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Robust Model Predictive Control via Random Convex Programming

G. C. Calafiore*, L. Fagiano*,**

***Abstract*— This paper proposes a new approach to design a robust model predictive control (MPC) algorithm for LTI discrete time systems. By using a randomization technique, the optimal control problem embedded in the MPC scheme is solved for a finite number of realizations of model uncertainty and additive disturbances. Theoretical results in random convex programming (RCP) are used to show that the designed controller achieves asymptotic closed loop stability and constraint satisfaction, with a guaranteed level of probability. The latter can be tuned by the designer to achieve a tradeoff between robustness and computational complexity. The resulting Randomized MPC (RMPC) technique requires quite mild assumptions on the characterization of the uncertainty and disturbances and it involves a convex optimization problem to be solved at each time step. The technique is applied here to a case study of an electro-mechanical positioning system.**

I. Introduction

Model Predictive Control (MPC) is a model based control technique which is receiving an ever-increasing attention, mainly thanks to its capability of handling in an effective way the presence of input and state constraints. In MPC, the control input $u_t \in \mathbb{R}^m$ is computed by solving, at each sampling time t, a constrained finite horizon optimal control problem (FHOCP), according to a receding horizon (RH) strategy [1]. A still very active research direction is the study of robust MPC approaches, that are able to guarantee stability and constraint satisfaction also in the presence of model uncertainty and external disturbances. In the last decade, a quite extensive literature has been developed, both in the case of linear time invariant (LTI) models and in the case of nonlinear models [1], [2]. As regards LTI models, uncertainty is typically taken into account by assuming that the system matrices $A(\theta)$, $B(\theta)$ depend on some unknown but bounded parameter vector $\theta \in \Theta \subset \mathbb{R}^g$ and belong to a bounded set Σ, and disturbances are modeled by an external input $\gamma_t \in \Gamma \subset \mathbb{R}^{m_\gamma}$ that affects the system dynamics through an input matrix $B_\gamma(\theta)$. Then, one of the most common approaches for robust MPC design aims to minimize the worst-case performance, while satisfying input and state constraints for all possible realizations of θ and γ_t. In this approach, a *deterministic* min-max optimization problem has to be solved on-line and a tractable (i.e. convex) solution can be obtained at the price of conservativeness, since typically an upper bound of the worst-case cost is minimized [3], [4], [5], [6]. The problem of robust MPC in the presence of additive bounded disturbances has been also studied, see e.g. [7], [8], [9], [10], [11] and the references therein, often assuming a convex set Γ and exact knowledge of the system A, B matrices. Some of these approaches minimize a nominal cost function (i.e. computed in absence of disturbance) and rely on constraint tightening to robustly enforce state and input constraint satisfaction [8]. Yet, the assumption of exact knowledge of the system matrices and the polytopic characterization of the disturbance set Γ may limit the applicability of these approaches. Finally, some works in the literature address the robust MPC design in the presence of both system uncertainty and additive disturbances [5], assuming convex/polytopic sets Σ and Γ so to achieve a convex formulation of the problem. In the last years, stochastic MPC techniques have been also developed (see e.g. [12], [13], [14] and the references therein). By assuming some known statistical description of the uncertain parameters and/or of the additive disturbance, these techniques employ a *deterministic* algorithm to address the robust MPC design problem. This way, the cost function becomes a random variable and the MPC algorithm aims to minimize its expectation.

In this paper, we address the design of robust MPC laws for LTI systems subject to both model uncertainty and additive disturbances, and we propose a new approach, named Random MPC (RMPC), that represents a shift of perspective, from a *deterministic* MPC algorithm to a *randomized* one, i.e. an algorithm that relies on some random choices. It has to be noted that a randomized approach for MPC has been studied also in [15], where the performance to be optimized is the expectation of the cost function, which is computed by using a Monte Carlo technique. However, the approach of [15] may result to be very computationally demanding and it can not handle in a straightforward way the presence of state constraints. On the contrary, the RMPC approach proposed here exploits results in Random Convex Programming (RCP) [16], [17], [18], [19] to design a control law that guarantees robust closed loop stability and constraint satisfaction to a given guaranteed level p of probability. Thus, the robust MPC problem is solved with the RMPC technique not in expectation, but in a probabilistically robust way. We prove here the stability and constraint satisfaction properties of RMPC, and we apply this new technique to a case-study of an electro-mechanical positioning system.

II. Problem Settings

Consider the following uncertain LTI system model:

$$x_{t+1} = A(\theta)x_t + B(\theta)u_t + B_\gamma(\theta)\gamma_t \tag{1}$$

where $t \in \mathbb{Z}$ is the discrete time variable, $x_t \in \mathbb{R}^n$ is the system state, $u_t \in \mathbb{R}^m$ is the control input, $\gamma_t \in \Gamma \subset \mathbb{R}^{m_\gamma}$

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*Politecnico di Torino, Dipartimento di Automatica e Informatica, Corso Duca degli Abruzzi 24 – 10129 Torino – Italy

**University of California at Santa Barbara, Department of Mechanical Engineering, Engineering II bldg. – 93106-5070 Santa Barbara – California

e–mail addresses: giuseppe.calafiore@polito.it, lorenzo.fagiano@polito.it

is an unmeasured disturbance vector, $\theta \in \Theta \subseteq \mathbb{R}^g$ is the vector of uncertain parameters, and $A(\theta)$, $B(\theta)$, $B_\gamma(\theta)$ are matrices of suitable dimensions. Let us make the following assumptions:

Assumption 1: The set Γ is bounded and contains the origin. Moreover, the set $\Sigma \doteq \{A(\theta),\, B(\theta),\, B_\gamma(\theta) : \theta \in \Theta\}$ is bounded. We assume γ and θ to have stochastic nature, and let $\mathbb{P}_\theta$ denote the probability measure on Θ, and $\mathbb{P}_\gamma$ the probability measure on Γ. Variables θ and γ are independent. ■

Assumption 2: The pair $A(\theta), B(\theta)$ is stabilizable for any $\theta \in \Theta$. ■

The control problem is to design a control law $\kappa(x)$ that is able to regulate the system state to the origin, subject to the following (possibly uncertain) input and output constraints:

$$x_t \in \mathbb{X}(\theta),\ u_t \in \mathbb{U}(\theta),\ \forall t \tag{2}$$

The next assumption characterizes the constraint sets:

Assumption 3: $\mathbb{X}(\theta) \subset \mathbb{R}^n$ and $\mathbb{U}(\theta) \subset \mathbb{R}^m$ are convex sets with respect to x and u, respectively, they contain the origin in their interiors and they are represented by:

$$\begin{array}{rcl} \mathbb{X}(\theta) & = & \{x \in \mathbb{R}^n : f_X(x,\theta) \preceq 0\} \\ \mathbb{U}(\theta) & = & \{u \in \mathbb{R}^m : f_U(u,\theta) \preceq 0\}, \end{array} \tag{3}$$

where $\preceq$ denotes element-wise inequalities, $f_X : \mathbb{R}^n \times \Theta \to \mathbb{R}^r$, $f_U : \mathbb{R}^m \times \Theta \to \mathbb{R}^q$ are convex functions with respect to x and u, respectively, and r, q are suitable integers. ■

Due to the presence of the generally non-zero unmeasured disturbance γ_t, regulation of the system state to the equilibrium $\overline{x} = 0$, $\overline{u} = 0$ can not be attained. Rather, we can require regulation to a neighborhood of the origin, described by a terminal set. The latter is formally characterized by the next definition.

Definition 2.1: (Robust positively invariant terminal set and terminal control law) A convex compact set $\mathbb{X}_f$ is said to be a robust positively invariant terminal set for system (1), or simply terminal set, if it contains the origin and

$$\begin{array}{r} \exists \kappa_f(x) : A(\theta)x + B(\theta)\kappa_f(x) + B_\gamma(\theta)\gamma \in \mathbb{X}_f, \\ \forall \theta \in \Theta,\ \forall \gamma \in \Gamma,\ \forall x \in \mathbb{X}_f \end{array}$$

A control law $\kappa_f(x)$ such that a set $\mathbb{X}_f$ is robustly positively invariant is called a terminal control law. We assume that

$$\mathbb{X}_f = \{x : f_{X_f}(x) \preceq 0\},$$

for some convex function $f_{X_f} : \mathbb{R}^n \to \mathbb{R}^l$, where l is a suitable integer. ■

The set $\mathbb{X}_f$ represents a limit to the regulation precision (i.e. distance from the origin) achievable by the control law. Thus, in order to obtain good regulation precision, the set $\mathbb{X}_f$ should be as small as possible. To this regard, there is a number of contributions in the literature, concerned with the computation of approximations of the (minimal) robust positively invariant terminal set $\mathbb{X}_f$, see e.g. [20] and the references therein. We consider the following assumption on the terminal set $\mathbb{X}_f$ and the related terminal control law $\kappa_f(x)$.

Assumption 4: A robust positively invariant terminal set $\mathbb{X}_f$ and an associated static linear terminal control law $\kappa_f(x) = K_f\, x$, $K_f \in \mathbb{R}^{m\times n}$, exist for system (1) and are known, moreover $\mathbb{X}_f \subseteq \mathbb{X}(\theta)$ and $\kappa_f(x) \in \mathbb{U}(\theta), \forall x \in \mathbb{X}_f$, $\forall \theta \in \Theta$. ■

Assumptions 1 and 2 are indeed necessary for Assumption 4 to hold.

A. Finite-horizon control problem and deterministic robust MPC

Let $N \in \mathbb{N}$ be a finite control horizon, chosen by the control designer, let $t \geq 0$ be the current time instant and let x_t be the system state observed at time t. We consider the predicted evolution of (1) for N steps forward, under a control law entirely determined at the current time t:

$$u_{t+j|t} \doteq K_f x_{t+j|t} + v_{j|t},\ j = 0, \ldots, N-1, \tag{4}$$

where $x_{t|t} = x_t$ and, for $j = 1, \ldots, N$,

$$x_{t+j|t} = A_{cl}(\theta)x_{t+j-1|t} + B(\theta)v_{j-1|t} + B_\gamma(\theta)\gamma_{j-1},$$

$A_{cl}(\theta) = A(\theta) + B(\theta)K_f$, and γ_j, $v_{j|t}$, $j = 0, \ldots, N-1$, are, respectively, a sequence of independent random variables identically distributed according to $\mathbb{P}_\gamma$, and the corrective control sequence computed at t. Prediction of the state trajectories in a closed-loop fashion is quite common in the context of robust MPC, since it allows significant improvements in feasibility, see e.g. [7], [21]. The previous recursion implies that

$$x_{t+j|t} = A_{cl}^j(\theta)x_t + \Phi_j(\theta)\mathcal{V}_t + \Upsilon_j(\theta)g, \tag{5}$$

where

$$\begin{array}{rcl} \Phi_j(\theta) & = & [A_{cl}^{j-1}(\theta)B(\theta)\ \cdots\ A_{cl}(\theta)B(\theta)\ B(\theta)\ 0\ \cdots\ 0], \\ \Upsilon_j(\theta) & = & [A_{cl}^{j-1}(\theta)B_\gamma(\theta)\ \cdots\ A_{cl}(\theta)B_\gamma(\theta)\ B_\gamma(\theta)\ 0\ \cdots\ 0], \\ \mathcal{V}_t & = & [v_{0|t}^\top\ \cdots\ v_{N-1|t}^\top]^\top \in \mathbb{R}^{Nm}, \\ g & = & [\gamma_0^\top\ \cdots\ \gamma_{N-1}^\top]^\top \in \Gamma^N \subset \mathbb{R}^{Nm_\gamma}. \end{array}$$

The uncertainty in the problem is due to θ (the plant parametric uncertainty) and g (the disturbance sequence), which we collect in the vector δ:

$$\delta \doteq (\theta, g) \in \Delta, \quad \Delta \doteq \Theta \times \Gamma^N.$$

The following (stochastic) cost function can now be defined:

$$J(x_t, \mathcal{V}_t, \delta) \doteq \sum_{j=0}^{N-1} d(x_{t+j|t}, \mathbb{X}_f) + \sum_{j=0}^{N-1} v_{j|t}^T R v_{j|t}, \tag{6}$$

where $d(x, \mathbb{X}_f)$ is the distance between x and the terminal set $\mathbb{X}_f$, computed in some norm $\|\cdot\|$:

$$d(x, \mathbb{X}_f) \doteq \min_{y \in \mathbb{X}_f} \|x - y\|,$$

and R is a symmetric positive definite matrix chosen by the control designer.

As a consequence of Assumption 1 and of the fact that $\gamma_0, \ldots, \gamma_{N-1}$ is an i.i.d. sequence, we have that the random quantity δ has a probability measure that we denote by $\mathbb{P}$, which is the product measure of $\mathbb{P}_\theta$ and the product measure $\mathbb{P}_\gamma^N$ (the measure over Γ^N):

$$\mathbb{P} = \mathbb{P}_\theta \times \mathbb{P}_\gamma^N.$$

The support of $\mathbb{P}$ is the set $\Theta \times \Gamma^N$. Since $A(\theta)$, $B(\theta)$, $B_\gamma(\theta)$ and g belong to bounded sets, one may at first disregard probabilistic information, and consider a deterministic design problem where the worst-case cost is minimized under robust constraints.

Definition 2.2: (Deterministic robust FHOCP) The deterministic robust FHOCP is:

$$\mathcal{P}(x_t) : \qquad \min_{\mathcal{V}_t} \max_{\delta\in\Delta} \; J(x_t, \mathcal{V}_t, \delta) \tag{7}$$

subject to

$$x_{t+j|t} \in \mathbb{X}(\delta),\ j = 1, \ldots, N-1,\ \forall \delta \in \Delta \tag{8}$$
$$u_{t+j|t} \in \mathbb{U}(\delta),\ j = 0, \ldots, N-1,\ \forall \delta \in \Delta \tag{9}$$
$$x_{t+N|t} \in \mathbb{X}_f,\ \forall \delta \in \Delta \tag{10}$$

In the deterministic robust MPC approach, problem $\mathcal{P}(x_t)$ is solved in a receding horizon fashion:

Algorithm 2.1: **Deterministic robust MPC algorithm**

1) At time t, observe x_t.
2) Solve the problem $\mathcal{P}(x_t)$. Let $(\mathcal{V}_t^*, z_t^*)$ be a solution.
3) Apply the control input $u_t^* = K_f\, x_t + v_{0|t}^*$, set $t = t+1$ and go to 1).

From a theoretical point of view, the application of Algorithm 2.1 yields a feedback control law with quite strong stabilizing properties, however in most practical cases the described deterministic robust MPC scheme can not be used. In fact, finding a solution to problem $\mathcal{P}(x_t)$ is not an easy task in general, because the constraints are typically semi-infinite and the dependence of the constraints on δ is generic and nonlinear.

We next write the optimization problem $\mathcal{P}(x_t)$ in a compact standard form. To this end, notice that, by introducing a slack variable z_t, we can use a linear objective z_t in the problem and add a constraint

$$J(x_t, \mathcal{V}_t, \delta) - z_t \le 0,\ \forall \delta \in \Delta. \tag{11}$$

By collecting the optimization variables $(\mathcal{V}_t, z_t)$ in vector $s_t \in \mathbb{R}^{mN+1}$, the cost z_t can be expressed as $z_t = c^\top s_t$, where $c = [0, \ldots, 0, 1]^\top$. Moreover, it can be noted that, for any fixed value of $\delta \in \Delta$, due to linearity of (4) and (5), the constraints (8)-(11) are convex in the decision variable s_t. Moreover, these constraints can be formally expressed compactly as $h(s_t, x_t, \delta) \le 0$, $\forall \delta \in \Delta$, where $h : \mathbb{R}^{mN+1} \times \mathbb{R}^n \times \Delta \to \mathbb{R}$ is a convex function in s_t, defined as follows

$$h(s_t, x_t, \delta) = \max_{j=0,\ldots,N-1} \max \big\{ f_X(x_{t+j|t}, \delta), f_U(u_{t+j|t}, \delta), f_{X_f}(x_{t+N|t}), J(x_t, \mathcal{V}_t, \delta) - z_t \big\}\,.$$

The deterministic robust FHOCP can hence be rewritten as

$$\mathcal{P}(x_t) : \qquad \min_{\mathcal{V}_t, z_t} c^\top s_t \tag{12}$$
$$\text{subject to} \quad h(s_t, x_t, \delta) \le 0,\ \forall \delta \in \Delta. \tag{13}$$

B. The scenario optimization problem

Suppose now we extract M i.i.d. random samples of the uncertainty $\delta \in \Delta$ according to $\mathbb{P}$, and collect the samples in the multi-sample $\omega = \{\delta^{(1)}, \ldots, \delta^{(M)}\}$. We consider the following *scenario* counterpart of the deterministic FHOCP, where instead of the possibly infinite set of constraints parameterized by $\delta \in \Delta$, we use a finite number M of randomly extracted scenarios.

Definition 2.3: (Scenario FHOCP) The convex optimization problem:

$$\mathcal{P}_M(x_t, \omega) : \qquad \min_{s_t} c^\top s_t \tag{14}$$
$$\text{subject to} \quad h(s_t, x_t, \delta^{(i)}) \le 0,\ i = 1, \ldots, M \tag{15}$$

is named the scenario FHOCP. We adopt the standard convention that the optimal objective value is $+\infty$ when the problem in unfeasible. ■

The solution $s_t^*(x_t, \omega)$ of $\mathcal{P}_M(x_t, \omega)$ contains the sequence of optimal control actions

$$\mathcal{V}_t^*(x_t, \omega) = \{v_{0|t}^*, \ldots, v_{N-1|t}^*\},$$

and the optimal cost $z_t^* = c^\top s_t^*(x_t, \omega)$. The following assumption is made on the scenario problem.

Assumption 5: There exists a feasibility set $\mathcal{F} \subseteq \mathbb{R}^n$ such that, for any $M > Nm + 1$ and for any $x \in \mathcal{F}$, the scenario problem $\mathcal{P}_M(x, \omega)$ admits a unique optimal solution with probability one. ■

Remark 2.1: Note that, according to Assumptions 4 and 5, it holds that $\mathbb{X}_f \subseteq \mathcal{F}$ and that the optimal solution of $\mathcal{P}_M(x_t, \omega)$, for $x_t \in \mathbb{X}_f$ is

$$\mathcal{V}_t^* = 0, \quad z^*(x_t) = 0,$$

independently of ω. Moreover, the set $\mathcal{F}$ coincides with the set of state values for which the deterministic robust FHOCP (7)-(10) is feasible.

III. Robust MPC: the Random Convex Programming approach

The RMPC algorithm is now introduced, and its main properties are derived. The main idea underlying RMPC is that a target guaranteed probability p of achieving closed loop stability is chosen by the designer, and the problem $\mathcal{P}_M(x_t, \omega)$ is then solved with a sufficiently high number M of randomly extracted scenarios (precisely how large M should be is specified next), so that the optimal solution has indeed at least a probability p of driving the state to the terminal set. Moreover, we solve problem $\mathcal{P}_M(x_t, \omega)$ according to a receding horizon scheme, in order to take advantage of the observation of the actual system state at each time step.

Note first that events related to ω are measured by the product probability $\mathbb{P}^M$ over Δ^M, hence also events related to $s_t^*(x_t, \omega)$ are measured by $\mathbb{P}^M$. Then, since $\delta \in \Delta$ has distribution $\mathbb{P}$ and it is independent of ω, events related to $h(s_t^*, x_t, \delta)$ are measured by $\mathbb{P}^{M+1}$. Now, problem $\mathcal{P}_M(x_t, \omega)$ belongs to the class of so-called Random Convex Programs (RCP) (see e.g. [16], [17], [18], [19], [22]) and, in particular, the results of [22] apply to this problem. We now restate these results in our context. Define the so-called expected constraint violation probability:

$$V \doteq \mathbb{P}^{M+1}\{(\omega, \delta) \in \Delta^{M+1} : h(s_t^*(\omega), x_t, \delta) > 0\}. \tag{16}$$

V is the probability with which the optimal solution s_t^*, computed on the basis of M sampled scenarios in ω, actually violates a constraint on a generic random $\delta \in \Delta$. The following key result holds, see Theorem 2.1 in [22].

Theorem 3.1: Under Assumption 5, if $x_t \in \mathcal{F}$, then it holds that

$$V \leq \frac{mN+1}{M+1}.$$

An obvious consequence of this theorem is that, for given $p \in (0,1)$, the condition

$$M \geq \frac{mN+1}{1-p} - 1 \tag{17}$$

implies that $V \leq 1-p$. Equivalently, condition (17) implies that $1 - V \geq p$, that is

$$\mathbb{P}^{M+1}\{(\omega,\delta) \in \Delta^{M+1} : h(s_t^*(\omega), x_t, \delta) \leq 0\} \geq p. \tag{18}$$

We can now state the following theorem.

Theorem 3.2: (Finite horizon robustness) Let Assumption 5 be satisfied, and let $x_t \in \mathcal{F}$. Given $p \in (0,1)$, let M be an integer satisfying (17), and let $s_t^* = (\mathcal{V}_t^*, z_t^*)$ be the optimal solution of problem $\mathcal{P}_M(x_t, \omega)$. Then, with probability no smaller than p, the corresponding optimal control sequence $u_{t+j}^* = K_f\, x_{t+j} + v_{j|t}^*$, $j = 0, \ldots, N-1$:

- **a)** steers the state of system (1) to the terminal set $\mathbb{X}_f$ in N steps; and
- **b)** satisfies the state constraints $f_X(x_{t+j}, \theta) \preceq 0$, $\forall j \in [1, N]$ and $f_U(u_{t+j}^*, \theta) \preceq 0$, $\forall j \in [0, N-1]$

The proof of this theorem follows immediately from Theorem 3.1: eq. (17) implies that, with probability at least p, the optimal solution s_t^* of the scenario problem satisfies $h(s_t^*(\omega), x_t, \delta) \leq 0$, which indeed implies that points (a) and (b) in the theorem hold with probability at least p.

A. Random MPC algorithm

We next describe a receding-horizon implementation of the scenario FHOCP.

Algorithm 3.1: **RMPC algorithm**

(Initialization) Given an initial state $x_0 \in \mathcal{F}$ of the system at time $t = 0$, extract a random multisample ω, solve problem $\mathcal{P}_M(x_0, \omega)$ and obtain the optimal control sequence

$$\mathcal{V}_0^* = \{v_{0|0}^*, v_{1|0}^* \ldots, v_{N-1|0}^*\},$$

and the optimal objective z_0^*. Apply control action $u_0^* = K_f x_0 + v_{0|0}^*$.

1) Let $t = t + 1$, observe x_t, and set

$$\begin{aligned}\tilde{\mathcal{V}}_t &= \{v_{1|t-1}^*, \ldots, v_{N-1|t-1}^*, 0\}\\ \tilde{z}_t &= \max\left(0, z_{t-1}^* - d(x_{t-1}, \mathbb{X}_f)\right);\end{aligned}$$

2) Consider problem $\mathcal{P}_M(x_t, \omega)$. If this problem is feasible, let $(\mathcal{V}_t^*, z_t^*)$ be its optimal solution, else let $z_t^* = +\infty$;

3) If $z_t^* > \tilde{z}_t$, then set

$$\mathcal{V}_t^* = \tilde{\mathcal{V}}_t; \quad z_t^* = \tilde{z}_t.$$

4) Apply the control input $u_t^* = K_f\, x_t + v_{0|t}^*$, go to 1).

The RMPC algorithm gives rise to a random feedback control law $u_t^* = \kappa(x_t, \omega) \doteq K_f x_t + v_{0|t}^*(x_t, \omega)$, which is in general nonlinear w.r.t. to both x_t and ω (for example, for a fixed value of ω, linear constraints (8)-(10) and quadratic cost (6), it can be shown that $\kappa(x, \omega)$ is a piecewise affine function of x, see e.g. [23]). Thus, the state equation of system (1) with the feedback law $\kappa(x_t, \omega)$ is:

$$x_{t+1} = A(\theta)x_t + B(\theta)\kappa(x_t, \omega) + \gamma_t, \tag{19}$$

i.e. the closed-loop system is an uncertain nonlinear system subject to the unknown additive disturbance γ_t and to the random parameter ω. Our interest is now focused on the properties of system (19) in terms of convergence of its trajectories to the terminal set $\mathbb{X}_f$ and of recursive state and input constraint satisfaction. These aspect are dealt with by the following result.

Theorem 3.3: (Random MPC) Let Assumptions 1-5 be satisfied, let u_t^* be the sequence of control inputs generated by Algorithm 3.1 for $t = 0, 1, \ldots$, and applied to system (1), and let $x(t)$ be the corresponding sequence of states. Then, for all $x_0 \in \mathcal{F}$:

- **(a)** with probability no smaller than p it holds that

$$\lim_{t \to \infty} d(x_t, \mathbb{X}_f) = 0;$$

- **(b)** at each time step $t > 0$, the state and input constraints $\{f_X(x_t, \theta) \preceq 0,\ f_U(u_t^*, \theta) \preceq 0\}$ are satisfied with probability no smaller than p.

Proof: See [24]. ■

IV. Application of RMPC to an electro-mechanical positioning system

Consider an electro-mechanical positioning system, in which an electric motor is linked to an input shaft. The latter is linked through a set of gears with an output shaft. τ is the continuous time variable, $V_m(\tau)$ is the input voltage for the electric motor, $\theta_i(\tau)$ and $\theta_o(\tau)$ are the angular positions of the input and output shafts respectively, $T_o(\tau)$ is the torsional momentum applied to the output shaft, whose torsional stiffness is indicated with K_θ. Finally, $T_e(\tau)$ is an external unknown torque applied to the output shaft. R_m is the motor electrical resistance, K_m is the motor torque/current constant, J_i and J_o are the moments of inertia of the input and output shafts, respectively, and β_i, β_o are the viscous friction coefficients of the shafts, finally ζ is the transmission ratio of the gears that connect the two shafts. All of the parameters, except for the transmission ratio, are uncertain and they are distributed according to a Beta probability density function $B(\alpha, \beta)$, with parameters $\alpha = \beta = 1.2$ and maximum, minimum and mean values shown in Table I. The external torque $T_e(\tau)$ is uniformly distributed between ± 1 Nm. The input variable is the voltage $V_m(\tau)$, the measured outputs are the input and output angular positions and speeds, $\theta_i(\tau)$, $\theta_o(\tau)$, $\dot{\theta}_i(\tau)$, $\dot{\theta}_o(\tau)$. The control problem is to track a constant reference position $\theta_{o,\text{ref}}$ issued by the user, in the presence of the disturbance $T_e(\tau)$, subject to a ± 220 V limit on the input voltage and a ± 80 Nm limit on the torsional momentum $T_o(\tau)$. First-principle laws of electric circuits and mechanics yield the following system

TABLE I

CASE STUDY. SYSTEM PARAMETERS FOR THE ELECTRO-MECHANICAL POSITIONING SYSTEM.

Symbol	Description	Mean value	Max. and min. values
R_m	Motor electrical resistance (Ω)	20	20±2
K_m	Motor constant (Nm/A)	10	10±1
K_θ	Output shaft torsional stiffness (Nm/rad)	1280	1280±128
J_i	Input shaft moment of inertia (kg·m^2)	0.5	0.5±0.1
J_0	Output shaft moment of inertia (kg·m^2)	25	25±2.5
β_i	Input shaft friction coefficient (Nm·s/rad)	0.1	0.1±0.01
β_o	Output shaft friction coefficient (Nm·s/rad)	25	25±2.5
ζ	Gear ratio (input/output)	20	-

model:

$$\begin{aligned}
J_i\ddot{\theta}_i(\tau)+\beta_i\dot{\theta}_i(\tau) &= \frac{K_m V_m(\tau)-K_m^2\dot{\theta}_i(\tau)}{R_m}+\frac{T_o(\tau)}{\zeta}\\
J_o\ddot{\theta}_o(\tau)+\beta_o\dot{\theta}_o(\tau) &= T_o(\tau)-T_e(\tau)\\
T_o(\tau) &= K_\theta\left(\frac{\theta_i(\tau)}{\zeta}-\theta_o(\tau)\right).
\end{aligned} \tag{20}$$

In order to re-formulate the control aim as regulation of the system state to zero, we consider as state variables the tracking errors $x_1(\tau)=\theta_i(\tau)-\zeta\,\theta_{o,\text{ref}}$, $x_2(\tau)=\theta_o(\tau)-\theta_{o,\text{ref}}$ and their derivatives, as input $u(\tau)=V_m(\tau)$ and as disturbance $\gamma(\tau)=T_e(\tau)$. The uncertain parameter θ is chosen as $\theta=[\theta_1,\ldots,\theta_7]^\top=[R_m,K_m,K_\theta,J_i,J_o,\beta_i,\beta_o]^\top$. By discretizing the equation (20) with forward Euler approximation and sampling time $T_s=0.1\,$s, a model of the form (1) is obtained, where:

$$A(\theta)=\begin{bmatrix}
1 & 0 & T_s & 0\\
0 & 1 & 0 & T_s\\
T_s\dfrac{\theta_3}{\theta_4\zeta^2} & -T_s\dfrac{\theta_3}{\theta_4\zeta} & 1-T_s\left(\dfrac{\theta_6+\dfrac{\theta_2^2}{\theta_1}}{\theta_4}\right) & 0\\
T_s\dfrac{\theta_3}{\theta_4\zeta} & -T_s\dfrac{\theta_3}{\theta_4} & 0 & 1-T_s\dfrac{\theta_7}{\theta_5}
\end{bmatrix},$$

$$B(\theta)=\begin{bmatrix}0\\0\\T_s\dfrac{\theta_2}{\theta_4\theta_1}\\0\end{bmatrix},\; B_\gamma=\begin{bmatrix}0\\0\\0\\-T_s\end{bmatrix}. \tag{21}$$

The dependance of the system's matrices on θ is clearly nonlinear, yet this problem can be straightforwardly approached with the RMPC technique. The constraint on the input voltage $V_m(\tau)$ and on the output shaft torsional momentum $T_o(\tau)$ can be expressed, on the basis of (20) as:

$$\begin{aligned}
|u_t|-\overline{V}_m &\le 0\\
\left|\theta_3\left(\frac{x_{t,1}}{\zeta}-x_{t,2}\right)\right|-\overline{T}_o &\le 0,
\end{aligned}$$

where $\overline{V}_m=220\,$V and $\overline{T}_o=80\,$Nm. It can be noted that the constraint on the torsional momentum is a function of θ_3. The following terminal control law and terminal set satisfying Assumption 4 are employed:

$$\begin{aligned}
\kappa(x) &= K\,x,\; K=[1.8\;\;-67.6410\;\;-0.8\;174.609]\\
\mathbb{X}_f &= \left\{x\in\mathbb{R}^4 : x^TQ_fx\le 1\right\},\\
Q_f &= \begin{bmatrix}0&0&0&0\\0&5.2559&-0.0293&0.7262\\0&-0.0293&0.0434&-0.6798\\0&0.7262&-0.6798&13.0878\end{bmatrix}.
\end{aligned} \tag{22}$$

The RMPC law is designed with $N=75$ and $R=0.01$. In order to reduce the computational complexity, a blocking strategy on the sequence $\mathcal{V}_t$ is employed, by keeping each five subsequent values of $v_{k|t}$, starting from $v_{0|t},\ldots,v_{4|t}$, constant. This way, the number of free control variables scales down from 75 to 15. By setting a desired guaranteed probability $p=0.95$ for the robust design, a value $M=320$ is obtained from (17). It has to be noted that the value of M does not depend on the dimension of the state variable or of the uncertainty/disturbance variables; it only depends on p and on the number of decision variables in the scenario FHOCP, i.e. the number m of inputs multiplied by the control horizon N, eventually reduced by a blocking strategy like in this case, plus the slack variable z. However, the number of constraints embedded in $h(s,x,\delta)$ depends linearly on n, m and N, so that the growth of the overall number of constraints in the scenario problem is $\sim(n\cdot m^2\cdot N^2/(1-p))$. 1,000 Monte Carlo simulations have been carried out by starting from initial conditions $\theta_i=\theta_o=\dot{\theta}_i=\dot{\theta}_o=0$ and setting different values of reference position $\theta_{o,\text{ref}}\in[-4,4]$. In all cases, the RMPC algorithm was able to drive the state to the set $\mathbb{X}_f$, thus suggesting that the actual probability of achieving robust stability may be higher than the prescribed bound.

As an example, Fig. 1 shows the time course of the output angular position θ_o^{CL}, obtained by applying the RMPC algorithm and setting a reference $\theta_{o,\text{ref}}=4\,$rad. The same

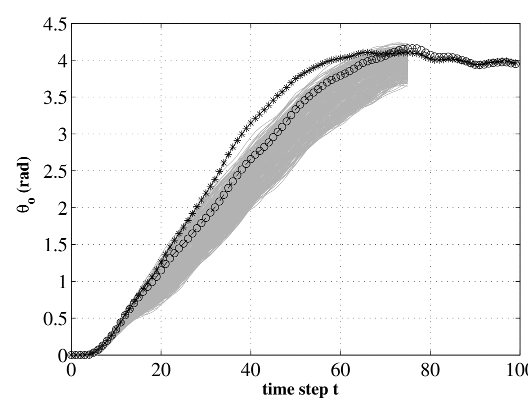

Fig. 1. Case study. Courses of $\theta_{o,t}^{\text{CL}}$ (solid line with '$*$'), of $\theta_{o,t}^{\text{OL}}$ (dashed line with '$\circ$') starting from $\theta_i=\theta_o=\dot{\theta}_i=\dot{\theta}_o=0$ and setting a reference $\theta_{o,\text{ref}}=4\,$rad, and courses of the output position $\theta_{o,k|0}$, $k=0,\ldots,75$ predicted at $t=0$ according to the M random extractions of δ (gray lines).

Fig. also shows the course of θ_o^{OL}, obtained by applying in "open loop" the optimal sequence $\mathcal{V}_0^*$ computed at $t=0$, as well as the trajectories predicted at $t = 0$ according to the M randomly-extracted scenarios. It can be noted that the system controlled by the RMPC law shows a faster transient towards a neighborhood of the reference position. The course of the input variable $V_{m,t} = u_t$ and of the torsional moment $T_{o,t}$, obtained either with the RMPC algorithm or with the open loop sequence $\mathcal{V}_0^*$, are shown in Figs. 2-3. Both these

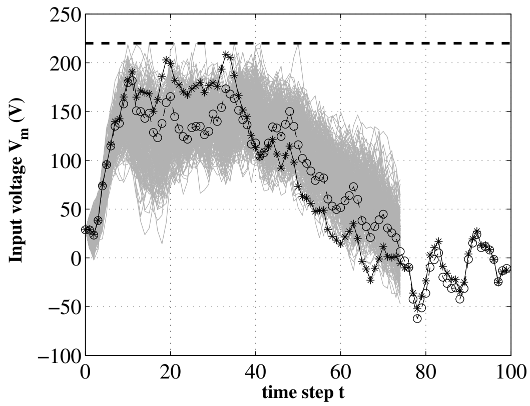

Fig. 2. Case study. Courses of input voltage V_m obtained with the RMPC law (solid line with '$*$') and with the open loop sequence $\mathcal{V}_0^*$ (dashed line with '$\circ$') starting from $\theta_i = \theta_o = \dot{\theta}_i = \dot{\theta}_o = 0$ and setting a reference $\theta_{o,\text{ref}} = 4\,\text{rad}$, and courses of inputs $V_{m,k|0}$, $k = 0, \ldots, 74$ predicted at $t = 0$ according to the M random extractions of δ (gray lines). The input constraint of 220 V is also shown with thick dashed line.

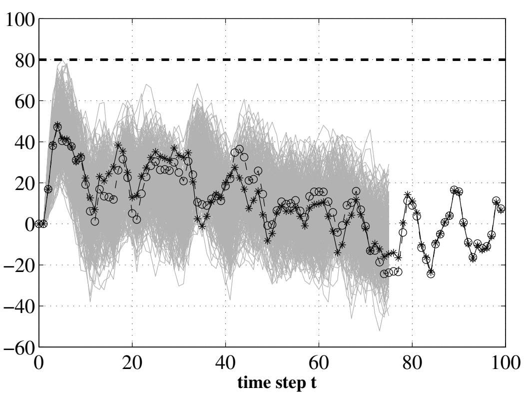

Fig. 3. Case study. Courses of output torsional moment T_o obtained with the RMPC law (solid line with '$*$') and with the open loop sequence $\mathcal{V}_0^*$ (dashed line with '$\circ$') starting from $\theta_i = \theta_o = \dot{\theta}_i = \dot{\theta}_o = 0$ and setting a reference $\theta_{o,\text{ref}} = 4\,\text{rad}$, and courses of the moment $T_{o,k|0}$, $k = 0, \ldots, 74$ predicted at $t = 0$ according to the M random extractions of δ (gray lines). The moment constraint of 80 Nm is also shown with thick dashed line.

variables satisfy the constraints, and it can be noted that several of the input trajectories predicted at $t = 0$ actually hit the constraints (see Fig. 2 between 10 and 50 time steps and Fig. 3 at $t = 5 - 10$). Moreover, it can be noted that the RMPC law, by re-optimizing the control sequence at each time step, is able to adopt a more aggressive control input, closer to the constraints (see Fig. 2 in the first 35 time steps).

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