Asymptotic stochastic characterization of phase and amplitude noise in free-running oscillators

ASYMPTOTIC STOCHASTIC CHARACTERIZATION OF PHASE AND AMPLITUDE NOISE IN FREE-RUNNING OSCILLATORS

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Starting from the definition of the stochastic differential equation for amplitude and phase fluctuations of an oscillator described by an ordinary differential equation, we study the associated Fokker–Planck equation by using tools from stochastic integral calculus, harmonic analysis and Floquet theory. We provide an asymptotic characterization of the relevant correlation functions, showing that within the assumption of a linear perturbative analysis for the amplitude fluctuations phase noise and orbital fluctuations at the same time are asymptotically statistically independent, and therefore the nonlinear perturbative analysis of phase noise recently derived still exactly holds even if orbital noise is taken into account.

Keywords: Oscillator; Fokker–Planck equation; stochastic integral calculus; orbital noise.

1. Introduction

Noise in autonomous circuits has been studied for decades [1–3] because of the large impact on the performance of many electronic and telecommunication systems. In presence of noise, the oscillator noiseless solution $\mathbf{x}_{\mathrm{S}}(t)$ (a periodic function of period T, leading to the oscillation frequency $f_0 = 1/T$) is plagued by fluctuations which, in turn, are decomposed into a perturbation of the phase (or, equivalently, of the time reference) of the output signal (phase noise) and into an orbital deviation (amplitude or orbital noise). We consider here the case of stable oscillators, both because this is the practically more important condition, and because the stability

of the limit cycle $\mathbf{x}_{\mathrm{S}}(t)$ is a prerequisite to allow for a perturbative noise analysis. Under this assumption, amplitude noise is quenched by the very nature of the oscillator making in many cases phase noise the dominant effect to be taken into practical consideration [4, 5].

Despite this rather long history, the detailed analysis of some specific features such as the noise spectrum arbitrarily close to the oscillator frequency harmonics led to the development of more complex, albeit rigorous, techniques based on the use of Floquet theory [6]. The nonlinear perturbative approach developed in [6] is based on two assumptions:

- Orbital deviations are completely neglected before starting the analysis of phase fluctuations;
- Phase noise is treated defining a time-reference fluctuation $\alpha(t)$ by means of a nonlinear differential equation based on the Floquet decomposition of the solution of the oscillator equations linearized around $\mathbf{x}_{\mathrm{S}}(t)$. The stochastic properties of $\alpha(t)$ are then rigorously derived making use of the associated Fokker–Planck equation [7].

In this contribution, we generalize the treatment in [6] by including into the stochastic analysis the effect of orbital deviations under the assumption that they represent a small deviation of the noiseless solution.

The paper is structured as follows: the stochastic differential system governing the fluctuations dynamics is discussed in Sec. 2, while its solution is derived in terms of the corresponding characteristic function in Sec. 3. The results of the mathematical derivation are given a geometric interpretation in Sec. 4. Finally, Sec. 5 is devoted to the conclusions.

2. Derivation of the Stochastic System

The oscillator is described by the autonomous ordinary differential equation:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} - \mathbf{f}(\mathbf{x}) = \mathbf{0}, \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector, $\mathbf{f}(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear function, and $\mathbf{x}_{\mathrm{S}}(t)$ the non-zero solution of (1). Noise is added to the right hand side of Eq. (1)

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} - \mathbf{f}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\mathbf{b}(t) \tag{2}$$

leading to the noisy solution $\mathbf{z}(t)$. Noise sources are represented by the set of p stochastic processes $\mathbf{b}(t) \in \mathbb{R}^p$, while $\mathbf{B}(\mathbf{z}) \in \mathbb{R}^{n\times p}$ is a solution-dependent matrix accounting for possible modulation of the noise sources. The noisy solution $\mathbf{z}(t)$ is decomposed as [6]

$$\mathbf{z}(t) = \mathbf{x}_{\mathrm{S}}(t + \alpha(t)) + \mathbf{y}(t), \tag{3}$$

where $\mathbf{y}(t)$ respresents the orbital deviation. The autocorrelation matrix of the noisy solution is therefore

$$\mathbf{R}_{\mathbf{z},\mathbf{z}}(t,\tau) = \mathrm{E}\{\mathbf{z}(t)\mathbf{z}^{\dagger}(t+\tau)\} = \mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{x}_\mathrm{S}}(t,\tau) + \mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{x}_\mathrm{S}}(t,\tau) + \mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau), \tag{4}$$

where $\mathrm{E}\{\cdot\}$ is the ensemble average operator, † denotes the complex conjugate and transpose operation, and

$$\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{x}_\mathrm{S}}(t,\tau) = \mathrm{E}\{\mathbf{x}_\mathrm{S}(t+\alpha(t))\mathbf{x}_\mathrm{S}^{\dagger}(t+\tau+\alpha(t+\tau))\}, \tag{5a}$$

$$\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,\tau) = \mathrm{E}\{\mathbf{x}_\mathrm{S}(t+\alpha(t))\mathbf{y}^{\dagger}(t+\tau)\}, \tag{5b}$$

$$\mathbf{R}_{\mathbf{y},\mathbf{x}_\mathrm{S}}(t,\tau) = \mathrm{E}\{\mathbf{y}(t)\mathbf{x}_\mathrm{S}^{\dagger}(t+\tau+\alpha(t+\tau))\}, \tag{5c}$$

$$\mathbf{R}_{\mathbf{y},\mathbf{y}}(t,\tau) = \mathrm{E}\{\mathbf{y}(t)\mathbf{y}^{\dagger}(t+\tau)\}. \tag{5d}$$

The first term corresponds to phase noise, Eq. (5d) to orbital noise and Eqs. (5b)–(5c) to phase-orbital correlation.

Defining the $n+1$ dimensional (column) vector stochastic process $\mathbf{Y}(t)$

$$\mathbf{Y}(t) = \begin{bmatrix} \alpha(t) \\ \mathbf{y}(t) \end{bmatrix} \tag{6}$$

the discussion in [6] (see Eq. (12) and Theorem 6.1) allows to obtain the dynamic equation for $\mathbf{Y}(t)$

$$\frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}t} = \mathbf{F}[\mathbf{Y}(t),t]\mathbf{Y}(t) + \mathbf{G}[\mathbf{Y}(t),t]\mathbf{b}(t), \tag{7}$$

where matrices $\mathbf{F} \in \mathbb{R}^{(n+1)\times(n+1)}$ and $\mathbf{G} \in \mathbb{R}^{(n+1)\times p}$ are

$$\mathbf{F}\left[\mathbf{Y}(t),t\right] = \begin{bmatrix} 0 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}(t+\alpha(t)) \end{bmatrix}, \tag{8}$$

$$\mathbf{G}\left[\mathbf{Y}(t),t\right] = \begin{bmatrix} \mathbf{v}_1^{\mathrm{T}}(t+\alpha(t))\mathbf{B}(t+\alpha(t)) \\ \displaystyle\sum_{k=2}^{n} \mathbf{u}_k(t+\alpha(t))\mathbf{v}_k^{\mathrm{T}}(t+\alpha(t))\mathbf{B}(t+\alpha(t)) \end{bmatrix}. \tag{9}$$

In Eqs. (8) and (9)

$$\mathbf{A}(t+\alpha(t)) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}_\mathrm{S}(t+\alpha(t))}, \quad \mathbf{B}(t+\alpha(t)) = \mathbf{B}[\mathbf{x}_\mathrm{S}(t+\alpha(t))] \tag{10}$$

are T periodic functions of time, as well as the n Floquet eigenvectors $\mathbf{u}_k(t) \in \mathbb{R}^n$ and $\mathbf{v}_k(t) \in \mathbb{R}^n$ associated, respectively, to the direct and adjoint linear systems derived linearizing (1) around the limit cycle $\mathbf{x}_\mathrm{S}(t)$ and corresponding to the Floquet exponent μ_k: details on Floquet theorem can be found in [6, 8]. Notice that we have assumed $k=1$ for the zero Floquet exponent (i.e., $\mu_1 = 0$) always present for the periodic solution of autonomous systems.

3. Analysis of the Characteristic Function

The statistical properties of $\mathbf{Y}(t)$ are assessed by studying the characteristic function $\varphi(\boldsymbol{\omega}, t)$ defined as the $(n+1)$-dimensional integral

$$\varphi(\boldsymbol{\omega}, t) = \mathrm{E}\{\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{Y}(t)}\} = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\boldsymbol{\omega}^{\mathrm{T}}\boldsymbol{\eta}}\, p_{\mathbf{Y}}(\boldsymbol{\eta}, t)\, \mathrm{d}\boldsymbol{\eta}, \tag{11}$$

where $\boldsymbol{\omega} = [\omega_1, \ldots, \omega_{n+1}]^{\mathrm{T}}$, i is the imaginary unit and $p_{\mathbf{Y}}(\boldsymbol{\eta}, t)$ is the probability density of the vector stochastic process $\mathbf{Y}(t)$.

Since Eqs. (8) and (9) are both T-periodic functions, they can be expanded into a Fourier series obtaining

$$\mathbf{F}[\mathbf{Y}(t), t] = \sum_j \tilde{\mathbf{F}}_j\, \mathrm{e}^{\mathrm{i}j\omega_0(t+\alpha(t))}, \tag{12}$$

$$\mathbf{G}[\mathbf{Y}(t), t] = \sum_j \tilde{\mathbf{G}}_j\, \mathrm{e}^{\mathrm{i}j\omega_0(t+\alpha(t))}, \tag{13}$$

where the sums span all integer values from $-\infty$ to $+\infty$, and $\omega_0 = 2\pi/T$. Inspection of Eqs. (8) and (9) allows to define

$$\tilde{\mathbf{F}}_j = \begin{bmatrix} 0 & \mathbf{0} \\ 0 & \tilde{\mathbf{A}}_j \end{bmatrix}, \tag{14}$$

where $\tilde{\mathbf{A}}_j \in \mathbb{C}^{n\times n}$ is the j-th harmonic component of $\mathbf{A}(t)$, and

$$\tilde{\mathbf{G}}_j = \begin{bmatrix} \tilde{\mathbf{V}}_{1_j}^{\mathrm{T}} \\ \sum_{k=2}^{n} \tilde{\mathbf{V}}_{k_j} \end{bmatrix}, \tag{15}$$

where $\tilde{\mathbf{V}}_{1_j}^{\mathrm{T}} \in \mathbb{C}^{1\times p}$ is the j-th Fourier component of $\mathbf{v}_1^{\mathrm{T}}(t)\mathbf{B}(t)$, and $\tilde{\mathbf{V}}_{k_j} \in \mathbb{C}^{n\times p}$ represents the j-th harmonic component of $\mathbf{u}_k(t)\mathbf{v}_k(t)^{\mathrm{T}}\mathbf{B}(t)$.

Lemma 1. *The characteristic function $\varphi(\boldsymbol{\omega}, t)$ satisfies the following equation*:

$$\frac{\partial\varphi(\boldsymbol{\omega}, t)}{\partial t} = \sum_{k=1}^{n+1} \omega_k \sum_h \tilde{\mathbf{F}}_{h_k}^T \nabla_{\boldsymbol{\omega}}\varphi(\boldsymbol{\omega} + h\overline{\boldsymbol{\omega}}_0, t)\mathrm{e}^{\mathrm{i}h\omega_0 t} - \sum_{k=1}^{n+1} \omega_k \sum_{h,r} \Bigg[\omega_0 \lambda \delta_{k,1} h \tilde{\mathbf{G}}_{h_k}^T \tilde{\mathbf{G}}_{-r_k}$$
$$+ \frac{1}{2}\sum_{j=1}^{n+1} \omega_j \tilde{\mathbf{G}}_{h_k}^T \tilde{\mathbf{G}}_{-r_j} \Bigg] \varphi(\boldsymbol{\omega} + (h-r)\overline{\boldsymbol{\omega}}_0, t)\mathrm{e}^{\mathrm{i}\omega_0(h-r)t}, \tag{16}$$

where $\nabla_{\boldsymbol{\omega}}$ is the (column) gradient operator with respect to the $\boldsymbol{\omega}$ variables, δ is Kronecker's symbol, $\overline{\boldsymbol{\omega}}_0 = [\omega_0, 0, \ldots, 0]^T$, and $\tilde{\mathbf{F}}_{h_k}^T, \tilde{\mathbf{G}}_{h_k}^T$ are the k-th rows of $\tilde{\mathbf{F}}_h, \tilde{\mathbf{G}}_h$, respectively.

Proof. See Appendix A. □

Once the governing equation for the characteristic function (16) has been derived, the next step is to verify whether a Gaussian random variable is obtained, at least asymptotically for $t \to +\infty$. Since we will not be able to prove the full Gaussianess of $\mathbf{Y}(t)$, we provide a weaker definition.

Definition 1. A vector stochastic variable $\mathbf{X}(t) = [X_1(t), \ldots, X_{n+1}(t)]^{\mathrm{T}}$ is called *weakly Gaussian* if its characteristic function has the form

$$\varphi(\boldsymbol{\omega}, t) = \exp\left(\mathrm{i}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{m}(t) - \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{K}(t)\boldsymbol{\omega}\right), \tag{17}$$

where $\mathbf{m}(t) \in \mathbb{R}^{n+1}$ and $\mathbf{K}(t) \in \mathbb{R}^{(n+1)\times(n+1)}$.

Remark 1. If $\mathbf{K}(t)$ is a symmetric matrix, then $\mathbf{X}(t)$ is a Gaussian vector random variable.

Remark 2. Since for any characteristic function

$$\left.\frac{\partial \varphi}{\partial \omega_j}\right|_{\boldsymbol{\omega}=\mathbf{0}} = -\mathrm{i}\mathrm{E}\{X_j(t)\}, \tag{18}$$

the expected value of $\mathbf{X}(t)$ is given by

$$\mathrm{E}\{\mathbf{X}(t)\} = \mathbf{m}(t). \tag{19}$$

Remark 3. Since for any characteristic function

$$\left.\frac{\partial^2 \varphi}{\partial \omega_j \partial \omega_k}\right|_{\boldsymbol{\omega}=\mathbf{0}} = -\mathrm{E}\{X_j(t)X_k(t)\}, \tag{20}$$

the covariance matrix of $\mathbf{X}(t)$ is given by

$$\mathrm{E}\{\mathbf{X}(t)\mathbf{X}^{\dagger}(t)\} - \mathbf{m}(t)\mathbf{m}^{\dagger}(t) = \frac{1}{2}[\mathbf{K}(t) + \mathbf{K}^{\mathrm{T}}(t)]. \tag{21}$$

Proposition 1. *If a vector stochastic variable $\mathbf{X}(t)$ is weakly Gaussian, then each component $X_j(t)$ is a Gaussian scalar stochastic variable.*

Proof. From Definition 1, it follows

$$\varphi(\overline{\boldsymbol{\omega}}_j, t) = \mathrm{E}\{\mathrm{e}^{\mathrm{i}\omega X_j(t)}\} = \mathrm{e}^{\mathrm{i}\omega m_j(t) - K_{j,j}(t)\omega^2/2}, \tag{22}$$

where $\overline{\boldsymbol{\omega}}_j$ is a null vector except for the j-th component, equal to ω. The previous equation is the characteristic function of a scalar Gaussian variable. □

Theorem 2. *The solution of Eq. (16) becomes, for $t \to +\infty$, the characteristic function of a weakly Gaussian stochastic variable*

$$\varphi^{\infty}(\boldsymbol{\omega}, t) = \exp\left(\mathrm{i}\boldsymbol{\omega}^{T}\mathbf{m}^{\infty} - \frac{1}{2}\boldsymbol{\omega}^{T}\mathbf{K}^{\infty}(t)\boldsymbol{\omega}\right), \tag{23}$$

meaning that $\varphi(\boldsymbol{\omega}, t) \sim \varphi^{\infty}(\boldsymbol{\omega}, t)$ for $t \to +\infty$. In Eq. (23) $\mathbf{m}^{\infty} = [m, 0, \ldots, 0]^{T}$ is time-independent and matrix $\mathbf{K}^{\infty}(t)$ is defined in Eqs. (B.19), (B.20), (B.25) and (B.33).

Proof. See Appendix B. □

Theorem 1 can be used to assess the asymptotic properties of the correlation function $\mathbf{R}_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,0)$ expressing, according to Eq. (5b), the correlation between $\mathbf{x}_\mathrm{S}(t+\alpha(t))$ and the orbital deviation $\mathbf{y}(t)$. Developing $\mathbf{x}_\mathrm{S}(t)$ in Fourier series, and denoting its Fourier coefficients as $\tilde{\mathbf{X}}_h$, the asymptotic correlation $\mathbf{R}^\infty_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,0)$ is easily expressed as a function of the asymptotic characteristic function

$$\begin{aligned}\mathbf{R}^\infty_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,0) &= \sum_h \tilde{\mathbf{X}}_h\,\mathrm{e}^{\mathrm{i}h\omega_0 t}\,\mathrm{E}\{\mathrm{e}^{\mathrm{i}h\omega_0\alpha(t)}\,\mathbf{y}^\dagger(t)\}\\ &= -\mathrm{i}\sum_h \tilde{\mathbf{X}}_h\,\mathrm{e}^{\mathrm{i}h\omega_0 t}\,\nabla^\mathrm{T}_{\boldsymbol{\omega}_\mathbf{y}}\varphi^\infty(\boldsymbol{\omega},t)\Big|_{\boldsymbol{\omega}_h},\end{aligned} \tag{24}$$

where $\nabla^\mathrm{T}_{\boldsymbol{\omega}_\mathbf{y}}$ is the (row) gradient operator acting on the $\omega_2,\ldots,\omega_{n+1}$ variables only, and $\boldsymbol{\omega}_h = [h\omega_0, 0, \ldots, 0]^\mathrm{T}$. Therefore, from Eq. (23) and Appendix B (see Eq. (B.4) for the definition of $\mathbf{K}^\infty_{:,\alpha}$ and $\mathbf{K}^\infty_{\alpha,:}$)

$$\mathbf{R}^\infty_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,0) = -\mathrm{i}\sum_h \frac{h\omega_0}{2}\tilde{\mathbf{X}}_h\,\mathrm{e}^{\mathrm{i}h\omega_0 t}\left(\mathbf{K}^\infty_{:,\alpha}(t)+\mathbf{K}^\infty_{\alpha,:}(t)\right)^\mathrm{T}\mathrm{e}^{\mathrm{i}h\omega_0 m}\,\mathrm{e}^{-h^2K^\infty_{\alpha,\alpha}(t)/2}, \tag{25}$$

which asymptotically tends to zero because of Eq. (B.19). This means that, after a proper system relaxation time t_0, the fluctuations along the limit cycle $\mathbf{x}_\mathrm{S}(t+\alpha(t))$ (i.e., the phase noise contribution) and the orbital deviation $\mathbf{y}(t)$ become statistically independent.

4. A Geometrical Interpretation of the Phase/Amplitude Decomposition

The decomposition of noise in phase and amplitude fluctuations as defined in Eq. (3) leads to the expressions for the (asymptotic) correlation functions derived in Theorem 1. At first glance, the asymptotically zero value of $\mathbf{R}^\infty_{\mathbf{x}_\mathrm{S},\mathbf{y}}(t,0)$ seems surprising since it appears to suggest a null correlation between amplitude noise and fluctuations along the orbit. A more detailed analysis, based on geometrical considerations, reveals that this is ultimately not the case.

In order to fix the ideas, and to make the discussion more concrete, we consider as an example the simple oscillator in [12, 13], although the discussion holds also in the general case. In polar coordinates, the defining system equations read

$$\dot{\rho} = \rho - \rho^2 + \beta\xi_\rho(t), \tag{26a}$$

$$\dot{\theta} = 1 + \rho + \beta\xi_\theta(t), \tag{26b}$$

where ρ and θ are, respectively, the radial and angular coordinates, β is a parameter, and $\xi(t)$ represent unit white Gaussian noise sources. The two-dimensional system

is converted in cartesian coordinates obtaining a system in the form of Eq. (2) where (z_i is the ith component of $\mathbf{z}$)

$$\mathbf{f}(\mathbf{z}) = \begin{bmatrix} z_1 - z_2 - (z_1 + z_2)\sqrt{z_1^2 + z_2^2} \\ z_1 + z_2 + (z_1 - z_2)\sqrt{z_1^2 + z_2^2} \end{bmatrix}, \tag{27}$$

$$\mathbf{B}(\mathbf{z}) = \beta \begin{bmatrix} z_1/\sqrt{z_1^2 + z_2^2} & -z_2 \\ z_2/\sqrt{z_1^2 + z_2^2} & z_1 \end{bmatrix}, \tag{28}$$

$$\mathbf{b} = [\xi_\rho, \xi_\theta]^{\mathrm{T}}. \tag{29}$$

The noiseless limit cycle is calculated for $\mathbf{b} = \mathbf{0}$, obtaining

$$\mathbf{x}_{\mathrm{S}}(t) = [\cos(2t), \sin(2t)]^{\mathrm{T}}. \tag{30}$$

For this simple system, Floquet analysis can be carried out analytically [12] yielding the two Floquet exponents $\mu_1 = 0$ and $\mu_2 = -1$ and the associated direct and adjoint Floquet eigenvectors:

$$\mathbf{u}_1(t) = 2 \begin{bmatrix} -\sin(2t) \\ \cos(2t) \end{bmatrix}, \quad \mathbf{v}_1(t) = \frac{1}{2} \begin{bmatrix} \cos(2t) - \sin(2t) \\ -\cos(2t) - \sin(2t) \end{bmatrix}, \tag{31}$$

$$\mathbf{u}_2(t) = \begin{bmatrix} -\cos(2t) - \sin(2t) \\ \cos(2t) - \sin(2t) \end{bmatrix}, \quad \mathbf{v}_2(t) = \begin{bmatrix} -\cos(2t) \\ -\sin(2t) \end{bmatrix}. \tag{32}$$

The eigenvectors were calculated using the standard orthonormalization condition $\mathbf{u}_j^{\mathrm{T}}(t)\mathbf{v}_k(t) = \mathbf{u}_j^{\mathrm{T}}(0)\mathbf{v}_k(0) = \delta_{jk}$.

These expressions allow us to provide an explicit form for the stochastic differential system (7)–(9), where $\mathbf{Y}(t) = [\alpha(t), y_1(t), y_2(t)]^{\mathrm{T}}$ ($y_i(t)$ is the i-th component of $\mathbf{y}(t)$) and

$$\mathbf{F}[\mathbf{Y}(t), t] = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 - \cos(4\hat{t}) - \sin(4\hat{t}) & -5 + \cos(4\hat{t}) - \sin(4\hat{t}) \\ 0 & 5 + \cos(4\hat{t}) - \sin(4\hat{t}) & -1 + \cos(4\hat{t}) + \sin(4\hat{t}) \end{bmatrix}, \tag{33}$$

$$\mathbf{G}[\mathbf{Y}(t), t] = \beta \begin{bmatrix} 1/2 & 1/2 \\ \cos(2\hat{t}) + \sin(2\hat{t}) & 0 \\ \sin(2\hat{t}) - \cos(2\hat{t}) & 0 \end{bmatrix}. \tag{34}$$

Notice that $\hat{t} = t + \alpha(t)$.

Starting from Eqs. (7) and (9) in [6], we have that the time reference (phase) fluctuation $\alpha(t)$ is governed by an equation independent of the orbital deviation:

$$\dot{\alpha}(t) = \frac{\beta}{2} \left[\xi_\rho(t) + \xi_\theta(t)\right]. \tag{35}$$

The very definition of α transforms it into a "secondary" source term for the equations setting the dynamics of the oscillator fluctuations along the noiseless

orbit $\mathbf{x}_{\mathrm{S}}(t)$

$$\dot{\mathbf{x}}_{\mathrm{S}}(t+\alpha(t)) = 2\begin{bmatrix} -\sin(2\hat{t}) \\ \cos(2\hat{t}) \end{bmatrix} \tag{36}$$

and for the orbital noise

$$\dot{\mathbf{y}}(t) = \frac{1}{2}\begin{bmatrix} -1-\cos(4\hat{t})-\sin(4\hat{t}) & -5+\cos(4\hat{t})-\sin(4\hat{t}) \\ 5+\cos(4\hat{t})-\sin(4\hat{t}) & -1+\cos(4\hat{t})+\sin(4\hat{t}) \end{bmatrix}\mathbf{y}(t)$$
$$+\beta\begin{bmatrix} \cos(2\hat{t})+\sin(2\hat{t}) \\ \sin(2\hat{t})-\cos(2\hat{t}) \end{bmatrix}\xi_\rho(t). \tag{37}$$

Notice that Eqs. (36) and (37) can be solved independently of each other: this is a direct consequence of the very definition of $\alpha(t)$ exploited in [6] on the basis of [10], in contrast with the decomposition proposed in [11] where the projection chosen to define the phase fluctuation $\alpha_{\mathrm{K}}(t)$ gives rise to a dynamic equation which in turn depends on the orbital deviation $\mathbf{y}$. This property makes it impossible to interpret $\alpha_{\mathrm{K}}(t)$ as a source term for the fluctuations of the state variables. Demir *et al.*'s approach [6], on the other hand, defines a stochastic variable (the time-reference fluctuation $\alpha(t)$) which makes (at least asymptotically) statistically independent $\mathbf{x}_{\mathrm{S}}(t+\alpha(t))$ and $\mathbf{y}(t)$ which, in turn, satisfy dynamic equations mathematically independent of each other. As a consequence oscillator fluctuations, through the projection on the subspaces defined by the Floquet eigenvectors, are completely decomposed into:

(1) A noise component pertaining to the space defined by the eigenvector $\mathbf{u}_1 = \dot{\mathbf{x}}_{\mathrm{S}}$, i.e., the phase noise defined through the fluctuations $\mathbf{x}_{\mathrm{S}}(t+\alpha(t))$ taking place only *along* the orbit;
(2) A second noise component (the orbital deviation $\mathbf{y}(t)$) pertaining, for each time instant t, to a hyperplane defined by the remaining $n-1$ Floquet eigenvectors. Notice that this hyperplane is always orthogonal to $\mathbf{v}_1$ but it is not, in general, orthogonal to $\mathbf{u}_1 = \dot{\mathbf{x}}_{\mathrm{S}}$ (i.e., to the oscillator orbit), and therefore depends on the fluctuations due to the stochastic variable α.

Ultimately this analysis allows to prove that the results in [6], although demonstrated neglecting orbital fluctuations altogether, still hold even if amplitude noise is accounted for, thus providing a sound foundation to the analysis in [14–16].

According to these remarks, notice that the asymptotic statistical independence between the fluctuations taking place for $\tau = 0$ in the $(n-1)$-dimensional hyperplane and along the orbit does not imply null phase-orbit correlations (i.e., between $\mathbf{x}_{\mathrm{S}}(t+\alpha(t))$ and $\mathbf{y}(t+\tau)$ for $\tau > 0$), since these are actually taken into account indirectly by means of the secondary source term defined above. In fact the expressions derived in [14] show that asymptotically with time

$$\mathbf{R}^{\infty}_{\mathbf{x}_{\mathrm{S}},\mathbf{y}}(t,\tau) = 0 \quad \text{if } \tau > 0 \quad \text{and} \quad \mathbf{R}^{\infty}_{\mathbf{y},\mathbf{x}_{\mathrm{S}}}(t,\tau) = 0 \quad \text{if } \tau \leq 0, \tag{38}$$

thus implying that $\mathbf{y}(t_1)$ is uncorrelated with $\mathbf{x}_{\mathrm{S}}(t_2+\alpha(t_2))$ for $t_2 > t_1$.

5. Conclusion

Exploiting a rigorous analysis based on the Fokker–Planck equation associated to the stochastic equation defining the dynamics of the phase and orbital fluctuations in oscillators, and making use of the Floquet decomposition proposed in [6], we have demonstrated that the phase noise and orbital fluctuations can be asymptotically decoupled, at least as far as a linearized approach for the amplitude noise is assumed, thus implying that the analysis in [6] still holds even for non-negligible $\mathbf{y}(t)$. Notice however that this does not correspond to a null correlation between phase and orbital noise for different observation times, since the latter is influenced by the $\alpha(t)$ process by means of an equivalent source term in the corresponding dynamic equation.

Appendix A. Proof of Lemma 1

To prove Lemma 1, we first consider a rigorous treatment of the stochastic differential equation (7). According to [6, 9], the Langevin equation (7) can be transformed into the Fokker–Planck equation for the probability density $p_{\mathbf{Y}}(\boldsymbol{\eta}, t)$ of $\mathbf{Y}(t)$

$$\frac{\partial p_{\mathbf{Y}}(\boldsymbol{\eta}, t)}{\partial t} = -\sum_{k=1}^{n+1} \frac{\partial}{\partial \eta_k} \left[\mathbf{F}_k^{\mathrm{T}}(\boldsymbol{\eta}, t) \boldsymbol{\eta} p_{\mathbf{Y}}(\boldsymbol{\eta}, t) \lambda \frac{\partial \mathbf{G}_k^{\mathrm{T}}(\boldsymbol{\eta}, t)}{\partial \eta_k} \mathbf{G}_k(\boldsymbol{\eta}, t) p_{\mathbf{Y}}(\boldsymbol{\eta}, t) \right] + \frac{1}{2} \sum_{k,j=1}^{n+1} \frac{\partial^2}{\partial \eta_k \partial \eta_j} [\mathbf{G}_k^{\mathrm{T}}(\boldsymbol{\eta}, t) \mathbf{G}_j(\boldsymbol{\eta}, t) p_{\mathbf{Y}}(\boldsymbol{\eta}, t)], \tag{A.1}$$

where $\boldsymbol{\eta} = [\eta_1, \ldots, \eta_{n+1}]^{\mathrm{T}}$, $\mathbf{F}_k^{\mathrm{T}}(\boldsymbol{\eta}, t)$ is the k-th row of matrix $\mathbf{F}[\mathbf{Y}(t), t]$ and $\mathbf{G}_k^{\mathrm{T}}(\boldsymbol{\eta}, t)$ is the k-th row of matrix $\mathbf{G}[\mathbf{Y}(t), t]$. Finally, λ is a real parameter as defined in [6].

Multiplying (A.1) times an arbitrary, limited and regular function $a(\boldsymbol{\eta})$ and integrating it over the entire $(n+1)$-dimensional $\boldsymbol{\eta}$ space allows, after some algebra, to derive

$$\frac{\partial}{\partial t} \mathrm{E}\{a\} = \mathrm{E}\left\{ \sum_{k=1}^{n+1} \frac{\partial a}{\partial Y_k} \left(\mathbf{F}_k^{\mathrm{T}} \mathbf{Y} + \lambda \frac{\partial \mathbf{G}_k^{\mathrm{T}}}{\partial Y_k} \mathbf{G}_k \right) \right\} + \frac{1}{2} \mathrm{E}\left\{ \sum_{k,j=1}^{n+1} \frac{\partial^2 a}{\partial Y_k \partial Y_j} \mathbf{G}_k^{\mathrm{T}} \mathbf{G}_j \right\}. \tag{A.2}$$

Choosing $a(\mathbf{Y}) = \exp(\mathrm{i}\boldsymbol{\omega}^{\mathrm{T}}\mathbf{Y})$, making use of the Fourier representations (12) and (13), and performing the derivatives the following expression is obtained:

$$\begin{aligned} \frac{\partial \varphi(\boldsymbol{\omega}, t)}{\partial t} = & \sum_{k=1}^{n+1} \mathrm{i}\omega_k \sum_h \tilde{\mathbf{F}}_{h_k}^{\mathrm{T}} \, \mathrm{e}^{\mathrm{i}h\omega_0 t} \, \mathrm{E}\{\mathbf{Y}(t) \, \mathrm{e}^{\mathrm{i}[\boldsymbol{\omega}^{\mathrm{T}} + h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}]\mathbf{Y}(t)}\} \\ & + \lambda \mathrm{i}\omega_1 \sum_{h,r} \mathrm{i}h\omega_0 \tilde{\mathbf{G}}_{h_1}^{\mathrm{T}} \tilde{\mathbf{G}}_{-r_1} \, \mathrm{e}^{\mathrm{i}\omega_0 (h-r)t} \, \mathrm{E}\{\mathrm{e}^{\mathrm{i}[\boldsymbol{\omega}^{\mathrm{T}} + (h-r)\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}]\mathbf{Y}(t)}\} \\ & + \frac{1}{2} \sum_{k,j=1}^{n+1} \mathrm{i}\omega_k \mathrm{i}\omega_j \sum_{h,r} \tilde{\mathbf{G}}_{h_k}^{\mathrm{T}} \tilde{\mathbf{G}}_{-r_j} \, \mathrm{e}^{\mathrm{i}\omega_0 (h-r)t} \, \mathrm{E}\{\mathrm{e}^{\mathrm{i}[\boldsymbol{\omega}^{\mathrm{T}} + (h-r)\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}]\mathbf{Y}(t)}\}, \end{aligned} \tag{A.3}$$

where $\overline{\boldsymbol{\omega}}_0 = [\omega_0, 0, \ldots, 0]^{\mathrm{T}}$, and $\tilde{\mathbf{F}}_{h_k}^{\mathrm{T}}$, $\tilde{\mathbf{G}}_{h_k}^{\mathrm{T}}$ are the k-th rows of $\tilde{\mathbf{F}}_h$, $\tilde{\mathbf{G}}_h$, respectively.

Since

$$\begin{aligned}\mathrm{E}\{\mathbf{Y}(t)\,\mathrm{e}^{\mathrm{i}[\boldsymbol{\omega}^{\mathrm{T}}+h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}]\mathbf{Y}(t)}\} &= -\mathrm{i}\nabla_{\boldsymbol{\omega}}\{\mathrm{e}^{\mathrm{i}[\boldsymbol{\omega}^{\mathrm{T}}+h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}]\mathbf{Y}(t)}\} \\ &= -\mathrm{i}\nabla_{\boldsymbol{\omega}}\varphi(\boldsymbol{\omega}+h\overline{\boldsymbol{\omega}}_0, t) \end{aligned} \tag{A.4}$$

Eq. (A.3) yields directly Eq. (16).

Appendix B. Proof of Theorem 1

To prove that Eq. (23) solves asymptotically Eq. (16), we substitute φ^{∞} into Eq. (16) obtaining the following expression:

$$\begin{aligned}&\mathrm{i}\boldsymbol{\omega}^{\mathrm{T}}\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} - \frac{1}{2}\boldsymbol{\omega}^{\mathrm{T}}\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}t}\boldsymbol{\omega} \\ &\quad = \boldsymbol{\omega}^{\mathrm{T}}\sum_h \mathrm{e}^{\mathrm{i}h\omega_0 t}\,\tilde{\mathbf{F}}_h\left[\mathrm{i}\mathbf{m} - \frac{1}{2}\left(\mathbf{K}+\mathbf{K}^{\mathrm{T}}\right)(\boldsymbol{\omega}+h\overline{\boldsymbol{\omega}}_0)\right] \\ &\qquad \times \mathrm{e}^{\mathrm{i}h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{m}}\,\mathrm{e}^{-h(h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{K}\overline{\boldsymbol{\omega}}_0+\boldsymbol{\omega}^{\mathrm{T}}\mathbf{K}\overline{\boldsymbol{\omega}}_0+\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{K}\boldsymbol{\omega})/2} \\ &\qquad - \sum_{h,r}\mathrm{e}^{\mathrm{i}(h-r)\omega_0 t}\,\boldsymbol{\omega}^{\mathrm{T}}\left(\lambda\omega_0 h\boldsymbol{\mathfrak{C}}^{(h,-r)} + \frac{1}{2}\boldsymbol{\mathfrak{G}}^{(h,-r)}\boldsymbol{\omega}\right) \\ &\qquad \times \mathrm{e}^{\mathrm{i}(h-r)\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{m}}\,\mathrm{e}^{-(h-r)^2\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{K}\overline{\boldsymbol{\omega}}_0/2}\,\mathrm{e}^{-(h-r)(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{K}\overline{\boldsymbol{\omega}}_0+\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{K}\boldsymbol{\omega})/2},\end{aligned} \tag{B.1}$$

where vector $\boldsymbol{\mathfrak{C}}^{(h,-r)} \in \mathbb{C}^{n+1}$ has elements

$$\mathfrak{C}_k^{(h,-r)} = \delta_{k,1}\tilde{\mathbf{G}}_{h_k}^{\mathrm{T}}\tilde{\mathbf{G}}_{-r_k}, \tag{B.2}$$

while matrix $\boldsymbol{\mathfrak{G}}^{(h,-r)} \in \mathbb{C}^{(n+1)\times(n+1)}$ is defined by the elements

$$\mathfrak{G}_{k,j}^{(h,-r)} = \tilde{\mathbf{G}}_{h_k}^{\mathrm{T}}\tilde{\mathbf{G}}_{-r_j}. \tag{B.3}$$

To proceed further, we decompose matrix $\mathbf{K}$ (and $\boldsymbol{\mathfrak{G}}^{(h,-r)}$) as follows:

$$\mathbf{K} = \begin{pmatrix} K_{\alpha,\alpha} & \mathbf{K}_{\alpha,:}^{\mathrm{T}} \\ \mathbf{K}_{:,\alpha} & \mathbf{K}_{\mathbf{y},\mathbf{y}} \end{pmatrix}, \tag{B.4}$$

where $K_{\alpha,\alpha}$ is a scalar, $\mathbf{K}_{\alpha,:}^{\mathrm{T}}$ a row vector and $\mathbf{K}_{:,\alpha}$ a column vector with n elements, and finally $\mathbf{K}_{\mathbf{y},\mathbf{y}}$ a $n \times n$ matrix. Accordingly, we have

$$\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{K}\overline{\boldsymbol{\omega}}_0 = \omega_0^2 K_{\alpha,\alpha}. \tag{B.5}$$

Furthermore, the exponential function can be developed in power series as a function of $\boldsymbol{\omega}$:

$$\mathrm{e}^{-q(\boldsymbol{\omega}^{\mathrm{T}}\mathbf{K}\overline{\boldsymbol{\omega}}_0+\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{K}\boldsymbol{\omega})/2} = 1 - q\frac{\omega_0}{2}(\mathbf{K}_{:,\alpha}^{\mathrm{T}} + \mathbf{K}_{\alpha,:}^{\mathrm{T}})\boldsymbol{\omega} + \cdots \tag{B.6}$$

Substituting Eqs. (B.5) and (B.6) into Eq. (B.1), equating the coefficients of order one in $\boldsymbol{\omega}$ we find

$$\frac{\mathrm{d}\mathbf{m}}{\mathrm{d}t} = -\mathrm{i}\sum_h \mathrm{e}^{\mathrm{i}h\omega_0 t}\,\tilde{\mathbf{F}}_h\left[\mathrm{i}\mathbf{m} - \frac{h}{2}\left(\mathbf{K}+\mathbf{K}^{\mathrm{T}}\right)\overline{\boldsymbol{\omega}}_0\right]\mathrm{e}^{\mathrm{i}h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{m}}\,\mathrm{e}^{-h^2\omega_0^2 K_{\alpha,\alpha}/2}$$
$$+\sum_{h,r}\mathrm{i}\lambda\omega_0 h\boldsymbol{\mathfrak{C}}^{(h,-r)}\,\mathrm{e}^{\mathrm{i}(h-r)\omega_0 t}\,\mathrm{e}^{\mathrm{i}(h-r)\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{m}}\,\mathrm{e}^{-(h-r)^2\omega_0^2 K_{\alpha,\alpha}/2}\,. \tag{B.7}$$

Similarly, the order two coefficients yield

$$\frac{\mathrm{d}\mathbf{K}}{\mathrm{d}t} = \sum_h \mathrm{e}^{\mathrm{i}h\omega_0 t}\,\tilde{\mathbf{F}}_h\Big\{\left(\mathbf{K}+\mathbf{K}^{\mathrm{T}}\right)$$
$$+h\omega_0\left[\mathrm{i}\mathbf{m} - \frac{h}{2}\left(\mathbf{K}+\mathbf{K}^{\mathrm{T}}\right)\overline{\boldsymbol{\omega}}_0\right]\left(\mathbf{K}^{\mathrm{T}}_{:,\alpha}+\mathbf{K}^{\mathrm{T}}_{\alpha,:}\right)\Big\}\,\mathrm{e}^{\mathrm{i}h\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{m}}\,\mathrm{e}^{-h^2\omega_0^2 K_{\alpha,\alpha}/2}$$
$$+\sum_{h,r}[\boldsymbol{\mathfrak{G}}^{(h,-r)} - \lambda\omega_0^2 h(h-r)\boldsymbol{\mathfrak{C}}^{(h,-r)}(\mathbf{K}^{\mathrm{T}}_{:,\alpha}+\mathbf{K}^{\mathrm{T}}_{\alpha,:})]$$
$$\times\,\mathrm{e}^{\mathrm{i}(h-r)\omega_0 t}\,\mathrm{e}^{\mathrm{i}(h-r)\overline{\boldsymbol{\omega}}_0^{\mathrm{T}}\mathbf{m}}\,\mathrm{e}^{-(h-r)^2\omega_0^2 K_{\alpha,\alpha}/2}\,. \tag{B.8}$$

According to [6], $K_{\alpha,\alpha}$ (the variance of the phase deviation $\alpha(t)$) is a (positive) linearly growing function of time. Therefore, Eqs. (B.7) and (B.8) can be asymptotically (for $t\to+\infty$) simplified by neglecting the terms with $h\neq 0$ in the simple sums, and the terms with $h\neq r$ in the double sums. The resulting equations are

$$\frac{\mathrm{d}\mathbf{m}^\infty}{\mathrm{d}t} = \tilde{\mathbf{F}}_0\mathbf{m}^\infty + \sum_h \mathrm{i}\lambda\omega_0 h\boldsymbol{\mathfrak{C}}^{(h,-h)}, \tag{B.9}$$

$$\frac{\mathrm{d}\mathbf{K}^\infty}{\mathrm{d}t} = \tilde{\mathbf{F}}_0(\mathbf{K}^\infty + \mathbf{K}^{\infty\mathrm{T}}) + \sum_h \boldsymbol{\mathfrak{G}}^{(h,-h)}. \tag{B.10}$$

Let us consider Eq. (B.9) first. Because of Eqs. (15), (B.2) and (B.3), it can be casted into the form

$$\frac{\mathrm{d}\mathbf{m}^\infty}{\mathrm{d}t} = \begin{bmatrix} \dfrac{\lambda}{T}\displaystyle\int_0^T \frac{\mathrm{d}(\mathbf{v}_1^{\mathrm{T}}\mathbf{B})}{\mathrm{d}t}\mathbf{B}^{\mathrm{T}}\mathbf{v}_1\,\mathrm{d}t \\ \tilde{\mathbf{A}}_0[m_2,\ldots,m_{n+1}]^{\mathrm{T}} \end{bmatrix}, \tag{B.11}$$

where the first element of the rhs vector is zero because both $\mathbf{v}_1(t)$ and $\mathbf{B}(t)$ are T-periodic functions. Thus, the general solution of Eq. (B.11) is

$$\mathbf{m}^\infty(t) = \begin{bmatrix} m \\ \mathrm{e}^{t\tilde{\mathbf{A}}_0}[m_2(0),\ldots,m_{n+1}(0)]^{\mathrm{T}} \end{bmatrix}, \tag{B.12}$$

where $m\in\mathbb{R}$ is constant. The initial values $m_j^\infty(0)$ $(j=2,\ldots,n+1)$ can be assumed zero, since they represent the average value at $t=0$ of the orbital deviation $\mathbf{y}(t)$, therefore we finally have:

$$\mathbf{m}^\infty(t) = [m,0,\ldots,0]^{\mathrm{T}}. \tag{B.13}$$

According to Eq. (B.4), Eq. (B.10) can be splitted into four differential equations, to be solved one after the other

$$\frac{\mathrm{d}K^{\infty}_{\alpha,\alpha}}{\mathrm{d}t} = \sum_h \mathfrak{G}^{(h,-h)}_{\alpha,\alpha}, \tag{B.14}$$

$$\frac{\mathrm{d}\mathbf{K}^{\infty}_{\alpha,:}}{\mathrm{d}t} = \sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:}, \tag{B.15}$$

$$\frac{\mathrm{d}\mathbf{K}^{\infty}_{:,\alpha}}{\mathrm{d}t} = \tilde{\mathbf{A}}_0[\mathbf{K}^{\infty}_{:,\alpha} + \mathbf{K}^{\infty}_{\alpha,:}] + \sum_h \mathfrak{G}^{(h,-h)}_{:,\alpha}, \tag{B.16}$$

$$\frac{\mathrm{d}\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}}}{\mathrm{d}t} = \tilde{\mathbf{A}}_0[\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}} + \mathbf{K}^{\infty\mathrm{T}}_{\mathbf{y},\mathbf{y}}] + \sum_h \mathfrak{G}^{(h,-h)}_{\mathbf{y},\mathbf{y}}. \tag{B.17}$$

The first two equations can be immediately solved, yielding a solution linear with time t. Equation (B.14) has been already derived in [6] neglecting the orbital deviation altogether (while here we use Eqs. (15) and (B.3))

$$\frac{\mathrm{d}K^{\infty}_{\alpha,\alpha}}{\mathrm{d}t} = \frac{1}{T}\int_0^T \mathbf{v}_1^{\mathrm{T}}\mathbf{B}\mathbf{B}^{\mathrm{T}}\mathbf{v}_1 \, \mathrm{d}t = c. \tag{B.18}$$

Assuming a zero initial condition, we find the same result as in [6]

$$K^{\infty}_{\alpha,\alpha}(t) = tc. \tag{B.19}$$

Similarly, Eq. (B.15) yields

$$\mathbf{K}^{\infty}_{\alpha,:}(t) = t\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:}, \tag{B.20}$$

where according to Eqs. (15) and (B.3)

$$\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:} = \frac{1}{T}\int_0^T [\mathbf{v}_1^{\mathrm{T}}\mathbf{B}\mathbf{G}_2, \ldots, \mathbf{v}_1^{\mathrm{T}}\mathbf{B}\mathbf{G}_{n+1}] \, \mathrm{d}t. \tag{B.21}$$

Turning to Eq. (B.16), we substitute Eq. (B.20) obtaining

$$\frac{\mathrm{d}\mathbf{K}^{\infty}_{:,\alpha}}{\mathrm{d}t} = \tilde{\mathbf{A}}_0\left[\mathbf{K}^{\infty}_{:,\alpha} + t\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:}\right] + \sum_h \mathfrak{G}^{(h,-h)}_{:,\alpha}. \tag{B.22}$$

The general solution of Eq. (B.22) reads

$$\begin{aligned}\mathbf{K}^{\infty}_{:,\alpha}(t) &= \mathrm{e}^{\tilde{\mathbf{A}}_0 t}\,\mathbf{K}^{\infty}_{:,\alpha}(0) + \mathrm{e}^{\tilde{\mathbf{A}}_0 t}\int_0^t \mathrm{e}^{-\tilde{\mathbf{A}}_0 s}\left[s\tilde{\mathbf{A}}_0\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:} + \sum_h \mathfrak{G}^{(h,-h)}_{:,\alpha}\right]\mathrm{d}s \\ &= \mathrm{e}^{\tilde{\mathbf{A}}_0 t}\,\mathbf{K}^{\infty}_{:,\alpha}(0) + [-t\mathbf{I} + \tilde{\mathbf{A}}_0^{-1}(\mathrm{e}^{\tilde{\mathbf{A}}_0 t} - \mathbf{I})]\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:} \\ &\quad + \tilde{\mathbf{A}}_0^{-1}(\mathrm{e}^{\tilde{\mathbf{A}}_0 t} - \mathbf{I})\sum_h \mathfrak{G}^{(h,-h)}_{:,\alpha},\end{aligned} \tag{B.23}$$

where $\mathbf{I}$ is the identity matrix. The exponential time-dependence of $\mathbf{K}^{\infty}_{:,\alpha}(t)$ can be readily eliminated (notice that the phase deviation $\alpha(t)$ is expected to grow

unbounded linearly with t [6]) by choosing as initial condition

$$\mathbf{K}^{\infty}_{:,\alpha}(0) = -\tilde{\mathbf{A}}_0^{-1}\left[\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:} + \sum_h \mathfrak{G}^{(h,-h)}_{:,\alpha}\right] \tag{B.24}$$

since substituting Eq. (B.24) into Eq. (B.23) yields a linear time dependence

$$\mathbf{K}^{\infty}_{:,\alpha}(t) = -(t\mathbf{I} + \tilde{\mathbf{A}}_0^{-1})\sum_h \mathfrak{G}^{(h,-h)}_{\alpha,:} - \tilde{\mathbf{A}}_0^{-1}\sum_h \mathfrak{G}^{(h,-h)}_{:,\alpha}. \tag{B.25}$$

Notice that, although $\mathbf{K}^{\infty}_{:,\alpha}(t)$ is linear in t, the sum $\mathbf{K}^{\infty}_{:,\alpha} + \mathbf{K}^{\infty}_{\alpha,:}$ is constant: in fact, substituting Eq. (B.25) into Eq. (B.16) we have that the left hand side and the second term at the rhs are constant, thus implying that $\mathbf{K}^{\infty}_{:,\alpha}(t) + \mathbf{K}^{\infty}_{\alpha,:}(t)$ is t-independent.

In order to solve Eq. (B.17) it is convenient to rewrite the matrix unknown $\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}}$ into a column vector $\boldsymbol{\mathcal{K}}$ of size n^2 built by collecting the columns of $\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}}$. A similar procedure on the constant term at the rhs of Eq. (B.17) allows to express it as

$$\frac{\mathrm{d}\boldsymbol{\mathcal{K}}}{\mathrm{d}t} = \boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0}\boldsymbol{\mathcal{K}} + \boldsymbol{\mathcal{V}}_{\mathfrak{G}_{\mathbf{y},\mathbf{y}}}, \tag{B.26}$$

where $\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0}$ is the $n^2 \times n^2$ matrix

$$\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0} = \mathrm{diag}\{\tilde{\mathbf{A}}_0, \ldots, \tilde{\mathbf{A}}_0\}(\mathbf{I} + \mathbf{P}), \tag{B.27}$$

$\mathbf{I}$ being the identity matrix of size n^2, and $\mathbf{P}$ a permutation matrix transforming $\boldsymbol{\mathcal{K}}$ into a vector corresponding to the collection of the rows of $\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}}$.[a] Notice that $\mathbf{I} + \mathbf{P}$ is not invertible, and a direct calculation allows to verify that its rank is $n_{\mathrm{r}} = n(n+1)/2$. The n_{r} non-zero eigenvalues of $\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0}$ are denoted as $\lambda_1, \ldots, \lambda_{n_{\mathrm{r}}}$.

The general solution of Eq. (B.26) is expressed as

$$\boldsymbol{\mathcal{K}}(t) = \mathrm{e}^{\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0} t}\boldsymbol{\mathcal{K}}(0) + \mathrm{e}^{\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0} t}\int_0^t \mathrm{e}^{-\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0} s}\boldsymbol{\mathcal{V}}_{\mathfrak{G}_{\mathbf{y},\mathbf{y}}}\,\mathrm{d}s, \tag{B.28}$$

where the integral cannot be directly calculated because $\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0}$ is not invertible. However, we can find an invertible matrix $\mathbf{Q}$ such that

$$\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0} = \mathbf{Q}\,\mathrm{diag}\{\lambda_1, \ldots, \lambda_{n_{\mathrm{r}}}, 0, \ldots, 0\}\mathbf{Q}^{-1}, \tag{B.29}$$

and since

$$\mathrm{e}^{\boldsymbol{\mathcal{D}}_{\tilde{\mathbf{A}}_0} t} = \mathbf{Q}\,\mathrm{diag}\{\mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_{n_{\mathrm{r}}} t}, 1, \ldots, 1\}\mathbf{Q}^{-1}, \tag{B.30}$$

Eq. (B.28) takes the form

$$\begin{aligned}\boldsymbol{\mathcal{K}}(t) = {} & \mathbf{Q}\,\mathrm{diag}\{\mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_{n_{\mathrm{r}}} t}, 1, \ldots, 1\}\mathbf{Q}^{-1}\boldsymbol{\mathcal{K}}(0) \\ & + \mathbf{Q}\,\mathrm{diag}\left\{\frac{\mathrm{e}^{\lambda_1 t} - 1}{\lambda_1}, \ldots, \frac{\mathrm{e}^{\lambda_{n_{\mathrm{r}}} t} - 1}{\lambda_{n_{\mathrm{r}}}}, t, \ldots, t\right\}\mathbf{Q}^{-1}\boldsymbol{\mathcal{V}}_{\mathfrak{G}_{\mathbf{y},\mathbf{y}}}.\end{aligned} \tag{B.31}$$

[a] This corresponds to the $\mathbf{K}^{\infty\mathrm{T}}_{\mathbf{y},\mathbf{y}}$ term in Eq. (B.17).

The oscillator limit cycle is orbitally stable, thus we choose again to avoid the exponential dependence by assuming:

$$\boldsymbol{\mathcal{K}}(0) = -\mathbf{Q}\,\mathrm{diag}\left\{\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_{n_\mathrm{r}}},0,\ldots,0\right\}\mathbf{Q}^{-1}\boldsymbol{\mathcal{V}}_{\mathfrak{G}_{\mathbf{y},\mathbf{y}}}, \tag{B.32}$$

in order to obtain a linear t-dependence of the solution

$$\boldsymbol{\mathcal{K}}(t) = \mathbf{Q}\,\mathrm{diag}\left\{-\frac{1}{\lambda_1},\ldots,-\frac{1}{\lambda_{n_\mathrm{r}}},t,\ldots,t\right\}\mathbf{Q}^{-1}\boldsymbol{\mathcal{V}}_{\mathfrak{G}_{\mathbf{y},\mathbf{y}}}, \tag{B.33}$$

and therefore $\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}}(t)$ and $\mathbf{K}^{\infty\mathrm{T}}_{\mathbf{y},\mathbf{y}}(t)$ are linear functions of time. Notice that this is only apparently inconsistent with the remark on orbital stability, since, as for Eq. (B.25), substituting Eq. (B.33) into Eq. (B.17) $\mathbf{K}^{\infty}_{\mathbf{y},\mathbf{y}}(t)+\mathbf{K}^{\infty\mathrm{T}}_{\mathbf{y},\mathbf{y}}(t)$ results to be t-independent.

To completely prove the theorem, we still need to address the issue of the higher order moments of the stochastic variable. We give here a sketch of the complete proof by considering a simplified case: we treat a one-dimensional case and include all the non significant constant terms into the Fourier coefficients used in Eq. (B.35). Including all the moments, the characteristic function has the form

$$\varphi(\omega,t) = \exp\left(\mathrm{i}\omega m - \frac{\omega^2}{2}K + \sum_{n=3}^{+\infty}\frac{\omega^n}{n!}H_n\right), \tag{B.34}$$

where H_n represents the n-th order moment. Let us assume first that $H_n = 0$ $\forall\, n > 3$, so that H_3 only needs to be accounted for. Substituting into Eq. (16) we find (here $\dot{x}$ represents the time derivative of $x(t)$)

$$\begin{aligned}
&\mathrm{i}\omega\dot{m} - \frac{\omega^2}{2}\dot{K} + \frac{\omega^3}{6}\dot{H}_3 \\
&\quad = \omega\sum_h \tilde{A}_h\left[\mathrm{i}m - (\omega+h)K + \frac{(\omega+h)^2}{2}H_3\right]\mathrm{e}^{\mathrm{i}ht}\times\mathrm{e}^{\mathrm{i}hm} - \frac{2h\omega+h^2}{2}K \\
&\qquad + \frac{h^3+3h\omega^2+3h^2\omega}{3}H_3 + \sum_{h,r}\left(\omega\tilde{B}_{h,r} + \frac{\mathrm{i}}{2}\omega^2\tilde{C}_{h,r}\right)\mathrm{e}^{\mathrm{i}(h-r)t}\,\mathrm{e}^{\mathrm{i}(h-r)m} \\
&\qquad - \frac{2(h-r)\omega+(h-r)^2}{2}K\times\mathrm{e}^{\frac{(h-r)^3+3(h-r)\omega^2+3(h-r)^2\omega}{3}H_3}.
\end{aligned} \tag{B.35}$$

Developing the exponentials in power series of ω, and equating the terms of order 1, 2 and 3 in ω we find, for $t\to+\infty$, the following three differential equations for

the first three moments:

$$\dot{m}^{\infty} = -\tilde{A}_0 m^{\infty} + \sum_h \tilde{B}_{h,h}, \tag{B.36}$$

$$\dot{K}^{\infty} = 2\tilde{A}_0 K^{\infty} + \mathrm{i} \sum_{h,r} \tilde{C}_{h,r}, \tag{B.37}$$

$$\dot{H}_3^{\infty} = 3\tilde{A}_0 H_3^{\infty}. \tag{B.38}$$

From Eq. (B.38) we can estimate

$$H_3^{\infty}(t) = \mathrm{e}^{3\tilde{A}_0 t}\, H_3^{\infty}(0), \tag{B.39}$$

therefore if $\tilde{A}_0 < 0$ $H_3^{\infty}(t)$ is asymptotically zero. On the other hand, if $\tilde{A}_0 > 0$ we choose $H_3^{\infty}(0) = 0$ to avoid a moment exponentially growing with t, thus concluding that $H_3(t) = 0$ asymptotically for $t \to +\infty$.

Similarly, for $n > 3$ the following equation holds

$$\dot{H}_n^{\infty}(t) = n\tilde{A}_0 H_n^{\infty} \quad n > 3, \tag{B.40}$$

therefore we can conclude that $H_n = 0$ $\forall n \geq 3$ asymptotically with t.

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