An efficient algorithm for the evaluation of master stability function in networks of coupled oscillators

An efficient algorithm for the evaluation of Master Stability Function in networks of coupled oscillators

Marco Righero, Fernando Corinto and Mario Biey

Electronics Dept., Politecnico di Torino, Corso Duca degli Abruzzi 24, I-10129, Torino, Italy
Email:{marco.righero, fernando.corinto, mario.biey}@polito.it

Abstract—An efficient methodology to study stable in-phase synchronization in networks of identical nonlinear oscillators is proposed. The problem of investigating synchronization properties is reduced to an eigenvalue problem by means of the joint application of the Master Stability Function and the Harmonic Balance technique. The proposed method permits to achieve huge reduction in computational time respect to traditional time-domain approaches. In addition, such method can be extended to study synchronization patterns in networks of nonlinear oscillators described by differential-algebraic equations.

1. Introduction

The existence of (locally) stable synchronous states in networks of coupled nonlinear oscillators is mainly shown by computing the spectrum of Lyapunov Exponents (LE). Unfortunately, specially when dealing with continuous time high-order systems, such a computation requires long CPU time and may show numerical instabilities [1–3].

In 1998, Pecora and Caroll [4] proposed a technique, subject to some constraints, to simplify this task. The problem of synchrony detection was split in two parts: one related to network topology and the other requesting the computation of LE of the *single* (generally low-order) uncoupled oscillator. This second part requires the evaluation of the so called Master Stability Function (MSF) [4]. Even if in this case we have to deal with low order nonlinear oscillators, the computational effort remains important because steady-state periodic solutions of the uncoupled oscillator are required. Such solutions may be obtained by means of time-domain methods by discarding transient behavior. On the other hand spectral methods (for instance Harmonic Balance - (HB)) provide an accurate approximation of steady-state periodic oscillations in nonlinear oscillators [5].

The main aim of this paper is to present an efficient method, based on the joint use of HB and MSF techniques, in order to evaluate synchronization properties in networks composed of coupled identical nonlinear oscillators with at least a stable *limit cycle*. In particular, the HB-based method allows one to identify limit cycles and, beyond the reduction in CPU time, makes it possible to investigate nonlinear oscillators described by implicit differential equations [6].

The manuscript is organized as follows. In Section II, a brief summary of the MSF approach is presented and in Section III the new algorithm is described in details. The impressive reduction in CPU time is presented through examples in Section IV. Some conclusions are drawn at the end of the paper.

2. The Master Stability Function

The Master Stability Function permits to study synchronization conditions for networks of coupled nonlinear systems [4]. We summarize the main ideas to point out how spectral methods can be successfully used to conceive efficient algorithms for evaluating synchronization on limit cycles.

We consider networks of N cells described by the model ($n = 1, \dots, N$)

$$\dot{x}_n = f(x_n) + \epsilon \sum_{n'=1}^{N} A_{nn'} H(x_{n'} - x_n), \qquad (1)$$

where $x_n \in \mathbb{R}^D$ is the D-dimensional state of the n^{th} cell, $f : \mathbb{R}^D \to \mathbb{R}^D$ accounts for the nonlinear self dynamics, $\epsilon \in \mathbb{R}$ is the global coupling strength, $A \in \{0,1\}^{N\times N}$ is the adjacency matrix describing the coupling among cells (so that $A_{nn'} = 1$ if cell n is connected to cell n' and $A_{nn'} = 0$ otherwise), and $H \in \mathbb{R}^{D\times D}$ select which variables are used in the coupling. Even if Eq. (1) gives a more explicit idea of how $x_{n'}$ is linked to x_n, for the following calculation it is better to use the laplacian matrix L associated to A (namely $L_{nn'} = A_{nn'}$ when $n \neq n'$, $L_{nn} = -\sum_{n'=1}^{N} A_{nn'}$) to describe the system

$$\dot{x}_n = f(x_n) + \epsilon \sum_{n'=1}^{N} L_{nn'} H x_{n'}. \qquad (2)$$

We are interested in studying conditions on the coupling strength ϵ and on the network topology described by L so that the synchronous manifold $x_1 = x_2 =$

$\dots x_n = \dots x_N$ is stable [1]. It is worth noting that the synchronous manifold corresponds to the in-phase periodic oscillation, i.e. there are zero phase shifts among all oscillators.

Denoting by x the common dynamics, we consider the variational equations associated to Eq. (2)

$$\dot{\delta}_n = Df(x)\delta_n + \epsilon \sum_{n'=1}^{N} L_{nn'} H \delta_{n'}, \tag{3}$$

where δ_n is an infinitesimal increment with respect to x of the n-th cell and Df is the Jacobian of f. By considering the linear transformation which diagonalizes ϵL, we get N uncoupled equations

$$\dot{\delta}_n = \big(Df(x) + (\sigma_n + \mathrm{i}\,\beta_n)H\big)\delta_n, \tag{4}$$

where $\sigma_1 + \mathrm{i}\,\beta_1, \dots \sigma_N + \mathrm{i}\,\beta_N$ are the N eigenvalues of ϵL, considered with their multiplicities. The stability of the synchronous manifold with respect to the direction identified by the eigenvalue $\sigma_n + \mathrm{i}\,\beta_n$ is then determined by evaluating the maximun Lyapunov exponent of Eq. (4) on the synchronous manifold $\dot{x} = f(x)$. If we consider $\sigma_n + \mathrm{i}\,\beta_n$ as a parameter in $\mathbb{C}$, we can perform the same calculation for every value of σ and β in a certain interval, to get a broad sweep of possible configurations. The result is the Master Stability Function (MSF) $\Lambda(\sigma, \beta)$. If, given ϵ and L, we have that $\Lambda(\sigma_n, \beta_n) < 0$ for each eigenvalue of ϵL, then the considered configuration and coupling strength give a (locally) stable synchronous manifold.

In the following we focus on symmetric coupling, so that all the eigenvalues of ϵL have vanishing imaginary part. It follows that we are interested in computing $\Lambda(\sigma, 0) = \Lambda(\sigma)$. In addition, ranking the eigenvalues in decreasing order, we have that $0 = \sigma_1 > \sigma_2 \geq \dots \geq \sigma_N$: the first eigenvalue is 0, as we consider laplacian matrices, and the second is strictly negative, since the corresponding graph is connected.

3. Efficient algorithm based on HB and MSF

The MSF is often evaluated using a numerical algorithm that computes the maximum Lyapunov exponent of Eq. (4) on the synchronous manifold. This approach usually requests a numerical integration of the differential equations, discarding the transient to reach the synchronous state, and further time to have the algorithm to estimate the Lyapunov exponents at the convergence. In [7] the HB technique was successfully employed to study bifurcation phenomena in limit cycle systems. In this Section, we present an alternative efficient algorithm, based on MSF and HB, using the numerical ideas developed in [6].

Before describing the algorithm, we should briefly recall how HB technique is used for analyzing limit cycles and their Lyapunov exponents.

Given a scalar function $x(t)$ which is T-periodic, it can be expressed, in an approximated form, as a truncated Fourier series

$$x(t) \simeq a_0 + \sum_{k=1}^{K} \big(a_k \cos(k\omega t) + b_k \sin(k\omega t)\big), \tag{5}$$

where a_0, a_k, and b_k are the Fourier coefficients and $\omega = 2\pi/T$. Using $M = 2K + 1$ equally spaced time samples $x(t_m)$ in $(0, T]$, with $t_m = mT/M$, $m = 1, \dots, M$, we can link the Fourier coefficients and the time samples $x(t_m)$ using an appropriate matrix Γ with inverse

$$\Gamma^{-1} = \begin{pmatrix} 1 & \gamma^C_{1,1} & \gamma^S_{1,1} & \cdots & \gamma^C_{1,K} & \gamma^S_{1,K} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & \gamma^C_{H,1} & \gamma^S_{H,1} & \cdots & \gamma^C_{H,K} & \gamma^S_{H,K} \end{pmatrix} \tag{6}$$

whose entries are given by

$$\gamma^C_{p,q} = \cos\left(\frac{q 2\pi p}{2K+1}\right),\ \gamma^S_{p,q} = \sin\left(\frac{q 2\pi p}{2K+1}\right). \tag{7}$$

We then have, $X^F = \Gamma X$ where (denoting transposition by an apex $'$) $X = [x(t_1), \dots, x(t_H)]'$, and $X^F = [a_0, a_1, b_1, \dots, a_K, b_K]'$. We can link the Fourier coefficient of the time derivative $\dot{x}(t)$ to the ones of $x(t)$ as $\dot{X}^F = \omega\Omega X^F$,where Ω is a $M \times M$ matrix full of 0 except $\Omega_{2k,2k+1} = k$ and $\Omega_{2k+1,2k} = -k$, for $k = 1, \dots, K$. As described in [7], this formalism can be used to look for the Fourier coefficients of limit cycles for a system $\dot{x} = f(x)$.

We recall now that, given a system of dimension D

$$\dot{\delta} = B(t)\delta, \tag{8}$$

with $B(t)$ a T periodic matrix, the corresponding D Floquet multipliers (FMs) $\lambda_d = \mathrm{e}^{\mu_d T}$ are such that $u_d(t)\,\mathrm{e}^{\mu_d t}$, with $u_d(t)$ T-periodic, are D linearly independent solutions of Eq. (8). The Lyapunov (or characteristic) Exponents are μ_d. If $B(t)$ in Eq. (8) is the Jacobian matrix of the vector field f on a limit cycle solution, then the FMs carry information about the stability of the limit cycle. Following [6], we can express the FMs in the HB setting as an eigenvalue problem: $\mathrm{e}^{\mu_d T}$ is a FM corresponding to the solution $u(t)$ (whose time samples form U and whose truncated Fourier coefficients form U^F, such that, with Γ_D a $DM \times DM$ block-diagonal matrix built up of D copies of Γ, $U^F = \Gamma_D U$) if μ_d and U are solution to the following eigenvalue problem

$$(\Gamma_D B_M \Gamma_D^{-1} - \omega\Omega_M)U^F = \mu_d U, \tag{9}$$

[1] Under suitable assumptions on function $f(\cdot)$, stable synchronous manifolds result to be $\rho_1\, x_1 = \rho_2\, x_2 = \dots = \rho_n\, x_n = \dots = x_N$, where $\rho_n \in \{-1, 1\}$. The results given in this manuscript can be generalized to such stable synchronous manifolds.

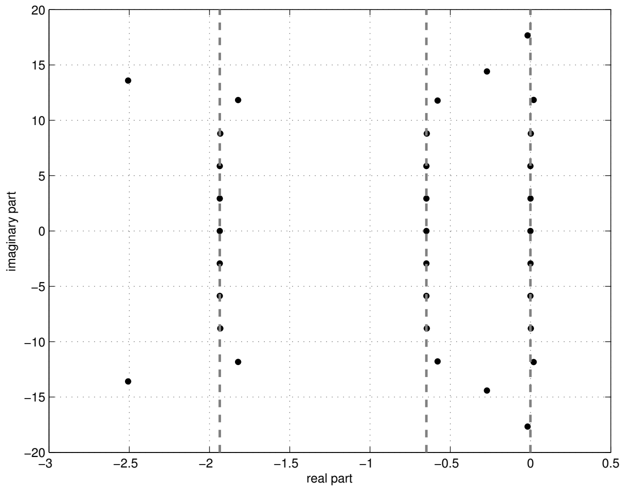

Figure 1: Location on the Gauss plan of the eigenvalues of the systems Eq. (9) for the Chua circuit ($D = 3$) approximated with $K = 5$.

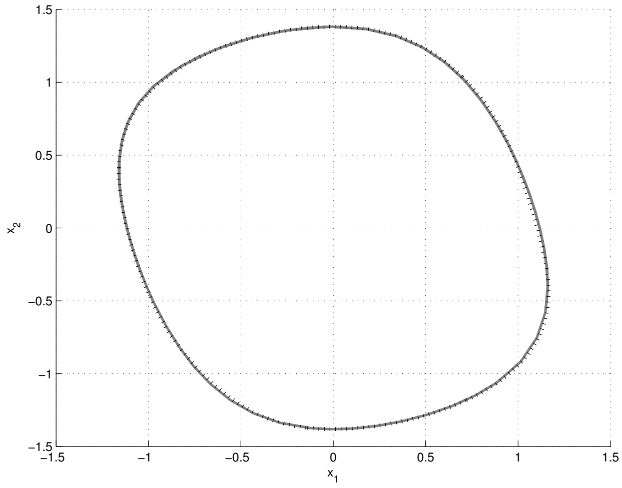

Figure 2: Limit cycle for the Van der Pol oscillator: solid grey line (HB approximation), dotted black line (time-domain simulation).

where B_M is a $DM \times DM$ matrix constructed expanding each element of B in a diagonal block of time samples $B(t_1), \dots B(t_M)$, and Ω_M is a $DM \times DM$ block-diagonal matrix built up of D copies of Ω. The solution of Eq. (9) gives DM different eigenvalues, which correspond to a subset of the infinite μ_d determining D independent FM $\mathrm{e}^{\mu_d T}$. These eigenvalues are distributed along D vertical lines in the complex plane (see Fig. 1 for an example with $D = 3$ and $K = 5$). As noted in [6], to get the more stable results, we should look for the ones with smaller imaginary part.

The proposed algorithm to determine the MSF $\Lambda(\sigma)$ for $\sigma \in [\sigma_*, \sigma^*]$ on a limit cycle is then made of four steps: (1) given the uncoupled oscillator $\dot{x} = f(x)$, find an accurate approximation of the limit cycle of interest X^F; (2) for $\sigma \in [\sigma_*, \sigma^*]$ solve Eq. (9) with B_M constructed using B given by $Df + \sigma H$ evaluated on the limit cycle X^F, obtaining DM eigenvalues; (3) among them, select the D eigenvalues with smaller imaginary part $\mu_1, \dots, \mu_D$; (4) $\Lambda(\sigma) = \max\{\Re(\mu_1), \dots, \Re(\mu_D)\}$.

4. Numerical results

In order to verify the accuracy of the proposed algorithm, we study the stability of synchronous states in two networks composed of simple and well-known oscillators: Van der Pol and Chua oscillators.

To this end, the MSF for Van der Pol and Chua oscillators are derived according to the proposed algorithm. Then the results are compared to those obtained by means of time-domain methods.

Van der Pol oscillator is described by the following differential equations

$$\begin{cases} \dot{x}_1 = 1/C(x_1 - kx_1^3 - x_2), \\ \dot{x}_2 = 1/Lx_1. \end{cases} \tag{10}$$

Assuming $C = 1, L = 4/5, k = 1$ we get a smooth limit cycle which can be approximated using $K = 5$ harmonics. Fig. 2 shows the approximation superimposed to the numerical integration in the time-domain. Chua's circuit (oscillator) can be described by the following set of normalized differential equations

$$\begin{cases} \dot{x}_1 = \alpha[-x_1 + x_2 - n(x_1)], \\ \dot{x}_2 = x_1 - x_2 + x_3, \\ \dot{x}_3 = -\beta x_2. \end{cases} \tag{11}$$

We assume $\alpha = 8, \beta = 15$, $n(x_1) = -8/7x_1 + 4/63x_1{}^3$ and we focus on one of the two asymmetric limit cycles [7], that can be accurately approximated using again $K = 5$ harmonics.

Besides $K = 5$, we also consider different choices of K (i.e. $K = 1$ and $K = 3$) to check how the algorithm is influenced by the number of harmonics. Steps (2)–(4) of the algorithm outlined in Sec. 3 are executed with $\sigma \in [-20, 0]$. Finally, the results are compared with those obtained by applying standard MSF approach in the time-domain.

Fig. 3 (Van der Pol oscillator) and Fig. 4 (Chua's oscillator) make clear that, in both cases MSFs are practically coincident with those derived in time-domain even if a few number of harmonics ($K = 5$) are exploited. It is also important to note that even $K = 1$ permits to identify accurately the synchronization region $\Lambda(\sigma) < 0$. The low number of harmonics necessary for a satisfactory approximation is mainly due to

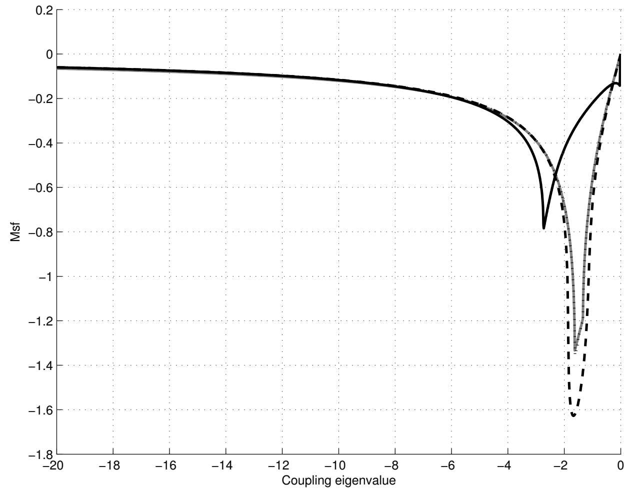

Figure 3: MSF for the Van der Pol oscillators: black lines are obtained with the HB approach described in Sec. 3 with $K = 1$ (solid), $K = 3$ (dashed), and $K = 5$ (dotted), whereas the solid grey line is obtained in the time-domain.

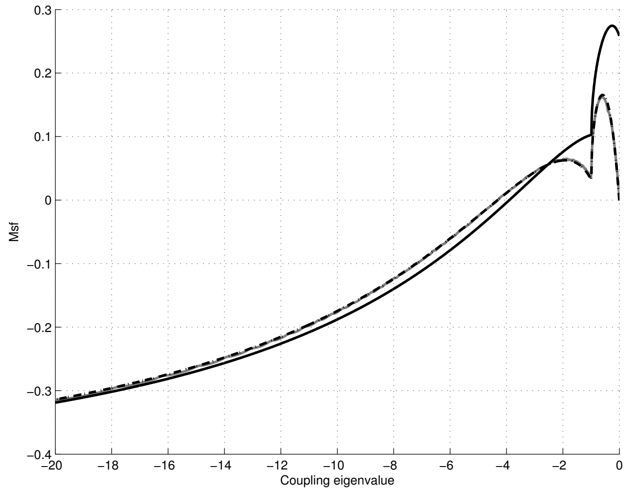

Figure 4: MSF for the Chua circuit: black lines are obtained with the HB approach described in Sec. 3 with $K = 1$ (solid), $K = 3$ (dashed), and $K = 5$ (dotted), whereas the solid grey line is obtained in the time-domain.

the properties of the oscillators chosen to test the proposed algorithm. Hovewer, it gives remarkable results also in the case of nonlinear oscillators having limit cycles whose approximation requires several harmonics.

The main advantages of the proposed algorithm are its numerical stability and short computation time. In the considered example, the reduction of CPU time [2] is about 90% compared to standard time-domain techniques [2, 3].

5. Conclusion

We have proposed an efficient method to study condition for stable local synchronization in networks of identical nonlinear oscillators exploiting the MSF approach in the frequency domain. Once limit cycles of uncoupled oscillators are approximated by means of HB technique, an eigenvalue problem is solved to study stability. The proposed method presents huge advantages in terms of computational time and can also be extended to oscillators described by implicit differential equations.

6. Aknowledgments

This work was partially supported by the CRT Foundation, under the project no. 2009.0570, by the Istituto Superiore Mario Boella and the regional government of Piedmont.

[2]The comparison is based on several numerical simulations run on standard desktop under MatLab.

References

[1] J. P. Eckmann and D. Ruelle, “Ergodic theory of strange attractors,” *Rev. Mod. Phys.*, vol. 57, pp. 617–656, 1985.

[2] A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, “Determining Lyapunov Exponents from a Time Series,” *Physica D*, vol. 16, pp. 285 – 317, 1985.

[3] K. Geist, U. Parlitz, and W. Lauterborn, “Comparison of different methods for computing Lyapunov exponents,” *Progress of Theoretical Physics*, vol. 83, no. 5, pp. 875–893, 1990.

[4] L. M. Pecora and T. L. Carroll, “Master stability functions for synchronized coupled systems,” *Phys. Rev. Letters*, vol. 80, no. 10, pp. 2109– 112, Mar. 1998.

[5] A. I. Mees, *Dynamics of feedback systems.* Wiley, 1981.

[6] F. L. Traversa, F. Bonani, and S. Donati Guerrieri, “A frequency-domain approach to the analysis of stability and bifurcation in nonlinear systems described by differential-algebraic equations,” *Int. J. Circ. Theor. Appl.*, vol. 36, pp. 421–439, 2008.

[7] M. Gilli, F. Corinto, and P. Checco, “Periodic oscillations and bifurcations in cellular nonlinear networks,” *IEEE Trans. Circ. Syst. I*, vol. 51, no. 5, pp. 948–962, 2004.