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# Congruence curves of the Goldstein-Petrich flows

E. Musso

ABSTRACT. We study the existence of contours which evolve retaining their shapes under the second Goldstein-Petrich flow. We present a proof of the existence, for each integer  $n \geq 2$ , of a 1-parameter family of non-congruent Goldstein-Petrich contours of  $\mathbb{R}^2$  with symmetry group of order  $n$ . Explicit algorithms to compute and visualize the contours and their evolution are given.

## 1. Introduction

In ref. [GP1], R.E. Goldstein and D.M. Petrich showed that the mKdV equation

$$(1) \quad \kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0$$

is associated to the flow on the space of unit-speed plane curves  $\mathbf{z} : \mathbb{R} \rightarrow \mathbb{C}$  defined by

$$(2) \quad \mathbf{z}_t = -\left(\frac{\kappa^2}{2} + i\kappa_s\right)\mathbf{z}_s, \quad |\mathbf{z}_s| = 1, \quad \kappa = -i\mathbf{z}_{ss}\bar{\mathbf{z}}_s$$

A simple closed curve which evolves retaining its shape under (2) is said to be a GP contour. The existence of GP contours was considered in [GP2, NSW] and examples of closed, non-simple congruence curves of the flow (2) have been examined by Chou and Qu in ref. [CQ]. In [Mu], we exhibited explicit numerical examples of GP contours. Based on these results we wish to prove the following theorem :

**Theorem 1.** *For every integer  $n \geq 2$  there exist  $q_n \in (0, 1)$  and a 1-parameter family  $\{\gamma_{[q,n]}\}_{q \in [0, q_n)}$  of non-congruent GP contours with symmetry group of order*

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*n.* The evolution of  $\gamma_{[q,n]}$  under the second Goldstein-Petrich flow is given by

$$(3) \quad \mathbf{z}_{[q,n]} : (s, t) \in \mathbb{R} \times \mathbb{R} \rightarrow \text{Exp}(t\mu_{[q,n]}) \cdot \gamma_{[q,n]}(s - v_{[q,n]}t) \in \mathbb{R}^2,$$

where  $\mu_{[q,n]} \in \mathfrak{e}(2)$  and  $v_{[q,n]} \in \mathbb{R}$  are the momentum and the wave velocity of  $\gamma_{[q,n]}$ . Moreover, there exist a countable set  $\mathcal{T}_n \subset [0, q_n)$  such that  $\mathbf{z}_{[q,n]}$  is periodic in time, for each  $q \in \mathcal{T}_n$ .

The material is organized as follows. Section 2 recalls the basic definitions and collects the preliminary results from the existing literature. Section 3 analyzes the explicit integration of GP contours and proves the Theorem. Section 4 develops the numerical algorithms for the construction and the visualization of the 1-parameter families of GP contours with assigned symmetry group.

## 2. Preliminaries

**2.1. Local motions.** Denote by  $J(\mathbb{R}, \mathbb{R})$  the *total jet space* of smooth  $\mathbb{R}$ -valued functions of one independent variable, endowed with its standard coordinates

$$(s, u_{(0)}, u_{(1)}, \dots, u_{(h)}, \dots).$$

If  $u : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function, its *prolongation* is defined by

$$j(u) : s \in \mathbb{R} \mapsto \left( s, u|_s, \frac{du}{ds}|_s, \dots, \frac{d^h u}{ds^h}|_s, \dots \right) \in J(\mathbb{R}, \mathbb{R}).$$

A map  $\mathfrak{w} : J(\mathbb{R}, \mathbb{R}) \rightarrow \mathbb{R}$  is said a *polynomial differential function* if there exists  $w \in \mathbb{R}[x_0, \dots, x_h]$  such that

$$\mathfrak{w}(\mathbf{u}) = w(u_{(0)}, u_{(1)}, \dots, u_{(h)}),$$

for each  $\mathbf{u} = (s, u_{(0)}, u_{(1)}, \dots, u_{(h)}, \dots) \in J(\mathbb{R}, \mathbb{R})$ . The algebra of polynomial differential functions,  $J[\mathbf{u}]$ , is endowed with the *total derivative*, defined by

$$D\mathfrak{w} = \sum_{p=0}^{\infty} \frac{\partial w}{\partial u_{(p)}} u_{(p+1)}.$$

A differential function  $\mathfrak{w} \in J[\mathbf{u}]$  is a *total divergence* if there exists  $\mathfrak{p} \in J[\mathbf{u}]$  such that  $\mathfrak{w} = D(\mathfrak{p})$ . The primitive  $\mathfrak{p}$  is unique up to an additive constant. By  $D^{-1}(\mathfrak{w})$  we denote the unique primitive of  $\mathfrak{w}$  which vanishes at  $\mathbf{u} = \mathbf{0}$ . There is another natural differential operator, known as the *Euler operator*, defined by

$$\delta(\mathfrak{w}) = \sum_{\ell=0}^{\infty} (-1)^\ell D^\ell \left( \frac{\partial \mathfrak{w}}{\partial u_{(\ell)}} \right).$$

We now recall three elementary properties :

- $\mathfrak{w} \in J[\mathbf{u}]$  is a total divergence if and only if  $\delta(\mathfrak{w}) = 0$ ;
- for each  $\mathfrak{w} \in J[\mathbf{u}]$ ,  $u_{(1)}\delta(\mathfrak{w})$  is a total divergence;
- for each  $\mathfrak{w} \in J[\mathbf{u}]$ ,  $u_{(0)}D(\delta(\mathfrak{w}))$  is a total divergence.

We let  $\mathcal{M}$  be the space of unit-speed curves  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2 \cong \mathbb{C}$ . The arc-length parameter and the curvature are denoted by  $s$  and  $k$  respectively. Tangent vectors to  $\mathcal{M}$  at  $\gamma$  are vector fields  $V = (v_1 + iv_2)\gamma'$  along  $\gamma$  satisfying  $v_1' = kv_2$ . For each  $\mathfrak{v} \in J[\mathbf{u}]$  such that  $\delta(u_{(0)}\mathfrak{v}) = 0$ , we define a cross section of  $T(\mathcal{M})$  by

$$\mathcal{V} : \gamma \in \mathcal{M} \rightarrow (D^{-1}(u_{(0)}\mathfrak{v}) + i\mathfrak{v})|_{j(k)\gamma'} \in T_\gamma(\mathcal{M}).$$

DEFINITION 1. We call  $\mathcal{V}$  the *local vector field* associated to  $\mathbf{v} \in J[\mathbf{u}]$ . If  $\mathbf{v} = D(\delta(\mathbf{w}))$ , then  $u_{(0)}\mathbf{v}$  is a total divergence and the corresponding local vector field is said to be the *Hamiltonian vector field* with energy  $\mathbf{w}$ . By a *local motion of plane curves* is meant an integral curve of a local vector field.

In other words, a local motion associated to  $\mathbf{v}$  is a smooth map

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} : (s, t) \in \mathbb{R} \times (a, b) \rightarrow \mathbb{C} \cong \mathbb{R}^2$$

such that

$$(4) \quad \mathbf{z}_t = (D^{-1}(u\mathbf{v}))|_{j_s(\kappa)} + i\mathbf{v}|_{j_s(\kappa)}\mathbf{z}_s, \quad |\mathbf{z}_s| = 1,$$

where

$$(5) \quad \kappa = -i\mathbf{z}_{ss}\bar{\mathbf{z}}_s$$

is the *curvature function*. The *Frenet frame* along  $\mathbf{z}$  is the map  $\mathcal{A} : \mathbb{R} \times (a, b) \rightarrow \mathbb{E}(2)$  defined by

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 0 \\ \mathbf{x} & \mathbf{x}' & -\mathbf{y}' \\ \mathbf{y} & \mathbf{y}' & \mathbf{x}' \end{pmatrix}.$$

If we set

$$(6) \quad \mathbf{u} = D^{-1}(u_{(0)}\mathbf{v}), \quad \mathbf{p} = D\mathbf{v} + u_{(0)}\mathbf{u},$$

then

$$(7) \quad \Theta := \mathcal{A}^{-1}d\mathcal{A} = \mathcal{K}|_{j_s(\kappa)}ds + \mathcal{P}(\mathbf{v})|_{j_s(\kappa)}ds,$$

where  $\mathcal{K}$  and  $\mathcal{P}(\mathbf{v})$  are the  $\epsilon(2)$ -valued differential functions

$$(8) \quad \mathcal{K} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -u_{(0)} \\ 0 & u_{(0)} & 0 \end{pmatrix}, \quad \mathcal{P}(\mathbf{v}) = \begin{pmatrix} 0 & 0 & 0 \\ \mathbf{u} & 0 & -\mathbf{p} \\ \mathbf{v} & \mathbf{p} & 0 \end{pmatrix}.$$

The Maurer-Cartan equation  $d\Theta + \Theta \wedge \Theta = 0$  yields

$$(9) \quad \kappa_t = D(D\mathbf{v} + u_{(0)}D^{-1}(u_{(0)}\mathbf{v}))|_{j_s(\kappa)}.$$

If the local vector field is Hamiltonian with energy  $\mathbf{w}$ , then (9) takes the form

$$(10) \quad \kappa_t = \mathcal{E}(\delta(\mathbf{w}))|_{j_s(\kappa)}$$

where

$$\mathcal{E} = (D^3 + D \cdot u_{(0)}D^{-1} \cdot u_{(0)}D)$$

is the canonical Hamiltonian structure of the mKdV hierarchy (cf. chapter 7 of ref. [O1]).

**2.2. The Goldstein-Petrich flows and the mKdV hierarchy.** According to [GP1] we consider the sequence  $\{\mathbf{v}_n\}_{n \in \mathbb{N}}$  of polynomial differential functions

$$(11) \quad \mathbf{v}_1 = -u_{(1)} \quad \mathbf{v}_n = D(D(\mathbf{v}_{n-1}) + u_{(0)}D^{-1}(u_{(0)}\mathbf{v}_{n-1})), \quad n \geq 2.$$

Then, the *mKdV hierarchy* is given by

$$(12) \quad u_t = \mathbf{v}_n|_{j_s(u)}, \quad n \geq 1.$$

Setting

$$\mathbf{w}_n = \int_0^1 D^{-1}(\mathbf{v}_{n+1})|_{\epsilon u} u d\epsilon, \quad n \geq 0,$$

we obtain another sequence  $\{\mathfrak{w}_n\}_{n \in \mathbb{N}}$  of polynomial differential functions such that

$$(13) \quad \mathfrak{v}_1 = D(\delta(\mathfrak{w}_0)), \quad \mathfrak{v}_n = D(\delta(\mathfrak{w}_{n-1})) = \mathcal{E}(\delta(\mathfrak{w}_{n-2})), \quad n \geq 2.$$

This leads to the bi-Hamiltonian representations of the mKdV hierarchy, namely

$$(14) \quad u_t = D(\delta(\mathfrak{w}_{n-1}))|_{j_s(u)} = \mathcal{E}(\delta(\mathfrak{w}_{n-2}))|_{j_s(u)}, \quad n \geq 2.$$

The first three equations of the mKdV hierarchy are

$$\begin{cases} u_t + u_s = 0 \\ u_t + \frac{3}{2}u^2u_s + u_{sss} = 0, \\ u_t + u_{sssss} + \frac{5}{2}u^2u_{sss} + 10u^2u_su_{ss} + \frac{5}{2}u_s^3 + \frac{15}{8}u^4u_s = 0. \end{cases}$$

DEFINITION 2. The local vector field  $\mathcal{V}_n$  associated to  $\mathfrak{v}_n$  is called the *n-th flow of Goldstein-Petrich*.

REMARK 1. The Goldstein-Petrich flow  $\mathcal{V}_n$  is Hamiltonian with energy  $\mathfrak{w}_{n-2}$ , for each  $n \geq 2$ . Moreover, the curvature function of a local motion of  $\mathcal{V}_n$  evolves accordingly to the *n-th* member of the mKdV hierarchy.

### 3. Goldstein-Petrich contours

**3.1. Congruence curves.** A unit-speed curve  $\gamma$  which moves without changing its shape under the Goldstein-Petrich flow  $\mathcal{V}_n$  is said to be a *congruence curve* of class *n*. From now we consider curves with non-constant curvature. Then,  $\gamma$  is a congruence curve of order *n* if and only if there exist  $B : (a, b) \rightarrow \mathbb{E}(2)$  and  $v : (a, b) \rightarrow \mathbb{R}$  such that

$$(15) \quad \mathbf{z} : (s, t) \in \mathbb{R} \times (a, b) \rightarrow B(t)\gamma(s - v(t))$$

is a local motion of  $\mathcal{V}_n$ .

LEMMA 2. *The function  $v$  is linear.*

PROOF. Equation (15) implies that the curvature function of  $\mathbf{z}$  is given by

$$\kappa(s, t) = k(s + v(t)),$$

where  $k$  is the curvature of  $\gamma$ . From  $\kappa_t = \mathfrak{v}_n|_{j_s(\kappa)}$  we find

$$(16) \quad k'|_{s+v(t)} \frac{dv}{dt}|_t = (\mathfrak{v}_n|_{j_s(k)})|_{s+v(t)}.$$

Taking  $s_0 \in \mathbb{R}$  such that  $k'|_{s_0} \neq 0$  and setting  $s = -v(t) + s_0$  in (16) we obtain

$$\frac{dv}{dt}|_t = \frac{v_n|_{j_s(k)}|_{s_0}}{k'|_{s_0}} = \text{constant}.$$

□

As a consequence, we assume that the evolution of a congruence curve is

$$(17) \quad \mathbf{z}(s, t) = B(t) \cdot \gamma(s - vt),$$

where the constant  $v \in \mathbb{R}$  is the *wave velocity*. The curvature of a congruence curve of class *n* and wave velocity  $v$  is a solution of the *stationary mKdV equation*

$$(18) \quad \mathfrak{v}_n|_{j(k)} + vk' = 0.$$

In analogy with (6) we put

$$\mathbf{u}_n = D^{-1}(u_{(0)}\mathfrak{v}_n), \quad \mathfrak{p}_n = D(\mathfrak{v}_n) + u_{(0)}\mathbf{u}_n$$

and we consider the  $\epsilon(2)$ -valued polynomial differential function

$$(19) \quad \mathcal{H}(\mathbf{v}_n) = \mathcal{P}(\mathbf{v}_n) + v\mathcal{K},$$

where  $\mathcal{K}$  and  $\mathcal{P}(\mathbf{v}_n)$  are defined as in (8). An easy inspection shows that  $k$  satisfies (18) if and only if

$$(20) \quad (\mathcal{H}(\mathbf{v}_n)|_{j(k)})' = [\mathcal{H}(\mathbf{v}_n), \mathcal{K}]|_{j(k)}.$$

This implies that there exists  $\mathbf{m} \in \epsilon(2)$  such that

$$(21) \quad A \cdot \mathcal{H}(\mathbf{v}_n)|_{j(k)} \cdot A^{-1} = \mathbf{m},$$

where  $A : \mathbb{R} \rightarrow \mathbb{E}(2)$  is the Frenet frame along  $\gamma$ . We call  $\mathbf{m}$  the *momentum* of  $\gamma$ .

**PROPOSITION 3.** *Let  $\gamma$  be a congruence curve of class  $n$ , with wave velocity  $v \in \mathbb{R}$  and momentum  $\mathbf{m}$ , then its evolution under  $\mathcal{V}_n$  is given by*

$$(22) \quad \mathbf{z}(s, t) = \text{Exp}(t\mathbf{m}) \cdot \gamma(s - vt).$$

**PROOF.** Let  $\mathbf{z}(s, t) = B(t)\gamma(s - vt)$  be the evolution of  $\gamma$  under  $\mathcal{V}_n$ . The Frenet frame of  $\mathbf{z}$  is

$$(23) \quad \mathcal{A}(s, t) = B(t)A_\gamma(s - vt),$$

where  $A$  is the Frenet frame along the curve  $\gamma$ . From (7) we have

$$(24) \quad \mathcal{A}^{-1}d\mathcal{A} = \mathcal{K}|_{j_s(\kappa)}ds + \mathcal{P}(\mathbf{v}_n)|_{j_s(\kappa)}dt.$$

Then, (21), (23) and (24) imply

$$B^{-1}|_t \frac{dB}{dt}|_t = A_\gamma(s - vt) \cdot (\mathcal{H}(\mathbf{v}_n)|_{j(k)})|_{s-vt} \cdot A_\gamma(s + v_\gamma t)^{-1} = \mathbf{m}.$$

This yields the required result.  $\square$

**3.2. Congruence curves of class 2.** The curvature of a congruence curve of class two satisfies

$$k''' + \left(\frac{3}{2}k^2 - v\right)k' = 0,$$

where  $v$  is the wave velocity. From this we get

$$(k')^2 = -\frac{1}{4}(k^4 + c_2k^2 + c_1k - c_0),$$

where  $c_2 = -4v$  and  $c_1, c_0$  are constants of integration. Solutions with  $c_1 = 0$  are plane elastic curves. Since closed planar elasticae are not simple [BG], we suppose  $c_1 \neq 0$ . Eventually scaling  $\gamma$  by a similarity factor, we normalize the curve by  $c_1 = 1$  and we assume that the curvature is a periodic solution of

$$(25) \quad (k')^2 = -\frac{1}{4}(k^4 + c_2k^2 + k + c_0).$$

In addition, we require that the polynomial

$$P(t|c_2, c_0) = t^4 + c_2t^2 + t + c_0$$

has two distinct real roots  $r_1 > r_2$  and two complex conjugate roots  $r_3$  and  $r_4$ , with  $\text{Im}(r_3) > 0$ . The coefficients  $c_2$  and  $c_0$  can be written in terms of the parameters  $p < 0$  and  $q \in (-1, 1)$  by

$$(26) \quad c_{0,p,q} = \frac{(1 + 4p^3q^2)(1 + 4p^3(q^2 - 1))}{16p^4}, \quad c_{2,p,q} = -\frac{1}{2p^2} + p(2q^2 - 1).$$

We set

$$(27) \quad g_{p,q} = -\frac{1}{2p} (1 + p^6 + p^3(4q^2 - 2))^{1/4}, \quad m_{p,q} = \frac{1}{2} + \frac{-1 + p^3(1 - 2q^2)}{2(1 + p^6 + p^3(4q^2 - 2))^{1/2}},$$

and we define

$$\begin{cases} A_{1,p,q} = \frac{1}{2p^2} \sqrt{1 - p^3 + 2q(-p)^{3/2}}(1 - 2q(-p)^{3/2}), \\ A_{2,p,q} = \frac{1}{2p^2} \sqrt{1 - p^3 - 2q(-p)^{3/2}}(1 + 2q(-p)^{3/2}), \\ B_{1,p,q} = \frac{1}{p} \sqrt{1 - p^3 + 2q(-p)^{3/2}}, \\ B_{2,p,q} = \frac{1}{p} \sqrt{1 - p^3 - 2q(-p)^{3/2}}. \end{cases}$$

We denote by  $\text{cn}(-|m)$  the Jacobi elliptic  $\text{cn}$ -function with parameter  $m \in (0, 1)$  and we put

$$(28) \quad \begin{cases} \alpha_{1,p,q} = A_{1,p,q} - A_{2,p,q}, & \alpha_{2,p,q} = -(A_{1,p,q} + A_{2,p,q}), \\ \beta_{1,p,q} = B_{1,p,q} - B_{2,p,q}, & \beta_{2,p,q} = -(B_{1,p,q} + B_{2,p,q}). \end{cases}$$

Then,

$$(29) \quad k_{p,q}(s) = \frac{\alpha_{1,p,q} \text{cn}(g_{p,q}s | m_{p,q}) + \alpha_{2,p,q}}{\beta_{1,p,q} \text{cn}(g_{p,q}s | m_{p,q}) + \beta_{2,p,q}}$$

is a periodic solution<sup>1</sup> of (25), with coefficients  $c_{0,p,q}$  and  $c_{2,p,q}$  and period

$$(30) \quad \omega_{p,q} = \frac{4}{g_{p,q}} \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - m_{p,q} \sin^2(\vartheta)}}.$$

For each  $p < 0$  and  $q \in (-1, 1)$  we let  $\gamma_{p,q} : \mathbb{R} \rightarrow \mathbb{R}^2$  be the unit-speed curve with curvature  $k_{p,q}$  such that

$$\gamma_{p,q}(0) = (-2p + 4q(-p)^{-1/2}, 0), \quad \gamma'_{p,q}(0) = (0, -1)^t.$$

Since  $k_{p,-q}(s) = k_{p,q}(s + \omega_{p,q})$ , the curves  $\gamma_{p,q}$  and  $\gamma_{p,-q}$  are congruent each to the other. If  $q = 0$ , the curvature is constant and  $\gamma_{p,0}$  is a circle with signed radius  $2p$ . The angular function

$$\theta_{p,q}(s) := \int_0^s k_{p,q}(u) du$$

can be computed in terms of elliptic integrals of the third kind<sup>2</sup>. As a result we obtain

$$(31) \quad \theta_{p,q}(s) = h_{1,p,q}s + h_{2,p,q}\Phi_{2,p,q}(s) + h_{3,p,q}\Phi_{3,p,q}(s),$$

the coefficients  $h_{j,p,q}$  and the functions  $\Phi_{i,p,q}$  are defined by

$$(32) \quad \begin{cases} h_{1,p,q} = \frac{\alpha_{1,p,q}}{\beta_{1,p,q}}, \\ h_{2,p,q} = \frac{\alpha_{2,p,q}\beta_{1,p,q} - \alpha_{1,p,q}\beta_{2,p,q}}{g_{p,q} \sqrt{(\beta_{2,p,q} - \beta_{1,p,q})(\beta_{1,p,q} + \beta_{2,p,q})(\beta_{1,p,q}^2(1 - m_{p,q}) - \beta_{2,p,q}^2 m_{p,q})}}, \\ h_{3,p,q} = -\frac{\alpha_{2,p,q}\beta_{1,p,q} - \alpha_{1,p,q}\beta_{2,p,q}}{g_{p,q}\beta_{1,p,q}\beta_{2,p,q}\sqrt{1 - m_{p,q}}} \end{cases}$$

<sup>1</sup>See ref. [BF], pg. 133

<sup>2</sup>See ref. [La], pg. 67-69.

and by

$$(33) \quad \begin{cases} \Phi_{2,p,q}(s) = \operatorname{arctanh} \left( \sqrt{\frac{(1-m_{p,q})\beta_{1,p,q}^2 + m_{p,q}\beta_{2,p,q}}{(\beta_{1,p,q}^2 - \beta_{2,p,q})(\beta_{1,p,q} + \beta_{2,p,q})}} \operatorname{sd}(g_{p,q}s|m_{p,q}) \right), \\ \Phi_{3,p,q}(s) = \Pi \left( \frac{\beta_{1,p,q}^2}{\beta_{2,p,q}^2}, \frac{1}{2}(\pi - 2\operatorname{am}(g_{p,q}s|m_{p,q}), \frac{m_{p,q}}{1-m_{p,q}}) \right) - \Pi \left( \frac{\beta_{1,p,q}^2}{\beta_{2,p,q}^2}, \frac{\pi}{2}, \frac{m_{p,q}}{1-m_{p,q}} \right), \end{cases}$$

where

$$\begin{cases} \Pi(n, \phi, m) = \int_0^\phi \frac{d\theta}{(1-n \sin^2(\theta))\sqrt{1-m \sin^2(\theta)}}, \\ \operatorname{am}(s, m) = \int_0^s \operatorname{dn}(u|m)du \end{cases}$$

are the integral of the third kind and the Jacobi amplitude respectively.

PROPOSITION 4. *The curve  $\gamma_{p,q}$  is given by*

$$(34) \quad \gamma_{p,q} = 2e^{i\theta_{p,q}} \left( (2k_{p,q}^2 + c_{2,p,q}) + 4i\kappa'_{p,q} \right).$$

PROOF. We set

$$(35) \quad \eta_{1,p,q} = -\frac{1}{2}k_{p,q} - \frac{1}{4}c_{2,p,q}, \quad \eta_{2,p,q} = -k'_{p,q}.$$

Then,

$$(36) \quad \mathcal{H}(\mathbf{v}_2)|_{j(k_{p,q})} = \begin{pmatrix} 0 & 0 & 0 \\ \eta_{1,p,q} & 0 & -1/8 \\ \eta_{2,p,q} & 1/8 & 0 \end{pmatrix}.$$

The Frenet frame field of a unit-speed curve  $\gamma$  with curvature  $k_{p,q}$  and initial condition  $\gamma'(0) = (1, 0)^t$  is

$$(37) \quad A = \begin{pmatrix} 1 & 0 & 0 \\ \gamma_1 & \cos(\theta_{p,q}) & -\sin(\theta_{p,q}) \\ \gamma_2 & \sin(\theta_{p,q}) & \cos(\theta_{p,q}) \end{pmatrix}.$$

Denote by

$$\mathbf{m} = \begin{pmatrix} 0 & 0 & 0 \\ m_1 & 0 & -m_3 \\ m_2 & m_3 & 0 \end{pmatrix}$$

the momentum of  $\gamma$ . From (21) we have

$$(38) \quad A^{-1} \cdot \mathcal{H}(\mathbf{v}_2)|_{j(k_{p,q})} \cdot A = \mathbf{m}.$$

Combining (36), (37) and (38) we obtain

$$\gamma = 8i(e^{i\theta_{p,q}}(\eta_{1,p,q} + i\eta_{2,p,q}) + (m_1 + im_2)).$$

Then,

$$\tilde{\gamma} = 8e^{i\theta_{p,q}}(\eta_{1,p,q} + i\eta_{2,p,q})$$

is a unit-speed curve with curvature  $k_{p,q}$  and initial conditions

$$\tilde{\gamma}(0) = (-2p + 4q(-p)^{-1/2}, 0)^t, \quad \tilde{\gamma}'(0) = (0, -1)^t.$$

This implies the required result.  $\square$

Since  $\eta_{1,p,q} + i\eta_{2,p,q}$  is periodic, with period  $\omega_{p,q}$ , we deduce :



COROLLARY 5. *The curve  $\gamma_{p,q}$  is closed if and only if*

$$(39) \quad \Lambda_{p,q} = \frac{1}{2\pi} \int_0^{\omega_{p,q}} k_{p,q}(u) du = \frac{\ell}{n} \in \mathbb{Q},$$

where  $\ell, n \in \mathbb{Z}$  are relatively prime integers, with  $\ell \geq 0$ .

REMARK 6. The integer  $\ell$  is the turning number,  $|n|$  is the order of the symmetry group. In particular, for a simple curve the integer  $\ell$  is 1. If  $q \neq 0$ , the elliptic curve parameterized by  $k_{p,q}$  and  $k'_{p,q}$  intersects the  $Ox$ -axis in two points. Then, the four vertex theorem implies  $|n| > 1$ .

**3.3. Proof of Theorem 1.** We fix a positive integer  $n > 1$ . We define the *characteristic curve*

$$\Sigma_n = \{(p, q) \in \mathbb{R}^{-1} \times (-1, 1) : \Lambda_{p,q} = -1/n\},$$

and we let  $\Sigma_n^+$  be the set of all  $(p, q) \in \Sigma_n$  such that  $q \geq 0$ . Since the function

$$\Lambda : (p, q) \in \mathbb{R}^- \times (-1, 1) \rightarrow \Lambda_{p,q} \in \mathbb{R}$$

satisfies

$$(40) \quad \Lambda_{p,q} = \Lambda_{p,-q}, \quad \Lambda_{p,0} = -\frac{1}{\sqrt{1-p^3}}, \quad \partial_p \Lambda|_{p,0} = -\frac{3p^2}{2(1-p^3)^{3/2}} < 0,$$

then there exist a maximal  $\epsilon_n \in (0, 1]$  and a unique real-analytic even function

$$(41) \quad \phi_n : (-\epsilon_n, \epsilon_n) \rightarrow \mathbb{R}^-$$

such that

$$\phi_n(0) = (1 - n^2)^{1/3}, \quad (\phi_n(q), q) \in \Sigma_n, \quad \forall q \in (-\epsilon_n, \epsilon_n).$$

We define

$$(42) \quad \gamma_{[q,n]} := \gamma_{\phi_n(q), q},$$

and we consider the one-parameter family  $\{\gamma_{[q,n]}\}_{q \in (-\epsilon_n, \epsilon_n)}$  of closed curves with curvature functions  $k_{[q,n]} = k_{\phi_n(q), q}$ . We let  $\omega_{[q,n]}$  be the period of  $k_{[q,n]}$ . Then,

$$\mathfrak{K}_{[n]} : (s, q) \in \mathbb{R} \times (-\epsilon_n, \epsilon_n) \rightarrow k_{[q,n]}(s) \in \mathbb{R}$$

is a real-analytic function, periodic in  $s$ , satisfying

$$\mathfrak{K}_{[n]}(s, 0) = (2\phi_n(q))^{-1} < 0.$$

It follows that there exists  $\epsilon'_n \in (0, \epsilon_n]$  such that  $\gamma_{[q,n]}$  is strictly convex and satisfies

$$\frac{1}{2\pi} \int_0^{\omega_{[q,n]}} k_{[q,n]}(u) du = -\frac{1}{n},$$

for each  $q \in (-\epsilon'_n, \epsilon'_n)$ . This implies (cf. [MN]) that  $\gamma_{[q,n]}$  is a simple curve, for every  $q \in (-\epsilon'_n, \epsilon'_n)$ . We set

$$q_n = \text{Sup}\{q \in (0, \epsilon_n) : \gamma_{[\tilde{q}, n]} \text{ is a simple curve, } \forall \tilde{q} \in (0, q)\}.$$

Then,  $\{\gamma_{[q,n]}\}_{q \in [0, q_n]}$  is a one-parameter family of simple congruence curves of class 2, with symmetry group of order  $n$ . Since the curves of the family have different

lengths, they are not congruent each to the other. The momentum  $\mathbf{m}_{[q,n]}$  and the wave velocity  $v_{[q,n]}$  of  $\gamma_{[q,n]}$  are given by

$$(43) \quad \mathbf{m}_{[q,n]} = \frac{1}{8} \begin{pmatrix} 0 & 0 & 0 \\ \mu_{[q,n]} & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mu_{[q,n]} = -2\phi_n(q) + \frac{4q}{\sqrt{-\phi_n(q)}}.$$

and by

$$(44) \quad v_{[q,n]} = \frac{1}{4} \left( \frac{1}{2\phi_n(q)^2} - \phi_n(q)(-1 + 2q^2) \right).$$

From (22), (43) and (44) we see that the evolution of  $\gamma_{[q,n]}$  is

$$(45) \quad \mathbf{z}_{[q,n]}(s, t) = e^{it/8} \gamma_{[q,n]}(s - v_{[q,n]}t) + \mu_{[q,n]} \rho(t),$$

where

$$\rho(t) = \sin(t/8) + i(1 - \cos(t/8)).$$

Therefore,  $\mathbf{z}_{[q,n]}(s, t)$  is periodic in time if and only if

$$\frac{4\pi v_{[q,n]}}{n\omega_{[q,n]}} \in \mathbb{Q}.$$

Since the function

$$(46) \quad T_n : q \in [0, q_n) \rightarrow \frac{4\pi v_{[q,n]}}{n\omega_{[q,n]}}$$

is non-constant and real-analytic, then there exists a countable set  $\mathcal{T}_n \subset [0, q_n)$  such that the evolution of  $\gamma_{[q,n]}$  is periodic, for all  $q \in \mathcal{T}_n$ .

**3.4. Example.** Consider the family  $\{\gamma_{[q,7]}\}_{q \in [0,1)}$ . The function  $\phi_7$  is defined for all  $q \in (-1, 1)$  and the upper part  $\Sigma_7^+$  of the characteristic curve is parameterized  $q \in [0, 1) \rightarrow (\phi_7(q), q) \in \Sigma_7^+$ . The approximate value of  $q_7$  is 0.8013658294677735. The behavior of the family is illustrated in the Figures 1, 2, 3 and 4.

REMARK 7. Numerical experiments show that the characteristic curve  $\Sigma_n$  is always the graph of the function  $\phi_n : (-1, 1) \rightarrow \mathbb{R}$ . Furthermore, there exists a well defined *separating value*  $q_n \in (0, 1)$  such that  $\gamma_{[q,n]}$  is simple if and only if  $|q| < q_n$ . The experimental evidence also suggest that each GP-contour is equivalent, up to a similarity of  $\mathbb{R}^2$ , to a curve of the form  $\gamma_{[q,n]}$ , with  $n \geq 2$  and  $q \in (0, q_n)$ .

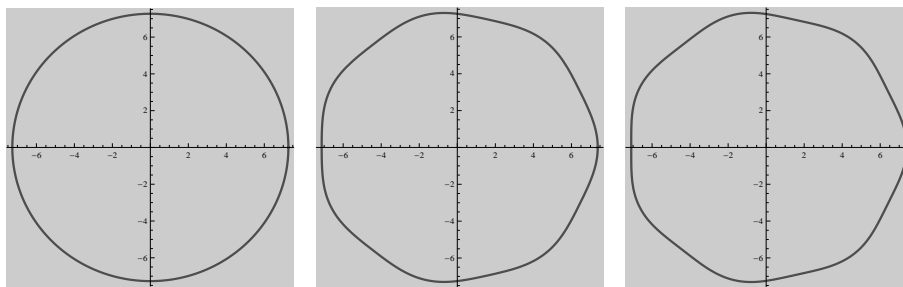
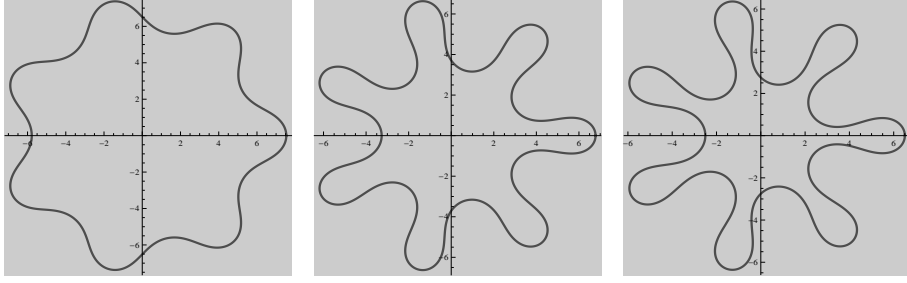


FIGURE 1. The curves  $\gamma_{[0,7]}$ ,  $\gamma_{[0.06,7]}$  and  $\gamma_{[q_1,7]}$ .

FIGURE 2. The curves  $\gamma_{[0.4,7]}$ ,  $\gamma_{[q_2,7]}$  and  $\gamma_{[0.75,7]}$ .

The numerical value of  $q_n$  can be found by the mean of the following procedure (see also Step 6 of Section 4) :

- compute  $q'_n$  such that  $c_2(\phi_n(q'_n), q'_n) = 0$ ;
- compute  $q''_n \in (q'_1, 1)$  such that

$$\eta_{1, \phi_n(q''_n), q''_n} \left( \frac{\omega_{[q''_n, n]}}{2} \right) = 0,$$

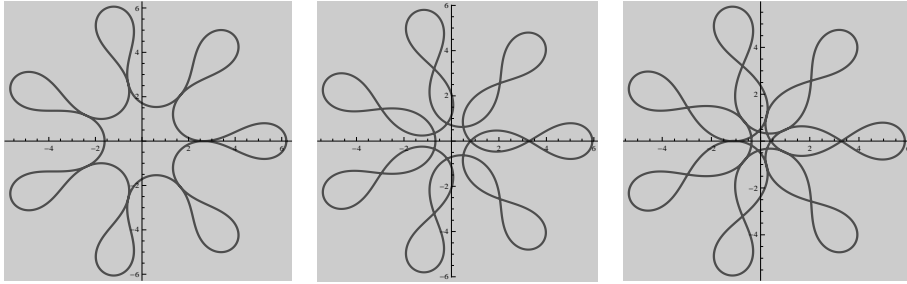
where  $\eta_{1,p,q}$  is defined as in (35).

- for each  $q \in (q'_1, q''_1]$  compute  $s_{[q,n]} \in [0, \frac{\omega_{[q''_n, n]}}{2}]$  such that

$$g_{\phi_n(q), q} \text{cn}(s_{[q,n]} | m_{\phi_n(q), q}) = - \frac{\alpha_{2, \phi_n(q), q} - \beta_{2, \phi_n(q), q} \sqrt{-c_{2, \phi_n(q), q}}}{\alpha_{1, \phi_n(q), q} - \beta_{1, \phi_n(q), q} \sqrt{-c_{2, \phi_n(q), q}}};$$

- $q_n$  is the unique zero of the function

$$q \in (q'_1, q''_1] \rightarrow |\gamma_{[q,n]}(s_{[q,n]}) - \gamma_{[q,n]}(-s_{[q,n]})|.$$

FIGURE 3. The curves  $\gamma_{[q^*,7]}$ ,  $\gamma_{[0.845,7]}$  and  $\gamma_{[q_3,7]}$ .

REMARK 8. Congruence curves which evolve periodically in time can be computed as follows: consider the function  $T_n$  and set  $I_n = \text{Im}(T_n)$ . For each  $\ell/h \in I_n \cap \mathbb{Q}$  there exists a unique  $\hat{q}_n(\ell, h) \in [0, q_n]$  such that  $T_n(\hat{q}_n(\ell, h)) = \ell/h$ . The explicit evaluation of  $\hat{q}_n(\ell, h)$  can be made via numerical routines (see Steps 9 and 10 of Section 4). Thus, the family of simple, closed congruence curves with symmetry group of order  $n$  and periodic evolution in time is  $\{\gamma_{[\hat{q}_n(u), n]}\}_{u \in I_n \cap \mathbb{Q}}$ .

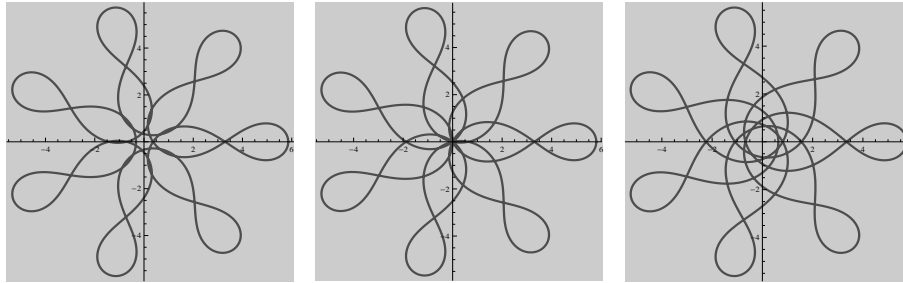


FIGURE 4. The curves  $\gamma_{[0.86,7]}$ ,  $\gamma_{[q_4,7]}$  and  $\gamma_{[0.9,7]}$ .

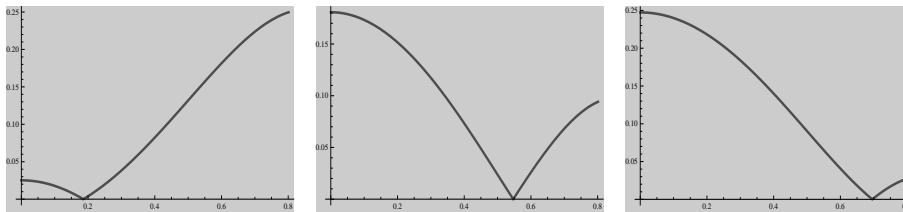


FIGURE 5. The functions  $T_7(q) - \ell/h$ , with  $\ell/h = -2/9, -1/15, 0$ .

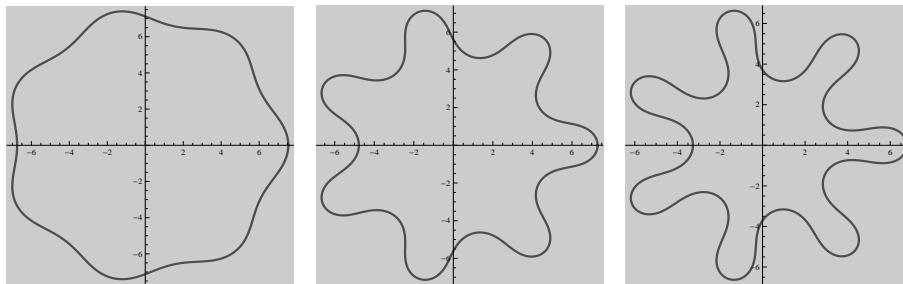


FIGURE 6. The curves  $\gamma_{[\hat{q}_7(-2/9),7]}$ ,  $\gamma_{[\hat{q}_7(-1/15),7]}$  and  $\gamma_{[\hat{q}_7(0),7]}$ .

#### 4. Numerical computations and visualization

In this section we show how to translate the results and the computations of Section 2 into numerical and graphical routines implemented the software *Mathematica 7.0*.

- **Step 1.** Define the coefficients  $c_{0,p,q}$ ,  $c_{2,p,q}$ ,  $\alpha_{j,p,q}$ ,  $\beta_{j,p,q}$ ,  $j = 1, 2$ ,  $g_{p,q}$  and  $m_{p,q}$  as in (26), (28) and (27) :

$$C0[p-, q-] := \frac{(1+4p^3q^2)(1+4p^3(-1+q^2))}{16p^4};$$

$$C2[p-, q-] := -\frac{1}{2p^2} + p(-1 + 2q^2);$$

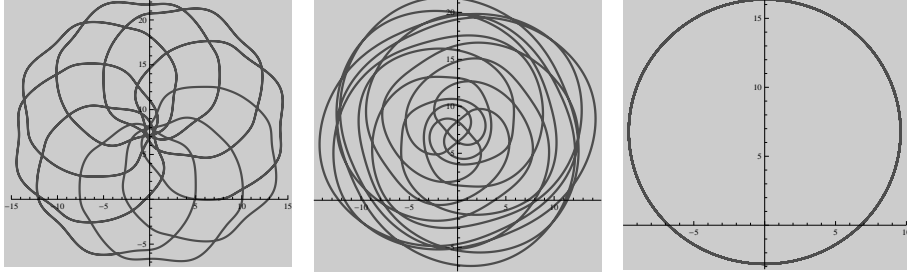


FIGURE 7. Trajectories of the points  $\gamma_{[\hat{q}_7(-\frac{2}{9}),7]}(0)$ ,  $\gamma_{[\hat{q}_7(-\frac{2}{15}),7]}(0)$  and  $\gamma_{[\hat{q}_7(0),7]}(0)$ .

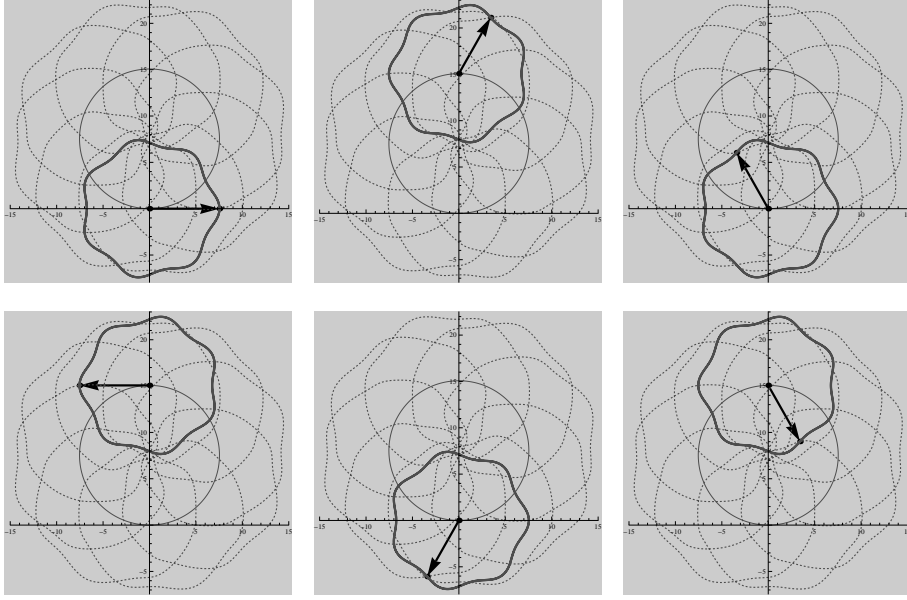


FIGURE 8. Evolution of the curve  $\gamma_{[\hat{q}_7(-\frac{2}{9}),7]}$ .

$$\begin{aligned} \alpha 1[p-, q-] &:= \frac{\sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}(1+2(-p)^{3/2}q)}}{2p} + \frac{\sqrt{1-p^3+2(-p)^{3/2}q}(1+2\sqrt{-ppq})}{2p^2}; \\ \alpha 2[p-, q-] &:= \frac{\sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}(1+2(-p)^{3/2}q)}}{2p} - \frac{\sqrt{1-p^3+2(-p)^{3/2}q}(1+2\sqrt{-ppq})}{2p^2}; \\ \beta 1[p-, q-] &:= \sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}} + \frac{\sqrt{1-p^3+2(-p)^{3/2}q}}{p}; \\ \beta 2[p-, q-] &:= \sqrt{\frac{1}{p^2} - p - \frac{2q}{\sqrt{-p}}} - \frac{\sqrt{1-p^3+2(-p)^{3/2}q}}{p}; \\ m[p-, q-] &:= \frac{-1+p^3(1-2q^2)+\sqrt{1+p^6+p^3(-2+4q^2)}}{2\sqrt{1+p^6+p^3(-2+4q^2)}}; \\ g[p-, q-] &:= -\frac{(1+p^6+p^3(-2+4q^2))^{1/4}}{2*p}; \end{aligned}$$

- **Step 2.** Define the curvature  $k_{p,q}$  and its period  $\omega_{p,q}$  (cf. (29) and (30)) :

$$\begin{aligned} k[s_-, p_-, q_-] &:= \frac{\alpha 1[p, q] * \text{JacobiCN}[g[p, q] * s, m[p, q]] + \alpha 2[p, q]}{\beta 1[p, q] * \text{JacobiCN}[g[p, q] * s, m[p, q]] + \beta 2[p, q]}; \\ \text{Dk}[s_-, p_-, q_-] &:= \text{Evaluate}[D[k[s, p, q], s]]; \\ \omega[p_-, q_-] &:= \frac{4}{g[p, q]} \text{EllipticK}[m[p, q]]; \end{aligned}$$

- **Step 3.** Compute the angular function  $\theta_{p,q}$  (cf. (31),(32) and (33)) :

$$\begin{aligned} h1[p_-, q_-] &:= \frac{\alpha 1[p, q]}{\beta 1[p, q]}; \\ h2[p_-, q_-] &:= \frac{(\alpha 2[p, q] \beta 1[p, q] - \alpha 1[p, q] \beta 2[p, q])}{g[p, q] * \sqrt{(-\beta 1[p, q] + \beta 2[p, q])(\beta 1[p, q] + \beta 2[p, q])}((-1 + m[p, q]) \beta 1[p, q]^2 - m[p, q] \beta 2[p, q]^2)}; \\ h3[p_-, q_-] &:= -\sqrt{\frac{1}{1 - m[p, q]}} * \frac{(\alpha 2[p, q] \beta 1[p, q] - \alpha 1[p, q] \beta 2[p, q])}{g[p, q] \beta 1[p, q] \beta 2[p, q]}; \\ \Phi 2[s_-, p_-, q_-] &:= \text{ArcTanh} \left[ \frac{\text{JacobiSD}[sg[p, q], m[p, q]]}{\sqrt{\frac{(\beta 1[p, q] - \beta 2[p, q])(\beta 1[p, q] + \beta 2[p, q])}{(-1 + m[p, q]) \beta 1[p, q]^2 + m[p, q] \beta 2[p, q]^2}}} \right]; \\ \Phi 3[s_-, p_-, q_-] &:= -\text{EllipticPi} \left[ \frac{\beta 1[p, q]^2}{\beta 2[p, q]^2}, \frac{m[p, q]}{-1 + m[p, q]} \right] + \\ \text{EllipticPi} &\left[ \frac{\beta 1[p, q]^2}{\beta 2[p, q]^2}, \frac{1}{2}(\pi - 2 \text{JacobiAmplitude}[sg[p, q], m[p, q]]), \frac{m[p, q]}{-1 + m[p, q]} \right]; \\ \theta[s_-, p_-, q_-] &:= h1[p, q] * s + h2[p, q] * \Phi 2[s, p, q] + h3[p, q] * \Phi 3[s, p, q]; \end{aligned}$$

- **Step 4.** Compute the function  $\phi_n$  and the one-parameter family of closed curves  $\{\gamma_{[q, n]}\}_{q \in [0, 1]}$  (cf. (35),(34),(41) and (42)) :

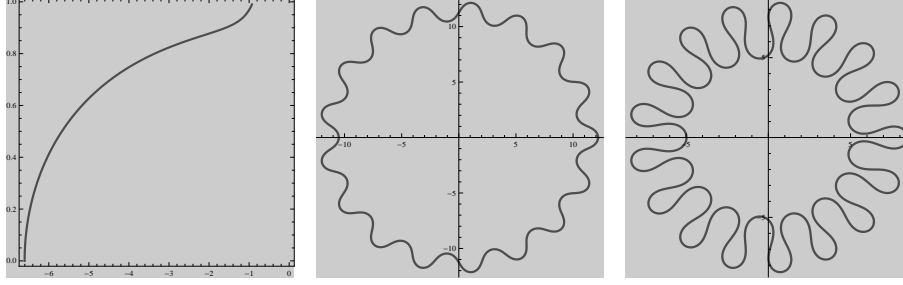
$$\begin{aligned} \Lambda[p_-, q_-] &:= \frac{1}{2\beta 1} \theta[\omega[p, q], p, q]; \\ \eta[s_-, p_-, q_-] &:= \left\{ \frac{C2[p, q]}{4} + \frac{k[s, p, q]^2}{2}, + \text{Dk}[s, p, q] \right\}; \\ R[s_-, p_-, q_-] &:= \{ \text{Cos}[\theta[s, p, q]], -\text{Sin}[\theta[s, p, q]] \}, \{ \text{Sin}[\theta[s, p, q]], \text{Cos}[\theta[s, p, q]] \}; \\ \varphi[q_-, n_-] &:= \text{Evaluate} [\text{FindRoot} [\Lambda[p, q] + \frac{1}{n} == 0, \{p, -\sqrt[3]{-1 + n^2}, -0.1\}, \text{Method} \rightarrow \text{"Brent"}]] [[1]][[2]]; \\ \gamma[s_-, q_-, n_-] &:= 8 * R[s, \varphi[q, n], q] * \eta[s, \varphi[q, n], q]; \end{aligned}$$

- **Step 5.** Visualize  $\Sigma_n$  and the curve  $\gamma_{[q, n]}$  :

$$\begin{aligned} \Sigma[n_-] &:= \text{ContourPlot} [\Lambda[p, q] == -1/n, \{p, -\sqrt[3]{-1 + n^2}, -0.01\}, \{q, 0.00001, 0.99\}, \\ \text{ContourStyle} &\rightarrow \{\text{GrayLevel}[0.3], \text{Thickness}[0.008]\}, \text{Background} \rightarrow \text{GrayLevel}[0.8], \\ \text{PlotPoints} &\rightarrow 50]; \\ \text{CURVE}[q_-, n_-] &:= \text{ParametricPlot} [\text{Evaluate}[\gamma[s, q, n]], \{s, 0, n * \omega[\varphi[q, n], q]\}, \\ \text{PlotStyle} &\rightarrow \{\text{GrayLevel}[0.3], \text{Thickness}[0.008]\}, \text{Background} \rightarrow \text{GrayLevel}[0.8], \\ \text{PlotRange} &\rightarrow \text{All}]; \end{aligned}$$

- **Step 6.** Specify the order of the symmetry group and compute  $q_n$  :

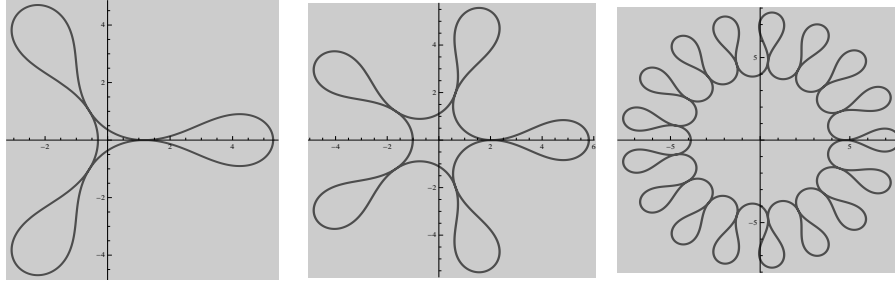
$$\begin{aligned} \text{nn} &:= 5 \\ \text{f1}[q_-] &:= \text{Abs}[C2[\varphi[q, \text{nn}], q]]; \\ \text{F1} &:= \text{Plot}[\text{f1}[q], \{q, 0, 0.99\}]; \\ \text{f2}[q_-] &:= \text{Abs} \left[ \frac{C2[\varphi[q, \text{nn}], q]}{4} + \frac{k[\frac{1}{2} * \omega[\varphi[q, \text{nn}], q], \varphi[q, \text{nn}], q]^2}{2} \right]; \\ \text{F2} &:= \text{Plot}[\text{f2}[q], \{q, 0, 0.99\}]; \\ \text{Q1} &:= \text{Evaluate}[\text{First}[\text{Sort}[\text{InputForm}[\text{F1}][[1, 1]][[1, 3]][[2]][[1]], \#1[[2]] < \#2[[2]] \&]]]; \\ \text{Q2} &:= \text{Evaluate}[\text{First}[\text{Sort}[\text{InputForm}[\text{F2}][[1, 1]][[1, 3]][[2]][[1]], \#1[[2]] < \#2[[2]] \&]]]; \\ \text{s1}[q_-] &:= \end{aligned}$$

FIGURE 9.  $\Sigma_{17}$  and the curves  $\gamma_{[0.5,17]}$ ,  $\gamma_{[0.8,17]}$ .

```

1
g[φ[q,nn],q] InverseJacobiCN [ - (α2[φ[q,nn],q]-β2[φ[q,nn],q]√-C2[φ[q,nn],q]/2)
α1[φ[q,nn],q]-β1[φ[q,nn],q]√-C2[φ[q,nn],q]/2,
m[φ[q, nn], q]];
f3[q_]:=Norm[γ[s1[q], q, nn] - γ[-s1[q], q, nn]];
steps:=7; initialpoint:=½(Q1[[1]] + Q2[[1]]); internalparameter[1]:=1/30;
internalparameter[2]:=20;
QQ[y_, δ_, k_]:=First[Sort[Table[{f3[q], q}, {q, y - δ, y + δ, 1/k}]]];
S[1, y_, δ_, k_]:=QQ[y, δ, k];
S[m_, y_, δ_, k_]:=S[m - 1, S[m - 1, y, δ, k][[2]], δ / (2m-1), k * (2m-1)];
Qn:=Evaluate[S[steps, initialpoint, internalparameter[1], internalparameter[2]]];

```

FIGURE 10. The "separating" curves  $\gamma_{[q_3,3]}$ ,  $\gamma_{[q_5,5]}$  and  $\gamma_{[q_{17},17]}$ .

- **Step 7.** Compute the evolution of the congruence curves :

$$\mu_1[q_-, n_-] := -2 * \varphi[q, n] + \frac{4 * q}{\sqrt{-\varphi[q, n]}}$$

$$\mu[q_-, n_-] := \frac{1}{8} \{ \{0, 0, 0\}, \{ \mu_1[q, n], 0, -1 \}, \{0, 1, 0\} \};$$

$$v[q_-, n_-] := \frac{1}{4} \left( \frac{1}{2 * \varphi[q, n]^2} - \varphi[q, n](2q^2 - 1) \right);$$

$$z[s_-, t_-, q_-, n_-] := \{ \{ \text{Cos} \left[ \frac{t}{8} \right], -\text{Sin} \left[ \frac{t}{8} \right] \}, \{ \text{Sin} \left[ \frac{t}{8} \right], \text{Cos} \left[ \frac{t}{8} \right] \} \} . \gamma[s - v[q, n] * t, q, n] + \mu_1[q, n] \{ \text{Sin}[t/8], -\text{Cos}[t/8] + 1 \};$$

- **Step 9.** Compute the function  $T_n$  :

$$T[q_-, n_-] := 4 \frac{\text{Pi} * v[q, n]}{n * \omega[\varphi[q, n], q]};$$

• **Step 10.** Specify the order of the symmetry group, take  $u/w \in I_7 \cap \mathbb{Q}$  and compute  $\hat{q}_n(u, w)$  :

```

nnn:=7; u:=-2; w:=9;
T1:=Plot [Abs [T[q, nnn] -  $\frac{u}{w}$ ], {q, 0, 0.99}];
QPA:=Evaluate[First[Sort[InputForm[T1][[1, 1]][[1, 3]][[2]][[1]], #1[[2]] < #2[[2]]&]];
steps:=6
initialpoint:=QPA[[1]];
internalparameter[1]:=1/30;
internalparameter[2]:=20;
QP1[y-,  $\delta$ -, k-]:=First [Sort [Table [{Abs [T[q, nnn] -  $\frac{u}{w}$ ], q}, {q, y -  $\delta$ , y +  $\delta$ , 1/k}]]];
SQP1[1, y-,  $\delta$ -, k-]:=QP1[y,  $\delta$ , k];
SQP1[m-, y-,  $\delta$ -, k-]:=SQP1 [m - 1, SQP1[m - 1, y,  $\delta$ , k][[2]],  $\delta / (2^{m-1})$ , k * (2^{m-1})];
QPA2:=Evaluate[SQP1[steps, initialpoint, internalparameter[1], internalparameter[2]]];
QP:=QPA2[[2]];

```

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