TWO CONDITIONAL RESULTS ABOUT PRIMES IN SHORT INTERVALS

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Abstract. In 1937 A. E. Ingham proved that $\psi(x+x^\theta)-\psi(x)\sim x^\theta$ for $x\to\infty$, under the assumption of the Lindelöf hypothesis for $\theta>1/2$. In this paper we examine how the above asymptotic formula holds by assuming in turn two different heuristic hypotheses. It must be stressed that both the hypotheses are weaker than the Lindelöf hypothesis.

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1. Introduction

Let $\psi(x)=\sum_{n\le x}\Lambda(n)$, where $\Lambda(n)$ is the von Mangoldt function. This paper is concerned with the asymptotic formula

$$\psi(x+x^\theta)-\psi(x)\sim x^\theta \qquad x\to\infty, \tag{1}$$

which estimates the number of primes in the interval $(x,x+x^\theta]$. If $\theta<1$ this type of interval is called short interval. The prime number theorem implies that (1) holds with $\theta\ge 1$. In 1930 G. Hoheisel [7] proved that there is a prime in each of the intervals of the form $(x,x+x^\theta]$, with a constant $\theta<1$. The best known unconditional result about the constant θ is due to M. N. Huxley [8] and asserts that (1) holds for $\theta>7/12$, which was slightly improved by D. R. Heath-Brown [5] to $\theta\ge 7/12-\varepsilon(x)$, for every $\varepsilon(x)\to 0$. If we assume some well-known hypotheses we can handle smaller θ. For instance A. E. Ingham [9, Theorem 4] proved that the asymptotic formula (1) holds for $\theta>1/2$, assuming the Lindelöf hypothesis, which states that the Riemann Zeta-function satisfies

$$\zeta(\sigma+it)\ll t^\eta \quad (\sigma\ge\frac{1}{2}, t\ge 2),$$

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for any $\eta > 0$.

In a previous paper, see D. Bazzanella [1], we proved that (1) holds for $\theta > 1/2$, under the assumption of the following unproved hypothesis.

Hypothesis 1. *There exist a constant X_0 and a function $\Delta(y,T)$ such that, for every $5/12 < \beta < 1/2$ and $\varepsilon > 0$, we have*

$$\int_X^{2X} \left|\psi\left(y+\frac{y}{T}\right) - \psi(y) - \frac{y}{T} + \Delta(y,T)\right|^4 \mathrm{d}y \ll X^{4+\varepsilon}T^{-3} \tag{2}$$

and

$$\Delta(y,T) \ll \frac{y}{T\ln y} \tag{3}$$

uniformly for $X \geq X_0, X^{5/12} \leq T \leq X^{\beta}$ and $X \leq y \leq 2X$.

Through the work of G. Yu [13, Lemma B] the above hypothesis was proved to be weaker than the Lindelöf hypothesis.

In this paper we give two new conditional results about the validity of (1) for $\theta > 1/2$. To state the theorems we need to use the counting functions $N(\sigma,T)$ and $N^{(k)}(\sigma,T)$. The former is defined as the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function which satisfy $\sigma \leq \beta \leq 1$ and $|\gamma| \leq T$, while $N^{(k)}(\sigma,T)$ is defined as the number of ordered sets of zeros $\rho_j = \beta_j + i\gamma_j$ $(1 \leq j \leq 2k)$, each counted by $N(\sigma,T)$, for which $|\gamma_1 + \cdots + \gamma_k - \gamma_{k+1} - \cdots - \gamma_{2k}| \leq 1$. We now state the heuristic hypotheses that we need to assume. The first new hypothesis is the natural generalization of Hypothesis 1.

Hypothesis 2. *There exist an integer $k \geq 1$, a constant X_0 and a function $\Delta(y,T)$ such that, for every $5/12 < \beta < 1/2$ and $\varepsilon > 0$, we have*

$$\int_X^{2X} |\psi(y+y/T) - \psi(y) - y/T + \Delta(y,T)|^{2k}\mathrm{d}y \ll X^{2k+\varepsilon}T^{1-2k}$$

and

$$\Delta(y,T) \ll y/(T\log y)$$

uniformly for $X \geq X_0$, $X^{5/12} \leq T \leq X^{\beta}$ and $X \leq y \leq 2X$.

The second new hypothesis is about the upper bound of the counting functions $N^{(k)}(\sigma,T)$. We start to observe that D. Bazzanella and A. Perelli [3] made the heuristic assumption that

$$N^{(2)}(\sigma,T) = N^*(\sigma,T) \ll \frac{N(\sigma,T)^4}{T} \qquad (1/2 \leq \sigma \leq 1)\,.$$

The above may be generalized and weakened to

$$N^{(k)}(\sigma,T) \ll \frac{N(\sigma,T)^{2k}}{T}T^{\varepsilon} \qquad (1/2 \leq \sigma \leq \overline{\sigma})\,,$$

with suitable $\overline{\sigma} < 1$ and arbitrarily small $\varepsilon > 0$. If we recall that the Lindelöf hypothesis implies that for every $\eta > 0$ we have

$$N(\sigma,T) \ll T^{2(1-\sigma)+\eta} \quad (1/2 \leq \sigma \leq 1),$$

see A. E. Ingham [9], we are led to claim the following.

Hypothesis 3. *For every $\eta > 0$ there exists an integer $k \geq 2$ such that*

$$N^{(k)}(\sigma, T) \ll T^{4k(1-\sigma)-1+\eta} \qquad (1/2 \leq \sigma \leq 5/6 + \eta)\,.$$

Our new conditional results are the following.

Theorem 1. *If we assume Hypothesis 2, then the asymptotic formula (1) holds for every $\theta > 1/2$.*

Theorem 2. *If we assume Hypothesis 3, then the asymptotic formula (1) holds for every $\theta > 1/2$.*

Note that despite Hypothesis 2 and 3 being weaker than the Lindelöf hypothesis, see G. Yu [13, Lemma B] and D. R. Heath-Brown [6, Lemma 1] respectively, the result obtained about the the asymptotic formula (1) is the same of A. E. Ingham [9, Theorem 4].

2. The basic lemma

The basic lemma is a result about the structure of the exceptional set for the asymptotic formula (1). Let X be a large positive number, $\delta > 0$ and let $|\quad|$ denote the modulus of a complex number or the Lebesgue measure of a set. We define

$$E_\delta(X, \theta) = \{X \leq x \leq 2X : |\psi(x + x^\theta) - \psi(x) - x^\theta| \geq \delta x^\theta\}.$$

It is clear that (1) holds if and only if for every $\delta > 0$ there exists $X_0(\delta)$ such that $E_\delta(X, \theta) = \emptyset$ for every $X \geq X_0(\delta)$. Hence for small $\delta > 0$ and X tending to ∞, the set $E_\delta(X, \theta)$ contains the exceptions, if any, to the expected asymptotic formula for the number of primes in short intervals. Moreover, we observe that

$$E_\delta(X, \theta) \subset E_{\delta'}(X, \theta) \quad \text{if} \quad 0 < \delta' < \delta.$$

We now provide a useful result about the exceptional set $E_\delta(X, \theta)$.

Lemma. *Let $0 < \theta < 1$, X be sufficiently large, $0 < \delta' < \delta$ with $\delta - \delta' \geq \exp(-\sqrt{\log X})$. If $x_0 \in E_\delta(X, \theta)$ then $E_{\delta'}(X, \theta)$ contains the interval $[x_0 - cX^\theta, x_0 + cX^\theta] \cap [X, 2X]$, where $c = (\delta - \delta')\theta/5$. In particular, if $E_\delta(X, \theta) \neq \emptyset$ then*

$$|E_{\delta'}(X, \theta)| \gg_\theta (\delta - \delta')X^\theta.$$

The lemma essentially says that if we have a single exception in $E_\delta(X, \theta)$, with a fixed δ, then we necessarily have an interval of exceptions in $E_{\delta'}(X, \theta)$, with δ' being a little smaller than δ. The interesting consequence of this lemma is that we can use a suitable bound for the exceptional set to prove the non-existence of the exceptions. The above lemma is part (i) of Theorem 1 of D. Bazzanella and A. Perelli, see [3].

3. Proof of the Theorems

We will always assume that n and X_n are sufficiently large as prescribed by the various statements, and $\varepsilon > 0$ is arbitrarily small and not necessarily the same at each occurrence. Our theorems assert that (1) holds with $\theta > 1/2$. For $\theta \geq 7/12$ the result follows unconditionally from the work of D. R. Heath-Brown [5] and so we consider only $1/2 < \theta < 7/12$. In order to prove the theorems we assume that (1) does not hold. Then there exists $\delta_0 > 0$ and a sequence $X_n \to \infty$ such that

$$\left|\psi(X_n + X_n^\theta) - \psi(X_n) - X_n^\theta\right| \geq \delta_0 X_n^\theta.$$

From the definition of the exceptional set, we then have $X_n \in E_{\delta_0}(X_n, \theta)$. The use of the Lemma with $\delta' = \delta_0/2$ leads to

$$|E_{\delta'}(X_n, \theta)| \gg X_n^\theta, \tag{4}$$

for every $1/2 < \theta < 7/12$.

On the other hand, assuming the suitable heuristic hypotheses, we can get an upper bound for $|E_{\delta'}(X_n, \theta)|$. If we consider $y \in E_{\delta'}(X_n, \theta)$ we get

$$|\psi(y + y^\theta) - \psi(y) - y^\theta| \gg X_n^\theta. \tag{5}$$

We divide the interval $[X_n, 2X_n]$ into $O(\ln^2 X_n)$ subintervals $J_i = [a_i, a_{i+1}]$, with

$$a_i = X_n + \frac{iX_n}{\log^2 X_n} \tag{6}$$

and define

$$E^i_{\delta'}(X_n, \theta) = E_{\delta'}(X_n, \theta) \cap J_i.$$

We let

$$T_i = a_i^{1-\theta} \tag{7}$$

and observe that Hypothesis 2 implies that there exist an integer $k \geq 1$, a constant X_0 and a function $\Delta(y, T)$ such that, for every i, we have

$$\int_{X_n}^{2X_n} |\psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i)|^{2k} \mathrm{d}y \ll X_n^{2k+\varepsilon} T_i^{1-2k} \tag{8}$$

and

$$\Delta(y, T_i) \ll y/(T_i \log y), \tag{9}$$

uniformly for $X_n \geq X_0$ and $X_n \leq y \leq 2X_n$. From the Brun–Titchmarsh theorem, see H. L. Montgomery and R. C. Vaughan [12], we can deduce that for every i we have

$$\psi(y + y^\theta) - \psi(y) - y^\theta = \psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i) + O\left(\frac{X_n^\theta}{\log X_n}\right),$$

for every $y \in J_i$. The above bound and (5) imply that

$$|\psi(y + y/T_i) - \psi(y) - y/T_i + \Delta(y, T_i)| \gg X_n^\theta,$$

for every $y \in E^i_{\delta'}(X_n, \theta)$. Thus we obtain

$$|E_{\delta'}(X_n,\theta)| \ll X_n^{-2k\theta} \sum_i \int_{E^i_{\delta'}(X_n,\theta)} |\psi(y+y/T_i) - \psi(y) - y/T_i + \Delta(y,T_i)|^{2k}\,\mathrm{d}y$$

$$\ll X_n^{-2k\theta} \sum_i \int_{X_n}^{2X_n} |\psi(y+y/T_i) - \psi(y) - y/T_i + \Delta(y,T_i)|^{2k}\,\mathrm{d}y.$$

By (8) we conclude that

$$|E_{\delta'}(X_n,\theta)| \ll X_n^{-2k\theta} \sum_i X_n^{2k+\varepsilon} T_i^{1-2k} \ll X_n^{1-\theta+\varepsilon}. \tag{10}$$

For $1/2 < \theta < 7/12$, when ε is sufficiently small and X_n is sufficiently large we have a contradiction between (10) and (4), and this completes the proof of Theorem 1.

To prove Theorem 2 we use the classical explicit formula, see H. Davenport [4, Chapter 17], to write

$$\psi(y+y/T_i) - \psi(y) - y/T_i = -\sum_{|\gamma| \le R_i} y^\rho\, \frac{e^{\delta_i \rho} - 1}{\rho} + O\left(\frac{X_n \log^2 X_n}{R_i}\right), \tag{11}$$

uniformly for $X_n \le y \le 2X_n$, where $\delta_i = 1 + T_i^{-1}$, $10 \le R_i \le X_n$ and $\rho = \beta + i\gamma$ runs over the non-trivial zeros of $\zeta(s)$. If we choose $R_i = T_i \log^3 X_n$ and recall (7) and (6) we have

$$X_n^{1-\theta} \log^3 X_n \ll R_i \ll X_n^{1-\theta} \log^3 X_n$$

and

$$\psi(y+y/T_i) - \psi(y) - y/T_i = -\sum_{|\gamma| \le R_i} y^\rho\, \frac{e^{\delta_i \rho} - 1}{\rho} + O\left(\frac{X_n^\theta}{\log X_n}\right).$$

We note also that

$$\left|\frac{e^{\delta_i \rho} - 1}{\rho}\right| = \left|\int_0^{\delta_i} e^{t\rho}\,\mathrm{d}t\right| \le \int_0^{\delta_i} e^{t\beta}\,\mathrm{d}t \le e\delta_i \ll \frac{1}{T_i}. \tag{12}$$

Follow the method of D. R. Heath-Brown we can prove that for $\theta > 1/2$ and every fixed $u > 5/6$ we have

$$\sum_{|\gamma| \le R_i,\, \beta > u} y^\rho\, \frac{e^{\delta_i \rho} - 1}{\rho} \ll \frac{X_n^\theta}{\log X_n},$$

see (12.79) in [10]. Thus we obtain

$$\psi(y+y/T_i) - \psi(y) - y/T_i = -\sum_{|\gamma| \le R_i,\, \beta \le u} y^\rho\, \frac{e^{\delta_i \rho} - 1}{\rho} + O\left(\frac{X_n^\theta}{\log X_n}\right),$$

for every i and $y \in J_i$. As before we observe that for every $y \in J_i$ we have

$$\psi(y+y^\theta) - \psi(y) - y^\theta = \psi(y+y/T_i) - \psi(y) - y/T_i + O\left(\frac{X_n^\theta}{\log X_n}\right)$$

and then

$$\psi(y+y^\theta)-\psi(y)-y^\theta=-\sum_{|\gamma|\le R_i,\,\beta\le u} y^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}+O\left(\frac{X_n^\theta}{\log X_n}\right),$$

for every i and $y\in J_i$. This implies that

$$|E_{\delta'}(X_n,\theta)|X_n^{2k\theta}\ll\sum_i\int_{X_n}^{2X_n}\left|\sum_{|\gamma|\le R_i,\,\beta\le u} x^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}\right|^{2k}\mathrm{d}x. \tag{13}$$

To estimate the $2k$-power integral we divide the interval $[0,u]$ into $O(\ln X_n)$ subintervals I_j of the form

$$I_j=\left[\frac{j}{\log X_n},\frac{j+1}{\log X_n}\right].$$

By Hölder inequality we obtain

$$\left|\sum_{|\gamma|\le R_i,\,\beta\le u} x^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}\right|^{2k}\ll(\ln X_n)^{2k-1}\ \sum_j\left|\sum_{|\gamma|\le R_i,\,\beta\in I_j} x^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}\right|^{2k}.$$

Following again the method of D. R. Heath-Brown, we write

$$\int_{X_n}^{2X_n}\left|\sum_{|\gamma|\le R_i,\,\beta\in I_j} x^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}\right|^{2k}\mathrm{d}x\ll$$

$$\sum_{\substack{\beta_1,\dots,\beta_{2k}\in I_j\\|\gamma_1|\le R_i,\dots,|\gamma_{2k}|\le R_i}}\frac{(2X_n)^{\rho_1+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1}-X_n^{\rho_1+\rho_2+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1}}{\rho_1\dots\rho_{2k}\ (\rho_1+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1)}$$

$$\times(e^{\delta_i\rho_1}-1)\cdots(e^{\delta_i\rho_k}-1)(e^{\delta_i\overline{\rho_{k+1}}}-1)\dots(e^{\delta_i\overline{\rho_{2k}}}-1)$$

$$\ll\frac{1}{T_i^{2k}}\ X_n^{1+2kj/\log X_n}\sum_{\substack{\beta_1,\dots,\beta_{2k}\ge j/\log X_n\\|\gamma_1|\le R_i,\dots,|\gamma_{2k}|\le R_i}}\frac{1}{|\rho_1+\cdots+\rho_k+\overline{\rho_{k+1}}+\cdots+\overline{\rho_{2k}}+1|}.$$

This implies

$$\int_{X_n}^{2X_n}\left|\sum_{|\gamma|\le R_i,\,\beta\le u} x^\rho\ \frac{e^{\delta_i\rho}-1}{\rho}\right|^{2k}\mathrm{d}x\ll\frac{1}{T_i^{2k}}\max_{\sigma\le u}X_n^{2k\sigma+1}M_k(\sigma,R_i), \tag{14}$$

where

$$M_k(\sigma,R_i)=\sum_{\substack{\beta_1,\dots,\beta_{2k}\ge\sigma\\|\gamma_1|\le R_i,\dots,|\gamma_{2k}|\le R_i}}\frac{1}{1+|\gamma_1+\cdots+\gamma_k-\gamma_{k+1}-\cdots-\gamma_{2k}|}$$

and

$$M_k(\sigma,R_i)\ll N^{(k)}(\sigma,R_i)\log X_n, \tag{15}$$

see [10, p. 336]. From (13), (14) and (15) we have

$$|E_{\delta'}(X_n,\theta)| \ll X_n^{1-2k+\varepsilon} \sum_i \max_{\sigma\le u} X_n^{2k\sigma} N^{(k)}(\sigma, R_i). \tag{16}$$

We now consider an arbitrarily small constant $\eta > 0$, let $u = 5/6 + \eta$ and assume Hypothesis 3. Thus for every $1/2 \le \sigma \le u$ we have

$$X_n^{2k\sigma} N^{(k)}(\sigma, R_i) \ll X_n^{2k\sigma} R_i^{4k(1-\sigma)-1+\eta} \ll X_n^{2k\sigma+(1-\theta)(4k(1-\sigma)-1)+\eta}.$$

For $\theta > 1/2$ the above upper bound attains its maximum at $\sigma = u$ and then from (16) we obtain

$$|E_{\delta'}(X_n,\theta)| \ll X_n^{\theta-k(2\theta-1)/3+\varepsilon} \tag{17}$$

For $1/2 < \theta < 7/12$, when ε is sufficiently small and X_n is sufficiently large we have a contradiction between (17) and (4), and this completes the proof of Theorem 2.

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