Enhancing robustness and synchronizability of networks homogenizing their degree distribution

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Abstract

A new family of networks, called *entangled*, has recently been proposed in the literature. These networks have optimal properties in terms of synchronization, robustness against errors and attacks, and efficient communication. They are built with an algorithm which uses modified simulated annealing to enhance a well-known measure of networks ability to reach synchronization among nodes. In this work, we suggest that class of networks similar to entangled networks can be produced by myopically changing some connections in a given network, or by just adding a few connections. We call this class of networks *weak-entangled*. Although entangled networks can be considered as a subset of weak-entangled networks, we show that both classes share similar properties, especially with respect to synchronization and robustness, and that they have similar structural properties.

Keywords:
complex networks, entangled networks, synchronization, vulnerability

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1. Introduction

Complex networks permeates our everyday life, due to their simplicity (a certain number of nodes representing individual sites and edges representing connections) and their ability to grasp the essence of many different systems. Commonly cited examples include social networks, technological networks, information networks, biological networks, communication networks, neural networks, ecological networks and other natural and man-made networks. Abundant study of their topology and models is presented in [1, 2, 3]. An important topic of interest in present research is to find an optimal topology in order to reach consensus [4], for efficient communication or transport networks [5], or to improve the performance of computational tasks [6]. It was shown in [7] that the optimal topology has an entangled structure, i.e. has an extremely homogeneous one. More precisely, degree, node distance, betweenness, and loop distributions are all within very narrow intervals, the average distances are short, and there is no well-defined community structure. This spectral properties In addition, this topic appears to be tightly connected [7] with collective behavior in complex networks, referring especially to the synchronous state, where all individual sites operate in unison [8]. This ability of a network to synchronize is commonly referred to as *synchronizability* [9] and [10].

In [7], Donetti *et al.* propose a stochastic algorithm, based on simulated annealing, for producing entangled networks. They show that these networks exhibit excellent performances in synchronization, are robust against errors and attacks and support efficient communication. In [11, 12], Jalili *et al.* present a rewiring algorithm, based on simulated annealing, which improves the synchronizability of the network. Another rewiring algorithm, which uses memory tabu search, is proposed in [13]. Jalili *et al.* in [14] use node and edge betweenness for weighting dynamical networks. In [15], Gorochowski *et al.* introduce a computational tool, called NETEVO, which evolves the network topology in order to improve its synchronizability. This tool uses a simulated annealing algorithm in order to direct the future evolution of the network. In addition to above mentioned research, recently in [16] Nishikawa *et al.* discovered that negative interactions and link removals, between nodes in a given network, can be used to improve synchronizability in both directed and undirected networks.

All of the mentioned strategies to build or rewire a network require a deep knowledge of the topology properties of the network. For instance, one needs to know the spectrum of its Laplacian matrix and the resulting eigenvectors or to calculate complex global measures such as node and edge betweenness. In order to understand how much each step of an optimization algorithm affects a network, one could use the approximations of [17], where the authors analytically studied the effect of a small perturbation —such as adding or removing edges— on the spectra of the adjacency or Laplacian matrices of a network. Again, some knowledge of the eigenvectors is needed to elaborate the approximations.

Motter *et al.* in [18], [19] and [20] show and verify that maximum synchronizability for networks with heterogeneous degree distribution can be achieved by using weighted and directed couplings between the nodes in the network. In this way, the enhanced synchronizability is determined by the mean degree and does not depend on the degree distribution and the network size. Their method differs from the others because it requires neither the spectrum of the adjacency nor the Laplacian matrix, instead it uses just local information, i.e. node degree.

In this manuscript, we are concerned with the issue of numerically investigating the effects of a simple and myopic perturbation of the topology of a network, intuitively aimed at enhancing its synchronizability and robustness [21]. More specifically, given a network with a fixed number of nodes N and an average connectivity $\langle k \rangle$, we try to rewire or to add some connections in the networks in order to obtain a class of networks which we call weak-entangled networks (see Section III), just by exploiting the homogenization of the degree of the nodes in the network. As a comparison for the synchronization performances, we consider the rewiring algorithm described in [11] (in this paper referred to as the RJH algorithm). The results show that the homogenization procedure (i.e. a very simple rewiring) greatly enhances the synchronizability and the robustness of a given network in the first few steps. This corresponds not only to fewer calculations, but also to fewer modifications in the existing (real or synthetic) network.

In addition, we investigate in more detail some of the topological properties of the obtained networks. We argue that their structure and characteristics are similar to those of entangled networks [7], and our numerical simulations with identical oscillators validate their good synchronization properties. Finally, we compare the vulnerability of the obtained networks with that of random, geometric, small-world and scale-free topologies by using the measures proposed in [22, 23].

The manuscript is organized as follows: in Section 2 we present a brief review of the existing work on synchronizability. In Section 3 we give the main motivations for the homogenization procedure. Results and comparisons with the RJH algorithm are given in Section 4, where we also inspect the structural properties and present results related to the vulnerability of the obtained networks. In Section 5 we consider networks composed of coupled oscillators to obtain some real examples of the effects of the proposed homogenization procedure on synchronization. Section 6 concludes this work.

2. A measure of synchronizability

A relevant contribution in determining the (local) stability of the synchronized states was given in [24, 9], by using the eigenvalues of the Laplacian matrix representing the network. We briefly recall the main ideas in the following. Consider a network of N identical dynamical systems with symmetric coupling. The equations of motion for the system are

$$\dot{x}_i = F(x_i) - \sigma \sum_{j=1}^{N} L_{ij} H(x_j), \tag{1}$$

where $F(\cdot)$ governs the dynamics of each isolated node, $H(\cdot)$ is the coupling function, $\sigma > 0$ is the overall coupling strength, and L is the Laplacian matrix associated to the network. The network admits a synchronous state

$$x_1 = x_2 = \ldots = x_N = \bar{x},$$

whose (local) stability is determined by the corresponding system of variational equations, together with the motion on the synchronous manifold

$$\dot{\bar{x}} = F(\bar{x}). \tag{2}$$

This system can be diagonalized into N blocks of the form

$$\dot{y} = [DF(\bar{x}) - \gamma DH(\bar{x})]y, \tag{3}$$

where y represents a mode of perturbation from the synchronized state; $\gamma = \sigma\lambda_i$, $i = 1, \ldots, N$; $\lambda_1 = 0 < \lambda_2 \leq ... \leq \lambda_N$ are the eigenvalues of L, all of them real as the matrix L is symmetric.

The master stability function (MSF) $\Lambda(\gamma)$ is defined as the largest Lyapunov exponent of the system defined by Eq. (2) and Eq. (3) as a function of γ [24]. This function, which generally is obtained by numerical methods, determines the stability of the synchronized state. In particular, the synchronized state is unstable if $\Lambda(\sigma\lambda_i) > 0$, for at least an index $i \in \{2, \ldots, N\}$.

For a large class of chaotic oscillatory systems, there exists a bounded interval (α_m, α_M) on which $\Lambda(\sigma\lambda) < 0$. In this case, there exist coupling strengths σ for which the synchronized state is linearly stable if and only if $\lambda_N/\lambda_2 < \alpha_M/\alpha_m$ [9].

The ratio $Q = \lambda_N/\lambda_2$ depends only on the network topology, while the ratio α_M/α_m depends on the dynamics of each node ($F(\cdot)$) and the coupling function ($H(\cdot)$). The lower the Q, the wider the interval of possible coupling strength σ such that the corresponding network has the synchronous state locally stable, so building networks which are highly synchronizable generally means building networks with a low Q. In practice, the MSF is not always negative only in a finite interval (see class-A networks in [25] and the dynamical systems of class Γ_0 and Γ_1 [27]). However, synchronizability in a variety of dynamical processes can be described by similar spectral properties, which tend to improve when the network is more homogeneous, see [26], or, simply put, other measures of synchronizability often go "hand in hand" with λ_N/λ_2 [11].

3. Rewiring procedure for weak-entangled networks

A rigorous definition of entangled structure is still lacking in the scientific community. However, besides the fact that the structure is extremely homogeneous (degree, node distance, betweenness and loop distribution) [7], this structure should also optimize synchronizability for many dynamical processes, which means that the synchronization measure Q for this structures should guarantee minimal value of Q, as mentioned in Section 2.

Finding this kind of networks is a NP-hard problem and it can be solved for small values for N and $\langle k \rangle$. On other hand, the number of nodes N in real networks is a very big number (in reality it tends to infinity). For instance, the number of current Facebook users is around $5 \cdot 10^8$ [28] and the number of sensor nodes that would be needed to secure, for example, the US-Mexico border is of magnitude 10^6 [29]. Thus, we need suboptimal, and sometimes very fast methods, for finding topologies with lower value of Q, topologies that have good synchronization properties and are robust against errors and attacks. We call these suboptimal topologies, weak-entangled.

The basic idea is to work with approximation of the eigenvalues λ_2 and λ_N. There exist many bounds on these eigenvalues, and some of the simplest to compute are related to the maximum and minimum degree of the nodes of the network. We have indeed [30]:

$$\begin{cases} 2 - N + 2k_{\min} & \leq \lambda_2 \quad \leq \dfrac{N}{N-1}k_{\min} \\ \dfrac{N}{N-1}k_{\max} & \leq \lambda_N \quad \leq 2k_{\max} \end{cases} \tag{4}$$

where N is the number of nodes in the network, $k_{\min}$ and $k_{\max}$ are minimum and maximum node degree, respectively. This indicates that having a small gap between the maximum and minimum degree could lead to having a small Q and hence better synchronization properties. However, the problem is not so simple, as ring networks —where every vertex is connected to its neighbors up to a certain distance $\langle k \rangle/2$— have bad synchronization properties, whereas random regular networks [31, 32] have good synchronization properties. In general, random networks have better synchronizability than regular lattices and (possibly) better than small-world networks when they are above their percolation transition [33, 9]. Furthermore, small-world networks have better synchronizability than scale-free networks [34]. In addition, Nishikawa *et al.* discovered that Q decreases when the heterogeneity of some measures of small-world networks declines, even if the average distance increases [35]. In [33] the authors found out that Q is proportional to the betweenness heterogeneity, and they conclude that a small value of the maximum betweenness centrality is an important factor for better synchronizability. The complete correlation between homogeneity and synchronizability for *any connected network* is given in [35]:

$$(1 - \frac{1}{N})\frac{k_{\max}}{k_{\min}} \leq Q \leq (N-1)k_{\max} l^e_{\max} D_{\max} \langle D \rangle, \tag{5}$$

where $D_{\max}$ is the maximum length of the shortest path between two nodes, $l^e_{\max}$ is the maximum normalized edge betweenness and $\langle D \rangle$ is the average path length. Eq. (5) confirms that homogeneous networks have high synchronizability, because in this case $k_{\max}$ and $l^e_{\max}$ are smaller. Thus, one can argue that the combination of small network distances and homogeneous distribution of connectivities and loads makes the network more synchronizable.

Having the previous considerations in mind, by homogenizing the degree of the nodes in the network we should be able to enhance synchronizability in a simple and fast manner, and to produce a topology which tends to an entangled one. This can be done, in one of the simplest and myopic ways, by disconnecting the highly connected vertices and connecting poorly connected ones, or by just adding a few edges between the poorly connected nodes, as it is shown at the end of Subsection 4.1.

More precisely, at each step, we look for the node with highest degree, say v_1, and, among its neighbors, for the one with relatively highest degree, say v_2. Having disconnected them, removing the link (v_1, v_2), we look for the two nodes with lowest degree, say w_1 and w_2, which are disconnected and we connect them, putting the link (w_1, w_2). Whenever there are multiple nodes satisfying the requirements, the tie is broken deterministically. If the network becomes disconnected when removing the edge (v_1, v_2), we look for the second most highly connected vertex, and so on.

Of course, we do not expect to obtain better synchronizability than the modified simulated annealing optimization algorithm in [7] or the RJH algorithm [11]; we just want to emphasize that even with small changes of the network topology, the synchronization properties of the network change very much, and that this can be very useful when trying to rewire real-world networks (where the number of nodes is very large).

The beneficial features of this approach are: it is faster and easier to implement than the algorithm for creating optimal topologies proposed in [7] and the rewiring algorithm proposed in [11], and it is very effective even in a reduced number of steps, as it will be shown in Section 4. In addition, our procedure exploits only local information, i.e. the degree of the nodes, and even though in this work the procedure is centralized (i.e. we search the node with maximum degree), the approach can be modified in several ways to produce decentralized algorithms.

4. Simulation results

4.1. Synchronization aspects

As a mean field approximation of starting networks, we focus on random network generated with the Erdős and Rényi model [36], with assigned mean degree $\langle k \rangle \in \{4, 6\}$ and total number of vertices $N \in \{50, 100, 200, 300, 500\}$. For each pair $\langle k \rangle, N$, the number of generated networks is 10 and all the results are averaged over the different realizations. Fig. 1 shows the relative averaged decrease of the synchronizability index Q for networks with different number of nodes and average node degree $\langle k \rangle = 4$ (upper panel, and $\langle k \rangle = 6$ (lower panel) against the number of iteration s. From Fig. 1 one can see that the degree homogenization produces topologies which have nearly 50% reduced Q just after a small number of iterations. Moreover, in the case of larger networks, i.e. larger N, the synchronizability at the end of the degree homogenization is increased even more. For instance, for a network with $N = 500$ and $\langle k \rangle = 4$ the synchronizability is increased by 75% (i.e. Q is reduced by 75%) and by 60% when $\langle k \rangle = 6$.

In addition, we compare the results obtained with the effect of degree homogenization with the results produced when using the RJH rewiring al-

N	$Q(0)$	$Q^*(\bar{s})$	$Q_{\mathrm{RJH}}(\bar{s})$	$Q_{\mathrm{RJH}}(end)$	$\bar{s}$
50	23.8344	10.5048	13.1488	7.5711	26
100	34.2478	13.3045	16.0255	9.4362	5
200	39.9462	13.9258	19.2149	11.7292	9
300	57.6587	15.0228	21.3068	12.9733	139

Table 1: Comparison of the degree homogenization procedure and the RJH algorithm described in [11], for networks with different sizes N and $\langle k \rangle = 4$ (data are averaged over 10 networks).

gorithm, which uses simulating annealing [11] and exploit more elaborate measures related to the network. The results are shown in Tab. (1) ($\langle k \rangle = 4$) and Tab. (2) ($\langle k \rangle = 6$), where $Q(0)$ is the starting value of the synchronizability index, $Q^*(\bar{s})$ is the final synchronizability index obtained by the homogenization procedure, $Q_{\mathrm{RJH}}(\bar{s})$ is the value of the synchronyzability index obtained by the RJH algorithm at the last iteration $\bar{s}$ of the simple homogenization procedure (the value of $\bar{s}$ is given in the last column) and $Q_{\mathrm{RJH}}(end)$ is the final synchronizability index obtained by the RJH algorithm after 1000 iterations. It is noticeable that the RJH algorithm at the end gives better results, but at the iteration $\bar{s}$ when the rewiring stops, the proposed procedure gives better results than the ones obtained by the RJH algorithm. The results are more evident in Fig. 2 for a network with $N = 200$ and $\langle k \rangle = 6$. When using the degree homogenization procedure, the decrease is sharper than using the RJH algorithm until the 50-th iteration, after which the ratio $Q(s)/Q(0)$ stabilizes, while for the RJH algorithm the ratio keep decreasing even after the 50-th iteration, and after the 170-th iteration the RJH algorithm outperforms the degree homogenization procedure, due to its ability to escape local minima and the cost function more focused on the synchronization properties.

We have also compared the synchronizability index Q of the topology produced by the degree homogenization procedure (summarized in the column $Q^*(\bar{s})$ of Tab. (1) and Tab. (2)) with the synchronizability index Q of some of existing network topologies. For a small-world network with $N = 50$, $\langle k \rangle = 6$, and $p = 0.1$ [37], we computed the average value of Q, which turned out to be around 16, thus the synchronizability of the network produced by

N	$Q(0)$	$Q^*(\bar{s})$	$Q_{\mathrm{RJH}}(\bar{s})$	$Q_{\mathrm{RJH}}(end)$	$\bar{s}$
50	13.2345	6.4538	6.1172	4.4019	30
100	16.1649	6.6658	8.3359	5.3253	49
200	24.4975	8.6263	10.2172	6.0894	96
300	27.5263	9.4384	11.8970	6.7391	140

Table 2: Comparison of the degree homogenization procedure and the RJH algorithm described in [11], for networks with different sizes N and $\langle k \rangle = 6$ (data are averaged over 10 networks).

the degree homogenization is nearly 2.5 times greater, while in the case of the scale-free network it is more than 4 times greater. The same analysis for the networks with $N = 200$ and $\langle k \rangle = 6$ shows that this degree homogenization produces networks that are around 2.1 times more synchronizable than the small-world counterpart and around 3.3 times more than the corresponding scale-free networks. Networks with better Q are those obtained with the random regular model [31, 32], which have $Q = 3.16$ if $N = 50$ and $\langle k \rangle = 6$ and $Q = 3.38$ if $N = 200$ and $\langle k \rangle = 6$.

Another analysis involved the behavior of λ_2 and λ_N through the degree homogenization process. Fig. 3 shows the relative averaged increase of λ_2 and the relative averaged decrease of λ_N with respect to their averaged initial values, i.e. the values they had before the degree homogenization started ($\lambda_2 = 0.761$ and $\lambda_N = 15.115$), throughout the degree homogenization process. In this case we analyze 10 networks with $N = 200$ and $\langle k \rangle = 6$. It is noticeable that the degree homogenization influences λ_2 more than λ_N, i.e. λ_2 is increased almost 70% with the respect to the initial value, and λ_N is decreased by 30%.

Another and very common type of proxy topology for real networks is the scale-free network topology. The degree homogenization procedure, briefly summarized in Section 3, might disconnect the network by disconnecting the most connected nodes (hubs) in the network. Thus, the degree homogenization procedure is not appropriate for this kind of topology. Instead, an even simpler way to increase synchronizability is to add edges between the least connected nodes in the network. Doing this we achieve, in an easy and fast manner, a topology which is easier to synchronize. In Fig. 4 on the x-axis is the iteration number (i.e. in this case the number of added edges between poorly connected nodes). By adding just 25 edges between the most poorly connected nodes, one might conclude from Fig. 4 that the synchronizability is increased 2 times. With this we want to show that it is possible to enhance synchronizability of the scale-free topology by just adding edges between the most poorly connected nodes. We will not make further analysis of the resulting network topology, because with this addition of edges we are still exploiting the homogenization of the nodes' degree in the network, and thus, this method will produce similar weak-entangled topology as homogenizing the random topology.

4.2. Some topological insights

In the next part we want to inspect the obtained homogenized networks (which we call weak-entangled networks) using some topological properties. The properties that we inspect in Fig. 5 are the following:

- **The network average clustering coefficient (CC)** [37]. This measure indicates the degree to which nodes in a network tend to cluster together and it is calculated as: $CC = \frac{1}{N}\sum_{i=1}^{N} C_i$. C_i is the local clustering coefficient of a node i and it is calculated as: $C_i = 2\Delta_i/(k_i-1)k_i$, where Δ_i is the number of the existing links between the neighbors of node i and k_i is the number of neighbors of node i.

- **Average path length ($\langle D\rangle$)**. This measure distinguishes an easily negotiable network from one which is inefficient and rather complicated. It is calculated as: $\langle D\rangle = \frac{1}{N(N-1)}\sum_{i,j} d(v_i, v_j)$, where $d(v_i, v_j)$ is the shortest distance between vertices v_i and v_j.

- **Maximum path length ($D_{\mathbf{max}}$)**. It is the maximum length of the shortest path between any two nodes. It is calculated as $\max_{i,j} d(v_i, v_j)$.

- **Maximum normalized node betweenness ($BC_{\mathbf{max}}$)** and **standard deviation of the normalized node betweenness ($BC_{\mathbf{dev}}$)**. The node betweenness [38] $BC(v)$ for a vertex v is calculated as: $BC(v_i) = \sum_{k\neq i\neq \ell, k\neq \ell} \frac{\sigma_{k\ell}(v_i)}{\sigma_{\ell k}}$, where $\sigma_{k\ell}$ is the number of shortest paths from v_k to v_ℓ, and $\sigma_{k\ell}(v_i)$ is the number of shortest paths from v_k to v_ℓ that pass through a vertex v_i.

- **Maximum normalized closeness centrality ($CLO_{\mathbf{max}}$)** and **standard deviation of the normalized closeness centrality ($CLO_{\mathbf{dev}}$)**

[39]. For a given node v_i the closeness centrality is calculated as: $CLO(v_i) = \frac{1}{\sum_j d(v_i,v_j)}$.

- **Maximum authority value using the Hypertext Induced Topic Selection (HITS) algorithm ($HITS_{\mathbf{max}}$)** and **the standard deviation of HITS authority values ($HITS_{\mathbf{dev}}$)** [40]. HITS is a link analysis algorithm that rates Web pages for their authority and hub values. Authority value estimates the value of the content of the page; hub value estimates the value of its links to other pages. Authority and hub values are defined in terms of one another in a mutual recursion. An authority value is computed as the sum of the scaled hub values that point to that page. A hub value is the sum of the scaled authority values of the pages it points to. Thus, the node pointed to by many links will have higher authority value. HITS, like PageRank, is an iterative algorithm based on the linkage of the documents on the web.

These measures are normalized with respect to their value obtained from the initial network and are plotted as a function of the iteration steps. Among all values just $\langle D\rangle$ remains almost unaltered (slightly increases) in the resulting network ($\langle D\rangle$ is around 5.5), which correspond to the results shown in [35], whereas all other properties have lower values in the resulting network. The BC_{dev} decreases the most, so that this property can be a good indicator for better synchronizable networks (see also [33]). The standard deviation of the authority rank (represented by $HITS_{\text{dev}}$) decrease by around 65%. In addition the maximum value of the betweenness centrality decreases by around 60%, which suggests that this could be an efficient topology for communication networks (see [5]). Other good indicators for networks with enhanced synchronizability are the standard deviation of the closeness centrality CLO_{dev} (it decreases by 55%) and $HITS_{\text{max}}$ (it decreases by 50%). These results are totally correlated to Eq. (5) which reasserts the fact that the proposed degree homogenization procedure produces networks with enhanced synchronizability and weak-entangled structure. In order to completely satisfy Eq. (5) the networks should have low value for k_{max} and the ratio $k_{\text{max}}/k_{\text{min}}$ should be close to 1. The main idea behind the rewiring is the homogenization of the nodes' degree, thus it satisfies both the conditions. In addition, the average clustering coefficient (CC) is decreased by 20% and the resulting network topology has a very small clustering coefficient, which is around 0.005.

The homogeneous structure —characterized by homogeneous degree, betweenness, closeness and authority (i.e. HITS) values— indicates that the topology produced by the degree homogenization belongs to the class of entangled (or interwoven) topologies [7], which are optimal in many senses, such as: synchronization, robustness and support for efficient communication. In order to prove the homogeneous structure of the obtained topology we give the average values of this 4 measures that each node has in Fig. 6, respectively, for a network with $N = 200$ and $\langle k \rangle = 4$ at the beginning (black line) and at the end of the degree homogenization process (gray line). Figure 6 clearly shows the homogeneous structure of the final topology. For the average degree values (Fig. 6 (a)) the average degree of each node at the end of the degree homogenization is nearly 4, which is not the case at the starting random topology (i.e. the values are between 2 and 7). The average values for the betweenness centrality, closeness centrality and HITS (Fig. 6(b,c,d)) at the end of degree homogenization are very close to 0.005 which is the mean value ($1/N$, $N = 200$).

In Fig. 7 we show the evolution of the second moment of the degree (DEGREE), betweenness (BC), closeness (CC) and authority (HITS) as the iterative process of degree homogenization takes place. It is obvious that the variance between these four measures at the end of the homogenization is practically 0, which is in close agreement with the previous results.

On the other hand, beyond the higher end-value of Q, our simulations and analyses show that the weak-entangled topology compared to the entangled topology obtained by Donetti [7] and the RJH algorithm [11], has slightly bigger average length path (around 2.0% bigger) and different loop distribution, i.e. the average loop size of the entangled topology is around 67.5 with standard deviation of 33.3, whereas weak-entangled topologies have slightly bigger average loop size of 74.5 with standard deviation of 40. The tests were performed on networks with $N = 200$ and $\langle k \rangle = 4$.

4.3. Vulnerability of the proposed networks

In this Subsection we show that the obtained networks, besides having enhanced synchronizability, also have a robust topology, as said before. As a vulnerability measure we use the maximal value of the pointwise vulnerability of the network [23] defined as

$$V = \max_i \frac{E - E(i)}{E},$$

Here E is global network efficiency [22], defined as

$$E = \frac{1}{N(N-1)} \sum_{i \neq j} \frac{1}{d(v_i, v_j)},$$

and $E(i)$ is the network efficiency after removal of the node v_i and all its edges. N is the total number of nodes and $d(v_i, v_j)$ is the minimal distance between the nodes v_i and v_j.

We compare the vulnerability of the networks produced by the degree homogenization with the Erdős-Rényi (ER) model of random networks [36], geometric random networks (GR) [41], Barabási-Albert (BA) model of scale-free networks [42], Watts-Strogatz (WS) model of small-world networks [37], and random regular networks (RRN) [31, 32]. The networks have 500 nodes and average connectivity 6, and data are averaged over 10 realizations. The networks obtained from the algorithm ($V = 0.0053$) are more than 1.6 times more robust than the WS small-world networks ($V = 0.0086$) and the ER random networks ($V = 0.0091$), 10 times more robust than the geometric random networks ($V = 0.0551$), and 30 times more robust than the BA scale-free networks ($V = 0.1597$). The only networks which are comparable robust are the random regular networks ($V = 0.0044$). However, we emphasize that they are obtained by building networks starting from scratch, whereas we are interested in rewiring existing networks. The proposed algorithm has no effects on this kind of networks —as all the vertices have the same degree— and so, given their low Q, observed in Section 4.1, they constitute a lower bound for the performance of the new algorithm. In Fig. 8, we plot the vulnerability of the new, homogenized networks on every 20 iteration, to show that the improvement in robustness is concentrated in the first iterations.

5. Synchronization in rewired networks

In this Section we test the proposed rewiring on networks with real identical oscillators. The oscillators we are using are Chua oscillators [43]. The systems are coupled in a linear way with varying coupling strength and the coupling matrix is obtained from the Laplacian matrix of the evolving networks through the iteration of the proposed algorithm.

The equations governing the motion are, for each $i = 1, \dots N$,

$$\begin{cases} \dot{x}_{i1} = -\alpha\big(x_{i1} + x_{i2} - n(x_{i1})\big) - \sigma \sum_{j=1}^{N} L_{ij} x_{j2}, \\ \dot{x}_{i2} = x_{i1} - x_{i2} + x_{i3} \\ \dot{x}_{i3} = -\beta x_{i2}, \end{cases} \tag{6}$$

with parameters α and β fixed at 8.5 and 15 respectively and $n(y) = -8/3y + 4/63y^3$.

The starting topologies are chosen at random (as in the examples of Section 4) with $N = 100$ and $\langle k \rangle \in \{4, 6\}$. As a figure of merit for synchronization we use the time average of the Mean Square Error (MSE)

$$\langle e \rangle = \frac{1}{\tau - t_0} \int_{t_0}^{\tau} e(t)\, \mathrm{d}\, t, \tag{7}$$

with

$$e(t) = \mathrm{std}(x_1(t))^2 + \mathrm{std}(x_2(t))^2 + \mathrm{std}(x_3(t))^2, \tag{8}$$

where

$$\begin{cases} x_1(t) = [x_{i1}(t), \dots, x_{N1}(t)], \\ x_2(t) = [x_{i2}(t), \dots, x_{N2}(t)], \\ x_3(t) = [x_{i3}(t), \dots, x_{N3}(t)], \end{cases} \tag{9}$$

$\mathrm{std}(\cdot)$ is the standard deviation, t_0 is a time instant such that the systems have reached steady-state behavior and τ is a time horizon large enough. Hence, the lower is the mean square error, the better is the synchronization achieved by the network.

We consider the cases with $N = 100, \langle k \rangle = 4$, $N = 100, \langle k \rangle = 6$, $N = 200, \langle k \rangle = 6$, and $N = 300, \langle k \rangle = 6$, in Fig. 9 (a), Fig. 9 (b), Fig. 9 (c), and Fig. 9 (d), respectively. We plot the time average of the MSE with respect to the coupling strength σ of Eq. (6) using the starting topology [line denoted with (0)], the topology after two iterations of the simple homogenization procedure [line denoted with (2)], and so on up to the topology after 10 steps [line denoted with (10)].

It is noticeable that the simple degree homogenization produces networks of oscillators which have better synchronizability than the starting networks in a few number of steps.

6. Concluding remarks and future work

We inspected some properties of a simple perturbation of networks in order to enhance synchronizability, robustness, and support for efficient communication (as in the case of entangled networks, [7]). The rewiring is very simple compared to others presented in the literature. However, it improves synchronizability and robustness of a given network in a small number of steps, just by exploiting the homogenization of the nodes' degree. Moreover, we show that for scale-free topology it is easier to simply add edges between the poorly connected nodes in order to enhance synchronizability. Time domain simulations with networks of coupled oscillators confirmed the enhanced synchronizability. In addition, we inspected the resulting networks in order to give insights about the characteristics one topology should generally have in order to be optimal and robust. The conclusion, as in [7], is that weak-entangled structures, i.e. very homogeneous structures, and democracy, i.e. low authority rank, are instrumental in obtaining synchronizability and robustness. We have shown that the networks obtained with the simple homogenization algorithm are more robust than other networks with similar average connectivity, such as random, geometric random, small-world and scale-free networks. Finally, our simple and myopic procedure is able to build networks with weak-entangled structure in a small number of steps, which can be very useful in practice, when the networks are dense and have a large number of nodes.

Future investigation of quantifying and explaining the good performances in the first iterations of the proposed rewiring could be useful to design more elaborated algorithms to build networks with more entangled structures. Another improvement would be to find how close our weak-entangled structure is to an optimal, i.e. entangled, topology.

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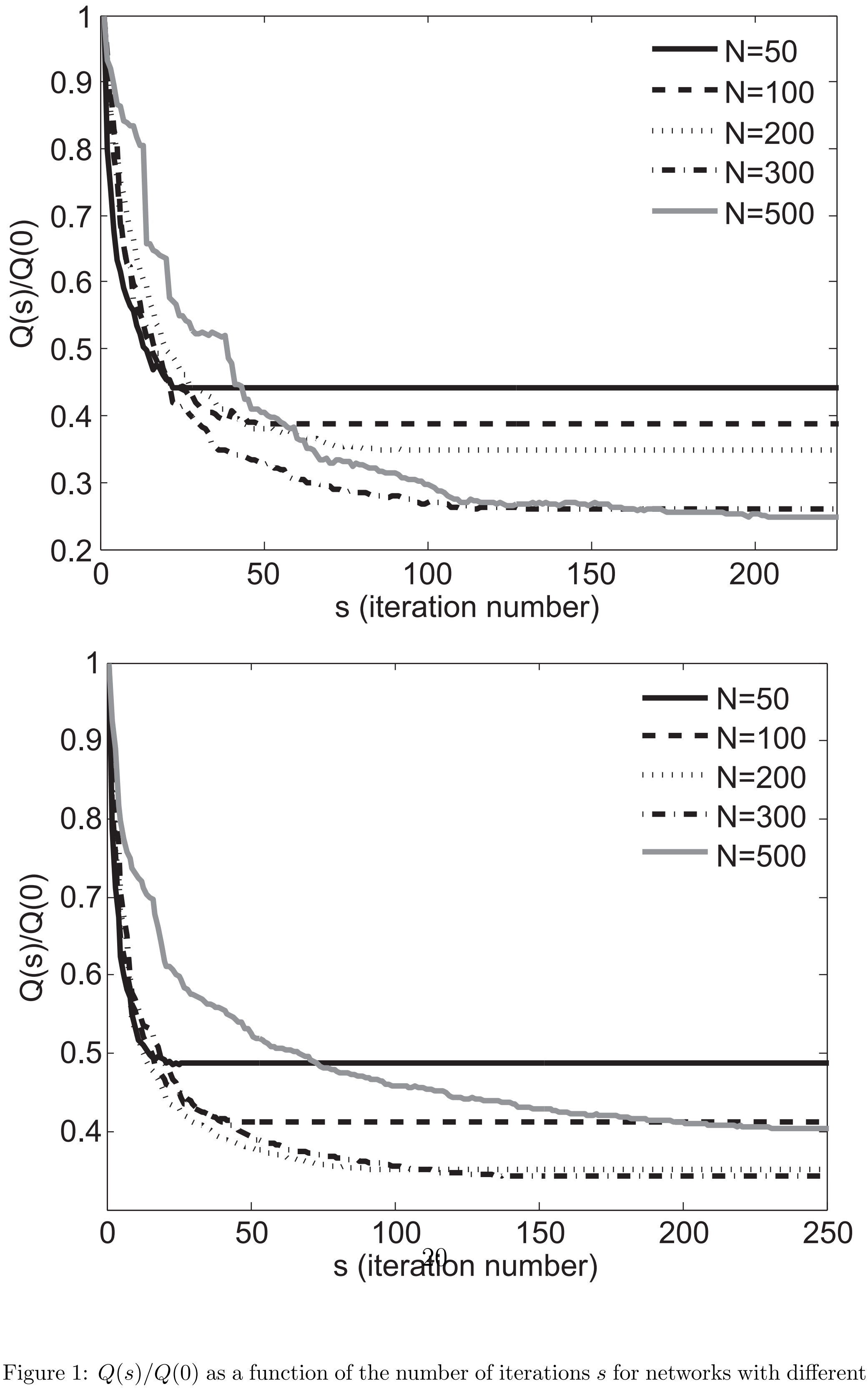


Figure 1: $Q(s)/Q(0)$ as a function of the number of iterations s for networks with different number of nodes N and $\langle k \rangle = 4$ for the upper panel, and $\langle k \rangle = 6$ for the lower panel (data are averaged over 10 networks).

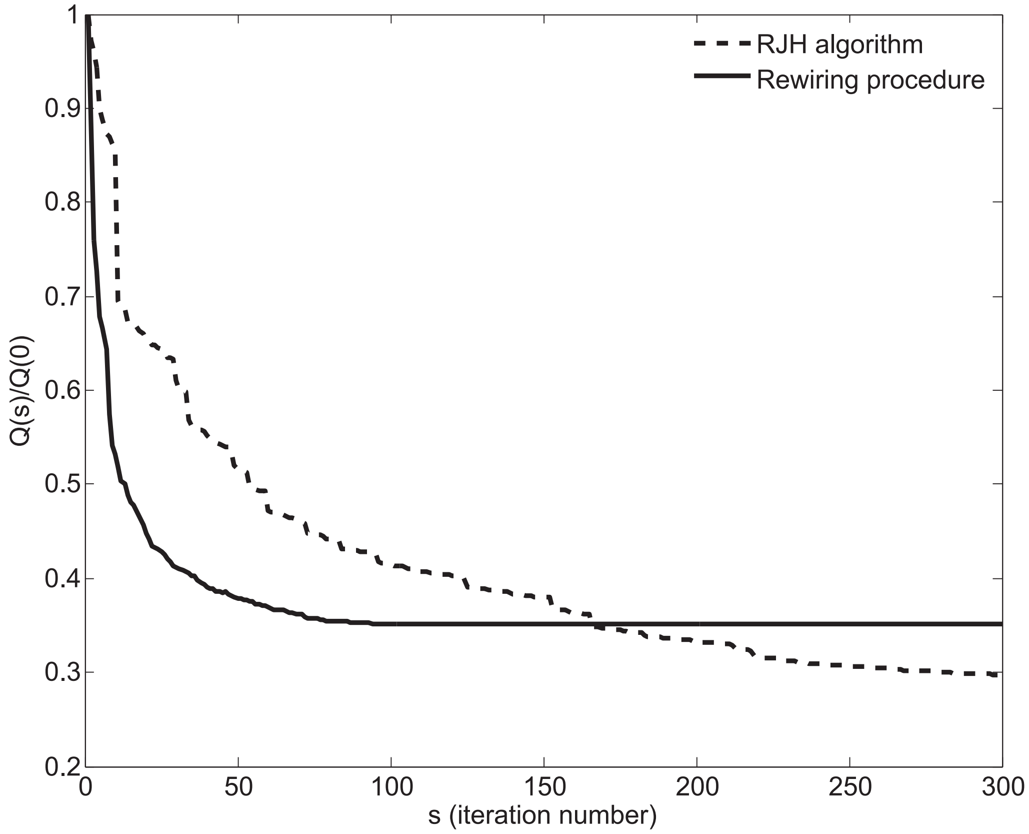


Figure 2: $Q(s)/Q(0)$ as a function of the number of iterations s for the degree homogenization procedure and the RJH algorithm described in [11], for a network with $N = 200$ and $\langle k \rangle = 6$ (data are averaged over 10 networks).

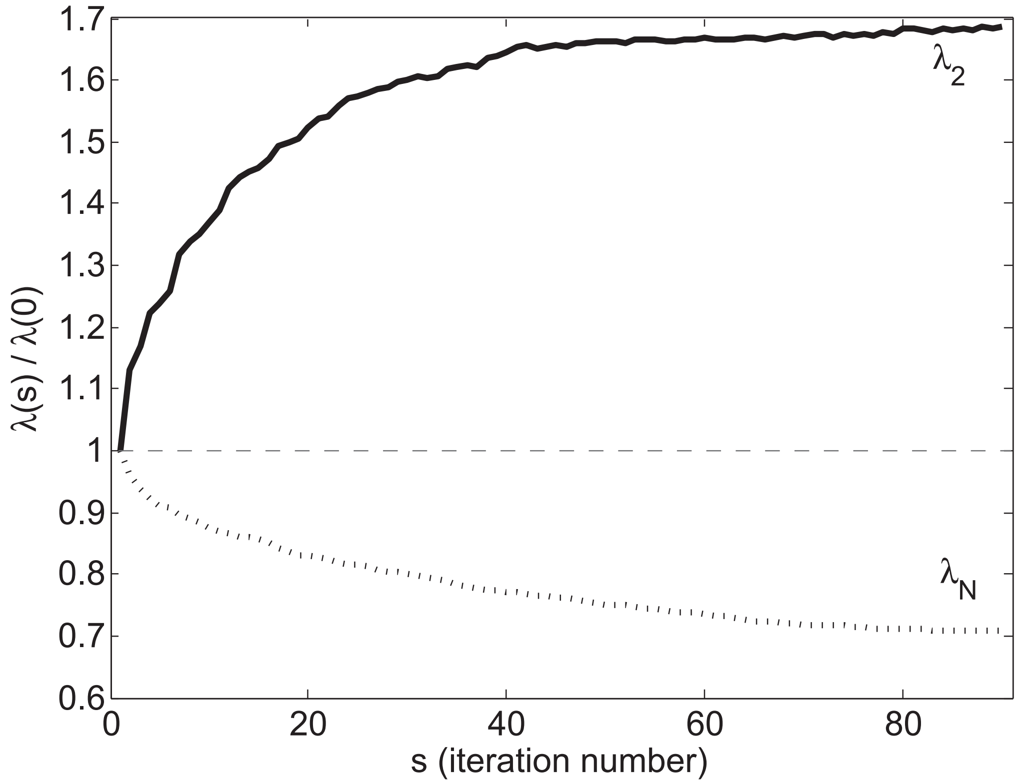


Figure 3: Behavior of the relative λ_2 and λ_N through the degree homogenization process for network with $N = 200$ and $\langle k \rangle = 6$ (data are averaged over 10 networks).

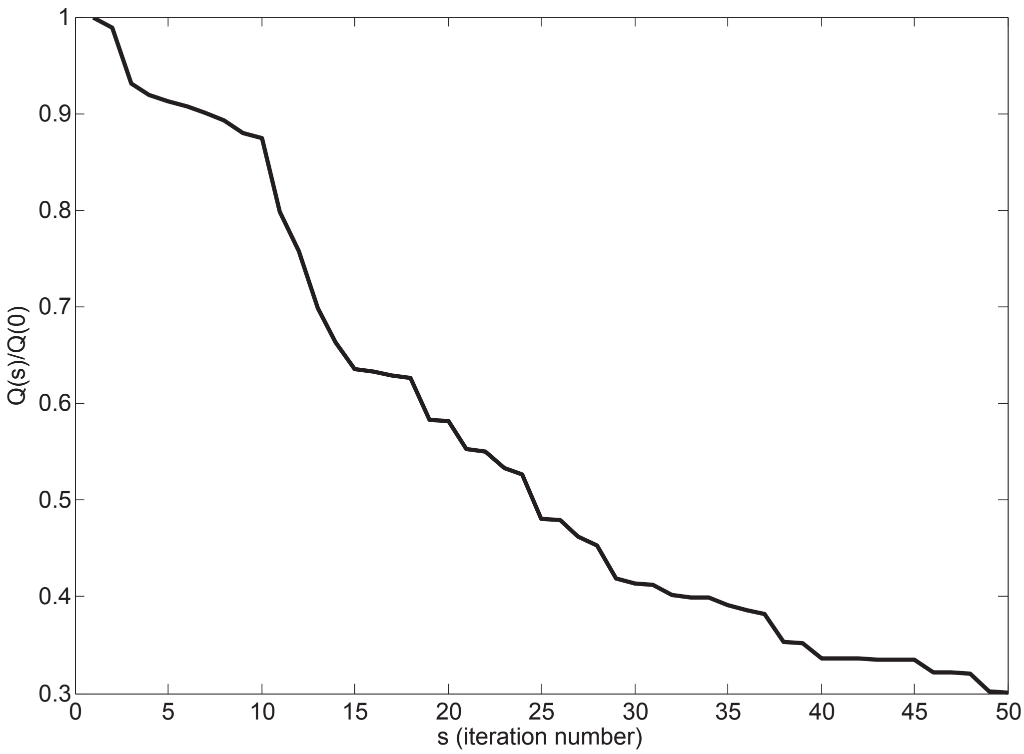


Figure 4: $Q(s)/Q(0)$ as a function of the number of iterations s for scale-free networks for $N = 200$ and $\langle k \rangle = 4$ (data are averaged over 10 networks).

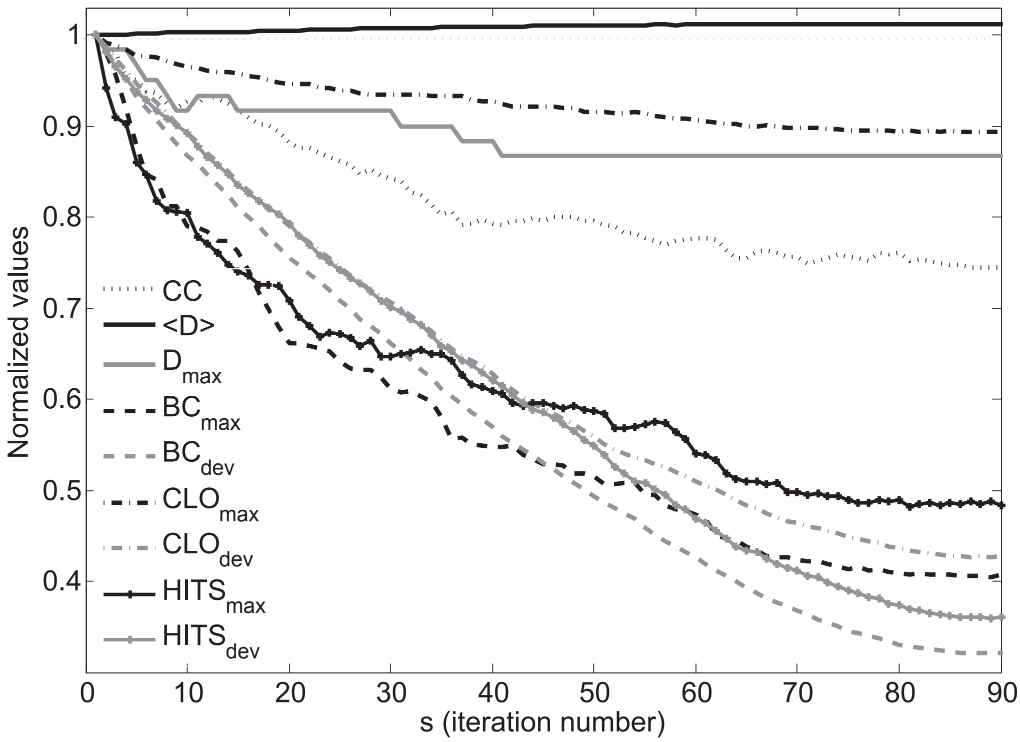
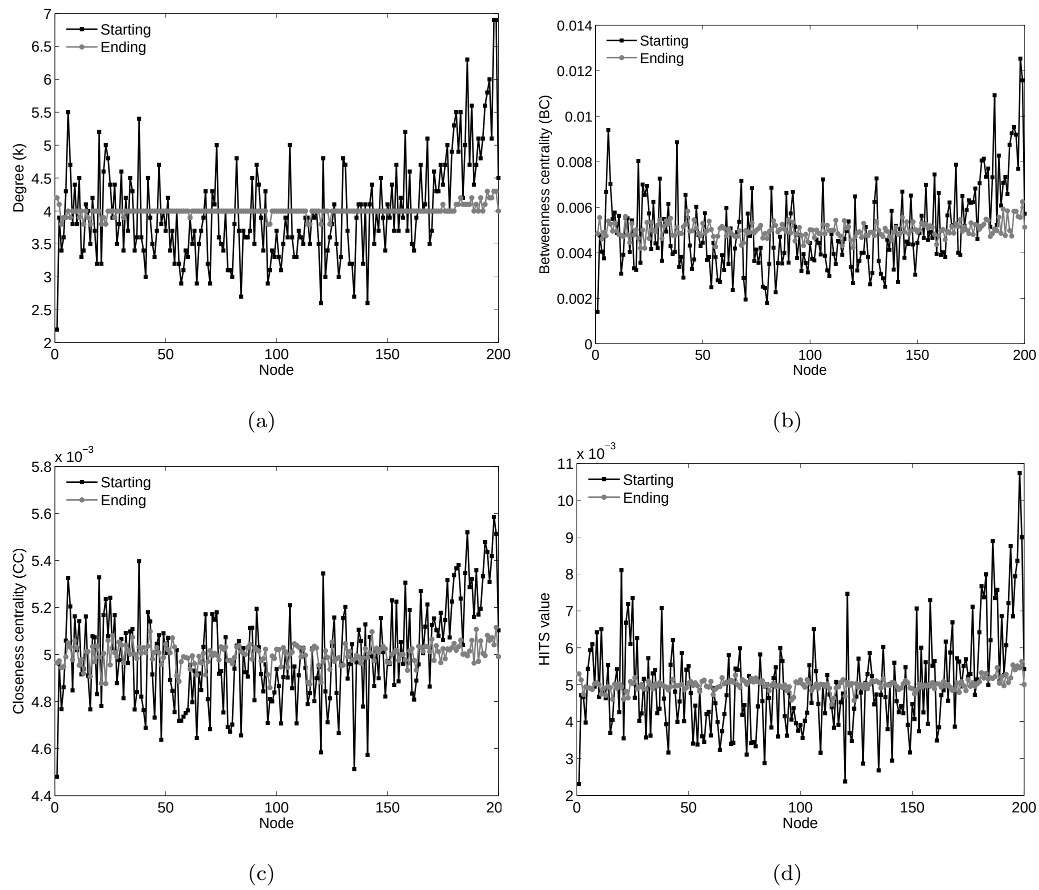


Figure 5: Topological properties of the obtained network with $N = 200$ and $\langle k \rangle = 6$ (data are averaged over 10 networks).



(a) (b) (c) (d)

Figure 6: Centrality values at the beginning (black line) and at the end of the degree homogenization (grey line) for a network with $N = 200$ and $\langle k \rangle = 4$ (data are averaged over 10 networks). The centrality values are: (a) Degree, (b) Betweenness, (c) Closeness and (d) HITS.

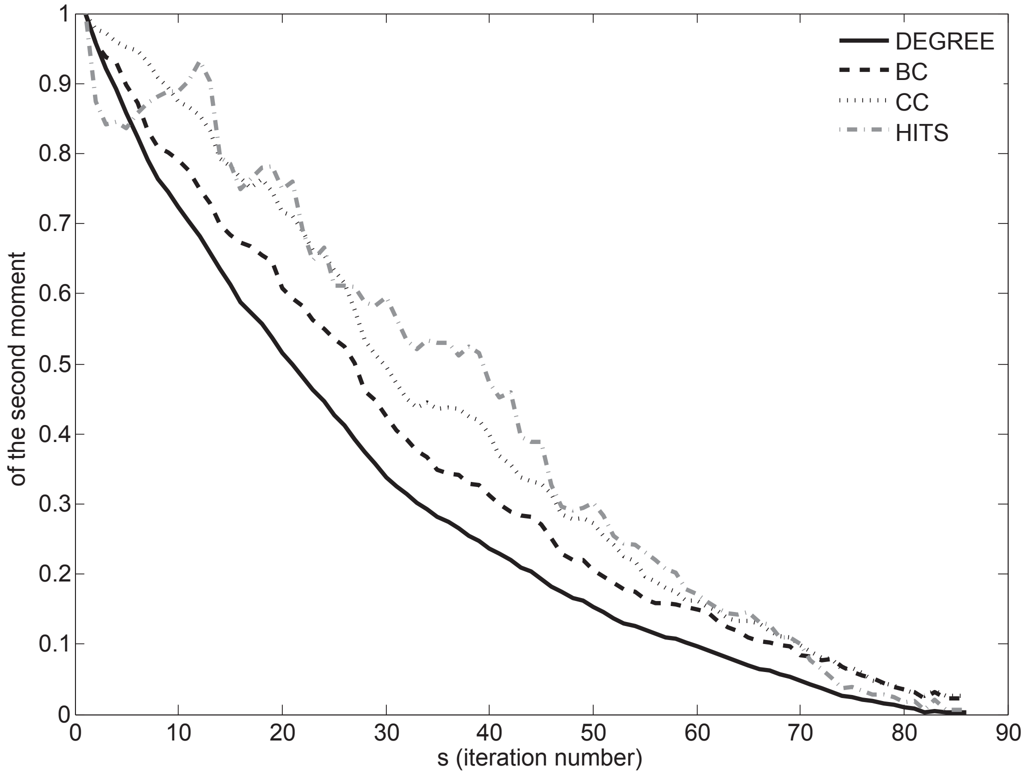


Figure 7: The evolution of the second moment of the degree (DEGREE), betweenness (BC), closeness (CC) and authority (HITS) for a network with $N = 200$ and $\langle k \rangle = 4$ (data are averaged over 10 networks).

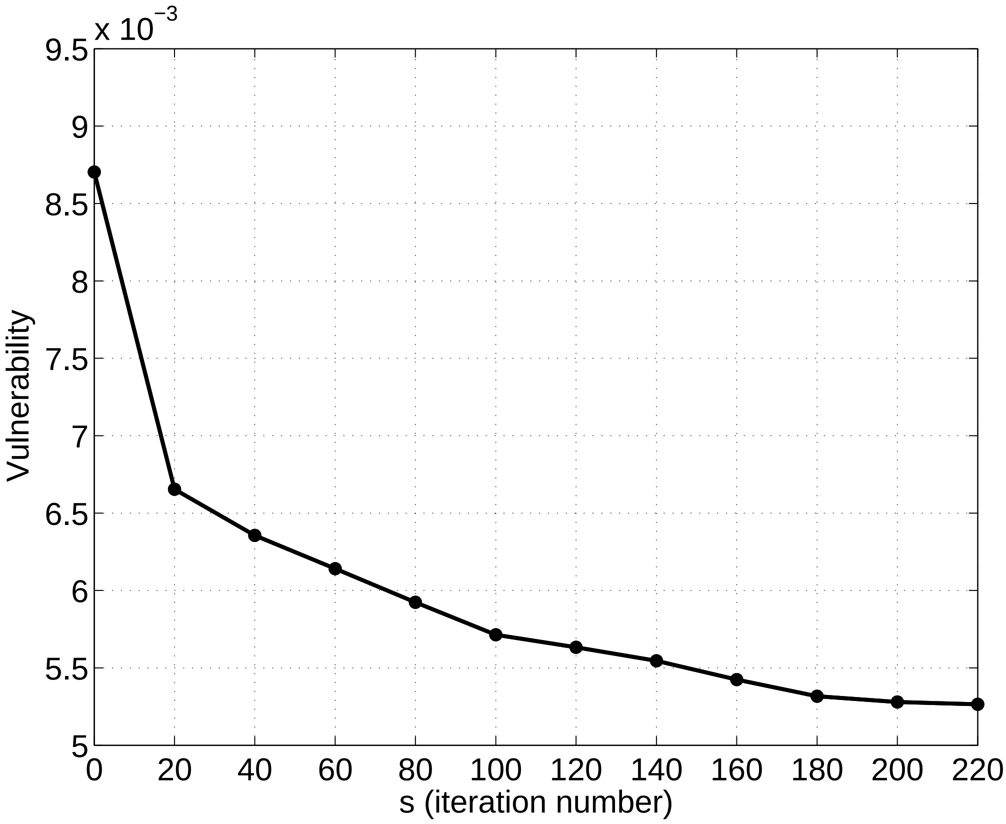
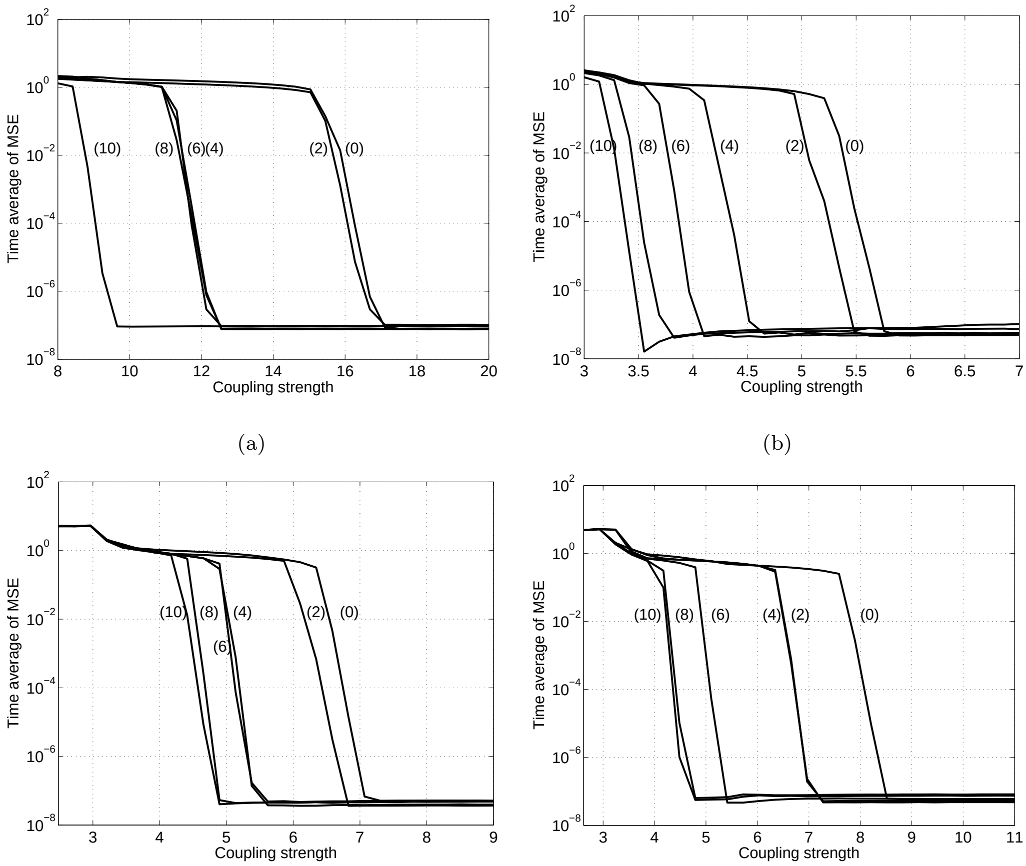


Figure 8: Vulnerability of the networks with respect to different iterations of the simple homogenization procedure.



(a) (b) (c) (d)

Figure 9: Time average of the means square error $\langle e \rangle$ with respect to the coupling strength σ for different iterations of the simple homogenization procedure. (0) denotes the error using the starting topology, (2) after 2 iterations, and so on up to (10) for 10 iterations. We observe the following cases: (a) $N = 100$ and $\langle k \rangle = 4$, (b) $N = 100$ and $\langle k \rangle = 6$, (c) $N = 200$ and $\langle k \rangle = 6$, (d) $N = 300$ and $\langle k \rangle = 6$.