k is open or potential if k is closed).

The updating of the multipliers $\{\lambda_k\}$ and $\{\eta_k\}$ is made as follows

- if $L_k \leq \rho_k \leq U_k$, leave $\lambda_k = \eta_k = 0$ (then $e^{-\beta\lambda_k} = e^{-\beta\eta_k} = 1$)
- if $\rho_k > U_k$, set $e^{-\beta\lambda_k} = U_k/\rho_k$ and $e^{-\beta\eta_k} = 1$
- if $\rho_k < L_k$, set $e^{-\beta\lambda_k} = 1$ and $e^{-\beta\eta_k} = L_k/\rho_k$

The rationale for the above updating mechanism is the following one. If the current size ρ_k of facility k does satisfy the lower and upper capacity constraints, then D_k is kept like it is. Otherwise, if ρ_k is greater than the upper capacity, D_k will be reduced by multiplying ρ_k by a proper coefficient $e^{-\beta\lambda_k} < 1$ (because $\lambda_k > 0$). If ρ_k is smaller than the lower capacity, D_k will be augmented by multiplying ρ_k by a proper coefficient $e^{-\beta\eta_k} > 1$ (because $\eta_k < 0$).
Given the updated $\{\lambda_k\}$ and $\{\eta_k\}$, the multipliers $\{\mu_j\}$ are then recalculated by (53) and the iterative procedure goes on until the upper and lower capacity constraints (19) and (20) are satisfied.
With the final values of $\{\mu_j\}$, $\{\lambda_k\}$, and $\{\eta_k\}$ one can calculate the optimal flows $\{s_{ij}^k\}$ for the given transshipment location $\{y_k\}$ by (52), then the optimum of C^2TLP_d by (45).

4.2 Opening a transshipment facility

The second procedure of our heuristic is for opening facilities with the aim of improving the given facility location.
By substituting (37) in (45), the optimum of C^2TLP_d can be written as follows

$$\min_{\mu,\lambda\geq 0,\eta\leq 0}\max_{y}\sum_{k\in K}\left[-f_k y_k+\lambda_k U_k y_k+\eta_k L_k y_k\right]-\sum_{j\in J}\mu_j Q_j$$
$$+1/\beta\sum_{i\in I}P_i\ln\sum_{k\in K}\sum_{j\in J}y_k e^{\beta(v^k_{ij}+\mu_j-\lambda_k-\eta_k)} \tag{57}$$

Let us consider in (57) the continuous relaxation of the binary variables $\{y_k\}$ in the interval [0,1] and the Lagrangian multipliers $\{\mu_j\}$, $\{\lambda_k\}$, and $\{\eta_k\}$ obtained by the procedure of Section 4.1.
By imposing to (57) the necessary first order conditions for $\{y_k\}$ one gets

$$\partial\left\{\sum_{k\in K}\left[-f_k y_k+\lambda_k U_k y_k+\eta_k L_k y_k\right]-\sum_{j\in J}\mu_j Q_j\right.$$
$$\left.+1/\beta\sum_{i\in I}P_i\ln\sum_{k\in K}\sum_{j\in J}y_k e^{\beta(v^k_{ij}+\mu_j-\lambda_k-\eta_k)}\right\}/\partial y_k=$$
$$-f_k+\lambda_k U_k+\eta_k L_k+1/\beta\sum_{i\in I}P_i\frac{\sum_{j\in J}e^{\beta(v^k_{ij}+\mu_j-\lambda_k-\eta_k)}}{\sum_{k'\in K}\sum_{j\in J}y_{k'}e^{\beta(v^{k'}_{ij}+\mu_j-\lambda_{k'}-\eta_{k'})}}=$$
$$-f_k+\lambda_k U_k+\eta_k L_k+\frac{1}{\beta}e^{-\beta\lambda_k}e^{-\beta\eta_k}\rho_k \tag{58}$$

Eq. (58) represents the impact on the optimum of a continuous variation of the location y_k. Let us consider only those closed facilities k for which $(-f_k+\lambda_k U_k+\eta_k L_k+\frac{1}{\beta}e^{-\beta\lambda_k}e^{-\beta\eta_k}\rho_k)>0$, because only they could improve the total utility (57) by "increasing" their y_k. Moreover, this improvement will be maximized when $y_k=1$. So, if one wants to open one of those facilities by improving the total utility as much as possible, the transshipment facility with the highest positive term $(-f_k+\lambda_k U_k+\eta_k L_k+\frac{1}{\beta}e^{-\beta\lambda_k}e^{-\beta\eta_k}\rho_k)$ will be the candidate to be opened, i.e. the facility r for which

$$r=argmax_{k:y_k=0}\left[-f_k+\lambda_k U_k+\eta_k L_k+\frac{1}{\beta}e^{-\beta\lambda_k}e^{-\beta\eta_k}\rho_k\right]^+$$

If we define the *revenue* of facility k as being the difference between its *profit* $(\lambda_k U_k+\frac{1}{\beta}e^{-\beta\lambda_k}e^{-\beta\eta_k}\rho_k)$ and its *cost* $(-f_k+\eta_k L_k)$, the facility with the highest revenue will be the candidate for opening.

Let us consider a closed facility k. Because of the Kuhn-Tucker conditions we know that if the facility potential current size ρ_k is such that $L_k\leq\rho_k\leq U_k$, then $\lambda_k=\eta_k=0$ and the facility revenue is given by $(-f_k+\frac{1}{\beta}\rho_k)$.
Vice versa, if $\lambda_k>0$ then the potential current size of that facility is over its upper capacity (so opening that facility should be highly recommended) and the term $\lambda_k U_k$ will encourage the choice of that facility to be the candidate for opening. In such a case, the constraints (19) would be saturated, i.e. $e^{-\beta\lambda_k}e^{-\beta\eta_k}\rho_k=U_k$, and the facility

revenue would become $[-f_k + U_k(\lambda_k + 1/\beta)]$ (as, of course, $\eta_k = 0$).
If $\eta_k < 0$ then the potential current size of that facility is under its lower capacity (so its opening should not be recommended) and the term $\eta_k L_k$ will discourage the choice of this facility to be the candidate for opening. In such a case, the constraints (20) would be saturated, i.e. $e^{-\beta\lambda_k} e^{-\beta\eta_k} \rho_k = L_k$, and the facility revenue would become $[-f_k + L_k(\eta_k + 1/\beta)]$ (as, of course, $\lambda_k = 0$).

We can note that the above criterion for opening facilities requires the possibility to calculate the size ρ_k for closed facilities, but, as seen above, this can be done easily by (56), which holds for all k's.

4.3 Closing down a transshipment facility

The third and last procedure of our heuristic is for closing down facilities.
We observe that a mechanism similar to that of Section 4.2 can be adopted for finding, within the open facilities, the candidate to be closed down. In such a case, the transshipment facility q for which

$$q = argmin_{k:y_k=1} \left[-f_k + \lambda_k U_k + \eta_k L_k + \frac{1}{\beta} e^{-\beta\lambda_k} e^{-\beta\eta_k} \rho_k \right]^{-} \tag{59}$$

will be the candidate to be closed down (provided that the total upper capacity of the remaining open facilities is not less than the total flow T).

We are now ready to put the three procedures together and build up the overall heuristic to solve C^2TLP_d.

4.4 The overall heuristic

As assumed previously, a transshipment location $\{y_k\}$ is given. Using the procedure developed in Section 4.1, we can calculate the Lagrangian multipliers and derive the optimum of the problem and the optimal flows $\{s_{ij}^k\}$. Then, we try to improve the given transshipment location by opening and closing down facilities. This process calls the above procedure for the Lagrangian multipliers calculation as a subroutine. We reiterate until no further improvements for the optimum are found.
More in detail, the heuristic to solve C^2TLP_d acts as follows

- Problem Feasibility check.
  If the total upper capacity is less than the total flow, i.e. $\sum_{k\in K} U_k < T$ or the minimum lower capacity is greater than the total flow, i.e. $min_{k\in K} L_k > T$, then STOP, the problem is infeasible.
- While the number of iterations is not greater than MAXITER (maximum number of iterations) and the overall computational time is not greater than MAXTIME (maximum computational time), apply the Core heuristic (see Subsection 4.4.1) which, by opening and closing operations, builds a solution.
- Keep the best solution as the optimal one.

4.4.1 Core heuristic

The core heuristic builds a solution according to the following steps

1. Open all facilities, i.e. $y_k = 1, k \in K$.
2. Compute the Lagrangian multipliers as in Section 4.1. Calculate the optimal flows and set the best solution $BestSol$ to the optimal flows.
3. Repeat the following steps
   (a) Decide whether to close down a transshipment facility or simultaneously close down and open two different facilities. The decision is taken according to a rule based on a randomized process and a short term search memory. This rule, called the *operation choosing rule*, is described in detail in Subsection 4.4.2.
   (b) Let q be the facility to be closed down and r the facility to be opened (if any). Close down q and open r.
   (c) Compute the Lagrangian multipliers and the optimal flows. Set the current solution $CurrSol$ to the optimal flows.
   (d) If no opening operation has been performed and the objective function of $BestSol$ is worse than that of $CurrSol$, then exit from the heuristic and return the value $BestSol$. Otherwise, set $BestSol$ to $CurrSol$.

4.4.2 Operation choosing rule

In our heuristic, we can decide whether either to close down a facility (we remind we start with all facilities open) or simultaneously close down and open two different facilities. The operation choosing rule uses both a dynamic random process guided by the search history and a short-term memory structure to take the above decision. The short-term memory structure is a list FL which forbids, after opening a facility, its closing down for a fixed amount m of iterations.
The rule works as follows:

- If we are at the first iteration of the overall heuristic, initialize the opening probability step $\delta_O = 0$, otherwise set $\delta_O = 2/|K|$, where $|K|$ is the number of potential transshipment locations. Empty the list FL, set its size equal to MAXFL and put $v_O = 0$.
- While the solution is feasible
  - Get a random number $v \in (0, 1]$.
  - If $v \geq v_O$, find the candidate q to be closed down as in Section 4.3 and check that $q \notin FL$. Increment v_O by δ_O and, for all facilities in FL, decrement by one the number of iterations for which they cannot be closed down. Remove from FL the facilities for which the number of iterations is 0.
  - Otherwise, set $v_O = \delta_O$, find the candidate q to be closed down and the candidate p to be opened, as in Sections 4.3 and 4.2, respectively. If their swap (i.e. closing down q and opening p) is feasible and improves the current optimum, make the swap, add p to FL and set to MAXFL the number of iterations for which p cannot be closed.

We can note that, in the first iteration of the overall heuristic, we only apply closing operations. The opening operations will be considered in next iterations of the heuristic.

In other words, we start with all facilities open and we close down one after another according to the criterion of Section 4.3. When a local optimum has been reached, a sort of local search is introduced through the swapping mechanism above described.

5 Computational results

In this section we compare C^2TLP_u and its deterministic approximation C^2TLP_d solved both exactly, by means of one of the best state-of-the-art nonlinear solvers, and by our heuristic. We consider three classes of instances, which contain identical instances except for the facility lower capacity. In particular, the lower capacity of Class I is set equal to zero. The lower capacity of Class II is set to 60% of the upper capacity, whilst for instances belonging to Class III such percentage rises up to 85%. For each class 10 instances are generated as follows

- the number of depots $|I|$ is drawn from $U[2, 3]$;
- the number of customers $|J|$ is drawn from $U[30, 40]$;
- the number of possible locations for the transshipments $|K|$ is drawn from $U[10, 20]$;
- supply P_i is drawn from $U[900, 1000]$;
- demand Q_j is drawn from $U[1, \sum_{i \in I} P_i / \mid J \mid]$. If necessary, the demand of the last customer is adjusted so that the total demand is equal to the total supply;
- upper capacity U_k is drawn from $U[0.5avU, 3avU]$, where $avU = \sum_{i \in I} P_i / \mid K \mid$;
- unit transportation cost c_{ij}^k is drawn from $U[1, 10]$, $k \in 1, .., |K|/2$, and $U[5, 10]$, $k \in |K|/2 + 1, .., |K|$;
- fixed cost $f_k = 0.3U_k \frac{TC}{IJK}$, $k \in 1, .., |K|/2$, $f_k = 0.03U_k \frac{TC}{IJK}$, $k \in |K|/2 + 1, .., |K|$, where TC is the total transportation cost;
- random utility $\tilde{u}^k$ is drawn from $U[1, 10]$.

The rationale for choosing the above values for c_{ij}^k and f_k is to have half of the facilities with high fixed cost and low mean unit transportation cost, whereas for the other half is the reverse. In this way, if one opens a facility in the first half of the potential locations, the high fixed cost and low mean transportation cost force the model to saturate the facility. On the other hands, in the second half of the potential locations the cost structure pushes the model to spread flows within the open facilities, possibly violating the lower capacity constraints.

Both C^2TLP_d and the heuristic need to know a proper value of the positive parameter β, which must be calibrated. This is done as follows.

Let us consider the standard Gumbel distribution $G(x) = \exp(-e^{-x})$. If one accepts an approximation error of 0.01, then $G(x) = 1 \iff x = 4.60$ and $G(x) = 0 \iff x = -1.52$. Let us consider the interval $[m, M]$ where the random utility $\tilde{u}^k$ is drawn from. The following equations hold

$$\beta(m - \zeta) = -1.52 \tag{60}$$

$$\beta(M - \zeta) = 4.60 \tag{61}$$

where ζ is the mode of the Gumbel distribution $G(x) = \exp\left(-e^{-\beta(x-\zeta)}\right)$.

From (60) and (61) one gets

$$\beta = \frac{6.12}{M - m}$$

In our case, as $M - m = 10 - 1 = 9$, we get $\beta = 0.68$.

Table 1 shows the number of origins, destinations and potential locations for the ten instances of each class. Let us note that these numbers are the same for all classes of instances because these classes only differ for the lower capacity values of the facilities.

INST	$\|\mathbf{I}\|$	$\|\mathbf{J}\|$	$\|\mathbf{K}\|$
1	2	35	10
2	2	38	16
3	3	37	16
4	2	33	17
5	2	36	16
6	3	35	18
7	2	31	16
8	2	38	11
9	3	34	16
10	3	40	20

Table 1: Detail of the instance parameters

The solution of C^2TLP_u is generated by means of the stochastic programming module provided in the XPress Optimization Suite [5]. The tests are performed by generating an appropriate number of scenarios for each instance. In order to tune this number, we start with 50 scenarios and increase them with step 50. Then we solve each instance 10 times, reinitializing every time the pseudo-random generator of the stochastic components with a different seed, and compute the standard deviation and the mean of the optima over the 10 runs. The appropriate number of scenarios is then fixed to the smallest value which ensures for each instance a maximum ratio less than 1% between the standard deviation and the mean [9]. According to our tests, this value is fixed to 200 scenarios, which show a maximum ratio of 0.34%. To solve the deterministic approximation C^2TLP_d we use the nonlinear solver BonMIN (release 1.1) within a time limit of 1000 seconds [1, 2]. The parameters are set to their default values, which show a satisfactory behavior both in accuracy and computational effort. The heuristic presented in Section 4.4 is implemented in Matlab 2007. After a preliminary testing phase, the parameters of the heuristic are set as follows

- MAXTIME= 10000 seconds
- MAXITER= 50
- Size of FL= 3.

All the tests were performed on a Pentium Quad Duo 2.4 GhZ workstation with 2 Gb of Ram.

Table 2 compares the optimal solution of the stochastic problem C^2TLP_u with its deterministic approximation C^2TLP_d. The table columns have the following meaning

- Column 1: instance class
- Column 2: mean objective function of C^2TLP_u
- Column 3: mean objective function of C^2TLP_d

- Column 4: percentage gap between the two objective functions with respect to the first one
- Column 5: mean computational time in seconds for C^2TLP_u
- Column 5: mean computational time in seconds for C^2TLP_d.

For each value, the mean results of the 10 random generated instances in each class are given in the first three rows, while the last row gives the overall mean over the three classes.
According to figures, the deterministic approximation of the stochastic model is quite good. In fact, it shows a mean gap less than 1%, while reducing the computational time of 70%. One can also notice how this gap slightly increases for higher values of the lower capacity. Moreover, the computational time of the stochastic model increases a lot in such cases, while the deterministic approximation is less affected.

The good behavior of C^2TLP_d is also confirmed by analyzing the number of open facilities, as well as the number of those facilities which are opened by both models. Table 3 shows these results. In particular, its columns have the following meaning

- Column 1: instance class
- Column 2: number of open facilities for C^2TLP_u
- Column 3: number of open facilities for C^2TLP_d
- Column 4: percentage of common open facilities. Such percentage is calculated as the ratio of the number of common facilities over the number of open facilities for C^2TLP_u.

The results show how the solutions of the stochastic model and its deterministic approximation share about 90% of the open facilities.

The results of our heuristic are summarized in Table 4, where the meaning of each column is as follows

- Column 1: instance class
- Columns 2 and 3: percentage gap between the heuristic (first iteration and best solution) and the deterministic approximation solved by BonMIN (we remind that the first iteration of the procedure starts with all facilities open and only closing down operations are performed)
- Columns 4, 5 and 6: computational time (s) of the heuristic (first iteration and best solution) and BonMIN
- Columns 7 and 8: total computional time (s) at the end of the heuristic and BonMIN.

The results show how the gap between the first and the best solution is increasing with the value of the lower capacity, giving a mean gap between the BonMIN solution and the first iteration solution of the heuristic of 3.12%. On the other hand, after applying the closing and opening operations (column *HeurBest*) the heuristic is able to reduce the overall gap to 0.82%. These results are more impressive considering that they can be obtained in a short computing time. In fact, the mean computing time of the heuristic is half of that of BonMIN. Moreover, if we consider the time when the best solution has been found, we discover that our heuristic needs only a mean computing time of 20 seconds, which is one fifth of the BonMIN computing time (and one twentieth of that of the stochastic model). Furthermore, the computing time of the heuristic could be further reduced by implementing an ad hoc fixed point method to compute the Lagrangian multipliers in Section 4.1. In fact, the present implementation uses the standard Matlab *fsolve* function, which becomes inefficient when the network size increases.

If we consider the impact of the opening operations on the final heuristic solution, we can see that a certain number of transshipment facilities which have been opened by the heuristic are still open in the final solution. So, it would seem that the first iteration has made some mistake in closing down these facilities, but it is not true. Actually, this situation is due to the fact that the criterion for closing down a facility is satisfied by more than one facility.

CLASS	OBJECTIVE FUNCTION			TIME	
	STOCH	DET	GAP %	STOCH	DET
1	15741.7	15836.3	0.60	73	65
2	16122.8	16291.9	1.05	452	145
3	16898.5	17062.2	0.97	796	178
MEAN	**16254.3**	**16396.8**	**0.87**	**440**	**129**

Table 2: Comparison between the stochastic model and its deterministic approximation: objective function and computational time

CLASS	OPEN FACILITIES		
	STOCH	DET	COMMON (%)
1	14.1	14.2	95.1
2	11.5	11.7	89.8
3	10.2	10.5	90.3
MEAN	**11.9**	**12.1**	**91.7**

Table 3: Comparison between the stochastic and its deterministic approximation: open facilities

CLASS	GAP %		OPT TIME (s)			TOT TIME (s)	
	HEURFIRST	HEURBEST	HEURFIRST	HEURBEST	DET	HEURBEST	DET
1	0.75	0.29	1.8	8.7	36.4	45.1	65.0
2	3.78	1.15	2.9	25.4	120.8	67.8	145.0
3	4.83	1.03	3.1	24.8	137.4	75.2	178.0
MEAN	**3.12**	**0.82**	**2.60**	**20.30**	**98.2**	**62.7**	**129.3**

Table 4: Comparison between the deterministic approximation and the heuristic

6 Conclusions

In this paper we have addressed the problem of locating transshipment facilities for freight transportation to minimize the total cost of different operations. This cost consists of a fixed cost of locating the facility, a transportation cost from origin to destination via the facility, and a cost for freight handling operations at the facility. In reality, the handling operations are organized in alternative operating paths and, given the finite capacity of the paths, congestion effects make the handling costs random variables, with unknown probability distribution.
To the authors' knowledge, this paper is among the first ones which

- integrate in a comprehensive model the two main levels of a transshipment network, i.e. the design level and the management level
- address the congestion effects inside the transshipment facilities, leading to a stochastic location-allocation problem.

Moreover, from a theoretical perspective, the paper shows that, under mild assumptions, the unknown probability distribution of the maximum random utility converges to a Gumbel distribution and the expected optimal flows are multinomial Logit functions.

Finally, from a computational point of view both the deterministic approximation of the stochastic model and the heuristic presented in the paper show a mean gap between the optimum values which is less than 1%.

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References

[1] Bonami, P., Biegler, L.T., Conn, A.R., Cornuéjols, G., Grossmann, I.E., Laird, C.D., Lee, J., Lodi, A., Margot, F., Sawaya, N., Wächter, A.: An algorithmic framework for convex mixed integer nonlinear programs. Discrete Optimization **5**, 186–204 (2008)

[2] Bonami, P., Lee, J.: BONMIN 1.1 Users' Manual. COIN-OR (2009)

[3] DeGroot, M., Schervish, M.: Probability and Statistics (Third Edition). AddisonWesley (2002)

[4] Domencich, T., McFadden, D.: Urban travel dynamics: a behavioral analysis. North Holland (1975)

[5] Dormer, A., Vazacopoulos, A., Verma, N., Tipi, H.: Supply Chain Optimization, chap. Modeling & Solving Stochastic Programming Problems in Supply Chain Management Using Xpress-SP, pp. 307–354. Springer-Verlag (2005)

[6] Galambos, J.: The asymptotic theory of extreme order statistics. John Wiley (1978)

[7] Gumbel, E.J.: Statistics of Extremes. Columbia University Press (1958)

[8] Hansen, W.: How accessibility shapes land use. Journal of the American Institute of Planners **25**, 73–76 (1959)

[9] Kaut, M., Wallace, S.W.: Evaluation of scenario generation methods for stochastic programming. Pacific Journal of Optimization **3**, 257–271 (2007)

[10] Ricciardi, N., Tadei, R., Grosso, A.: Optimal facility location with random throughput costs. Computers and Operations Research **29**, 593–607 (2002)

[11] Snyder, L.V., Daskin, M.S., Teo, C.: The stochastic location model with risk pooling. European Journal of Operational Research **179**, 1221–1238 (2007)

[12] Tadei, R., Ricciardi, N., Perboli, G.: The stochastic p-median problem with unknown cost probability distribution. Operations Research Letters **37**, 135–141 (2009)