The stochastic generalized bin packing problem

The Stochastic Generalized Bin Packing Problem

Guido Perboli[a,b], Roberto Tadei[a,*], Mauro M. Baldi[a]

[a] *Politecnico di Torino, Turin, Italy*
[b] *CIRRELT, Montreal, Canada*

Abstract

The Generalized Bin Packing Problem (GBPP) is a recently introduced packing problem where, given a set of bins characterized by volume and cost and a set of items characterized by volume and profit (which also depends on bins), we want to select a subset of items to be loaded into a subset of bins which maximizes the total net profit, while satisfying the volume and bin availability constraints. The total net profit is given by the difference between the total profit of the loaded items and the total cost of the used bins. In this paper we consider the stochastic version of the GBPP (S-GBPP), where the item profits are random variables to take into account the profit oscillations due to the handling operations for bin loading. The probability distribution of these random variables is assumed to be unknown. By using the asymptotic theory of extreme values a deterministic approximation for the S-GBPP is derived.

Keywords: generalized bin packing; random item profit; extreme values; deterministic approximation.

1. Introduction

Given a set of items characterized by volume and random profit and a set of bins characterized by volume and cost, the Stochastic Generalized Bin Packing Problem (S-GBPP) chooses a subset of items to be loaded into a subset of bins which maximizes the expected total net profit, given by the difference between the expected total profit of the loaded items and the total cost of the used bins, while satisfying the volume and bin availability constraints. The item profits, which also depend on bins where the items will be loaded, are random variables with unknown probability distribution. They are composed by a deterministic profit plus a random term, which represents the profit oscillations due to the handling operations needed for loading the items into the bins.

The S-GBPP frequently arises in real-life applications, in particular in logistics, where the freight consolidation is essential to optimize the delivery process. In this case, a series of handling operations for bin loading must be performed at the logistic platforms and these operations could significantly affect the final total profit of the loading [1].

In this paper we introduce a stochastic model for the S-GBPP. In most papers dealing with uncertainty, the probability distribution of the random variables is given and their expected value can then be calculated. This is not the case of the S-GBPP, where the probability distribution of the random item profit is unknown, because it is difficult to be measured in practice and any assumption on its shape would be arbitrary.

We show that, by using some results of the asymptotic theory of extreme values [2], the probability distribution of the maximum random profit of any item becomes a Gumbel (or double exponential) probability distribution and the total expected profit of the loaded items can be easily calculated. By using this result a deterministic approximation of the S-GBPP is derived.

The rest of the paper is organized as follows. Section 2 reviews the related literature. The Generalized Bin Packing Problem and its stochastic counterpart are presented in Sections 3 and 4, respectively. Section 5 derives the

*Corresponding author:
Department of Control and Computer Engineering, Politecnico di Torino
Corso Duca degli Abruzzi 24, 10129 Torino (Italy)
Tel.: +39 0110907032, Fax: +39 0110907099 E-mail: roberto.tadei@polito.it

formulation of the probability distribution of the maximum profit of any item, which can be then computed by using the asymptotic approximation introduced in Section 6. Section 7 gives the deterministic approximation of the original stochastic problem. Finally, the conclusion of our work is reported in Section 8.

2. Literature review

The S-GBPP is the stochastic version of the Generalized Bin Packing Problem (GBPP), introduced in [3] as a generalization of the Variable Cost and Size Bin Packing Problem [4]. An important feature of the GBPP is that it generalizes not only the Variable Cost and Size Bin Packing Problem, but also many other packing problems present in the literature, such as the Bin Packing Problem, the Multiple Homogeneous, and the Heterogeneous Knapsack Problem [3, 5]. The literature on the GBPP is quite limited, due to its recent introduction. For this reason, in the following we will also consider some relevant literature on similar problems, which are special cases of the GBPP and S-GBPP.

The Knapsack Problem is a deeply studied Combinatorial Optimization problem where, given a set of items characterized by volume and profit, we want to find their subset such that the profit is maximized and the sum of the items does not exceed the knapsack volume. Usually, in the Stochastic Knapsack Problem the source of uncertainty is the item profit, and strong hypotheses on its probability distribution are made [6, 7]. Some papers also consider the item volume stochasticity. In [8] an on-line application, where the probability distribution of the random item volume is known in advance, is given. The authors present some policies to decide whether to load the items into the knapsack, showing how an adaptive loading policy outperforms a non-adaptive one. In [9] a variant of the Stochastic Knapsack Problem with normally distributed volumes is presented. The authors derive a two-stage Stochastic Knapsack Problem where, contrary to the single-stage Stochastic Knapsack Problem, items can be added to or removed from the knapsack at the moment the actual volumes become known (second stage).

In the classical Bin Packing Problem we have a set of bins with given volume and a set of items to be compulsorily loaded [5]. The problem aims to load all items in the minimum number of bins. In the Stochastic Bin Packing Problem the source of uncertainty is usually the item volume [10, 11, 12]. These papers mainly deal with on-line algorithms for solving the Stochastic Bin Packing Problem where strong hypotheses on the probability distribution of the random terms are usually done.

3. The Generalized Bin Packing Problem

The GBPP considers a set of items characterized by volume and profit and sets of bins of various types characterized by volume and cost.

Contrary to the classical Bin Packing Problem, the profit of an item also depends on the bin into it will be loaded. The formal definition of the GBPP is given as follows

- I: set of items
- J: set of bins
- T: set of bin types
- $\sigma(j) = t \in T$: type of bin $j \in J$
- L_t : minimum number of bins of type $t \in T$
- U_t : maximum number of bins of type $t \in T$
- C_j: non negative cost of bin $j \in J$
- W_j: non negative volume of bin $j \in J$
- $U \leq \sum_{t \in T} U_t$: maximum number of available bins of any type
- w_i: non negative volume of item $i \in I$
- r_{ij}: non negative profit of loading item $i \in I$ into bin $j \in J$.

The item-to-bin accommodation rules of the GBPP are as follows

- each item can be loaded into one bin at most
- for each used bin of any type, the total volume of the loaded items must not be greater than the bin volume
- for each bin type t, the number of used bins of that type must be within the lower L_t and upper bound U_t

- the total number of used bins of any type cannot exceed the maximum number of available bins U.

The objective is to select the items to be loaded and the necessary bins in order to maximize the total net profit, given by the difference between the total profit of the loaded items and the total cost of the used bins, while satisfying the volume and the bin availability constraints.

There are two possible formulations of the GBPP, the assignment formulation and the set-covering formulation [3]. Here we consider the former, which extends the assignment model of the BPP [5] to the GBPP.

Let us consider the variables

- x_{ij}: item-to-bin assignment boolean variable which is equal to 1 if item $i \in I$ is loaded into bin $j \in J$, 0 otherwise
- y_j: bin selection boolean variable which is equal to 1 if bin $j \in J$ is used, 0 otherwise.

The assignment model of the GBPP becomes

$$\max_{\{x\},\{y\}} \quad \sum_{i \in I} \sum_{j \in J} r_{ij} x_{ij} - \sum_{j \in J} C_j y_j \tag{1}$$

$$\text{subject to} \quad \sum_{j \in J} x_{ij} \leq 1 \qquad i \in I \tag{2}$$

$$\sum_{i \in I} w_i x_{ij} \leq W_j \qquad j \in J \tag{3}$$

$$\sum_{i \in I} x_{ij} \leq |I| y_j \qquad j \in J \tag{4}$$

$$\sum_{j \in J : \sigma(j) = t} y_j \geq L_t \qquad t \in T \tag{5}$$

$$\sum_{j \in J : \sigma(j) = t} y_j \leq U_t \qquad t \in T \tag{6}$$

$$\sum_{j \in J} y_j \leq U \tag{7}$$

$$x_{ij} \in \{0, 1\} \qquad i \in I, j \in J \tag{8}$$

$$y_j \in \{0, 1\} \qquad j \in J \tag{9}$$

The objective function (1) maximizes the total net profit, given by the difference between the total profit of the loaded items and the total cost of the used bins.

Constraints (2) ensure that each item is loaded into one bin at most. Constraints (3) limit the bin capacity. Constraints (4) prevent to load items into not used bins. Constraints (5) and (6) give bounds to the minimum and maximum number of used bins per type, respectively, whilst constraint (7) limits the total number of used bins, regardless of their type. Finally, (8)-(9) are the integrality constraints.

4. The Stochastic Generalized Bin Packing Problem

In the S-GBPP the item profit of the GBPP becomes a random variable. In fact, it is composed by a deterministic profit (the one of the GBPP) plus a random term, which represents the profit oscillations due to the handling costs for loading items into bins. We assume that such profit oscillations randomly depend on the handling scenarios which are adopted for bin loading. These random profit oscillations are very difficult to be measured in practice, so that their probability distribution must be assumed as unknown.

The data and variables of the S-GBPP are the same of the GBPP, but some new data and variables must be considered as follows

- S: set of handling scenarios for bin loading
- θ^{jl}: random profit oscillation of loading bin j under handling scenario $l \in S$.

Let us assume, as it is usually done in this context, that θ^{jl} are independent and identically distributed (i.i.d.) random variables with a common probability distribution

$$F(x) = Pr\{\theta^{jl} \leq x\} \tag{10}$$

The main feature of our approach consists, as stated above, in considering the probability distribution $F(x)$ as unknown.

Without loosing in generality, the random variables θ^{jl} can be scaled by a constant a as follows

$$\tilde{\theta}^{jl} = \theta^{jl} - a \qquad j \in J, l \in S \tag{11}$$

The probability distribution of $\tilde{\theta}^{jl}$ then becomes

$$Pr\{\tilde{\theta}^{jl} \leq x\} = Pr\{\theta^{jl} - a \leq x\} = Pr\{\theta^{jl} \leq x + a\} = F(x + a) \tag{12}$$

Let $\tilde{r}_{ij}(\tilde{\theta}^{jl})$ be the random profit of loading item i into bin j under handling scenario l given by

$$\tilde{r}_{ij}(\tilde{\theta}^{jl}) = r_{ij} + \tilde{\theta}^{jl} \qquad i \in I, j \in J, l \in S \tag{13}$$

Let us define with $\overline{\theta}^j$ the maximum of the random profit oscillations of loading bin j among the alternative handling scenarios $l \in S$

$$\overline{\theta}^j = \max_{l \in S} \tilde{\theta}^{jl} \qquad j \in J \tag{14}$$

$\overline{\theta}^j$ is still of course a random variable with unknown probability distribution given by

$$B_j(x) = \Pr\left\{\overline{\theta}^j \leq x\right\} \qquad j \in J \tag{15}$$

As, for any bin j, $\overline{\theta}^j \leq x \Longleftrightarrow \tilde{\theta}^{jl} \leq x,\ l \in S$ and $\tilde{\theta}^{jl}$ are independent, using (12) one gets

$$B_j(x) = \prod_{l \in S} \Pr\left\{\tilde{\theta}^{jl} \leq x\right\} = \prod_{l \in S} F(x + a) = [F(x + a)]^{|S|} \qquad j \in J \tag{16}$$

We assume that the bin loading policy is efficiency-based so that, for any item i and bin j, among the alternative handling scenarios $l \in S$, the one which maximizes the random profit $\tilde{r}_{ij}(\tilde{\theta}^{jl})$ will be selected.

Then, the random profit of loading item i into bin j becomes

$$\overline{r}_{ij}(\overline{\theta}^j) = \max_{l \in S} \tilde{r}_{ij}(\tilde{\theta}^{jl}) = r_{ij} + \max_{l \in S} \tilde{\theta}^{jl} = r_{ij} + \overline{\theta}^j \qquad i \in I, j \in J \tag{17}$$

The S-GBPP can be formulated as follows

$$\max_{\{y\}} \qquad \left\{\mathbb{E}_{\{\overline{\theta}\}}\left[\max_{\{x\}} \sum_{i \in I} \sum_{j \in J} \overline{r}_{ij}(\overline{\theta}^j) x_{ij}\right] - \sum_{j \in J} C_j y_j\right\} \tag{18}$$

$$\text{subject to} \qquad (2) - (9) \tag{19}$$

The objective function (18) maximizes the expected total net profit, given by the difference between the expected total profit of the loaded items and the total cost of the used bins.

Let us consider the Lagrangian relaxation of problem (18)-(19), obtained by relaxing the capacity constraints (3) by means of the non negative multipliers μ_j, $j \in J$

$$\min_{\{\mu\geq 0\}}\max_{\{y\}} \quad \left\{\mathbb{E}_{\{\bar{\theta}\}}\left[\max_{\{x\}}\sum_{i\in I}\sum_{j\in J}\max\left(0,\bar{r}_{ij}(\bar{\theta}^{j})-\mu_j w_i\right)x_{ij}\right]-\sum_{j\in J}(C_j y_j-\mu_j W_j)\right\} \tag{20}$$

$$\text{subject to} \qquad (2)\ \text{ and }\ (4)-(9) \tag{21}$$

The term $max\left(0,\bar{r}_{ij}(\bar{\theta}^{j})-\mu_j w_i\right)$ in (20) is named the shadow random profit of loading item i into bin j, due to the presence of the shadow prices μ_j.

Problem (20)-(21) gives an upper bound on the optimal value of problem (18)-(19), but we know that, when the strong duality conditions are satisfied, the two problems are equivalent.

For any item i, let us consider bin $j = i^*$ (for the sake of simplicity, we assume it is unique), which gives the maximum shadow random profit for item i over all the bins. This quantity represents an upper bound of the actual maximum shadow random profit of item i and becomes

$$\bar{r}_i(\bar{\theta}^{i^*})-\mu_{i^*}w_i=\max_{j\in J}\left[\max\left(0,\bar{r}_{ij}(\bar{\theta}^{j})-\mu_j w_i\right)\right] \qquad i\in I \tag{22}$$

If item i is loaded, it will select bin j which maximizes its shadow random profit, i.e.

$$x_{ij}=\begin{cases}1, & \text{if} \quad j=i^* \ \text{ and } \ \bar{r}_i(\bar{\theta}^{i^*})-\mu_{i^*}w_i>0\\ 0, & \text{otherwise}\end{cases} \tag{23}$$

Using (22), (23), and the linearity of the expected value operator $\mathbb{E}$, a valid upper bound of the objective function (20) becomes

$$\begin{aligned}&\min_{\{\mu\geq 0\}}\max_{\{y\}}\left\{\sum_{i\in I}\mathbb{E}_{\{\bar{\theta}^{*}\}}\left[\bar{r}_i(\bar{\theta}^{i^*})-\mu_{i^*}w_i\right]-\sum_{j\in J}(C_j y_j-\mu_j W_j)\right\}=\\ &=\min_{\{\mu\geq 0\}}\max_{\{y\}}\left\{\sum_{i\in I}\hat{r}_i-\sum_{j\in J}(C_j y_j-\mu_j W_j)\right\}\end{aligned} \tag{24}$$

where

$$\hat{r}_i=\mathbb{E}_{\{\bar{\theta}^{*}\}}\left[\bar{r}_i(\bar{\theta}^{i^*})-\mu_{i^*}w_i\right] \qquad i\in I \tag{25}$$

The calculation of $\hat{r}_i$ in (25) requires to know the probability distribution of the maximum shadow random profit of item i, i.e. $\bar{r}_i(\bar{\theta}^{i^*})-\mu_{i^*}w_i$, which will be introduced in the next section.

5. Formulation of the probability distribution of the maximum shadow random profit of any item

By (17) and (22), let

$$G_i(x)=Pr\left\{\bar{r}_i(\bar{\theta}^{i^*})-\mu_{i^*}w_i\leq x\right\}=Pr\left\{\max_{j\in J}\left[r_{ij}+\bar{\theta}^{j}-\mu_j w_i\right]^{+}\leq x\right\} \qquad i\in I \tag{26}$$

be the probability distribution of the maximum shadow random profit of item i.

As, for any item i, $\max_{j\in J}\left[r_{ij}+\bar{\theta}^{j}-\mu_j w_i\right]^{+}\leq x \iff \left[r_{ij}+\bar{\theta}^{j}-\mu_j w_i\right]^{+}\leq x, \quad j\in J$, and the random variables $\bar{\theta}^{j}$ are independent (because θ^{jl} are independent), due to (15) and (16), $G_i\{x\}$ in (26) becomes a function of the number

$|S|$ of handling scenarios for bin loading

$$\begin{aligned} G_i(x,|S|) &= Pr\left\{\max_{j\in J}\left[r_{ij}+\overline{\theta}^j-\mu_j w_i\right]^+ \le x\right\} = \prod_{j\in J} Pr\left\{\left[r_{ij}+\overline{\theta}^j-\mu_j w_i\right]^+ \le x\right\} \\ &= \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} Pr\left\{\overline{\theta}^j \le x - r_{ij}+\mu_j w_i\right\} = \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} B_j\left(x-r_{ij}+\mu_j w_i\right) \\ &= \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} \left[F\left(x-r_{ij}+\mu_j w_i+a\right)\right]^{|S|} \qquad i\in I \end{aligned} \tag{27}$$

Let us now set the value of the constant a in (27) equal to the root of the equation

$$1-F(a)=1/|S| \tag{28}$$

Let us assume that $|S|$ is large enough to use the asymptotic approximation $\lim_{|S|\to+\infty} G_i(x,|S|)$ as a good approximation of $G_i(x)$, i.e.

$$G_i(x)=\lim_{|S|\to+\infty} G_i(x,|S|)) \qquad i\in I \tag{29}$$

The calculation of the limit in (29) would require to know the shape of the probability distribution $F(.)$, which is still unknown. To overcome this problem, in the next section we will show that under a mild assumption on the shape of the unknown probability distribution $F(.)$, the limit in (29) will tend towards a specific functional form.

6. The asymptotic approximation of the probability distribution of the maximum shadow random profit of any item

The method we use to calculate the asymptotic approximation of the probability distribution of the maximum shadow random profit of item i, $G_i(x)$, derives from the asymptotic theory of extreme values [2] and is based on the following observation.

Under the assumption that $F(.)$ is asymptotically exponential in its right tail, i.e. there is a constant $\beta>0$ such that

$$\lim_{z\to+\infty}\frac{1-F(x+z)}{1-F(z)}=e^{-\beta x} \tag{30}$$

the limit in (29) tends towards a specific functional form, which is a Gumbel (or double exponential) probability distribution [13].

(30) is a very mild assumption for the unknown probability distribution $F(x)$ as we observe that many probability distributions show such behavior, among them the widely used distributions Exponential, Normal, Lognormal, Gamma, Gumbel, Laplace, and Logistic.

Let us consider the following theorem, which provides the desired asymptotic approximation for $G_i(x)$.

Theorem 1. *Under assumption* (30)*, the asymptotic approximation of the probability distribution $G_i(x), i\in I$, becomes the following Gumbel probability distribution*

$$G_i(x)=\lim_{|S|\to+\infty} G_i(x,|S|))=\exp\left(-A_i e^{-\beta x}\right) \qquad i\in I \tag{31}$$

where

$$A_i=\sum_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} e^{\beta(r_{ij}-\mu_j w_i)} \qquad i\in I \tag{32}$$

is the "accessibility", in the sense of Hansen [14], of item i to the set of bins.

Proof. By (27) and (29) one has

$$\begin{aligned} G_i(x) &= \lim_{|S|\to+\infty} G_i(x,|S|)) = \lim_{|S|\to+\infty} \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} \left[F\left(x-r_{ij}+\mu_j w_i+a\right)\right]^{|S|} = \\ &= \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} \lim_{|S|\to+\infty} \left[F\left(x-r_{ij}+\mu_j w_i+a\right)\right]^{|S|} \end{aligned} \tag{33}$$

Let us consider in (33)

$$\lim_{|S|\to+\infty} F\left(x-r_{ij}+\mu_j w_i+a\right) \tag{34}$$

As by (28) $\lim_{|S|\to+\infty} a = +\infty$ (because $F(a)$ tends to 1), from (30) one obtains

$$\lim_{|S|\to+\infty} \frac{1-F(x-r_{ij}+\mu_j w_i+a)}{1-F(a)} = e^{-\beta(x-r_{ij}+\mu_j w_i)} \tag{35}$$

Then, by (28) and (35), (34) becomes

$$\lim_{|S|\to+\infty} F(x-r_{ij}+\mu_j w_i+a) = \lim_{|S|\to+\infty}\left(1-[1-F(a)]e^{-\beta(x-r_{ij}+\mu_j w_i)}\right) = \lim_{|S|\to+\infty}\left(1+\frac{-e^{-\beta(x-r_{ij}+\mu_j w_i)}}{|S|}\right) \tag{36}$$

Substituting (36) into (33) one gets

$$G_i(x) = \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} \lim_{|S|\to+\infty}\left(1+\frac{-e^{-\beta(x-r_{ij}+\mu_j w_i)}}{|S|}\right)^{|S|} \tag{37}$$

and, by reminding that $\lim_{n\to+\infty}(1+\frac{x}{n})^n = \exp(x)$, using (32) one finally gets

$$G_i(x) = \prod_{j\in J: r_{ij}+\overline{\theta}^j-\mu_j w_i>0} \exp\left(-e^{-\beta(x-r_{ij}+\mu_j w_i)}\right) = \exp\left(-A_i e^{-\beta x}\right) \tag{38}$$

□

7. The deterministic approximation of the S-GBPP

Using the probability distribution $G_i(x)$ given by (31), we are now able to calculate $\hat{r}_i$ in (25) as follows

$$\hat{r}_i = \int_{-\infty}^{+\infty} x dG_i(x) = \int_{-\infty}^{+\infty} x \exp\left(-A_i e^{-\beta x}\right) A_i e^{-\beta x}\beta dx \qquad i\in I \tag{39}$$

Substituting for $v = A_i e^{-\beta x}$ one gets

$$\begin{aligned} \hat{r}_i &= -1/\beta \int_0^{+\infty} \ln(v/A_i) e^{-v} dv = \\ &= -1/\beta \int_0^{+\infty} e^{-v}\ln v dv + 1/\beta \ln A_i \int_0^{+\infty} e^{-v} dv = \\ &= \gamma/\beta + 1/\beta \ln A_i = \\ &= 1/\beta(\ln A_i + \gamma) \end{aligned} \tag{40}$$

where $\gamma = -\int_0^{+\infty} e^{-v}\ln v\, dv \simeq 0.5772$ is the Euler constant.

By (40) and but the constant $\frac{\gamma}{\beta}|I|$, the objective function (24) becomes

$$\min_{\{\mu \geq 0\}} \max_{\{y\}} \left\{ 1/\beta \sum_{i \in I} \ln A_i - \sum_{j \in J} (C_j y_j - \mu_j W_j) \right\} \tag{41}$$

By defining the *total accessibility* of items to the set of bins as

$$\Phi = \prod_{i \in I} A_i \tag{42}$$

(41) becomes

$$\min_{\{\mu \geq 0\}} \max_{\{y\}} \left\{ \frac{1}{\beta} \sum_{i \in I} \ln A_i - \sum_{j \in J} (C_j y_j - \mu_j W_j) \right\} = \min_{\{\mu \geq 0\}} \max_{\{y\}} \left\{ \frac{1}{\beta} \ln \prod_{i \in I} A_i - \sum_{j \in J} (C_j y_j - \mu_j W_j) \right\} =$$
$$= \min_{\{\mu \geq 0\}} \max_{\{y\}} \left\{ \frac{1}{\beta} \ln \Phi - \sum_{j \in J} (C_j y_j - \mu_j W_j) \right\} \tag{43}$$

which gives the desired deterministic approximation of the upper bound of the S-GBPP.

By comparing (43) with (24), it is interesting to observe that the total expected shadow profit of the loaded items, $\sum_{i \in I} \hat{r}_i$, is proportional to the logarithm of the total accessibility of those items to the set of bins Φ.

Let us note that (43) requires to know a proper value of the positive constant β, which is the same parameter that appears in the Gumbel probability distribution (31). This can be obtained by calibration as follows.

Let us consider the standard Gumbel distribution $G(x) = \exp(-e^{-x})$. If one accepts an approximation error of 0.01, then $G(x) = 1 \Longleftrightarrow x = 4.60$ and $G(x) = 0 \Longleftrightarrow x = -1.52$.

Let us consider the interval $[m, M]$, where the shadow random profits $\overline{r}_i(\overline{\theta}^{i^*}) - \mu_{i^*} w_i$ are drawn from.

The following equations hold

$$\beta(m - \zeta) = -1.52, \quad \beta(M - \zeta) = 4.60 \tag{44}$$

where ζ is the mode of the Gumbel distribution $G(x) = \exp\left(-e^{-\beta(x-\zeta)}\right)$.

From (44) one finally gets

$$\beta = \frac{6.12}{M - m} \tag{45}$$

8. Conclusion

In this paper we have addressed a new packing problem where, given a set of bins characterized by volume and cost and a set of items characterized by volume and profit (which also depends on bins), we want to select a subset of items to be loaded into a subset of bins which maximizes the expected total net profit, while satisfying the volume and bin availability constraints. The above expected total net profit is given by the difference between the expected total profit of the loaded items and the total cost of the used bins.

To the authors' knowledge this paper is the first one which introduces stochasticity into the Generalized Bin Packing Problem.

The paper shows that, under a mild assumption on the shape of the probability distribution of the random profit oscillations, the unknown probability distribution of the maximum random profit of any item converges to a Gumbel probability distribution and a nonlinear deterministic approximation of the original stochastic problem can be then derived.

[1] R. Tadei, G. Perboli, F. Della Croce, A heuristic algorithm for the auto-carrier transportation problem, Transportation Science 36 (1) (2002) 55–62.

[2] J. Galambos, The asymptotic theory of extreme order statistics, John Wiley, New York, 1978.
[3] M. Baldi, T. Crainic, G. Perboli, R. Tadei, The generalized bin packing problem, Tech. rep., CIRRELT, CIRRELT-2010-21 (2010, submitted).
[4] T. G. Crainic, G. Perboli, W. Rei, R. Tadei, Efficient lower bounds and heuristics for the variable cost and size bin packing problem, Computers & Operations Research 38 (2011) 1474–1482, doi:10.1016/j.cor.2011.01.001.
[5] S. Martello, P. Toth, Knapsack Problems - Algorithms and computer implementations, John Wiley & Sons, Chichester, UK, 1990.
[6] A. Goel, P. Indyk, Stochastic load balancing and related problems, in: 40th Annual Symposium on Foundations of Computer Science, 1999, pp. 579–586.
[7] K. M. Ross, D. H. K. Tsang, Stochastic knapsack problem, IEEE Transactions on Communications 37 (1989) 740–747.
[8] B. Dean, M. Goemans, J. Vondrak, Approximating the stochastic knapsack problem: The benefit of adaptivity, Mathematics of Operations Research 33 (2008) 945–964.
[9] S. Kosuch, A. Lisser, On two-stage stochastic knapsack problems, Discrete Applied Mathematics doi:10.1016/j.dam.2010.04.006.
[10] G. S. Lueker, Bin packing with items uniformly distributed over intervals [a,b], in: Proceedings of 24th Annual Symposium on Foundations of Computer Science, IEEE Comput. Soc. Press., 1983, pp. 289–297.
[11] W. T. Rhee, M. Talagrand, On-line bin packing of items of random size, part i, SIAM Journal on Computing 18 (1993) 438–445.
[12] W. T. Rhee, M. Talagrand, On-line bin packing of items of random size, part ii, SIAM Journal on Computing 18 (1993) 473–486.
[13] E. J. Gumbel, Statistics of Extremes, Columbia University Press, 1958.
[14] W. Hansen, How accessibility shapes land use, Journal of the American Institute of Planners 25 (1959) 73–76.