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Abstract: We determine the two-centered generic charge orbits of magical $\mathcal{N} = 2$ and maximal $\mathcal{N} = 8$ supergravity theories in four dimensions. These orbits are classified by seven U-duality invariant polynomials, which group together into four invariants under the horizontal symmetry group $\mathrm{SL}(2, \mathbb{R})$. These latter are expected to disentangle different physical properties of the two-centered black-hole system. The invariant with the lowest degree in charges is the symplectic product $\langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle$, known to control the mutual non-locality of the two centers.

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Contents

1 Introduction

Multi-centered black-hole solutions of supergravity theories in $d=4$ space-time dimensions have recently received much attention, especially in connection to the classification of non-perturbative string BPS states and their brane interpretation [1–3]. A generalisation of the *attractor Mechanism* [4–8] (for a review, see e.g. [9]) has been shown to occur, as firstly pointed out by Denef [10], called split attractor flow for BPS $\mathcal{N}=2$ black holes [1–3, 10–12].

Attempts to generally classify the two-centered solutions of supergravity theories with symmetric scalar manifolds and electric-magnetic duality (U-duality[1]) symmetry given by classical Lie groups have been considered [16–18]. In particular, within the framework of the *minimal coupling* [19] of vector multiplets to $\mathcal{N}=2$ supergravity, it was shown in [17] that different physical properties, such as marginal stability and split attractor flow solutions, can be classified by duality-invariant constraints, which in this case involve two dyonic black-hole charge vectors, and not only one.

This leads one to consider the mathematical issue of the classification of orbits of two (or more) dyonic charge vectors in the context of multi-centered black-hole physics. For the theories treated in [17, 18], the charge vector lies in the fundamental representation of $U(1,n)$ (*minimally coupled* $\mathcal{N}=2$ supergravity [19]) and in the spinor-vector representation of $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(q,n)$, corresponding to reducible cubic $\mathcal{N}=2$ sequence [20–22] for $q=2$, and to matter-coupled $\mathcal{N}=4$ supergravity for $q=6$.

In [17], the two-centered U-invariant polynomials of the *minimally coupled* theory were constructed, and shown to be four (dimension of the adjoint of the two-centered horizontal symmetry $U(2)$). The same was done for the aforementioned cubic sequence in [18], where

[1]Here U-duality is referred to as the "continuous" symmetries of [13, 14]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [15].

the number of U-invariants were computed to be seven for $n \geqslant 2$, six for $n = 1$ and five for the irreducible t^3 model.

It is the aim of the present investigation to generalise these results to four-dimensional supergravity theories with symmetric *irreducible* scalar manifolds, in particular to the $\mathcal{N} = 8$ maximal theory and to the $\mathcal{N} = 2$ magical models.

We find that when the stabilizer of a two-centered charge orbit is *non-compact*, the corresponding orbit is *not* unique. As we will consider in section 4, this feature is also exhibited by the classification of the orbits of two non-lightlike vectors in a pseudo-Euclidean space $E_{p,q}$ of dimension $p+q$ and signature (p, q). A prominent role is played by an *emergent* horizontal symmetry $\mathrm{SL}_h(2, \mathbb{R})$, whose invariants classify all possible two-vector orbits.

In this respect, the aforementioned t^3 model, whose U-duality group is $\mathrm{SL}(2, \mathbb{R})$, provides a simple yet interesting example, because it may be obtained both as rank-1 truncation of the *reducible* symmetric models and as first, non-generic element of the sequence of *irreducible* $\mathcal{N} = 2$ symmetric models, which contains the four rank-3 magical supergravity theories mentioned above. The two-centered configurations and the generic (BPS) orbit $\mathcal{O} = \mathrm{SL}(2, \mathbb{R})$ of t^3 model were studied in section 7 of [18], in which it was pointed out that, as it occurs also for the one-centered case [23], no stabilizer for the two-centered orbit exists.[2] The five components of the spin $s = 2$ horizontal tensor $\mathbf{I}_{abcd}$ (defined in (3.12) below, and explicitly given by (3.15)–(3.19)) form a complete basis of duality-invariant polynomials [18]; as a consequence, the counting (2.2) for $p = 2$-centered black hole solutions in the t^3 model simply reads $5 + 3 - 0 = 4 \times 2$, because $I_{p=2} = 5$ and $\dim_{\mathbb{R}}(\mathcal{G}_p) = 0$. Moreover, there exist only two independent $[\mathrm{SL}_h(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})]$-invariant polynomials, which can be taken to be the symplectic product $\mathcal{W}$ (of order two in charges, defined in (3.9) below) and $\mathbf{I}_6$ (of order six in charges, defined in (3.24) below); an alternative choice of basis for the $\mathrm{SL}(2, \mathbb{R})$-invariant polynomials is thus e.g. given by three components of $\mathbf{I}_{abcd}$ out of the five (3.15)-(3.19), and the two horizontal invariants $\mathcal{W}$ and $\mathbf{I}_6$.

The plan of the paper is as follows.

In section 2 we give a group theoretical method (based on progressive branchings of symmetry groups, considered as complex groups) to find the multi-centered charge orbits of a theory with a symmetric scalar manifold; we then apply it to all *irreducible* symmetric cases. The analysis of this section will not depend on the real form of the stabilizer of the orbit, and the results will then hold both for BPS and all the non-BPS orbits of the given model. In section 3 we propose a complete basis for U-duality polynomials in the presence of two dyonic black-hole charge vectors in irreducible symmetric models, and we also consider the role of the horizontal symmetry in this framework. section 4 extends the analysis of section 2 to different non-compact real forms of the stabilizer of one-centered charge orbits related to Jordan algebras over the octonions, namely to $\mathcal{N} = 8$ theory (whose $\frac{1}{8}$-BPS one-centered stabilizer is $E_{6(2)}$) and for exceptional magical $\mathcal{N} = 2$ theory (whose BPS and non-BPS $\mathcal{I}_4 > 0$ one-centered stabilizers are the compact $E_{6(-78)}$ and the non-compact $E_{6(-14)}$, respectively).

Possible extensions of the present investigation may also cover composite configurations

[2] As it holds for the magical $J_3^{\mathbb{R}}$ model, see table I.

with "small" constituents, as well as a detailed study of the multi-centered charge orbits in $\mathcal{N}=5,6$-extended supergravity theories.

2 Little group of p charge vectors in irreducible symmetric models

We consider a p-center black hole solution in a Maxwell-Einstein supergravity theory in $d=4$ space-time dimensions.

The p dyonic black-hole charge vectors can be arranged as

$$\mathbf{Q}_a \equiv \left\{\mathcal{Q}_a^M\right\}_{M=1,\dots,f}, \tag{2.1}$$

where $\mathcal{Q}_a^M$ sits in the irreducible representation $(\mathbf{p}, \mathbf{Sympl}\,(G_4))$ of the group $\mathrm{SL}_h\,(p,\mathbb{R}) \times G_4$. $\mathbf{p}$ is the fundamental representation (spanned by the index $a=1,\dots,p$) of the horizontal symmetry group [18] $\mathrm{SL}_h\,(p,\mathbb{R})$ (see section 3), while $\mathbf{Sympl}\,(G_4)$ is the symplectic irreducible representation of the black-hole charges, spanned by the index $M=1,\dots,f$ of the U-duality group G_4, where $f \equiv \dim_{\mathbb{R}}(\mathbf{Sympl}\,(G_4))$.

Suppose there are I_p independent G_4-invariant polynomials constructed out of $\mathbf{Q}_a$, and let $\mathcal{G}_p$ denote the little group of the system of charges, defined as the largest subgroup of G_4 such that $\mathcal{G}_p\,\mathbf{Q}_a = \mathbf{Q}_a\ \forall a$. Then, the following relation[3] holds [18]:

$$I_p + \dim_{\mathbb{R}}(G_4) - \dim_{\mathbb{R}}(\mathcal{G}_p) = f\,p. \tag{2.2}$$

Some preliminary general observations are in order:

- The group theoretical analysis of the present section does not depend on the real form of G_4 and $\mathcal{G}_p$. We will then generally consider the complex groups. From a physical point of view, the BPS and non-BPS cases in various supergravity theories correspond to different choices of non-compact real forms of $\mathcal{G}_p$ (and of G_4, as well). However, for BPS orbits in $\mathcal{N}=2$ symmetric models, and in particular for magical models, the stabilizer is always the compact form of the relevant group (see table 1).
- We shall generally assume $\mathbf{Q}_1$ to be in a representation corresponding to a "large" black hole,[4] namely such that the quartic invariant $\mathcal{I}_4\left(\mathcal{Q}_1^4\right) \neq 0$.
- We shall consider "generic" orbits, in which all I_p invariants are independent.
- There are two relevant cases, corresponding to different behaviors in the counting of invariants:

  a) The largest subgroup commuting with $\mathcal{G}_p$ inside G_4 is $\mathrm{U}(1) \subset G_4$, so that $\mathcal{G}_p \times \mathrm{U}(1) \subset G$.

  b) A $\mathrm{U}(1)$ commuting with $\mathcal{G}_p$ inside G_4 does not exist.

[3] A necessary but not sufficient condition for eq. (2.2) to hold is $p<f$, such that the p dyonic charge vectors can all be taken to be linearly independent.

[4] Multi-center configurations with "small" constituents [12, 24, 25] can be treated as well, and they will be considered elsewhere.

$J_3^{\mathbb{A}}$	$\mathcal{O}_{p=2,BPS} = \frac{Conf\left(J_3^{\mathbb{A}}\right)}{\mathcal{G}_{p=2}\left(J_3^{\mathbb{A}}\right)}$
$J_3^{\mathbb{O}}$	$\frac{E_{7(-25)}}{\mathrm{SO}(8)}$
$J_3^{\mathbb{H}}$	$\frac{\mathrm{SO}^*(12)}{[SU(2)]^3}$
$J_3^{\mathbb{C}}$	$\frac{SU(3,3)}{[\mathrm{U}(1)]^2}$
$J_3^{\mathbb{R}}$	$Sp\,(6,\mathbb{R})$

Table 1. BPS generic charge orbits of 2-centered extremal black holes in $\mathcal{N}=2$, $d=4$ magical models. $Conf\left(J_3^{\mathbb{A}}\right)$ denotes the "conformal" group of $J_3^{\mathbb{A}}$ (see e.g. [26], and refs. therein). By introducing $\mathbb{A}=\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$, it is worth remarking that the stabilizer group $\mathcal{G}_{p=2}\left(J_3^{\mathbb{A}}\right)$ and the automorphism group $Aut\,(\mathbf{t}\,(\mathbb{A}))$ of the *normed triality* $\mathbf{t}\,(\mathbb{A})$ in dimension $\dim_{\mathbb{R}}\mathbb{A}=1,2,4,8$ (given e.g. in eq. (5) of [27]) share the same Lie algebra. In other words, $\mathfrak{g}_{p=2}\left(J_3^{\mathbb{A}}\right)\sim\mathfrak{tri}\,(\mathbb{A})$, where $\mathfrak{tri}\,(\mathbb{A})$ denotes the Lie algebra of $Aut\,(\mathbf{t}\,(\mathbb{A}))$ itself (see e.g. eq. (21) of [27]).

In the case b), all the singlets in the decomposition of $G_4\to\mathcal{G}_p$ correspond to p-center G_4-invariant polynomials of $\mathbf{Sympl}\,(G_4)$. On the other hand, in the case a) the number of singlets corresponds to the number of p-center G_4-invariant polynomials, plus one if some of them are charged with respect to U(1), because one of the singlets can still be acted on by the corresponding U(1)-grading.

- The general method for working out $\mathcal{G}_p$ and thus I_p, having solved the problem for $p-1$ centers, is to consider the p^{th} charge vector $\mathbf{Q}_p$ as transforming in a (reducible) representation of the little group $\mathcal{G}_{p-1}$ of the former $p-1$ charges, and solve the corresponding one-charge-vector problem.

In the next subsections we will consider the cases $p=1$ and $p=2$ in all irreducible symmetric cases pertaining to supergravity theories in $d=4$ dimensions (with the exception of the rank-1 t^3 model, treated in [18]). In the case $p=1$, we will retrieve the well known result $I_{p=1}=1$, whereas in the $p=2$ case we will obtain $I_{p=2}=7$ in all cases under consideration.

2.1 $J_3^{\mathbb{O}}$ ($\mathcal{N}=2$), $J_3^{\mathbb{O}_s}$ ($\mathcal{N}=8$)

Let us start considering the exceptional case, based on the Euclidean degree-3 Jordan algebra $J_3^{\mathbb{O}}$ on the octonions $\mathbb{O}$. Since, as mentioned earlier, we actually work with complex groups, this case pertains also to maximal $\mathcal{N}=8$ supergravity, based on the Euclidean degree-3 Jordan algebra $J_3^{\mathbb{O}_s}$ on the split octonions $\mathbb{O}_s$.

In the complex field, $G_4=E_7$ and $\mathbf{Sympl}\,(E_7)=\mathbf{Fund}\,(E_7)=\mathbf{56}$.

- Let us first solve the one-center problem ($p=1$). $\mathcal{G}_1$ is a real form of E_6; the **56** branches with respect to E_6 as follows (subscripts denote the $U\,(1)$-charges throughout):

$$\mathbf{56} \to \mathbf{1}_{-3} + \mathbf{27}_{-1} + \overline{\mathbf{27}}_{+1} + \mathbf{1}_{+3}\,, \tag{2.3}$$

and correspondingly the charge vector $\mathbf{Q}_1$ (defined as $\left(p^\Lambda, q_\Lambda\right)$ throughout) decomposes as follows:

$$\mathbf{Q}_1 = (p^0, \mathbf{p}_{\mathbf{27}}, q_0, \mathbf{q}_{\overline{\mathbf{27}}})\,. \tag{2.4}$$

Note that the branching (2.3) contains two E_6-singlets, and $E_7 \supset E_6 \times \mathrm{U}(1) = \mathcal{G}_1 \times \mathrm{U}(1)$. According to the previous discussion, one of the singlets can be freely acted on by the U(1). Thus, by acting with $G_4/\mathcal{G}_1 = E_7/E_6$, the 1-center charge vector $\mathbf{Q}_1$ can be reduced as follows:

$$\mathbf{Q}_1 \overset{E_7/E_6}{\longrightarrow} (I^{(1)}, \mathbf{0}_{\mathbf{27}}, \pm I^{(1)}, \mathbf{0}_{\overline{\mathbf{27}}})\,. \tag{2.5}$$

One is then left with only one independent singlet charge $I^{(1)}$ related to the 1-center quartic invariant $\mathcal{I}_4\left(\mathcal{Q}_1^4\right)$; therefore, $I_1 = 1$, as expected. This analysis is consistent with the general formula (2.2), which in this case reads:

$$I_1 + \dim_{\mathbb{R}}(E_7) - \dim_{\mathbb{R}}(E_6) = 1 + 133 - 78 = 56\,. \tag{2.6}$$

- Let us now proceed to deal with the two charge-vector problem ($p = 2$). The second charge vector is denoted as $\mathbf{Q}_2 \equiv (m^\Lambda, e_\Lambda)$ throughout. Having solved the problem for $p = 1$, we can decompose $\mathbf{Q}_2$ with respect to $\mathcal{G}_1 = E_6$ using (2.3), obtaining the decomposition

$$\mathbf{Q}_2 = (I^{(2)}, \mathbf{m}_{\mathbf{27}}, I^{(3)}, \mathbf{e}_{\overline{\mathbf{27}}}), \tag{2.7}$$

and then determine the corresponding little group inside E_6. The little group of the irreducible representation **27** of E_6 is F_4, under which

$$\mathbf{27} \to \mathbf{1} + \mathbf{26}\,, \tag{2.8}$$

and correspondingly

$$\mathbf{m}_{\mathbf{27}} \to (I^{(4)}, \mathbf{m}_{\mathbf{26}}); \quad \mathbf{e}_{\overline{\mathbf{27}}} \to (I^{(5)}, \mathbf{e}_{\mathbf{26}}). \tag{2.9}$$

Note in particular that F_4 is a maximal (symmetric) subgroup of E_6, so that all singlets correspond to extra E_7-invariant polynomials, and that $\mathbf{m}_{\mathbf{26}}$ can be set to zero through the action of $\mathcal{G}_1/F_4 = E_6/F_4$, thus yielding the result:

$$\mathbf{Q}_2 \overset{E_6/F_4}{\longrightarrow} (I^{(2)}, I^{(4)}, \mathbf{0}_{\mathbf{26}}, I^{(3)}, I^{(5)}, \mathbf{e}_{\mathbf{26}}). \tag{2.10}$$

- The **26** of F_4 has little group SO(8), which does not commute with a U(1) in F_4. Under this non-maximal embedding, the **26** branches as

$$\mathbf{26} \to \mathbf{1} + \mathbf{1} + \mathbf{8}_{\mathbf{v}} + \mathbf{8}_{\mathbf{s}} + \mathbf{8}_{\mathbf{c}}\,, \tag{2.11}$$

and correspondingly

$$\mathbf{e}_{\mathbf{26}} \to (I^{(6)}, I^{(7)}, \mathbf{e}_{\mathbf{8}_{\mathbf{v}}}, \mathbf{e}_{\mathbf{8}_{\mathbf{s}}}, \mathbf{e}_{\mathbf{8}_{\mathbf{c}}})\,. \tag{2.12}$$

Therefore, by acting with $F_4/\mathcal{G}_2 = F_4/\mathrm{SO}(8)$, $\mathbf{Q}_2$ can then be put in the form

$$\mathbf{Q}_2 \overset{F_4/\mathrm{SO}(8)}{\longrightarrow} (I^{(2)}, I^{(4)}, \mathbf{0}_{\mathbf{26}}, I^{(3)}, I^{(5)}, I^{(6)}, I^{(7)}, \mathbf{0}_{\mathbf{8}_{\mathbf{v}}}, \mathbf{0}_{\mathbf{8}_{\mathbf{s}}}, \mathbf{0}_{\mathbf{8}_{\mathbf{c}}}). \tag{2.13}$$

In conclusion, we found that the little group of a 2-centered black-hole solution is $\mathcal{G}_2 = \mathrm{SO}(8)$, and the corresponding 2-centered charge orbits correspond to different real forms of the quotient of complex groups

$$\mathcal{O}_{p=2} = \frac{G_4}{\mathcal{G}_2} = \frac{E_7}{\mathrm{SO}\,(8)}. \tag{2.14}$$

The E_7-invariant polynomials for a 2-centered configuration are seven: $I_2 = 7$; indeed, the general formula (2.2) gives:

$$I_2 + \dim_{\mathbb{R}}(E_7) - \dim_{\mathbb{R}}(\mathrm{SO}(8)) = 7 + 133 - 28 = 112 = 2 \cdot 56. \tag{2.15}$$

2.2 $J_3^{\mathbb{H}}$ ($\mathcal{N} = 2 \leftrightarrow \mathcal{N} = 6$)

This model is based on the Euclidean degree-3 Jordan algebra $J_3^{\mathbb{H}}$ on the quaternions $\mathbb{H}$, and it is "dual" to $\mathcal{N} = 6$ "pure" theory, because these theories share the same bosonic sector [20, 21, 28–31].

In the complex field $G_4 = \mathrm{SO}\,(12)$, and $\mathbf{Sympl}\,(\mathrm{SO}\,(12)) = \mathbf{32}$, the chiral spinor irreducible representation of $\mathrm{SO}\,(12)$.

- Let us first solve the problem for $p = 1$. $\mathcal{G}_1$ is a real form of SU(6), the relevant (maximal symmetric) embedding is

$$\mathrm{SO}(12) \supset \mathrm{SU}(6) \times \mathrm{U}(1) = \mathcal{G}_1 \times \mathrm{U}(1), \tag{2.16}$$

and the $\mathbf{32}$ accordingly branches

$$\mathbf{32} \rightarrow \mathbf{1}_{-3} + \mathbf{15}_{-1} + \overline{\mathbf{15}}_{+1} + \mathbf{1}_{+3}\,, \tag{2.17}$$

corresponding to the charge decomposition

$$\mathbf{Q}_1 = (p^0, \mathbf{p_{15}}, q_0, \mathbf{q_{15}})\,. \tag{2.18}$$

The analysis here is completely analogous to the exceptional case above. The branching (2.17) contains two SU(6)-singlets, but, by virtue of (2.16), one of the singlets can be freely acted on by the U(1). By acting with $G_4/\mathcal{G}_1 = \mathrm{SO}\,(12)\,/\mathrm{SU}(6)$, $\mathbf{Q}_1$ can be reduced to

$$\mathbf{Q}_1 \overset{\mathrm{SO}(12)/\mathrm{SU}(6)}{\longrightarrow} (I^{(1)}, \mathbf{0_{15}}, \pm I^{(1)}, \mathbf{0_{15}})\,, \tag{2.19}$$

so that $I_1 = 1$, corresponding to the 1-center quartic invariant $\mathcal{I}_4\left(\mathcal{Q}_1^4\right)$ only. Indeed, the general formula (2.2) yields

$$I_1 + \dim_{\mathbb{R}}(\mathrm{SO}(12)) - \dim_{\mathbb{R}}(\mathrm{SU}(6)) = 1 + 66 - 35 = 32\,. \tag{2.20}$$

- Let us consider now the 2-centered case ($p = 2$). Having solved the problem for $p = 1$, we further decompose $\mathbf{Q}_2$ with respect to $\mathcal{G}_1 = \mathrm{SU}(6)$:

$$\mathbf{Q}_2 = \left(I^{(2)}, \mathbf{m_{15}}, I^{(3)}, \mathbf{e_{15}}\right), \tag{2.21}$$

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and find the corresponding little group. The little group of the **15** of SU(6) is $USp\,(6)$, under which such a representation branches as follows:

$$\mathbf{15} \longrightarrow \mathbf{1} + \mathbf{14}, \tag{2.22}$$

yielding the charge decompositions

$$\mathbf{m_{15}} \longrightarrow \left(I^{(4)}, \mathbf{m_{14}}\right); \quad \mathbf{e_{15}} \longrightarrow \left(I^{(5)}, \mathbf{e_{14}}\right). \tag{2.23}$$

Since $USp\,(6)$ is maximally (and symmetrically) embedded in $SU\,(6)$, all singlets correspond to extra SO (12)-invariant polynomials, and $\mathbf{m_{14}}$ can be set to zero through the action of $\mathcal{G}_1/USp\,(6) = SU\,(6)\,/\mathrm{USp}(6)$, thus yielding the result:

$$\mathbf{Q}_2 \overset{SU(6)/\mathrm{USp}(6)}{\longrightarrow} \left(I^{(2)}, I^{(4)}, \mathbf{0_{14}}, I^{(3)}, I^{(5)}, \mathbf{e_{14}}\right). \tag{2.24}$$

- The **14** (rank-2 antisymmetric) of USp(6) has little group $[SU\,(2)]^3$, which does not commute with a $U\,(1)$ in USp(6). The **14** correspondingly branches as

$$\mathbf{14} \longrightarrow (\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{2},\mathbf{2}) + (\mathbf{2},\mathbf{2},\mathbf{2})\,, \tag{2.25}$$

and thus

$$\mathbf{e_{14}} \longrightarrow \left(I^{(6)}, I^{(7)}, \mathbf{e_{(1,2,2)}}, \mathbf{e_{(2,2,2)}}\right). \tag{2.26}$$

Therefore, by acting with $\mathrm{USp}(6)/\mathcal{G}_2 = \mathrm{USp}(6)/\ [SU\,(2)]^3$, $\mathbf{Q}_2$ can then be put in the form

$$\mathbf{Q}_2 \overset{\mathrm{USp}(6)/[SU(2)]^3}{\longrightarrow} (I^{(2)}, I^{(4)}, \mathbf{0_{14}}, I^{(3)}, I^{(5)}, I^{(6)}, I^{(7)}, \mathbf{0_{(1,2,2)}}, \mathbf{0_{(2,2,2)}}). \tag{2.27}$$

In conclusion, we found that the little group of a 2-centered black-hole solution is $\mathcal{G}_2 = [SU\,(2)]^3$, and the corresponding 2-centered charge orbit reads (in complexified form)

$$\mathcal{O}_{p=2} = \frac{G_4}{\mathcal{G}_2} = \frac{\mathrm{SO}\,(12)}{[SU\,(2)]^3}. \tag{2.28}$$

The SO (12)-invariant polynomials for a 2-centered configuration are seven: $I_2 = 7$; indeed, the general formula (2.2) gives:

$$I_2 + \dim_{\mathbb{R}}(\mathrm{SO}\,(12)) - \dim_{\mathbb{R}}([SU\,(2)]^3) = 7 + 66 - 9 = 64 = 2 \cdot 32. \tag{2.29}$$

2.3 $J_3^{\mathbb{C}}$ ($\mathcal{N} = 2$), $M_{1,2}\,(\mathbb{O})$ ($\mathcal{N} = 5$)

Let us now consider the model based on the Euclidean degree-3 Jordan algebra $J_3^{\mathbb{C}}$ on $\mathbb{C}$. Since, as mentioned earlier, we actually deal with groups on the complex field, this case pertains also to "pure" $\mathcal{N} = 5$ supergravity, which is based on $M_{1,2}\,(\mathbb{O})$, the Jordan triple system (not upliftable to $d = 5$) generated by 2×1 matrices over $\mathbb{O}$ [20, 21].

In the complex field $G_4 = SU\,(6)$, and **Sympl** $(SU\,(6)) = \mathbf{20}$, the real self-dual rank-3 antisymmetric irreducible representation.

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- Let us first solve the problem for $p = 1$. $\mathcal{G}_1$ is a real form of $\mathrm{SU}(3) \times \mathrm{SU}(3)$, the relevant (maximal symmetric) embedding is

$$\mathrm{SU}(6) \supset \mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1) = \mathcal{G}_1 \times \mathrm{U}(1), \tag{2.30}$$

and the **20** accordingly branches as

$$\mathbf{20} \rightarrow (\mathbf{1},\mathbf{1})_{-3} + (\mathbf{3},\overline{\mathbf{3}})_{-1} + (\overline{\mathbf{3}},\mathbf{3})_{+1} + (\mathbf{1},\mathbf{1})_{+3}\,, \tag{2.31}$$

corresponding to the charge decomposition

$$\mathbf{Q}_1 \rightarrow (p^0, \mathbf{p}_{(\mathbf{3},\overline{\mathbf{3}})}, q_0, \mathbf{q}_{(\overline{\mathbf{3}},\mathbf{3})})\,. \tag{2.32}$$

The analysis here is analogous to the cases treated above. The branching (2.31) contains two $[\mathrm{SU}(3) \times \mathrm{SU}(3)]$-singlets, but, by virtue of (2.30), one of the singlets can be freely acted on by the U(1). By acting with $G_4/\mathcal{G}_1 = SU(6)\,/\,[\mathrm{SU}(3) \times \mathrm{SU}(3)]$, $\mathbf{Q}_1$ can be reduced to

$$\mathbf{Q}_1 \overset{SU(6)/[\mathrm{SU}(3)\times\mathrm{SU}(3)]}{\longrightarrow} (I^{(1)}, \mathbf{0}_{(\mathbf{3},\overline{\mathbf{3}})}, \pm I^{(1)}, \mathbf{0}_{(\overline{\mathbf{3}},\mathbf{3})})\,, \tag{2.33}$$

so that $I_1 = 1$, which corresponds to $\mathcal{I}_4\left(\mathcal{Q}_1^4\right)$ only. Indeed, formula (2.2) yields

$$I_1 + \dim_{\mathbb{R}}(SU(6)) - \dim_{\mathbb{R}}(\mathrm{SU}(3) \times \mathrm{SU}(3)) = 1 + 35 - 16 = 20\,. \tag{2.34}$$

- Let us consider now the 2-centered case ($p = 2$). Having solved the problem for $p = 1$, we further decompose $\mathbf{Q}_2$ with respect to $\mathcal{G}_1 = \mathrm{SU}(3) \times \mathrm{SU}(3)$:

$$\mathbf{Q}_2 = (I^{(2)}, \mathbf{m}_{(\mathbf{3},\overline{\mathbf{3}})}, I^{(3)}, \mathbf{e}_{(\overline{\mathbf{3}},\mathbf{3})}), \tag{2.35}$$

and find the corresponding little group. The little group of the $(\mathbf{3},\overline{\mathbf{3}})$ of $\mathrm{SU}(3)\times\mathrm{SU}(3)$ is the *diagonal* SU(3),which is maximal in $\mathrm{SU}(3) \times \mathrm{SU}(3)$ (see e.g. [32]), under which such a representation branches as follows:

$$(\mathbf{3},\overline{\mathbf{3}}) \rightarrow \mathbf{1} + \mathbf{8}\,, \tag{2.36}$$

yielding the charge decompositions

$$\mathbf{m}_{(\mathbf{3},\overline{\mathbf{3}})} \rightarrow (I^{(4)}, \mathbf{m}_8); \quad \mathbf{e}_{(\overline{\mathbf{3}},\mathbf{3})} \rightarrow (I^{(5)}, \mathbf{e}_8).$$

The maximality of the embedding of the diagonal SU(3) in $\mathrm{SU}(3) \times \mathrm{SU}(3)$ implies all singlets to correspond to extra $SU(6)$-invariant polynomials, and $\mathbf{m_8}$ can be set to zero through the action of $\mathcal{G}_1/\mathrm{SU}(3) = [\mathrm{SU}(3) \times \mathrm{SU}(3)]\,/\mathrm{SU}(3)$, thus yielding the result:

$$\mathbf{Q}_2 \overset{[\mathrm{SU}(3)\times\mathrm{SU}(3)]/\mathrm{SU}(3)}{\longrightarrow} \left(I^{(2)}, I^{(4)}, \mathbf{0_8}, I^{(3)}, I^{(5)}, \mathbf{e_8}\right). \tag{2.37}$$

- The $\mathbf{8}$ (adjoint) of SU(3) has little group $[\mathrm{U}(1)]^2$, which does not commute with any U(1) in SU(3). The $\mathbf{8}$ correspondingly branches as

$$\mathbf{8} \rightarrow \mathbf{1}_{0,0} + \mathbf{1}_{0,0} + \mathbf{1}_{0,2} + \mathbf{1}_{0,-2} + \mathbf{1}_{3,1} + \mathbf{1}_{3,-1} + \mathbf{1}_{-3,1} + \mathbf{1}_{-3,-1}\,, \tag{2.38}$$

and thus

$$\mathbf{e_8} \longrightarrow (I^{(6)}, I^{(7)}, \mathrm{e}_{0,2}, \mathrm{e}_{0,-2}, \mathrm{e}_{3,1}, \mathrm{e}_{3,-1}, \mathrm{e}_{-3,1}, \mathrm{e}_{-3,-1}). \tag{2.39}$$

Therefore, by acting with $\mathrm{SU}(3)/\mathcal{G}_2 = \mathrm{SU}(3)/\left[\mathrm{U}(1)\right]^2$, $\mathbf{Q}_2$ can then be put in the form

$$\mathbf{Q}_2 \overset{\mathrm{SU}(3)/[\mathrm{U}(1)]^2}{\longrightarrow} (I^{(2)}, I^{(4)}, \mathbf{0_8}, I^{(3)}, I^{(5)}, I^{(6)}, I^{(7)}, \mathbf{0}_6), \tag{2.40}$$

where $\mathbf{0}_6$ collectively denotes the six charges pertaining to the $[\mathrm{U}(1)]^2$-charged representations $\mathbf{1}_{0,2}$, $\mathbf{1}_{0,-2}$, $\mathbf{1}_{3,1}$, $\mathbf{1}_{3,-1}$, $\mathbf{1}_{-3,1}$, $\mathbf{1}_{-3,-1}$ in the right-hand side of (2.38).

In conclusion, we found that the little group of a 2-centered black-hole solution is $\mathcal{G}_2 = [\mathrm{U}(1)]^2$, and the corresponding 2-centered charge orbit reads (in complexified form)

$$\mathcal{O}_{p=2} = \frac{G_4}{\mathcal{G}_2} = \frac{SU\,(6)}{[\mathrm{U}(1)]^2}. \tag{2.41}$$

The $SU\,(6)$-invariant polynomials for a 2-centered configuration are seven: $I_2 = 7$; indeed, the general formula (2.2) gives:

$$I_2 + \dim_{\mathbb{R}}(SU\,(6)) - \dim_{\mathbb{R}}([\mathrm{U}(1)]^2) = 7 + 35 - 2 = 40 = 2 \cdot 20. \tag{2.42}$$

2.4 $J_3^{\mathbb{R}}$ ($\mathcal{N} = 2$)

Finally, we consider the model based on the Euclidean degree-3 Jordan algebra $J_3^{\mathbb{R}}$ on $\mathbb{R}$.

In the complex field $G_4 = USp\,(6)$, and $\mathbf{Sympl}\,(USp\,(6)) = \mathbf{14'}$, the real self-dual rank-3 antisymmetric irreducible representation of $USp\,(6)$ (not to be confused with the rank-2 antisymmetric irreducible representation $\mathbf{14}$ considered in section 2.2).

- Let us first solve the problem for $p = 1$. $\mathcal{G}_1$ is a real form of SU(3), the relevant (maximal symmetric) embedding is

$$\mathrm{USp}(6) \supset \mathrm{SU}(3) \times \mathrm{U}(1) = \mathcal{G}_1 \times \mathrm{U}(1), \tag{2.43}$$

and the $\mathbf{14'}$ accordingly branches as

$$\mathbf{14'} \rightarrow \mathbf{1}_{-3} + \mathbf{6}_{-1} + \overline{\mathbf{6}}_{+1} + \mathbf{1}_{+3}, \tag{2.44}$$

corresponding to the charge decomposition

$$\mathbf{Q}_1 \rightarrow \left(p^0, \mathbf{p_6}, q_0, \mathbf{q_6}\right). \tag{2.45}$$

Once again, the analysis here is analogous to the cases treated above. The branching (2.44) contains two SU(3)-singlets, but, by virtue of (2.43), one of the singlets can be freely acted on by the U(1). By acting with $G_4/\mathcal{G}_1 = USp(6)/\mathrm{SU}(3)$, $\mathbf{Q}_1$ can be reduced to

$$\mathbf{Q}_1 \overset{USp(6)/\mathrm{SU}(3)}{\longrightarrow} \left(I^{(1)}, \mathbf{0_6}, \pm I^{(1)}, \mathbf{0_6}\right), \tag{2.46}$$

so that $I_1 = 1$, which corresponds to $\mathcal{I}_4\left(\mathcal{Q}_1^4\right)$ only. Indeed, formula (2.2) yields

$$I_1 + \dim_{\mathbb{R}}(USp(6)) - \dim_{\mathbb{R}}(\mathrm{SU}(3)) = 1 + 21 - 8 = 14\,. \tag{2.47}$$

- Let us consider now the 2-centered case ($p = 2$). Having solved the problem for $p = 1$, we further decompose $\mathbf{Q}_2$ with respect to $\mathcal{G}_1 = \mathrm{SU}(3)$:

$$\mathbf{Q}_2 = (I^{(2)}, \mathbf{m_6}, I^{(3)}, \mathbf{e_6}), \tag{2.48}$$

and find the corresponding little group. The little group of the **6** of SU(3) is SO(3),which is maximal in SU(3), under which such a representation branches as follows:

$$\mathbf{6} \to \mathbf{1} + \mathbf{5}, \tag{2.49}$$

yielding the charge decompositions

$$\mathbf{m_6} \to (I^{(4)}, \mathbf{m_5}); \quad \mathbf{e_6} \to (I^{(5)}, \mathbf{e_5}). \tag{2.50}$$

The maximality of SO(3) in SU(3) implies all singlets to corresponds to extra $USp(6)$-invariant polynomials, and $\mathbf{m_6}$ can be set to zero through the action of $\mathcal{G}_1/\mathrm{SO}(3) = \mathrm{SU}(3)/\mathrm{SO}(3)$, thus yielding the result:

$$\mathbf{Q}_2 \overset{\mathrm{SU}(3)/\mathrm{SO}(3)}{\longrightarrow} (I^{(2)}, I^{(4)}, \mathbf{0_5}, I^{(3)}, I^{(5)}, \mathbf{e_5}). \tag{2.51}$$

- Note, however, that the little group of the **5** (rank-2 symmetric traceless) irreducible representation of SO(3) is the identity, so that $\mathcal{G}_2 = \mathbb{I}$. The **5** then trivially branches into five singlets, three of which can be rotated to zero through the action of $\mathrm{SO}(3)/\mathcal{G}_2 = \mathrm{SO}(3)$:

$$\mathbf{Q}_2 \overset{\mathrm{SO}(3)}{\longrightarrow} (I^{(2)}, I^{(4)}, \mathbf{0_5}, I^{(3)}, I^{(5)}, I^{(6)}, I^{(7)}, \mathbf{0}_3), \tag{2.52}$$

where $\mathbf{0}_3$ collectively denotes such three singlets set to zero.

In conclusion, we found that the little group of a 2-centered black-hole solution is the identity itself: $\mathcal{G}_2 = \mathbb{I}$, and the corresponding 2-centered charge orbit reads (in compact form)

$$\mathcal{O}_{p=2} = \frac{G_4}{\mathcal{G}_2} = USp(6)\,. \tag{2.53}$$

The $USp(6)$-invariant polynomials for a 2-centered configuration are seven: $I_2 = 7$; indeed, the general formula (2.2) yields:

$$I_2 + \dim_{\mathbb{R}}(\mathrm{USp}(6)) - \dim_{\mathbb{R}}(\mathbb{I}) = 7 + 21 - 0 = 28 = 2 \cdot 14. \tag{2.54}$$

3 Invariant structures and the role of the horizontal symmetry $\mathbf{SL}_h(2,\mathbb{R})$

We now propose a candidate for a complete basis of G_4-invariant polynomials for the $p=2$ case, highlighting the role of the horizontal symmetry group [18] in the classification of multi-center invariant structures.

Our treatment applies at least to the *irreducible* cubic geometries of *symmetric* scalar manifolds of $d=4$ supergravity theories [22] (which, with the exception of the rank-1 t^3 model,[5] are the ones considered in the counting analysis of section 2):

1. $\mathcal{N}=2$ magical Maxwell-Einstein supergravities ($J_3^{\mathbb{A}}$, $\mathbb{A}=\mathbb{O},\mathbb{H},\mathbb{C},\mathbb{R}$), with the case $J_3^{\mathbb{H}}$ encompassing also $\mathcal{N}=6$ "pure" supergravity [20, 21, 28–31];
2. $\mathcal{N}=5$ "pure" supergravity ($M_{1,2}(\mathbb{O})$);
3. $\mathcal{N}=8$ "pure" supergravity ($J_3^{\mathbb{O}_s}$).

The simplest invariant structures of a simple Lie group G (such as the U-duality group G_4 of an *irreducible* symmetric model) are the Killing-Cartan metric $g_{\alpha\beta}$, the structure constants $f_{\alpha\beta\gamma}$ and the symplectic metric $\mathbb{C}_{MN}$ (the Greek indices are in the adjoint representation of G_4, $\mathbf{Adj}(G_4)$, while the capital indices are in $\mathbf{Sympl}(G_4)$). It is well known that the entries of the generators in $\mathbf{Sympl}(G_4)$

$$t_{\alpha|MN}\equiv t_{\alpha|M}{}^{P}\mathbb{C}_{PN}=t_{\alpha|(MN)} \tag{3.1}$$

are invariant structures, symmetric in the symplectic indices (for the notation, see [33]).

In particular, one can construct the so-called K-tensor[6] [34]

$$\mathbb{K}_{MNPQ}\equiv-\frac{1}{3\tau}t^{\alpha}_{(MN}t_{\alpha|PQ)}=-\frac{1}{3\tau}\left(t^{\alpha}_{MN}t_{\alpha|PQ}-\tau\,\mathbb{C}_{M(P}\mathbb{C}_{Q)N}\right)=\mathbb{K}_{(MNPQ)}, \tag{3.2}$$

where τ is a G_4-dependent constant defined as

$$\tau\equiv\frac{2d}{f(f+1)}, \tag{3.3}$$

with $d\equiv\dim_{\mathbb{R}}\mathbf{Adj}(G_4)$ and $f\equiv\dim_{\mathbb{R}}(\mathbf{Sympl}(G_4))$. From its definition (3.2), the K-tensor is a completely symmetric rank-4 G_4-invariant tensor of $\mathbf{Sympl}(G_4)$.

In the presence of a single-centered black-hole background ($p=1$), associated to a dyonic black-hole charge vector $\mathcal{Q}^M$ in $\mathbf{Sympl}(G_4)$, the unique independent G_4-invariant polynomial reads [34]

$$\mathcal{I}_4\left(\mathcal{Q}^4\right)\equiv\mathbb{K}_{MNPQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q=-\frac{1}{3\tau}t^{\alpha}_{MN}t_{\alpha|PQ}\mathcal{Q}^M\mathcal{Q}^N\mathcal{Q}^P\mathcal{Q}^Q. \tag{3.4}$$

[5] As mentioned above, the irreducible rank-1 cubic case (the so-called $\mathcal{N}=2$, $d=4$ t^3 model, associated to the trivial degree-1 Jordan algebra $\mathbb{R}$) has been treated in [18].

[6] With respect to the treatment given in [34], we fix the overall normalization constant of the K-tensor to the value $\xi=-\frac{1}{3\tau}=-\frac{f(f+1)}{6d}$, as needed for consistency reasons.

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On the other hand, in the presence of a multi-centered black-hole solution ($p \geqslant 2$), the horizontal symmetry $\mathrm{SL}_h(p,\mathbb{R})$ [18] plays a crucial role in organizing the various G_4-covariant and G_4-invariant structures.

In the following treatment we will consider the 2-centered case ($p = 2$), the index $a = 1,2$ spanning the fundamental representation (spin $s = 1/2$) $\mathbf{2}$ of the horizontal symmetry $\mathrm{SL}_h(2,\mathbb{R})$.

By using the symplectic representation (3.1) of the generators of G_4, one can introduce the tensor (homogeneous quadratic in charges)

$$T_{\alpha|ab} \equiv t_{\alpha|MN}\mathcal{Q}_a^M\mathcal{Q}_b^N = T_{\alpha|(ab)} = \begin{pmatrix} T_{\alpha|11} & T_{\alpha|12} \\ T_{\alpha|12} & T_{\alpha|22} \end{pmatrix}, \tag{3.5}$$

lying in $(\mathbf{3}, \mathbf{Adj}(G_4))$ of $\mathrm{SL}_h(2,\mathbb{R}) \times G_4$, where $\mathbf{3}$ is the rank-2 symmetric (spin $s = 1$) representation of $\mathrm{SL}_h(2,\mathbb{R})$. In *irreducible* models, $T_{\alpha|ab}$ is the analogue of the so-called $\mathbb{T}$-tensor, introduced in [18] for *reducible* theories. Under the centers' exchange $1 \leftrightarrow 2$, $T_{\alpha|11} \leftrightarrow T_{\alpha|22}$, while $T_{\alpha|12}$ is invariant.

Interestingly, one can prove that the quantity

$$\mathbf{N} \equiv g^{\alpha\beta}\left(T_{\alpha|11}T_{\beta|22} - T_{\alpha|12}T_{\beta|12}\right) \tag{3.6}$$

is *not* independent from lower order invariants. Indeed, *at least* in the aforementioned irreducible cases, it holds that

$$t^{\alpha}_{M[N}t_{\alpha|P]Q} = \frac{\tau}{2}\left[\mathbb{C}_{M(P}\mathbb{C}_{Q)N} - \mathbb{C}_{M(N}\mathbb{C}_{Q)P}\right]. \tag{3.7}$$

Thus, from (3.5) and (3.6), it follows that

$$\mathbf{N} = 2t^{\alpha}_{M[N}t_{\alpha|P]Q}\mathcal{Q}_1^M\mathcal{Q}_1^N\mathcal{Q}_2^P\mathcal{Q}_2^Q = -\frac{1}{3}\left[\mathbb{C}_{M(P}\mathbb{C}_{Q)N} - \mathbb{C}_{M(N}\mathbb{C}_{Q)P}\right]\mathcal{Q}_1^M\mathcal{Q}_1^N\mathcal{Q}_2^P\mathcal{Q}_2^Q = \frac{1}{2}\mathcal{W}^2, \tag{3.8}$$

where

$$\mathcal{W} \equiv \langle \mathcal{Q}_1, \mathcal{Q}_2 \rangle \equiv \frac{1}{2}\mathbb{C}_{MN}\epsilon^{ab}\mathcal{Q}_a^M\mathcal{Q}_b^N \tag{3.9}$$

is the *symplectic product* of the charge vectors $\mathcal{Q}_1$ and $\mathcal{Q}_2$, which is a singlet $(\mathbf{1}, \mathbf{1})$ of $\mathrm{SL}_h(2,\mathbb{R}) \times G_4$ (manifestly antisymmetric under $1 \leftrightarrow 2$).

An important difference between the *reducible* models (studied in [18]) and the *irreducible* treated in the present investigation is that, while the former generally have a non-vanishing horizontal invariant polynomial $\mathcal{X}$, the latter have it vanishing identically. Indeed, the analogue of $\mathcal{X}$ (defined by eq. (4.13) of [18]) for irreducible models can be defined as

$$\mathcal{X}_{irred} \equiv \mathbf{N} - \frac{1}{2}\mathcal{W}^2 = 0, \tag{3.10}$$

where result (3.8) was used in the last step.

By using the K-tensor (3.2), one can also define the tensor (homogeneous cubic in charges)

$$\mathcal{Q}_{M|abc} \equiv \mathbb{K}_{MNPQ}\mathcal{Q}_a^N\mathcal{Q}_b^P\mathcal{Q}_c^Q = \mathcal{Q}_{M|(abc)}, \tag{3.11}$$

lying in $(\mathbf{4}, \mathbf{Sympl}(G_4))$ of $\mathrm{SL}_h(2,\mathbb{R}) \times G_4$, where $\mathbf{4}$ is the rank-3 symmetric representation (spin $s = 3/2$) of $\mathrm{SL}_h(2,\mathbb{R})$. Under $1 \leftrightarrow 2$, it holds that $\mathcal{Q}_{M|111} \leftrightarrow \mathcal{Q}_{M|222}$ and $\mathcal{Q}_{M|112} \leftrightarrow \mathcal{Q}_{M|122}$.

By further contracting with a 2-centered charge vector, one can introduce the tensor (homogeneous quartic in charges)

$$\mathbf{I}_{abcd} \equiv \mathbb{K}_{MNPQ}\mathcal{Q}_a^M \mathcal{Q}_b^N \mathcal{Q}_c^P \mathcal{Q}_d^Q = \mathbf{I}_{(abcd)}, \tag{3.12}$$

lying in $(\mathbf{5}, \mathbf{1})$ of $\mathrm{SL}_h(2,\mathbb{R}) \times G_4$, where $\mathbf{5}$ is the rank-4 symmetric representation (spin $s = 2$) of $\mathrm{SL}_h(2,\mathbb{R})$. Under $1 \leftrightarrow 2$, $\mathbf{I}_{1111} \leftrightarrow \mathbf{I}_{2222}$, $\mathbf{I}_{1112} \leftrightarrow \mathbf{I}_{1222}$, while $\mathbf{I}_{1122}$ is invariant.

Trivially, $\widetilde{\mathcal{Q}}_{abc} \equiv \mathcal{Q}_{M|abc}$ and $\mathbf{I}_{(abcd)}$ are related by[7]

$$\mathbf{I}_{abcd} = \mathcal{Q}_{M|abc}\mathcal{Q}_d^M = \mathbb{C}^{MN}\mathcal{Q}_{M|abc}\mathcal{Q}_{N|d} = \left\langle \widetilde{\mathcal{Q}}_{abc}, \mathcal{Q}_d \right\rangle; \tag{3.13}$$

$$\mathcal{Q}_{M|abc} = \frac{1}{4}\frac{\partial \mathbf{I}_{abcd}}{\partial \mathcal{Q}_d^M}. \tag{3.14}$$

Note that only the completely symmetric part $\mathcal{Q}_{M|(abc}\mathcal{Q}_{d)}^M$ survives the contraction in (3.13), because $\mathcal{Q}_{M|abc}\mathcal{Q}_d^M \epsilon^{cd} = 0$ from the symmetry of the K-tensor (3.2) and the definition (3.11) of $\mathcal{Q}_{M|abc}$ itself.

In order to generate G_4-invariant polynomials, one can:

1. multiply and contract on $\mathbf{Adj}(G_4)$ the three components of the quadratic tensor $T_{\alpha|ab}$ defined by (3.5), *or*

2. contract all four components of $\mathcal{Q}_{M|abc}$ defined by (3.11) with three 2-center charge vectors, in all possible ways, *or*

3. contract all five components of $\mathbf{I}_{abcd}$ defined by (3.12) with four 2-center charge vectors, in all possible ways.

By virtue of the various relations considered above, these three approaches give equivalent results, which we now specify for the sake of clarity:

$$\begin{aligned}\mathbf{I}_{+2}\left(\mathcal{Q}_1^4\right) &\equiv \mathcal{I}_4\left(\mathcal{Q}_1^4\right) \equiv \mathbf{I}_{1111} \\ &= \left\langle \widetilde{\mathcal{Q}}_{111}, \mathcal{Q}_1 \right\rangle = \mathbb{K}_{MNPQ}\mathcal{Q}_1^M \mathcal{Q}_1^N \mathcal{Q}_1^P \mathcal{Q}_1^Q = -\frac{1}{3\tau}T_{11}^{\alpha}T_{\alpha|11}; \end{aligned}\tag{3.15}$$

$$\begin{aligned}\mathbf{I}_{+1}\left(\mathcal{Q}_1^3\mathcal{Q}_2\right) &\equiv \mathbf{I}_{1112} \\ &= \left\langle \widetilde{\mathcal{Q}}_{111}, \mathcal{Q}_2 \right\rangle = \left\langle \widetilde{\mathcal{Q}}_{112}, \mathcal{Q}_1 \right\rangle = \mathbb{K}_{MNPQ}\mathcal{Q}_1^M \mathcal{Q}_1^N \mathcal{Q}_1^P \mathcal{Q}_2^Q = -\frac{1}{3\tau}T_{11}^{\alpha}T_{12|\alpha}; \end{aligned}\tag{3.16}$$

[7]We remark that relation (3.14) characterizes $\widetilde{\mathcal{Q}}_{abc}$ as the 2-center generalisation of the so-called *Freudenthal dual* of the dyonic charge vector $\mathcal{Q}^M$, introduced (with a different normalisation) in [36]. Thus, $\widetilde{\mathcal{Q}}_{abc}$ can be regarded as the (polynomial) 2-*center Freudenthal dual* of the dyonic charge vector $\mathcal{Q}_d$.

Furthermore, eqs. (3.9), (3.13) and (3.24) yield that, under the formal interchange $\mathcal{Q}_a^M \leftrightarrow \mathbb{C}^{MN}\mathcal{Q}_{N|abc}$, $\mathbf{I}_{abcd}$ is invariant and $\mathcal{W} \leftrightarrow \mathbf{I}_6$.

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$$\begin{aligned}
\mathbf{I}_0\left(\mathcal{Q}_1^2\mathcal{Q}_2^2\right) &\equiv \mathbf{I}_{1122}\\
&= \left\langle \widetilde{\mathcal{Q}}_{112},\mathcal{Q}_2\right\rangle = \left\langle \widetilde{\mathcal{Q}}_{122},\mathcal{Q}_1\right\rangle = \mathbb{K}_{MNPQ}\mathcal{Q}_1^M\mathcal{Q}_1^N\mathcal{Q}_2^P\mathcal{Q}_2^Q\\
&= -\frac{1}{9\tau}\left(T_{11}^{\alpha}T_{22|\alpha}+2T_{12}^{\alpha}T_{12|\alpha}\right) = -\frac{1}{3\tau}\left(T_{11}^{\alpha}T_{22|\alpha}+\tau\mathcal{W}^2\right); \qquad (3.17)\\
\mathbf{I}_{-1}\left(\mathcal{Q}_1\mathcal{Q}_2^3\right) &\equiv \mathbf{I}_{1222}\\
&= \left\langle \widetilde{\mathcal{Q}}_{122},\mathcal{Q}_2\right\rangle = \left\langle \widetilde{\mathcal{Q}}_{222},\mathcal{Q}_1\right\rangle = \mathbb{K}_{MNPQ}\mathcal{Q}_1^M\mathcal{Q}_2^N\mathcal{Q}_2^P\mathcal{Q}_2^Q = -\frac{1}{3\tau}T_{22}^{\alpha}T_{12|\alpha}; \qquad (3.18)\\
\mathbf{I}_{-2}\left(\mathcal{Q}_2^4\right) &\equiv \mathcal{I}_4\left(\mathcal{Q}_2^4\right) \equiv \mathbf{I}_{2222}\\
&= \left\langle \widetilde{\mathcal{Q}}_{222},\mathcal{Q}_2\right\rangle = \mathbb{K}_{MNPQ}\mathcal{Q}_2^M\mathcal{Q}_2^N\mathcal{Q}_2^P\mathcal{Q}_2^Q = -\frac{1}{3\tau}T_{22}^{\alpha}T_{22|\alpha}. \qquad (3.19)
\end{aligned}$$

The subscripts in the G_4-invariant polynomials $\mathbf{I}_{+2}$, $\mathbf{I}_{+1}$, $\mathbf{I}_0$, $\mathbf{I}_{-1}$ and $\mathbf{I}_{-2}$ defined by (3.15)–(3.19) denote the polarization with respect to the horizontal symmetry $SL_h(2,\mathbb{R})$, inherited from the components of $\mathbf{I}_{abcd}$ (3.12); indeed, the five G_4-invariant polynomials (3.15)–(3.19) sit in the rank-4 symmetric representation (spin $s=2$) $\mathbf{5}$ of $SL_h(2,\mathbb{R})$ itself [18].

In order to proceed further, it is worth mentioning the decomposition [34]

$$t_{\alpha|M}{}^{N}t_{\beta|NQ} = -t_{\alpha|MP}t_{\beta|NQ}\mathbb{C}^{PN} = \frac{1}{2n}g_{\alpha\beta}\mathbb{C}_{MQ}+\frac{1}{2}f_{\alpha\beta}{}^{\gamma}t_{\gamma|MQ}+S_{(\alpha\beta)[MQ]}, \qquad (3.20)$$

where

$$S_{\alpha\beta|MN} = S_{(\alpha\beta)|[MN]} \qquad (3.21)$$

denotes an invariant primitive tensor of G_4. From (3.20), the following identity for the K-tensor can be derived [34] (recall footnote 6):

$$\begin{aligned}
\mathbb{K}_{MNPQ}\mathbb{K}_{RSTU}\mathbb{C}^{QR} &= -\frac{(f+1)}{6d}\mathbb{K}_{(MN|(ST}\mathbb{C}_{U)|P)}+\frac{(f+1)}{18d}\mathbb{C}_{(M|(S|}\mathbb{C}_{|N||T|}\mathbb{C}_{|P)|U)}\\
&\quad+\frac{f^2(f+1)^2}{72d^2}f_{\alpha\beta\gamma}t^{\alpha}{}_{(MN}t^{\beta}{}_{P)(S}t^{\gamma}{}_{TU)}\\
&\quad-\frac{f^2(f+1)^2}{36d^2}t^{\alpha}{}_{(MN}S_{\alpha\beta|P)(S}t^{\beta}{}_{TU)}, \qquad (3.22)
\end{aligned}$$

where

$$S_{\alpha\beta|12} \equiv S_{\alpha\beta|MN}\mathcal{Q}_1^M\mathcal{Q}_2^N = S_{\alpha\beta|MN}\mathcal{Q}_1^{[M}\mathcal{Q}_2^{N]} = -S_{\alpha\beta|21}. \qquad (3.23)$$

A G_4-invariant polynomial homogeneous sextic in charges can then be defined as follows:

$$\begin{aligned}
\mathbf{I}_6\left(\mathcal{Q}_1^3\mathcal{Q}_2^3\right) &\equiv \frac{1}{8}\left\langle \widetilde{\mathcal{Q}}_{abc},\widetilde{\mathcal{Q}}_{def}\right\rangle \epsilon^{ad}\epsilon^{be}\epsilon^{cf} = \frac{1}{8}\mathbb{C}^{MN}\mathcal{Q}_{M|abc}\mathcal{Q}_{N|def}\epsilon^{ad}\epsilon^{be}\epsilon^{cf}\\
&= \frac{1}{4}\left\langle \widetilde{\mathcal{Q}}_{111},\widetilde{\mathcal{Q}}_{222}\right\rangle+\frac{3}{4}\left\langle \widetilde{\mathcal{Q}}_{122},\widetilde{\mathcal{Q}}_{112}\right\rangle\\
&= \frac{1}{4}\mathbb{K}_{MNPQ}\mathbb{K}_{RSTU}\mathbb{C}^{QR}\left(\mathcal{Q}_1^M\mathcal{Q}_1^N\mathcal{Q}_1^P\mathcal{Q}_2^S\mathcal{Q}_2^T\mathcal{Q}_2^U+3\mathcal{Q}_1^M\mathcal{Q}_2^N\mathcal{Q}_2^P\mathcal{Q}_1^S\mathcal{Q}_1^T\mathcal{Q}_2^U\right)\\
&= \frac{(f+1)}{36d}\mathcal{W}^3+\frac{f^2(f+1)^2}{144d^2}f_{\alpha\beta\gamma}T_{11}^{\alpha}T_{12}^{\beta}T_{22}^{\gamma}\\
&\quad+\frac{f^2(f+1)^2}{108d^2}\left(T_{12}^{\alpha}T_{12}^{\beta}-T_{11}^{\alpha}T_{22}^{\beta}\right)S_{\alpha\beta|12}. \qquad (3.24)
\end{aligned}$$

Note that $\mathbf{I}_6$ is manifestly antisymmetric under $1 \leftrightarrow 2$. The first line of (3.24) is manifestly $[\mathrm{SL}_h(2,\mathbb{R}) \times G_4]$-invariant, the second and third lines provide explicit expressions, and in the fourth line the "master" identity (3.22) was exploited.

If the symplectic product $\mathcal{W} \neq 0$ (defined in (3.9)), the two charge vectors $\mathcal{Q}_1^M$ and $\mathcal{Q}_2^M$ are *mutually non-local*. The concept of *mutual non-locality* is very important in the treatment of marginal stability in multi-center black holes (see e.g. [1–3, 10–12, 24, 25, 35]).

The above treatment suggests that a candidate for a complete basis of G_4-invariant polynomials in the irreducible cases under consideration is given by the seven polynomials:

$$(\mathcal{W},\ \mathbf{I}_{+2},\ \mathbf{I}_{+1},\ \mathbf{I}_0,\ \mathbf{I}_{-1},\ \mathbf{I}_{-2},\ \mathbf{I}_6), \tag{3.25}$$

respectively defined by (3.9), (3.15)–(3.19) and (3.24). The corresponding candidate for a complete basis of $[\mathrm{SL}_h(2,\mathbb{R}) \times G_4]$-invariant polynomials in the irreducible cases under consideration is then given by the four polynomials

$$\left(\mathcal{W},\ \mathbf{I}_6,\ \mathrm{Tr}\left(\mathbf{I}^2\right),\ \mathrm{Tr}\left(\mathbf{I}^3\right)\right), \tag{3.26}$$

where [18]

$$\mathrm{Tr}\left(\mathbf{I}^2\right) = \mathbf{I}_{+2}\mathbf{I}_{-2} + 3\mathbf{I}_0^2 - 4\mathbf{I}_{+1}\mathbf{I}_{-1}; \tag{3.27}$$

$$\mathrm{Tr}\left(\mathbf{I}^3\right) = \mathbf{I}_0^3 + \mathbf{I}_{+2}\mathbf{I}_{-1}^2 + \mathbf{I}_{-2}\mathbf{I}_{+1}^2 - \mathbf{I}_{+2}\mathbf{I}_{-2}\mathbf{I}_0 - 2\mathbf{I}_{+1}\mathbf{I}_0\mathbf{I}_{-1}. \tag{3.28}$$

Indeed, the spin $s = 2$ representation $\mathbf{5}$ of $\mathrm{SL}_h(2,\mathbb{R})$, whose components are the G_4-invariant polynomials $\mathbf{I}_{+2}$, $\mathbf{I}_{+1}$, $\mathbf{I}_0$, $\mathbf{I}_{-1}$ and $\mathbf{I}_{-2}$ (defined by (3.15)–(3.19)), can be re-arranged as a 3×3 symmetric traceless matrix $\mathbf{I}$ [18]. (3.27) and (3.28) (respectively homogeneous of order eight and twelve in charges) are the only independent $\mathrm{SL}_h(2,\mathbb{R})$-singlets which can be built out of such a 3×3 symmetric matrix $\mathbf{I}$, due to its tracelessness [18]. Note that $\mathrm{Tr}(\mathbf{I}^2)$ and $\mathrm{Tr}(\mathbf{I}^3)$ are both invariant under $1 \leftrightarrow 2$.

It is worth pointing out that the analysis of sections 2 and 3 can be easily generalised to $p \geqslant 3$ centers. The two-centered representation of spin $s = J/2$ of $\mathrm{SL}_h(2,\mathbb{R})$ is then replaced by the completely symmetric rank-J tensor representation $\mathcal{R}_J$ of $\mathrm{SL}_h(p,\mathbb{R})$ ($J = 1,2,3,4$ are the values relevant for the above analysis). On the other hand, $\mathcal{W}$ and $\mathbf{I}_6$ generally sit in the $\left(\widetilde{\mathcal{R}}_2, \mathbf{1}\right)$ representation of $\mathrm{SL}_h(p,\mathbb{R}) \times G_4$, where $\widetilde{\mathcal{R}}_2$ is the rank-2 antisymmetric representation of $\mathrm{SL}_h(p,\mathbb{R})$ (which, in the case $p = 2$, becomes a singlet). However, due to the tree structure of the split flow in multi-center supergravity solutions [1–3, 10–12], to consider only the case $p = 2$ does not imply any loss in generality (as far as marginal stability issues are concerned).

4 Two-centered orbits with non-compact stabiliser: the $\mathcal{N} = 8$ BPS and octonionic $\mathcal{N} = 2$ non-BPS cases

For $\mathcal{N} = 2$ BPS two-centered extremal black holes, the stabiliser of the supporting charge orbit is always *compact*, so the orbit is unique (see table 1 for magical models). This is no longer the case when the stabiliser is non-compact, as it holds for $\mathcal{N} = 2$ two-centered solutions with two non-BPS centers characterised by $\mathcal{I}_4\left(\mathcal{Q}_1^4\right) > 0$ and $\mathcal{I}_4\left(\mathcal{Q}_2^4\right) > 0$, and for $\mathcal{N} \geqslant 3$ two-centered solutions with two $\frac{1}{\mathcal{N}}$-BPS centers. These are interesting cases, in which a *split attractor flow* through a wall of marginal stability has been shown to occur [37, 38].

We will consider here the $\frac{1}{8}$- BPS two-centered orbits in the maximal $\mathcal{N} = 8$ theory (based on $J_3^{\mathbb{O}_s}$) and the non-BPS two-centered orbits (of the aforementioned type) in the exceptional $\mathcal{N} = 2$ magic model, based on $J_3^{\mathbb{O}}$. These two cases can be obtained by repeating the analysis of section 2.1 and choosing suitable non-compact real forms of G_4 and $\mathcal{G}_2$.

The 1-centered charge orbits respectively read [23, 39]:

$$\mathcal{N} = 8,\ \frac{1}{8}\text{-BPS} : \mathcal{O}_{p=1} = \frac{E_{7(7)}}{E_{6(2)}}; \tag{4.1}$$

$$\mathcal{N} = 2,\ J_3^{\mathbb{O}}\ \text{nBPS}\ \mathcal{I}_4 > 0 : \mathcal{O}_{p=1} = \frac{E_{7(-25)}}{E_{6(-14)}}. \tag{4.2}$$

In the maximal case, the chain of relevant group branchings reads

$$\mathcal{N} = 8,\ \frac{1}{8}\text{-BPS} : E_{7(7)} \longrightarrow E_{6(2)} \longrightarrow F_{4(4)} \longrightarrow \mathrm{SO}(5,4) \longrightarrow \begin{cases} \mathrm{SO}(4,4) \\ or \\ \mathrm{SO}(5,3) \end{cases}, \tag{4.3}$$

such that two $\frac{1}{8}$-BPS, $\mathcal{N} = 8$, 2-centered charge orbits exist:

$$\mathcal{O}_{\mathcal{N}=8,\frac{1}{8}\text{-BPS},p=2,\mathbf{I}} = \frac{E_{7(7)}}{\mathrm{SO}(4,4)} \tag{4.4}$$

$$\mathcal{O}_{\mathcal{N}=8,\frac{1}{8}\text{-BPS},p=2,\mathbf{II}} = \frac{E_{7(7)}}{\mathrm{SO}(5,3)}. \tag{4.5}$$

In the $\mathcal{N} = 2$ exceptional case, the chain of relevant group branchings reads

$$\mathcal{N} = 2, J_3^{\mathbb{O}}\ \text{nBPS} : E_{7(-25)} \longrightarrow E_{6(-14)} \longrightarrow F_{4(-20)} \longrightarrow \begin{cases} \mathrm{SO}(9) \longrightarrow \mathrm{SO}(8) \\ or \\ \mathrm{SO}(8,1) \longrightarrow \begin{matrix} \mathrm{SO}(8) \\ or \\ \mathrm{SO}(7,1) \end{matrix} \end{cases}, \tag{4.6}$$

such that two non-BPS, $\mathcal{N} = 2$, 2-centered charge orbits exist:

$$\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{I}} = \frac{E_{7(-25)}}{\mathrm{SO}(8)} \tag{4.7}$$

$$\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{II}} = \frac{E_{7(-25)}}{\mathrm{SO}(7,1)}. \tag{4.8}$$

As it holds for the stabilizer of $\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{BPS},p=2}$ (see table 1), the Lie algebra $\mathfrak{so}(8)$ of the stabilizer of $\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{I}}$ (4.7) is nothing but the Lie algebra $\mathfrak{tri}(\mathbb{O})$ of the automorphism group $Aut\left(\mathfrak{t}\left(\mathbb{O}\right)\right)$ of the *normed triality* over the octonionic division algebra

$\mathbb{O}$ (see e.g. eq. (21) of [27]). It is here worth observing that the Lie algebra $\mathfrak{so}(4,4)$ of the stabilizer of $\mathcal{O}_{\mathcal{N}=8,\frac{1}{8}\text{-BPS},p=2,\mathbf{I}}$ (4.4) enjoys an analogous interpretation as the Lie algebra $\mathfrak{tri}(\mathbb{O}_s)$ of the automorphism group $Aut(\mathbf{t}(\mathbb{O}_s))$ of the *normed triality* over the *split* form $\mathbb{O}_s$ of the octonions. On the other hand, a similar interpretation seems not to hold for the stabilizer of $\mathcal{O}_{\mathcal{N}=8,\frac{1}{8}\text{-BPS},p=2,\mathbf{II}}$ (4.5) as well as for the stabilizer of $\mathcal{O}_{\mathcal{N}=2,J_3^{\mathbb{O}},\text{nBPS},p=2,\mathbf{II}}$ (4.8).

We expect the $\mathcal{N}=8$ orbits (4.4) and (4.5), as well as the $\mathcal{N}=2$ orbits (4.7) and (4.8), to be defined by different constraints on the four $\text{SL}_h(2,\mathbb{R})\times G_4$ invariant polynomials given by eq. (3.26); we leave this interesting issue for further future investigation.

Here, we confine ourselves to present parallel results on pseudo-orthogonal groups, which may shed some light on the whole framework. Let us consider two vectors $\mathbf{x}$ and $\mathbf{y}$ in a pseudo-Euclidean $(p+q)$-dimensional space $E_{p,q}$ with signature (p,q) and $p>1$, $q>1$. The norm of a vector is defined as, say

$$\mathbf{x}^2 \equiv x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2, \tag{4.9}$$

and the scalar product as

$$\mathbf{x}\cdot\mathbf{y} \equiv x_1y_1+\ldots+x_py_p-x_{p+1}y_{p+1}-\ldots-x_{p+q}y_{p+q}. \tag{4.10}$$

The one-vector orbits (for non-lightlike vectors) are well

$$\mathcal{O}_{p=1,\text{timelike}} = \frac{\text{SO}(p,q)}{\text{SO}(p-1,q)} \quad \text{if} \quad \mathbf{x}^2>0; \tag{4.11}$$

$$\mathcal{O}_{p=1,\text{spacelike}} = \frac{\text{SO}(p,q)}{\text{SO}(p,q-1)} \quad \text{if} \quad \mathbf{x}^2<0. \tag{4.12}$$

It is intuitively clear that the two-vector orbits do depend on the nature of the vectors themselves. Let us start and consider two timelike vectors ($\mathbf{x}^2>0$ and $\mathbf{y}^2>0$), whose one-center orbits are separately given by $\mathcal{O}_{p=1,\text{timelike}}$. It is straightforward to show that the two-center orbits supporting this configuration are

$$\frac{\text{SO}(p,q)}{\text{SO}(p-2,q)} \quad \text{if} \quad \mathbf{x}^2\mathbf{y}^2>(\mathbf{x}\cdot\mathbf{y})^2; \tag{4.13}$$

$$\frac{\text{SO}(p,q)}{\text{SO}(p-1,q-1)} \quad \text{if} \quad \mathbf{x}^2\mathbf{y}^2<(\mathbf{x}\cdot\mathbf{y})^2. \tag{4.14}$$

If both vectors are spacelike ($\mathbf{x}^2<0$ and $\mathbf{y}^2<0$), the two-center orbits read

$$\frac{\text{SO}(p,q)}{\text{SO}(p,q-2)} \quad \text{if} \quad \mathbf{x}^2\mathbf{y}^2>(\mathbf{x}\cdot\mathbf{y})^2; \tag{4.15}$$

$$\frac{\text{SO}(p,q)}{\text{SO}(p-1,q-1)} \quad \text{if} \quad \mathbf{x}^2\mathbf{y}^2<(\mathbf{x}\cdot\mathbf{y})^2. \tag{4.16}$$

Finally, if one vector is timelike and the other one is spacelike (say, $\mathbf{x}^2>0$ and $\mathbf{y}^2<0$), the two-center orbit is unique:

$$\frac{\text{SO}(p,q)}{\text{SO}(p-1,q-1)}, \tag{4.17}$$

because in this case $\mathbf{x}^2\mathbf{y}^2 < (\mathbf{x}\cdot\mathbf{y})^2$ always holds.

By introducing the $SL_h(2,R) \times SO(p,q)$ invariant polynomial (see [40–43] and the last ref. of [4–7])

$$\mathbf{I}_4(\mathbf{x},\mathbf{y}) \equiv \mathbf{x}^2\mathbf{y}^2 - (\mathbf{x}\cdot\mathbf{y})^2, \tag{4.18}$$

all orbits (4.13)–(4.17) can actually be recognised to correspond to only three orbits (namely (4.13), (4.15), and (4.14)=(4.16)=(4.17)), respectively defined by the $[SL_h(2,R) \times SO(p,q)]$-invariant constraints: $\mathbf{I}_4 > 0$ (with $\mathbf{x}^2 > 0$ and $\mathbf{y}^2 > 0$); $\mathbf{I}_4 > 0$ (with $\mathbf{x}^2 < 0$ and $\mathbf{y}^2 < 0$); $\mathbf{I}_4 < 0$. Note that in the compact case (Euclidean signature: $q = 0$) $\mathbf{I}_4 > 0$ due to the Cauchy-Schwarz triangular inequality, and the two-vector orbit is unique: $\frac{SO(p)}{SO(p-2)}$. This is in analogy with the results (obtained in the complex field) discussed in section 2.

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