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# STAR CONFIGURATION POINTS AND GENERIC PLANE CURVES

ENRICO CARLINI AND ADAM VAN TUYL

ABSTRACT. Let  $\ell_1, \dots, \ell_l$  be  $l$  lines in  $\mathbb{P}^2$  such that no three lines meet in a point. Let  $\mathbb{X}(l)$  be the set of points  $\{\ell_i \cap \ell_j \mid 1 \leq i < j \leq l\} \subseteq \mathbb{P}^2$ . We call  $\mathbb{X}(l)$  a star configuration. We describe all pairs  $(d, l)$  such that the generic degree  $d$  curve in  $\mathbb{P}^2$  contains a  $\mathbb{X}(l)$ .

## 1. INTRODUCTION

The problem of studying subvarieties of algebraic varieties is a crucial one in algebraic geometry, e.g., the case of divisors. The study of subvarieties of hypersurfaces in  $\mathbb{P}^n$  has a particularly rich history. For example, one can look for the existence of  $m$  dimensional linear spaces on generic hypersurfaces of degree  $d$  leading to the theory of Fano varieties and to the well known formula relating  $n, m$  and  $d$  (e.g., see [10, Theorem 12.8]).

Because linear spaces are complete intersections, it is natural to look for the existence of complete intersection subvarieties on a generic hypersurface. The case of codimension two complete intersections was first studied by Severi [15], and later generalized and extended to higher codimensions by Noether, Lefschetz [11] and Groethendieck [9]. Recently, in [3], secant varieties were used to give a complete solution for the existence of complete intersections of codimension  $r$  on generic hypersurfaces in  $\mathbb{P}^n$  when  $2r \leq n + 2$ . Fewer results are known if the codimension of the complete intersection is large, i.e., the codimension is close to the dimension of the ambient space. In [16] the case of complete intersection curves is studied and completely solved. The case of complete intersection points on generic surfaces in  $\mathbb{P}^3$  is considered in [4] where some asymptotic results are presented. The case of complete intersection points in  $\mathbb{P}^2$  is a special case of [3], and it is completely solved.

Taking our inspiration from [3, 4], we examine the problem of determining when special configurations of points lie on a generic degree  $d$  plane curve in  $\mathbb{P}^2$ . We shall focus on **star configurations** of points. Consider  $l$  lines in the plane  $\ell_1, \dots, \ell_l \subset \mathbb{P}^2$  such that  $\ell_i \cap \ell_j \cap \ell_k = \emptyset$ . The set of points  $\mathbb{X}(l)$  consisting of the  $\binom{l}{2}$  pairwise intersections of the lines  $\ell_i$  is called a star configuration. In this paper, we address the following question

**Question 1.1.** *For what pairs  $(d, l)$  does the generic degree  $d$  plane curve contain a star configuration  $\mathbb{X}(l) \subseteq \mathbb{P}^2$ ?*

The name star configuration was suggested by A.V. Geramita since  $\mathbb{X}(5)$  is the ten points of intersections when drawing a star using five lines (see Figure 6.2.4 in [6]). The configurations  $\mathbb{X}(l)$  appeared in the work of Geramita, Migliore, and Sabourin [8] as the

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support of a set of double points whose Hilbert function exhibited an extremal behaviour. More recently, Cooper, Harbourne, and Teitler [6] computed the Hilbert function of any homogeneous set of fatpoints supported on  $\mathbb{X}(l)$ . Bocci and Harbourne [1] used star configurations (and their generalizations) to compare the symbolic and regular powers of an ideal. Further properties of star configurations continue to be uncovered; e.g., ongoing work of Geramita, Harbourne, Migliore [7].

We can answer Question 1.1 because we can exploit the rich algebraic structure of star configurations. In particular, we will require the fact that one can easily write down a list of generators for  $I_{\mathbb{X}(l)}$ , the defining ideal of  $\mathbb{X}(l)$ , as well as the fact that the Hilbert function of  $\mathbb{X}(l)$  is the same as the Hilbert function of  $\binom{l}{2}$  generic points in  $\mathbb{P}^2$ . Using these properties, among others, we give the following solution to Question 1.1:

**Theorem. 6.3** *Let  $l \geq 2$ . Then the generic degree  $d$  plane curve contains a star configuration  $\mathbb{X}(l)$  if and only if*

- (i)  $l = 2$  and  $d \geq 1$ , or
- (ii)  $l = 3$  and  $d \geq 2$ , or
- (iii)  $l = 4$  and  $d \geq 3$ , or
- (iv)  $l = 5$  and  $d \geq 5$ .

Our proof is broken down into a number of cases. It will follow directly from the generators of  $I_{\mathbb{X}(l)}$  that Question 1.1 can have no solution for  $d < l - 1$ . Using a simple dimension counting argument, Theorem 3.1 shows that for  $l \geq 6$ , there is no solution to Question 1.1. The cases  $l = 2$  and  $l = 3$  are trivial cases, so the bulk of the paper will be devoted to the cases that  $l = 4$  and  $l = 5$ .

To prove these cases, we rephrase Question 1.1 into a purely ideal theoretic question (see Lemma 4.3). Precisely, we construct a new ideal  $I$  from the generators of  $I_{\mathbb{X}(l)}$  and the linear forms defining the lines  $\ell_1, \dots, \ell_l$ . We then show that Question 1.1 is equivalent to determining whether  $I_d = (\mathbb{C}[x, y, z])_d$ . We then answer this new algebraic reformulation.

The proofs for the cases  $(d, l) = (3, 4)$  and  $(4, 5)$  are, we believe, especially interesting. To prove that Question 1.1 is true for  $(3, 4)$ , we exploit the natural group structure of the plane cubic curve to find a  $\mathbb{X}(4)$  on the curve. To show the non-existence of a solution for  $(4, 5)$ , we require the classical theory of **Lüroth quartics**. Lüroth quartics are the plane quartics that pass through a  $\mathbb{X}(5)$ . Lüroth quartics have the property of forming a hypersurface in the space of plane quartics. The existence of this hypersurface is the obstruction for the existence of a solution when  $(d, l) = (4, 5)$ .

Our paper is structured as follows. In Section 2, we describe the needed algebraic properties of star configurations. In Section 3, we give some asymptotic results. In Section 4, in preparation for the last two sections, we rephrase Question 1.1 into an equivalent algebraic problem. Sections 5 and 6 deal with the cases  $l = 4$  and  $l = 5$ , respectively.

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## 2. STAR CONFIGURATIONS

Throughout this paper, we set  $S = \mathbb{C}[x, y, z]$  and we denote its  $d$ th homogeneous piece with  $S_d$ . Moreover, we fix standard monomial bases in each  $S_d$  in such a way that  $\mathbb{P}S_d$  will be identified with  $\mathbb{P}^{N_d}$  where  $N_d = \binom{d+2}{2} - 1$ . We recall the relevant definitions concerning star configurations of points in  $\mathbb{P}^2$ .

Let  $l \geq 2$  be an integer. The scheme  $\mathbb{X}(l) \subset \mathbb{P}^2$  is said to be a **star configuration** if it consists of  $\binom{l}{2}$  distinct points which are the pairwise intersections of  $l$  distinct lines, say  $\ell_1, \dots, \ell_l$ , where no three lines pass through the same point. We will also call  $\mathbb{X}(l)$  a **star configuration set of points**.

Note that when  $l = 2$ , then  $\mathbb{X}(2)$  consists of a single point. When  $l = 3$ , then  $\mathbb{X}(3)$  is any set of three points in  $\mathbb{P}^2$ , provided the three points do not lie on the same line. It is clear that any degree  $d \geq 1$  plane curve contains a point. Furthermore, when  $d \geq 2$ , the generic degree  $d$  plane curve will contain three points not lying on a line. These remarks take care of the trivial cases of Theorem 6.3, which we summarize as a lemma:

**Lemma 2.1.** *The generic degree  $d$  plane curve contains a star configuration  $\mathbb{X}(l)$  with  $l = 2$ , respectively  $l = 3$ , if and only if  $d \geq 1$ , respectively  $d \geq 2$ .*

Given a  $\mathbb{X}(l)$ , for each  $i = 1, \dots, l$ , we let  $L_i$  denote a linear form in  $S_1$  defining the line  $\ell_i$ . The defining ideal of  $I_{\mathbb{X}(l)}$  is then given by

$$I_{\mathbb{X}(l)} = \bigcap_{i \neq j} (L_i, L_j).$$

We will sometimes write a point of  $\mathbb{X}(l)$  as  $p_{i,j}$ , where  $p_{i,j}$  is the point defined by the ideal  $(L_i, L_j)$ , i.e.,  $p_{i,j}$  is the point of intersection of the lines  $\ell_i$  and  $\ell_j$ . The following lemma describes the generators of  $I_{\mathbb{X}(l)}$ ; we will exploit this fact throughout the paper.

**Lemma 2.2.** *Let  $l \geq 2$ , and let  $\mathbb{X}(l)$  denote the star configuration constructed from the lines  $\ell_1, \dots, \ell_l$ . If  $L_i$  is a linear form defining  $\ell_i$  for  $i = 1, \dots, l$ , then*

$$I_{\mathbb{X}(l)} = (\hat{L}_1, \dots, \hat{L}_l) \text{ where } \hat{L}_i = \prod_{j \neq i} L_j.$$

For a proof of this fact, see [2, Claim in Proposition 3.4]. From this description of the generators, we see that  $I_{\mathbb{X}(l)}$  is generated in degree  $l - 1$ . Thus, for  $d < l - 1$ , there are no plane curves of degree  $d$  that contain a  $\mathbb{X}(l)$  since  $(I_{\mathbb{X}(l)})_d = (0)$ . Hence, as a corollary, we get some partial information about Question 1.1.

**Corollary 2.3.** *If  $d < l - 1$ , then the generic degree  $d$  curve does not contain a  $\mathbb{X}(l)$ .*

It is useful to recall that star configuration points have the same Hilbert function as generic points. More precisely, by [8, Lemma 7.8], we have

**Lemma 2.4.** *Let  $HF(\mathbb{X}(l), t) = \dim_{\mathbb{C}}(S/I_{\mathbb{X}(l)})_t$  denote the Hilbert function of  $S/I_{\mathbb{X}(l)}$ . Then*

$$HF(\mathbb{X}(l), t) = \min \left\{ \binom{t+2}{2}, \binom{l}{2} \right\} \text{ for all } t \geq 0.$$

## 3. AN ASYMPTOTIC RESULT

In [3] it is shown that the generic degree  $d$  plane curve contains a 0-dimensional complete intersection scheme of type  $(a, b)$  whenever  $a, b \leq d$  (actually the result holds for *any* degree  $d$  plane curve). Thus, arbitrarily large complete intersections can be found on generic plane curves of degree high enough. But the same does not hold for star configurations as shown by Theorem 3.1.

Using the previous description of  $I_{\mathbb{X}(l)}$  we introduce a quasi-projective variety parametrizing star configurations. Namely, we consider

$$\mathcal{D}_l \subset \underbrace{\check{\mathbb{P}}^2 \times \cdots \times \check{\mathbb{P}}^2}_l$$

such that  $(\ell_1, \dots, \ell_l) \in \mathcal{D}_l$  if and only if no three of the lines  $\ell_i$  are passing through the same point; here,  $\check{\mathbb{P}}^2$  denotes the dual projective space.

**Theorem 3.1.** *If  $l > 5$  is an integer, then the generic degree  $d$  plane curve does not contain any star configuration  $\mathbb{X}(l)$ .*

*Proof.* It is enough to consider the case  $d \geq l - 1$  (see Corollary 2.3). Let  $\mathbb{P}S_d$  be the space parametrizing degree  $d$  plane curves and define the following incidence correspondence

$$\Sigma_{d,l} = \{(\mathcal{C}, \mathbb{X}(l)) : \mathcal{C} \supset \mathbb{X}(l)\} \subset \mathbb{P}S_d \times \mathcal{D}_l.$$

We also consider the natural projection maps

$$\psi_{d,l} : \Sigma_{d,l} \longrightarrow \mathcal{D}_l \text{ and } \phi_{d,l} : \Sigma_{d,l} \longrightarrow \mathbb{P}S_d.$$

Clearly we have that  $\phi_{d,l}$  is dominant if and only if Question 1.1 has an affirmative answer.

Using a standard fiber dimension argument, we see that

$$\dim \Sigma_{d,l} \leq \dim \mathcal{D}_l + \dim \psi_{d,l}^{-1}(\mathbb{X}(l))$$

for a generic star configuration  $\mathbb{X}(l)$ , where  $\dim \mathcal{D}_l = 2l$  and by Lemma 2.4

$$\dim \psi_{d,l}^{-1}(\mathbb{X}(l)) = \dim_{\mathbb{C}} (I_{\mathbb{X}(l)})_d - 1 = \binom{d+2}{2} - \binom{l}{2} - 1.$$

Hence the map  $\phi_{d,l}$  is dominant only if  $\dim \Sigma_{d,l} - \dim \mathbb{P}S_d \geq 0$  and this is equivalent to

$$2l - \binom{l}{2} = \frac{l(5-l)}{2} \geq 0.$$

Thus the result is proved. □

By Lemma 2.1 and the above result, we only need to treat the cases  $l = 4$  and  $5$ . We postpone these cases to first rephrase Question 1.1 into an equivalent algebraic question.

## 4. RESTATEMENT OF QUESTION 1.1

We derive some technical results, moving our Question 1.1 back and forth between questions in algebra and questions in geometry. First, we notice the following trivial fact:

**Lemma 4.1.** *Let  $\{F = 0\}$  be an equation of the degree  $d$  curve  $\mathcal{C} \subset \mathbb{P}^2$ . Then  $\mathcal{C}$  contains a star configuration  $\mathbb{X}(l)$  only if*

$$F = \sum_{i=1}^l M_i \hat{L}_i$$

where the forms  $M_i$  have degree  $d - l + 1$  and the forms  $\hat{L}_i$  are defined as  $\hat{L}_i = \prod_{j \neq i} L_j$  for some linear forms  $L_1, \dots, L_l$ .

Hence, it is natural to perform the following geometric construction. We define a map of affine varieties

$$\Phi_{d,l} : \underbrace{S_1 \times \cdots \times S_1}_l \times \underbrace{S_{d-l+1} \times \cdots \times S_{d-l+1}}_l \longrightarrow S_d$$

such that

$$\Phi_{d,l}(L_1, \dots, L_l, M_1, \dots, M_l) = \sum_{i=1}^l M_i \hat{L}_i$$

We then rephrase our problem in terms of the map  $\Phi_{d,l}$ :

**Lemma 4.2.** *Let  $d, l$  be nonnegative integers with  $d \geq l - 1$ . Then the following are equivalent:*

- (i) *Question 1.1 has an affirmative answer for  $d$  and  $l$ ;*
- (ii) *the map  $\Phi_{d,l}$  is a dominant map.*

*Proof.* Lemma 4.1 proves that (i) implies (ii). To prove the other direction, it is enough to show that for a generic form  $F$ , the fiber  $\Phi_{d,l}^{-1}(F)$  contains a set of  $l$  linear forms defining a star configuration. More precisely, define  $\Delta \subset S_1 \times \cdots \times S_1 \times S_{d-l+1} \times \cdots \times S_{d-l+1}$  as follows:

$$\Delta = \left\{ (L_1, \dots, L_l, M_1, \dots, M_l) \mid \begin{array}{l} \text{there exists } a \neq b \neq c \text{ such that} \\ L_a, L_b, L_c \text{ are linearly dependent} \end{array} \right\}.$$

Then we want to show that  $\Phi_{d,l}^{-1}(F) \not\subset \Delta$ .

We proceed by contradiction, assuming that the generic fiber of  $\Phi_{d,l}$  is contained in  $\Delta$ . Then  $\Delta$  would be a component of the domain of  $\Phi_{d,l}$ . Thus a contradiction as the latter is an irreducible variety being the product of irreducible varieties.  $\square$

Using the map  $\Phi_{d,l}$  we can now translate Question 1.1 into an ideal theoretic question.

**Lemma 4.3.** *Let  $d, l$  be non-negative integers. Consider generic forms  $L_1, \dots, L_l \in S_1$  and  $M_1, \dots, M_l \in S_{d-l+1}$ . Set*

$$\hat{L}_i = \prod_{j \neq i} L_j \text{ and } \hat{L}_{i,j} = \prod_{h \neq i, h \neq j} L_h, \text{ for } i \neq j.$$

Also set

$$\begin{aligned} Q_1 &= M_2 \hat{L}_{1,2} + \cdots + M_l \hat{L}_{1,l} = \sum_{i \neq 1} M_i \hat{L}_{1,i}, \\ &\vdots \\ Q_l &= M_1 \hat{L}_{l,1} + \cdots + M_{l-1} \hat{L}_{l,l-1} = \sum_{i \neq l} M_i \hat{L}_{l,i}. \end{aligned}$$

With this notation, form the ideal

$$I = (\hat{L}_1, \dots, \hat{L}_l, Q_1, \dots, Q_l) \subset S.$$

Then the following are equivalent:

- (i) Question 1.1 has an affirmative answer for  $d$  and  $l$ ;
- (ii)  $I_d = S_d$ .

*Proof.* Using Lemma 4.2 we just have to show that  $\Phi_{d,l}$  is a dominant map. In order to do this we will determine the tangent space to the image of  $\Phi_{d,l}$  in a generic point  $q = \Phi_{d,l}(p)$ , where  $p = (L_1, \dots, L_l, M_1, \dots, M_l)$  and we denote with  $T_q$  this affine tangent space.

The elements of the tangent space  $T_q$  are obtained as

$$\left. \frac{d}{dt} \right|_{t=0} \Phi_{d,l}(L_1 + tL'_1, \dots, L_l + tL'_l, M_1 + tM'_1, \dots, M_l + tM'_l)$$

when we vary the forms  $L'_i \in S_1$  and  $M'_i \in S_{d-l+1}$ . By a direct computation we see that the elements of  $T_q$  have the form

$$\begin{aligned} &M'_1 \hat{L}_1 + \cdots + M'_l \hat{L}_l + L'_1(M_2 \hat{L}_{1,2} + \cdots + M_l \hat{L}_{1,l}) + \cdots + \\ &+ L'_j(M_1 \hat{L}_{1,j} + \cdots + M_l \hat{L}_{j,l}) + \cdots + L'_l(M_1 \hat{L}_{1,2} + \cdots + M_{l-1} \hat{L}_{1,l-1}), \end{aligned}$$

where  $\hat{L}_i = \prod_{j \neq i} L_j$  and  $\hat{L}_{i,j} = \prod_{h \neq i, h \neq j} L_h$ , for  $i \neq j$ .

Since the  $L'_i \in S_1$  and  $M'_i \in S_{d-l+1}$  can be chosen freely, we obtain that  $I_d = T_q$ .  $\square$

*Remark 4.4.* Lemma 4.3 is an effective tool to give a positive answer to each special issue of Question 1.1. Given  $d$  and  $l$  we construct the ideal  $I$  as described by choosing forms  $L_i$  and  $M_i$ . Then we compute  $\dim_{\mathbb{C}} I_d$  using a computer algebra system, e.g., CoCoA. If  $\dim_{\mathbb{C}} I_d = \dim_{\mathbb{C}} S_d$ , by upper semi-continuity of the dimension, we have proved that our question has an affirmative answer for these given  $d$  and  $l$ . In fact, the dimension can decrease only on a proper closed subset. On the contrary, if  $\dim_{\mathbb{C}} I_d < \dim_{\mathbb{C}} S_d$  we do not have any proof. Question 1.1 can both have a negative (for any choice of forms the inequality holds) or an affirmative answer (our choice of forms is too special and another choice will give an equality). But, if we choose our forms generic enough, we do have a strong indication that the answer to Question 1.1 should be negative.

**Example 4.5.** We use CoCoA [5] to consider the example  $(d, l) = (4, 4)$ .

```
--we define the ring we want to use
Use S:=QQ[x,y,z];
--we choose our forms
L1:=x;
```

```

L2:=y;
L3:=z;
L4:=x+y+z;
M1:=x+y-z;
M2:=-x+2y+2z;
M3:=2x-y-z;
M4:=x+y+2z;
--we build the forms Qi
Q1:=M2*L3*L4+M3*L2*L4+M4*L2*L3;
Q2:=M1*L3*L4+M3*L1*L4+M4*L1*L3;
Q3:=M1*L3*L4+M2*L1*L4+M4*L1*L2;
Q4:=M1*L2*L3+M2*L1*L3+M3*L1*L2;
--we define the ideal I
I:=Ideal(L2*L3*L4,L1*L3*L4,L1*L2*L4,L1*L2*L3,Q1,Q2,Q3,Q4);
--we compute the Hilbert function of S/I in degree 4
Hilbert(S/I,4);

```

Evaluating the code above we get

```

Hilbert(S/I,4);
0
-----

```

and this means that  $\dim_{\mathbb{C}} S_4 - \dim_{\mathbb{C}} I_4 = 0$ . Hence we showed that the generic plane quartic contains a  $\mathbb{X}(4)$ .

## 5. THE $l = 4$ CASE

In this section we consider Question 1.1 when  $l = 4$ . We begin with a special instance of Question 1.1, that is, when  $d = 3$ , since it has a nice geometric proof which takes advantage of the group structure on the curve.

**Lemma 5.1.** *Let  $\mathcal{C} \subset \mathbb{P}^2$  be a smooth cubic curve. Then there exists a star configuration  $\mathbb{X}(4) \subset \mathcal{C}$ .*

*Proof.* We will use the group law on  $\mathcal{C}$  and hence we fix a point  $p_0 \in \mathcal{C}$  serving as identity. Then we choose a point  $p_2 \in \mathcal{C}$  such that  $2p_2 = p_0$  and a generic point  $p_1 \in \mathcal{C}$ . Consider the line joining  $p_1$  and  $p_2$  and let  $p_3$  be the third intersection point with  $\mathcal{C}$ . Joining  $p_3$  and  $p_0$  and taking the third intersection we get the point  $p_1 + p_2$ . Now join  $p_1 + p_2$  and  $p_2$  and let  $p_4$  be the third intersection point. Then joining  $p_4$  and  $p_0$  we get  $p_1 + 2p_2 = p_1$ . Hence, the points  $p_0, p_1, p_2, p_1 + p_2, p_3, p_4 \in \mathcal{C}$  are a star configuration.  $\square$

We now consider the general situation:

**Theorem 5.2.** *Let  $\mathcal{C} \subset \mathbb{P}^2$  be a generic degree  $d \geq 3$  plane curve. Then there exists a star configuration  $\mathbb{X}(4) \subset \mathcal{C}$ .*

*Proof.* The case  $d = 3$  is treated in Lemma 5.1, while for  $d > 3$  we will use Lemma 4.3 and the notation introduced therein. For  $d = 4$ , we produced an explicit example in Example 4.5 where  $I_4 = S_4$ , and this is enough to conclude by semi-continuity.



To deal with the general case  $d \geq 5$ , we use the structure of the coordinate ring of a star configuration. Given four generic linear forms  $L_1, L_2, L_3, L_4$  we consider the star configuration they define, say  $\mathbb{X} = \{p_{i,j} : 1 \leq i < j \leq 4\}$ . The coordinate ring of  $\mathbb{X}$  is

$$A = \frac{S}{(\hat{L}_1, \dots, \hat{L}_4)}.$$

When  $d \geq 2$ ,  $\dim_{\mathbb{C}} A_d = 6$ . To prove that  $I_d = S_d$ , we want to find linear forms  $N_1, \dots, N_6$  such that the six forms  $N_i Q_i$  are linearly independent in  $A_d$ . To check whether elements in  $A$  are linearly independent it is enough to consider their evaluations at the points  $p_{i,j}$ . Consider the following evaluation matrix

$$(5.1) \quad \begin{array}{c|cccc} & Q_1 & Q_2 & Q_3 & Q_4 \\ \hline p_{1,2} & M_2 L_3 L_4 & M_1 L_3 L_4 & 0 & 0 \\ p_{1,3} & M_3 L_2 L_4 & 0 & M_1 L_2 L_4 & 0 \\ p_{1,4} & M_4 L_2 L_3 & 0 & 0 & M_1 L_2 L_3 \\ p_{2,3} & 0 & M_3 L_1 L_4 & M_2 L_1 L_4 & 0 \\ p_{2,4} & 0 & M_4 L_1 L_3 & 0 & M_2 L_1 L_3 \\ p_{3,4} & 0 & 0 & M_4 L_1 L_2 & M_3 L_1 L_2 \end{array}$$

obtained by evaluating each  $Q_i$  at the points of the configuration where we denote, by abuse of notation,  $M_h L_i L_j(p_{m,n})$  with  $M_h L_i L_j$ . For example, since  $Q_2 = M_1 L_3 L_4 + M_3 L_1 L_4 + M_4 L_1 L_3$ ,  $Q_2(p_{2,3}) = M_3 L_1 L_4(p_{2,3})$  because  $L_3$  vanishes at  $p_{2,3}$ , while  $Q_2(p_{1,4}) = 0$  since  $L_4$  and  $L_1$  vanish at  $p_{1,4}$ . Now choose the forms  $M_i$  with  $\deg M_i = d - 3 \geq 2$  so that

$$M_2(p_{2,3}) = M_3(p_{3,4}) = M_4(p_{2,4}) = 0$$

and no other vanishing occurs at the points of the star configuration. Notice that this is possible as  $\dim_{\mathbb{C}} A_d = 6$  for  $d \geq 2$ . The matrix (5.1) can be represented as

$$(5.2) \quad \begin{array}{c|cccc} & Q_1 & Q_2 & Q_3 & Q_4 \\ \hline p_{1,2} & * & * & 0 & 0 \\ p_{1,3} & * & 0 & * & 0 \\ p_{1,4} & * & 0 & 0 & * \\ p_{2,3} & 0 & * & 0 & 0 \\ p_{2,4} & 0 & 0 & 0 & * \\ p_{3,4} & 0 & 0 & * & 0 \end{array}$$

where  $*$  denotes a non-zero scalar. As this matrix has rank four, then the forms  $Q_i$  are linearly independent in  $A$ .

To represent  $N_i Q_i$  in  $A$  it is enough to multiply the  $j$ -th element of the  $i$ -th column of (5.1) by the evaluation of  $N_i$  at  $p_{j,i}$ . Hence, if we choose the forms  $N_i$  such that they do not vanish at any point of  $\mathbb{X}$ , the evaluation matrix of the  $N_i Q_i$  has the same non-zero pattern as (5.2). Thus the forms  $N_1 Q_1, \dots, N_4 Q_4$  are linearly independent in  $A$ .

To complete the proof we need two more forms, and we choose  $L_1Q_3$  and  $L_1Q_2$  whose evaluation matrix at the points  $p_{i,j}$  is

$$(5.3) \quad \begin{array}{c|cccccc} & p_{1,2} & p_{1,3} & p_{1,4} & p_{2,3} & p_{2,4} & p_{3,4} \\ \hline L_1Q_3 & 0 & 0 & 0 & 0 & 0 & * \\ L_1Q_2 & 0 & 0 & 0 & * & 0 & 0 \end{array}$$

These rows are linearly independent with the columns of (5.2), completing the proof.  $\square$

## 6. THE $l = 5$ CASE

It remains to consider the case that  $l = 5$ . The case  $l = 5$  and  $d = 4$  was classically studied. A quartic containing a star configuration of ten points, that is, a  $\mathbb{X}(5)$ . For more details, see [12, 13], and for a modern treatment we refer to [14]. In particular, we require the following property of Lüroth quartics:

**Theorem** (Theorem 11.4 of [14]). *Lüroth quartics form a hypersurface of degree 54 in the space of plane quartics.*

**Corollary 6.1.** *Question 1.1 has a negative answer for  $(d, l) = (4, 5)$ .*

*Proof.* We notice that the previous theorem is enough to give a negative answer to our question for  $l = 5$  and  $d = 4$ . In fact, a generic plane quartic is not a Lüroth quartic, and hence, no star configuration  $\mathbb{X}(5)$  can be found on it.  $\square$

For the remaining values  $d > 4$ , we give an affirmative answer to Question 1.1:

**Theorem 6.2.** *The generic degree  $d > 4$  plane curve contains a star configuration  $\mathbb{X}(5)$ .*

*Proof.* For the case  $d = 5$  we produce an explicit example and then we conclude by semi-continuity. For the general case, we produce a proof similar to Theorem 5.2.

For the  $d = 5$  case, let our linear forms be given by

$$L_1 = x, \quad L_2 = y, \quad L_3 = z, \quad L_4 = x + y + z, \quad \text{and} \quad L_5 = 2x - 3y + 5z.$$

To construct the polynomials  $Q_1, \dots, Q_5$ , we make use of the following forms:

$$M_1 = x+y-z, \quad M_2 = -x+2y+2z, \quad M_3 = 2x-y-z, \quad M_4 = 3x+y-z, \quad \text{and} \quad M_5 = 4x-4y+3z.$$

Form the ideal  $I$  as in Lemma 4.3. Using CoCoA [5] to compute  $I_5$ , we find that  $I_5 = S_5$ .

For the  $d \geq 6$  case, we use the notation of Lemma 4.3, and fix linear forms  $L_1, \dots, L_5$ , and use  $p_{i,j}$  to denote the ten points of  $\mathbb{X}(5)$  defined by the linear forms  $L_i$ . We construct the evaluation table



where  $*$  denotes a non-zero scalar. One can verify that (6.3) has rank ten and hence the forms in (6.2) are linearly independent in  $A$  and the thesis follows.  $\square$

We can now prove our main theorem.

**Theorem 6.3.** *Let  $l \geq 2$ . Then the generic degree  $d$  plane curve contains a star configuration  $\mathbb{X}(l)$  if and only if*

- (i)  $l = 2$  and  $d \geq 1$ , or
- (ii)  $l = 3$  and  $d \geq 2$ , or
- (iii)  $l = 4$  and  $d \geq 3$ , or
- (iv)  $l = 5$  and  $d \geq 5$ .

*Proof.* Combine Lemmas 2.1 and 5.1, Corollaries 2.3 and 6.1, and Theorems 3.1, 5.2, and 6.2.  $\square$

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