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Monomials as sums of powers: the Real binary case / Boji, M.; Carlini, Enrico; Geramita, A. V.. - In: PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY. - ISSN 0002-9939. - 139:(2011), pp. 3039-3043. [10.1090/S0002-9939-2011-11018-9]

*Availability:*

This version is available at: 11583/2440683 since:

*Publisher:*

AMERICAN MATHEMATICAL SOCIETY

*Published*

DOI:10.1090/S0002-9939-2011-11018-9

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# MONOMIALS AS SUMS OF POWERS: THE REAL BINARY CASE

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ABSTRACT. We generalize an example, due to Sylvester, and prove that any monomial of degree  $d$  in  $\mathbb{R}[x_0, x_1]$ , which is not a power of a variable, cannot be written as a linear combination of fewer than  $d$  powers of linear forms.

## 1. INTRODUCTION

It is well-known, and easy to prove, that if  $k$  is a field of characteristic zero and  $R = k[x_0, \dots, x_n] = \bigoplus_{i=0}^{\infty} R_i$  is the standard graded polynomial algebra, then the  $k$ -vector space  $R_d$  (for any  $d$ ) has a basis consisting of polynomials  $\{L_1^d, \dots, L_s^d\}$  where  $s = \binom{d+n}{n} = \dim_k R_d$  and the  $L_i$  are pairwise linearly independent forms in  $R_1$ . It follows that every form in  $R_d$  is a  $k$ -linear combination of at most  $s$   $d^{\text{th}}$  powers of linear forms and, if  $k$  is algebraically closed, simply a sum of at most  $s$   $d^{\text{th}}$  powers of linear forms. We will call such a way of writing  $F \in R_d$  a *Waring expansion of  $F$*  because of the echo of Waring's problem from number theory. We will further refer to such an expression as a *minimal Waring expansion for  $F$*  if the number of summands in such an expression for  $F$  is minimal among all such representations.

If  $n > 0$  and  $d = 2$  it is a classical fact that although  $s = \binom{n+2}{2}$  every quadratic form has a Waring expansion involving  $\leq n + 1 < s$  summands and that, in general, i.e. for  $[F]$  belonging to a non-empty Zariski open subset of  $\mathbb{P}(R_2)$  a minimal Waring expansion for  $F$  has exactly  $n + 1$  summands.

These observations have led to a series of problems, usually called **Waring Problems**, which ask for information on minimal Waring expansions for forms of degree  $d$  in  $R$ .

The long outstanding problem of finding the number of summands in a minimal Waring expansion of the generic form of degree  $d$  was solved, after being open for almost 100 years, by J. Alexander and A. Hirschowitz (see [AH95]), when  $k$  is an algebraically closed field.

Of course, solving this problem for the generic form of degree  $d$  does not always give information about any specific form of degree  $d$  and the

problem of finding the length of the minimal Waring expansion for specific forms has also been a continuing source of interesting speculations and lovely results. E.g. it was Sylvester ([Har92]) who first observed that although for  $R = \mathbb{C}[x_0, x_1]$ , the generic form of degree  $d$  has a Waring Expansion with  $s = \lceil \frac{d+1}{2} \rceil$  summands, the monomial  $x_0 x_1^{d-1}$  has  $d$  summands in its minimal Waring expansion (the maximum possible).

The Waring problem for specific forms has been considered in depth by B. Reznick in his monograph (see [Rez92]) and by Comas and Seiguer who, to our knowledge, were the first to resolve the problem completely and algorithmically in  $\mathbb{C}[x_0, x_1]$  in their unpublished work ([CS01]).

It is interesting to note that although the Waring problem is a very interesting and stimulating problem in purely algebraic terms, it has a surprising number of intimate connections with problems in areas as seemingly disparate as algebraic geometry and communication theory (see for example [RS00],[CC03] and [CM96])

Indeed, if  $k = \mathbb{R}$ , the field of real numbers, the connection with real world problems is very direct. This has prompted a re-examination of the Waring problem for  $R = \mathbb{R}[x_0, x_1]$ , and a recent very suggestive paper of Comon and Ottaviani (see [CO09]) considered this very problem for degrees  $d \leq 5$ .

Our main result in this paper follows the line of Sylvester's examples and concerns the minimal Waring expansion for monomials in  $\mathbb{R}[x_0, x_1]$ . We first give a new proof of the fact that the minimal Waring expansion of the monomial  $x_0^a x_1^b$  in  $\mathbb{C}[x_0, x_1]$  with  $0 < a \leq b$  has  $b + 1$  summands. In sharp contrast to this we show that in  $\mathbb{R}[x_0, x_1]$  every monomial of degree  $d$  (except  $x_0^d$  and  $x_1^d$ ) has  $d$  summands in its minimal Waring expansion.

## 2. BASIC RESULTS

Let  $S = k[x_0, x_1]$  and  $T = k[y_0, y_1]$ . We make  $S$  into a  $T$ -module using differentiation, i.e. we think of  $y_0 = \partial/\partial x_0$  and  $y_1 = \partial/\partial x_1$ . We refer to a polynomial in  $T$  as  $\partial$  instead of using capital letters. In particular, for any form  $F$  in  $S_d$  we define the ideal  $F^\perp \subseteq T$  as follows:

$$F^\perp = \{\partial \in T : \partial F = 0\}.$$

The following *Apolarity Lemma* is due to Iliev and Ranestad [IR01].

**Lemma 2.1.** *A homogeneous form  $F \in S$  can be written as*

$$F(x_0, x_1) = \sum_{i=1}^r \alpha_i (L_i)^d, \quad L_i \text{ pairwise linearly independent, } \alpha_i \in k$$

*i.e. has a Waring expansion with  $r$  summands, if and only if the ideal  $F^\perp$  contains the product of  $r$  distinct linear forms.*

### 3. BINARY MONOMIALS: THE COMPLEX CASE

The complex case is straightforward for monomials.

**Proposition 3.1.** *Let  $M = x_0^a x_1^b$  be a monomial in  $\mathbb{C}[x_0, x_1]$ . If  $0 < a \leq b$ , then  $M$  has a minimal Waring expansion with  $b + 1$  summands, *i.e. is a sum of  $b + 1$  powers of linear forms and no fewer.**

*Proof.* Let  $I = M^\perp = (y_0^{a+1}, y_1^{b+1})$  and notice that the linear system defined by  $I_{b+1}$  is base point free on  $\mathbb{P}^1 = \mathbb{P}S_1$ . Applying Bertini's Theorem, we get that the generic element of  $I_{b+1}$  defines a set of  $b + 1$  distinct points and hence it is the product of  $b + 1$  distinct linear forms. Thus the apolarity lemma yields that  $M$  is the sum of  $b + 1$  powers of linear forms. If  $r < b + 1$ , then  $r$  powers do not suffice as no element in  $I_r = (y_0^{a+1})_r$  is a product of  $r$  distinct linear forms.  $\square$

### 4. BINARY MONOMIALS: THE REAL CASE

We can also ask for a real Waring expansion of a monomial  $M$ . More precisely, we want to write

$$M(x_0, x_1) = \sum_{i=1}^r \alpha_i (L_i)^d, \quad \alpha_i \in \{1, -1\}$$

where the linear forms  $L_i$  are in  $\mathbb{R}[x_0, x_1]$ . In order to do this, we have to increase the number of summands in Proposition 3.1.

The following elementary facts will be extremely useful.

**Lemma 4.1.** *Consider the degree  $d$  polynomial*

$$F(x) = c_d x^d + \dots c_1 x + c_0 \in \mathbb{R}[x].$$

*If  $c_i = c_{i-1} = 0$  for some  $1 \leq i \leq d$ , then  $F(x)$  does not have  $d$  real roots.*

*Proof.* The proof is obvious if  $i = 1$  or  $i = d$ , so we may as well assume that  $1 < i < d$ .

Consider all the pairs  $(c_r, c_s)$  of non-zero coefficients such that  $r > s$  and  $c_j = 0$  if  $r > j > s$ . Let  $\alpha$  be number of pairs such that  $r - s$  is odd and  $\beta$  the number of pairs such that  $r - s$  is even. Notice that, by hypothesis,  $\alpha + 2\beta < d - 1$

Now we apply Descartes' rule of signs. For a pair  $(c_r, c_s)$  such that  $r - s$  is odd we get a real root of  $F(x)$ . For a pair  $(c_r, c_s)$  such that  $r - s$  is even we get either two real roots of  $F(x)$  or none.

In conclusion, the number of real roots of  $F(x)$  is at most  $\alpha + 2\beta$  and we are done.  $\square$

**Lemma 4.2.** *For each  $i < d$  there exists a degree  $d$  polynomial  $F(x) = c_d x^d + \dots c_1 x + c_0 \in \mathbb{R}[x]$  having  $d$  real roots and such that  $c_i = 0$ .*

*Proof.* Choose  $a_1, \dots, a_d \in \mathbb{R}$  and consider the polynomial  $F(x) = (x - a_1) \cdot \dots \cdot (x - a_d)$ . This polynomial can also be written as

$$F(x) = \sum_{i=0}^d E_i(a_1, \dots, a_d) x^i,$$

where  $E_i$  is the degree  $i$  elementary symmetric function in its arguments. The vanishing of the  $i$ -th coefficient of  $F(x)$  can be written as

$$E_i(a_1, \dots, a_{d-1}) + a_d E_{i-1}(a_1, \dots, a_{d-1}) = 0.$$

Hence, if we choose the  $a_1, \dots, a_{d-1} > 0$  and distinct there exists a unique, negative value of  $a_d$  such that the coefficient of  $x^i$  in  $F(x)$  is zero. As the roots of  $F(x)$  are  $a_1, \dots, a_d$  the polynomial has  $d$  real, distinct roots.  $\square$

Using the previous results we immediately get a lower bound on the number of summands in the minimal Waring expansion of a monomial in  $\mathbb{R}[x_0, x_1]$ .

**Lemma 4.3.** *Let  $M = x_0^a x_1^b$  be a monomial in  $\mathbb{R}[x_0, x_1]$ . If  $0 < a \leq b$ , then  $M$  does not have a Waring expansion with  $r \leq a + b - 1$  real summands.*

*Proof.* Let  $I = M^\perp = (y_0^{a+1}, y_1^{b+1})$ . The general degree  $r$  element in  $I$  has the form  $F(y_0, y_1) =$

$$c_r y_0^r + c_{r-1} y_0^{r-1} y_1 + \dots + c_{a+1} y_0^{a+1} y_1^{r-a-1} + c_{r-b-1} y_0^{r-b-1} y_1^{b+1} + \dots + c_0 y_1^r.$$

If  $a+1 \geq r-b+2$ , then by Lemma 4.1  $F(y_0, y_1)$  is not the product of  $r$  real linear forms. The conclusion follows by the apolarity lemma.  $\square$

**Proposition 4.4.** *Let  $M = x_0^a x_1^b$  be a monomial in  $\mathbb{R}[x_0, x_1]$ . If  $0 < a \leq b$ , then  $M$  has a minimal Waring expansion with  $a + b$  summands which are powers of real linear forms.*

*Proof.* We have that  $M^\perp = I = (y_0^{a+1}, y_1^{b+1})$ . Notice that  $I_{a+b}$  is the subspace of  $T_{a+b}$  of polynomials which are missing all the monomials having factor  $y_0^a$  or  $y_1^b$ . Thus, Lemma 4.2 and the apolarity lemma yield the result.  $\square$

## 5. ACKNOWLEDGEMENT

This project started with a research visit supported by a grant for visiting researchers from the Göran Gustafsson Foundation.

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