Essential and inessential elements of a standard basis

Essential and inessential elements of a standard basis

Giannina Beccari, Carla Massaza *

Dipartimento di Matematica, Politecnico Torino, Corso Duca degli Abruzzi, 24, 10129 Torino, Italy

A B S T R A C T

In this paper we introduce the concept of *inessential element* of a standard basis $\mathcal{B}(\mathfrak{I})$, where $\mathfrak{I}$ is any homogeneous ideal of a polynomial ring. An inessential element is, roughly speaking, a form of $\mathcal{B}(\mathfrak{I})$ whose omission produces an ideal having the same saturation as $\mathfrak{I}$; it becomes useless in any dehomogenization of $\mathfrak{I}$ with respect to a linear form. We study the properties of $\mathcal{B}(\mathfrak{I})$ linked to the presence of inessential elements and give some examples.

1. Introduction

The systems of generators of a given ideal $\mathfrak{I} \subset K[y_0, \ldots, y_n] = S$, satisfying given conditions, are widely studied. In the special case of homogeneous ideals, it is well known that there exist systems of generators, called standard bases, satisfying the following condition: their elements of degree d are forms defining a K-basis of the vector space $\mathfrak{I}_d/(\mathfrak{I}_{d-1}S_1)$, for every $d \in \mathbb{N}$ [3,2,9]. The standard bases are minimal among the systems of generators of a homogeneous ideal, but they are not the only interesting ones (for instance, Gröbner bases are not, in general, minimal, but they are of interest for other reasons). However, in this paper we will consider only standard bases of homogeneous ideals. The elements of each of them may be of two different kinds: essential generators and inessential generators. A generator g is called inessential, with respect to a basis $\mathcal{B}(\mathfrak{I})$ containing it, if it lies in the saturation of the ideal generated by $\mathcal{B}(\mathfrak{I}) - \{g\}$; this means that any dehomogenization $\mathfrak{I}_*$ of $\mathfrak{I}$ with respect to a linear form is generated by the image of $\mathcal{B}(\mathfrak{I}) - \{g\}$. A generator g not lying in the saturation of $\mathcal{B}(\mathfrak{I}) - \{g\}$ is called essential. We needed this concept in our attempt of considering the elements of a standard basis of $\mathfrak{I}$ as separators [1] with respect to a convenient ideal $\mathfrak{L} \subset \mathfrak{I}$; in fact, we found that such an interpretation is possible iff the generators are essential. Not all homogeneous ideals, and in particular ideals of 0-dimensional schemes in which we were originally interested, have a basis whose elements are all essential. This fact suggested it would be helpful to study the concept of essentiality independently from its use in the link between separating sequences and generators of a sub-ideal and for ideals of any height. So, we focused our attention not only on the standard bases with the maximum number of essential elements, but also on those whose inessential elements are all contained in the saturation of the ideal generated by the essential ones.

Our paper has the following format. In Section 2 we recall necessary definitions, notation, and facts. Section 3 contains the definition of essential and inessential elements, with equivalent formulations and some examples. In Section 4 we consider the special case of perfect height 2 ideals. In this situation, the essentiality or the inessentiality of a generator can be read as a property of the ideal generated by the entries of its corresponding column in any Hilbert matrix of $\mathfrak{I}$ [8,10].

In Section 5 we come back to the study of the general situation. We show that any saturated homogeneous ideal has at least one basis with the maximum (resp. minimum) number of essential generators in any degree; so, the two sequences of those numbers are numerical sequences linked to the ideal and their elements are, degree by degree, less than or equal to the corresponding graded Betti numbers. We will call those bases e-maximal (resp. e-minimal) and give an algorithm of construction of one of them starting from any standard basis. An e-maximal (resp. e-minimal) basis is characterized by the fact that its inessential (resp. essential) elements have their typical property with respect to every standard basis containing them. The e-maximal bases were the first object of our interest, as we were looking for bases with the greatest number of generators to be viewed as elements of a separating sequence. We give just a few examples of e-maximal bases, as we are planning to devote to them another paper. In this second paper we will study a family of perfect height 2 ideals for which it is possible to compute the number of the essential elements contained in an e-maximal basis, starting from some properties of their generators in minimal degree. From another point of view, an e-minimal basis seems to be of interest when we dehomogenize with respect to a linear form; in fact, an inessential element becomes useless as a generator of the dehomogenized ideal. However, from this point of view, the notion of inessential set (generalizing the one of inessential element) turns out to be more suitable. In fact, the standard bases giving rise to a basis of minimal cardinality, after a dehomogenization with respect to a generic linear form, are the ones containing an inessential set of maximal cardinality. So, the last part of Section 5 is devoted to such bases and to the ones (E-bases) whose set of inessential elements is an inessential set.

2. Background and notation

Let $S = K[y_0, \dots, y_n]$, where K is an algebraically closed field, be the coordinate ring of $\mathbb{P}^n$, $\mathfrak{I} = \bigoplus \mathfrak{I}_d, d \in \mathbb{N}$, a homogeneous ideal of S, and $\mathfrak{M} = (y_0, \dots, y_n)$ be the irrelevant maximal ideal. We recall the following definition.

Definition 2.1 (*[3]*). A *standard basis* $\mathcal{B}(\mathfrak{I})$ of $\mathfrak{I}$ is an ordered set of forms of S, generating $\mathfrak{I}$, such that its elements of degree d define a K-basis of $\mathfrak{I}_d/(\mathfrak{I}_{d-1}S_1)$.

It is well known [3] that the number of generators of $\mathcal{B}(\mathfrak{I})$, in a given degree d, depends only on the ideal $\mathfrak{I}$; it is the dth Betti number of $\mathfrak{I}$, at the first level. The cardinality of $\mathcal{B}(\mathfrak{I})$ will be denoted $\nu(\mathfrak{I})$.

When there is no matter of misunderstanding, we will use the notation $(f_1, \dots, f_r)$ to denote the ideal generated by the standard basis $(f_1, \dots, f_r)$, instead of the heavier notation $(f_1, \dots, f_r)S$. Moreover, if we need to point out a subset T of $\mathcal{B}(\mathfrak{I})$, we use the non-standard notation.

$\mathcal{B}(\mathfrak{I}) = (t_1, \dots, t_m, s_1, \dots, s_p)$, where $T_1 = (t_1, \dots, t_m)$ and $T_2 = (s_1, \dots, s_p)$ inherit the original ordering of $\mathcal{B}(\mathfrak{I})$.

If $\mathfrak{I}$ is perfect of height 2 [2,8,3] , it is useful to consider, for every basis $\mathcal{B}(\mathfrak{I})$, a *Hilbert matrix* [8] as follows. We set

> $\mathcal{B}(\mathfrak{I}) = (g_1, \dots, g_t)$, where $\deg g_j \le \deg g_{j+1}$ for $j = 1 \dots (t-1)$, and let $s_i = (a_{i1}, \dots, a_{it})$, $i = 1 \dots (t-1)$, denote the ith element of a basis of syzygies with respect to $\mathcal{B}(\mathfrak{I})$, where $\deg s_i = \deg(a_{ij}g_j)$ for $j = 1 \dots t$ and $\deg s_i < \deg s_{i+1}$.

With this notation $M(\mathfrak{I}) = (a_{ij})$, $i = 1 \dots (t-1)$, $j = 1 \dots t$, is a Hilbert matrix of $\mathfrak{I}$ related to $\mathcal{B}(\mathfrak{I})$. Moreover ([8]) $(-1)^j g_j$ is the minor of $M(\mathfrak{I})$ obtained by deleting its jth column C_j. We say that g_j is the generator linked to the column C_j or that C_j is its corresponding column. The ideal generated by C_j will be denoted $\mathfrak{I}_{C_j}$.

We will be mainly interested in saturated ideals, whose definition we recall.

Definition 2.2 (*[5]*). The *saturation* of a homogeneous ideal $\mathfrak{I} \subset S$ is $\mathfrak{I}^{sat} = \{F \in S \mid F\mathfrak{M}^t \subset \mathfrak{I}$ for some $t \in \mathbb{N}\}$. The ideal $\mathfrak{I}$ is saturated iff $\mathfrak{I}^{sat} = \mathfrak{I}$ or, equivalently , if $\mathfrak{M}$ is not associated to $\mathfrak{I}$.

A process of computation of $\mathfrak{I}^{sat}$, starting from $\mathfrak{I}$, can be found in [5]. To every projective scheme V of $\mathbb{P}^n$ we can associate a unique saturated ideal $\mathfrak{I}$, which is usually denoted $\mathfrak{I}(V)$.

Now we pass from the projective situation to the affine one. For every linear form $L \in S$, the ideal $\mathfrak{I}_*$, obtained from a homogeneous ideal $\mathfrak{I}$ by dehomogenization with respect to L, is the image of $\mathfrak{I}$ in the localization of S with respect to L. With a change of coordinates, it is possible to choose $L = y_0$; in this situation, the localization of S is isomorphic to $R = K[x_1, \dots, x_n]$ under the map associating to every form $F(y_0, \dots, y_n) \in S$ the polynomial $F_*(x_1, \dots, x_n) = F(1, x_1, \dots, x_n) \in R$ (see [4, 11]) and $\mathfrak{I}_*$ can be identified with the image of $\mathfrak{I}$ under that morphism. Vice versa, the homogenization $\mathfrak{L}^* \subset S$ of any ideal $\mathfrak{L} \subset R$ is the ideal generated by $F(x_1, \dots, x_n)^* = y_0^d F(y_1/y_0, \dots, y_n/y_0)$, where $F(x_1, \dots, x_n)$ is any polynomial of $\mathfrak{L}$ and d is its degree. Let us observe that $(\mathfrak{L}^*)_* = \mathfrak{L}$, while $(\mathfrak{I}_*)^* = \mathfrak{I}$ only if y_0 is regular for $S/\mathfrak{I}$. The operation of dehomogenization can be made on a set of generators of $\mathfrak{I}$, but the analogous is not true for the homogenization.

Let us recall the following definition.

Definition 2.3. A basis of an ideal $\mathfrak{L}$ in a polynomial ring $K[x_1, \dots, x_n]$ is a set of generators that fails to generate $\mathfrak{L}$ if one of its elements is omitted.

Two different bases of $\mathfrak{L}$ may have a different cardinality.

3. Equivalent conditions and examples

Let $\mathfrak{I}$ be a homogeneous ideal of $S = K[y_0, \dots, y_n]$ such that $\mathfrak{I}^{sat} \subsetneq \mathfrak{M} = (y_0, \dots, y_n)$, and f be an element of a standard basis $\mathcal{B}(\mathfrak{I})$. Denote by $\mathfrak{H}_{(f,\mathcal{B})}$ the ideal generated by $\mathcal{B}(\mathfrak{I}) - \{f\}$.

Definition 3.1. An element f of a standard basis $\mathcal{B}(\mathfrak{I})$ of $\mathfrak{I}$ is called an *inessential generator of* $\mathfrak{I}$ with respect to $\mathcal{B}(\mathfrak{I})$ iff $f \in (\mathfrak{H}_{(f,\mathcal{B})})^{sat}$. Otherwise, we say that f is an *essential generator of* $\mathfrak{I}$ with respect to $\mathcal{B}(\mathfrak{I})$.

The following proposition gives conditions equivalent to inessentiality.

Proposition 3.1. *Let $\mathfrak{H} = (g_1, \ldots, g_r)$ be a homogeneous ideal of $S = K[y_0, \ldots, y_n]$ such that $\mathfrak{H}^{sat} \subsetneq \mathfrak{M} = (y_0, \ldots, y_n)$. Let f be any form of $\mathfrak{M}$ and $\mathfrak{I} = (\mathfrak{H}, f)$. The following facts are equivalent.*

(i) *There exists $t \in \mathbb{N}$ such that $f\mathfrak{M}^t \subseteq \mathfrak{H}$ (in other words, $f \in \mathfrak{H}^{sat}$).*
(ii) *$S/\mathfrak{I}$ and $S/\mathfrak{H}$ have the same Hilbert polynomial (see [5,9,6,7]).*
(iii) *There exists a linear form $z \in S$ regular for $S/\mathfrak{H}^{sat}$ such that a dehomogenization with respect to z gives $\mathfrak{H}_* = \mathfrak{I}_*$.*
(iv) *$\mathfrak{H}_* = \mathfrak{I}_*$ for every dehomogenization with respect to any linear form $z \in \mathfrak{M}$.*

Proof. The equivalence between (i) and (ii) is obvious.

(i) $\Rightarrow$ (iv) Condition (i) implies $fz^t \in \mathfrak{H}$ for every $z \in \mathfrak{M}$. As a consequence, $f_* \in \mathfrak{H}_*$ or, equivalently, $\mathfrak{H}_* = \mathfrak{I}_*$.

(iv) $\Rightarrow$ (iii) The condition $\mathfrak{H}^{sat} \neq \mathfrak{M}$ assures the existence of an element z regular for $S/\mathfrak{H}^{sat}$ (as the union of its associated primes cannot be $\mathfrak{M}$), so that the implication is obvious.

(iii) $\Rightarrow$ (i) $f_* \in \mathfrak{H}_*$ means $f_* = \Sigma \alpha_i (g_i)_*$, $\alpha_i \in R = K[x_1, \ldots, x_n]$ (see Section 2), so that $(\exists t)\ fz^t = \Sigma \beta_i g_i \in \mathfrak{H} \subset \mathfrak{H}^{sat}$, $\beta_i \in S$. As z is regular for $S/\mathfrak{H}^{sat}$, we get $f \in \mathfrak{H}^{sat}$. □

Remarks. 1. Let us observe that it is sufficient to verify condition (iv) for a set of linear forms generating $\mathfrak{M}$; equivalently, the condition $f \in \mathfrak{H}^{sat}$ is verified iff, for every linear form L of a set of generators of $\mathfrak{M}$, there exists $n \in \mathbb{N}$ such that $fL^n \in \mathfrak{H}$.

2. In (iii) the condition "z is regular for $S/\mathfrak{H}^{sat}$" cannot be replaced by "z is regular for $S/\mathfrak{I}$", as we can see in the following example.

Let: $S = K[x, y, z]$, $\mathfrak{I} = (g_1, g_2, g_3, g_4)$, $\mathfrak{H} = (g_1, g_2, g_4)$, $f = g_3$, where

$$g_1 = x^5, \qquad g_2 = xy^5, \qquad g_3 = y^7, \qquad g_4 = y^3(-x^4 - y^2z^2)$$

are the maximal minors of the matrix

$$\begin{pmatrix} 0 & x^3 & z^2 & -y^2 \\ 0 & y^2 & -x & 0 \\ y^3 & z^2 & 0 & -x \end{pmatrix}.$$

It is easy to verify that z is regular for $S/\mathfrak{I}$ and that, in the dehomogenization with respect to z, we have: $(g_3)_* = y^7 = -y^2(g_4)_* - x^3(g_2)_*$ which implies that $(g_3)_* \in \mathfrak{H}_*$. However, g_3 does not satisfy the equivalent conditions of Proposition 3.1 (see Proposition 4.1 for a quicker check). The reason is that z is not regular for $\mathfrak{H}^{sat}$. In fact, $z^2y^5 = -g_4 - x^4y^3$ so that $z(zy^7) = -g_4y^2 - g_2x^3 \in \mathfrak{H} \subseteq \mathfrak{H}^{sat}$ and $zy^7 \notin \mathfrak{H}^{sat}$, as $zy^k \notin \mathfrak{H}, \forall k \in \mathbb{N}$.

3. It may be difficult to compute $\mathfrak{H}^{sat}$, and hence condition (iv) and Remark 1. become of some interest. □

We give a statement equivalent to the essentiality of $f \in \mathfrak{I}_d$, with respect to a standard basis $\mathcal{B}(\mathfrak{I}) = \{f, \mathcal{B}_1\}$.

Proposition 3.2. *Let $\mathcal{B} = \{f, \mathcal{B}_1\}$ be a standard basis of an ideal $\mathfrak{I} \subset S = K[y_0, \ldots, y_n]$, where $f \in \mathfrak{I}_d$. The following facts are equivalent:*

(i) *f is essential with respect to $\mathcal{B}$;*
(ii) *there exists a set $\{L_1^*, \ldots, L_n^*, N\}$ of linear forms, generating $\mathfrak{M}$, such that:*
  (a) *N is a regular form both for $S/\mathfrak{H}^{sat}$, where $\mathfrak{H} = (\mathcal{B}_1)S$, and for $S/\mathfrak{I}^{sat}$;*
  (b) *$\forall t \in \mathbb{N}, fN^t \notin \mathfrak{L}$, where $\mathfrak{L} = (\mathcal{B}_1, L_1^*f, \ldots, L_n^*f)$.*

Proof. (i) $\Rightarrow$ (ii) As the union of the primes associated to $\mathfrak{H}^{sat}$ or to $\mathfrak{I}^{sat}$ cannot coincide with $\mathfrak{M}$, we choose $N \in \mathfrak{M}$ regular both for $S/\mathfrak{H}^{sat}$ and $S/\mathfrak{I}^{sat}$. Let us remark that

(+) N does not divide f; otherwise $f = f_1N^u \in \mathfrak{I}$ and N regular for $S/\mathfrak{I}$ would imply $f_1 \in \mathfrak{I}$, so that f would not satisfy the condition $f \notin (\mathfrak{I}_{d-1})S$.

A dehomogenization with respect to N gives that $\mathfrak{I}_* = (f_*, \mathfrak{H}_*)$ and $\mathfrak{H}_*$ are different (see Proposition 3.1(iii)) and, as a consequence, there exists a maximal ideal $\mathfrak{P} = (L_1, \ldots, L_n) \subset R = S_*$ such that

$$(*) \qquad \mathfrak{I}_*R_{\mathfrak{P}} \neq \mathfrak{H}_*R_{\mathfrak{P}}.$$

Let us set $\mathfrak{L} = (\mathfrak{H}, fL_1^*, \ldots, fL_n^*)$ and consider $\mathfrak{L}_* = (\mathfrak{H}_*, f_*L_1, \ldots, f_*L_n)$; we prove that

$$\mathfrak{L}_*R_{\mathfrak{P}} \neq \mathfrak{I}_*R_{\mathfrak{P}}.$$

Otherwise, in $R_{\mathfrak{P}}$, we would have

$$f_* = \left(\sum_{i=1}^n a_iL_i\right) f_* + h_* \quad \text{for some } h_* \in \mathfrak{H}_*R_{\mathfrak{P}},$$

so that $(1 - \sum_{i=1}^n a_iL_i)f_* \in \mathfrak{H}_*R_{\mathfrak{P}}$. As $1 - \sum_{i=1}^n a_iL_i$ is invertible in $R_{\mathfrak{P}}$, this implies $f_* \in \mathfrak{H}_*R_{\mathfrak{P}}$, a contradiction to $(*)$. As a consequence, we have $\mathfrak{I}_* \neq \mathfrak{L}_*$. Hence, $(\forall t)\ fN^t \notin \mathfrak{L}$, for, otherwise, we would find, by dehomogenization with respect to N, that $f_* \in \mathfrak{L}_*$, which implies $\mathfrak{I}_* = \mathfrak{L}_*$.

(ii) $\Rightarrow$ (i) This implication comes immediately as (b) implies: $(\forall t)\ fN^t \notin (\mathcal{B}_1)S$ and, as a consequence, $(\forall t)\ f\mathfrak{M}^t \not\subset \mathfrak{H}$; so, condition (ii)(b) is sufficient to imply (i). □

Remark 1. Clearly it is enough to verify condition (b) for $t \gg 0$.

Remark 2. If $\dim S/\mathfrak{I} = 1$, condition (b) can be replaced by the following: (b′) f is a *separator* for $S/\mathfrak{L}$(see [1]). In fact, in this case the definition of separator is meaningful and condition (b) can be restated as: $\dim_K (S/\mathfrak{L})_t = \dim_K (S/\mathfrak{I})_t + 1,\ t \geq d$. This relation, with condition (+), is equivalent to saying that f is a separator for $S/\mathfrak{L}$. □

With the same notation of Proposition 3.2 and with the assumption that f is an essential generator, we can state the following proposition.

Proposition 3.3. *If $\mathfrak{I}$ is saturated, then so is $\mathfrak{L}$.*

Proof. Let us prove that $\mathfrak{L} = \mathfrak{L}^{sat}$. If not, we could find an element $u \in \mathfrak{M}$, $u \notin \mathfrak{L}$ and a number $s \in \mathbb{N}$ such that $u\mathfrak{M}^s \subset \mathfrak{L} \subset \mathfrak{I}$. As $\mathfrak{I}$ is saturated, u must be in $\mathfrak{I}$; as a consequence, $u = aN^t f + j$, $a \in K^*$, $j \in \mathfrak{L}$. But this implies $fN^t\mathfrak{M}^s \subset \mathfrak{L} \subset \mathfrak{I}$, a contradiction, as N is regular for $S/\mathfrak{I}$ and f is essential. □

The essentiality of an element f depends on the basis in which it is considered, as we can see in the following example.

Example 3.1. Let $\mathfrak{I} \subseteq K[x, y, z]$ be the ideal generated by the maximal minors of the matrix

$$M = \begin{pmatrix} z & 0 & 0 & -x \\ 0 & x & 0 & -y \\ 0 & 0 & y & -z \end{pmatrix}.$$

A standard basis of $\mathfrak{I}$ is $\mathcal{B}(\mathfrak{I}) = (g_1, g_2, g_3, f)$, where

$$g_1 = x^2y, \qquad g_2 = y^2z, \qquad g_3 = xz^2, \qquad f = xyz.$$

We can easily check that f is inessential with respect to $\mathcal{B}(\mathfrak{I})$. In fact: $fx = x^2yz = zg_1$, $fy = xy^2z = xg_2$, $fz = xyz^2 = yg_3$, so that $f\mathfrak{M} \subset (g_1, g_2, g_3) = \mathfrak{H}$.

Let us produce a new basis, with respect to which f is essential. Choose in $\mathbb{P}^2$ a point not lying on $f = 0$, for instance $P(1, 1, 1)$, and replace g_1, g_2, g_3 with generators vanishing at P, so obtaining the new basis $\mathcal{B}'(\mathfrak{I}) = (g_1 - f, g_2 - f, g_3 - f, f)$. Clearly $\mathfrak{H}' = (g_1 - f, g_2 - f, g_3 - f)$ is such that $(\mathfrak{H}')^{sat} \neq \mathfrak{I}$, as the underlying schemes differ for one point.

We observe that it is also possible to produce a standard basis $\mathcal{B}''(\mathfrak{I})$ such that every element of it is inessential: it is enough to replace, in M, the first three columns with the sum of each of them with the fourth (see Proposition 4.1). □

There are also situations in which a generator of $\mathfrak{I}$ is inessential with respect to any basis containing it and every basis contains at least an inessential element in the degree of f. This is demonstrated in the following example.

Example 3.2. Let $\mathfrak{I} \subset K[x, y, z]$ be the (saturated) ideal generated by the maximal minors of the matrix

$$M = \begin{pmatrix} y^2 & 0 & 0 & -x \\ 0 & z^2 & 0 & -y \\ 0 & 0 & x^2 & -z \end{pmatrix}.$$

We have: $\mathcal{B}(\mathfrak{I}) = (g_1, g_2, g_3, f)$, where

$$g_1 = x^3z^2, \qquad g_2 = x^2y^3, \qquad g_3 = y^2z^3, \qquad f = x^2y^2z^2.$$

It is immediate to see that $fx = y^2g_1$, $fy = z^2g_2$, $fz = x^2g_3$, so that $f\mathfrak{M} \subseteq \mathfrak{H} = (g_1, g_2, g_3)$.

In this case, every standard basis of $\mathfrak{I}$ must contain an element f' in degree 6, giving rise to the same ideal $\mathfrak{H}$; moreover, f' must satisfy the relation $f' = kf + h$, $k \in K$, $h \in \mathfrak{H}$. As a consequence, $f'\mathfrak{M} \subseteq \mathfrak{H}$ is still verified, so that f' is inessential with respect to any basis containing it. □

Taking into account the situation described in Example 3.2, we give the following definition.

Definition 3.2. An element $f \in \mathfrak{I}_d$ is *strongly inessential* (*s.i.*) iff $f \notin (\mathfrak{I}_{d-1})S$ and it is inessential with respect to any standard basis containing it. Analogously, an element $f \in \mathfrak{I}_d$ is *strongly essential* (*s.e.*) iff $f \notin (\mathfrak{I}_{d-1})S$ and it is essential with respect to any standard basis containing it.

Proposition 3.4. *A strongly inessential generator cannot have the minimal degree $\alpha(\mathfrak{I})$.*

Proof. Let us consider a basis $\mathcal{B}(\mathfrak{I}) = (f, h_1, \ldots, h_r)$, where $\deg f \leq \deg h_1 \leq \cdots \leq \deg h_r$. We prove that if f is inessential then it is possible to replace each h_i with an h'_i such that the behaviour of h'_i with respect to essentiality is equal to the one of h_i and f is essential with respect to $\mathcal{B}'(\mathfrak{I}) = (f, h'_1, \ldots, h'_r)$. To this aim, we choose a point $P \in \mathbb{P}^n$ and a linear form z such that $f(P) \neq 0$ and $z(P) \neq 0$. In every linear system $h_i + \lambda_i z^{t_i} f$, where $t_i = \deg h_i - \deg f$ and $\lambda_i \in K$, there is a form $h'_i = h_i + a_i z^{t_i} f$ such that $h'_i(P) = 0$. As a consequence, f is essential with respect to $\mathcal{B}'(\mathfrak{I})$. □

Corollary 3.1. *If $\mathfrak{I}$ is generated in minimal degree, then $\mathfrak{I}$ admits a standard basis of essential elements.*

Proof. If $\mathcal{B} = (f_1, \ldots, f_r)$ is any standard basis for $\mathfrak{I}$ and if f_i is its first inessential generator, then, thanks to Proposition 3.4, we can find a basis $\mathcal{B}' = (f_1', \ldots, f_i', \ldots, f_r')$, where $f_j' = f_j + a_j f_i$ with $a_i = 0$, to which $f_i = f_i'$ is essential. Moreover, it is easy to check that f_j' is still essential for $j < i$ (see Lemma 5.1). So, at any step, the basis $\mathcal{B}$ can be replaced by another basis with one more essential element. □

With a reasoning very similar to the one of Proposition 3.4, we can prove the following proposition.

Proposition 3.5. *Let $\mathcal{B}(\mathfrak{I}) = (h_1, \ldots, h_m, f, g_1, \ldots, g_k)$, where deg $h_1 \le \cdots \le \deg h_m < \deg f \le \deg g_1 \le \cdots \le \deg g_k$ and f is inessential with respect to $\mathcal{B}(\mathfrak{I})$. If there exists a point $P \in \mathbb{P}^n$ such that $h_i(P) = 0,\ i = 1 \ldots m$, and $f(P) \neq 0$, then there exist $g_1', \ldots, g_k'$ such that f is essential for $\mathcal{B}'(\mathfrak{I}) = (h_1, \ldots, h_m, f, g_1', \ldots, g_k')$.*

Let us observe that the requirement of Proposition 3.5 implies $f \notin (h_1, \ldots, h_m)^{sat}$; on the other hand, $f \in (h_1, \ldots, h_m)^{sat}$ implies f is *s.i.*, but the converse is not true (which we will see in Example 4.1, where $g_3 \notin (g_1, g_2)^{sat}$, and g_3 is *s.i.*).
More generally, we would like to investigate the following problems:

A. Given a standard basis $\mathcal{B}(\mathfrak{I})$, find all its elements of a given degree which are essential with respect to it.
B. Check how the "nature" (essentiality–inessentiality) of f varies with the basis containing it.
C. Check how the number of essential elements in a given degree varies with the chosen basis.

4. The case of perfect height 2 ideals

If $\mathfrak{I}$ is a perfect codimension 2 ideal (for instance, the ideal of a 0-dimensional scheme in $\mathbb{P}^2$), we can give an answer to both problems A. and B. in terms of a Hilbert–Burch matrix $M(\mathfrak{I})$ with respect to $\mathcal{B}(\mathfrak{I})$. If f_r is the rth element of $\mathcal{B}(\mathfrak{I})$, let us denote $\mathfrak{I}_{C_r} \subset S = K[y_0, \ldots, y_n]$ the ideal generated by the entries of the rth column of $M(\mathfrak{I})$. Also, let $\mathfrak{M}$ be the irrelevant maximal ideal of S. With this notation, we can state the following proposition.

Proposition 4.1. *Let $\mathfrak{I}$ be a perfect codimension 2 ideal of S. Then $f_r \in \mathcal{B}(\mathfrak{I})$ is inessential for $\mathcal{B}(\mathfrak{I})$ iff the following condition is satisfied*

$$\exists t \in \mathbb{N} \text{ such that } \mathfrak{M}^t \subseteq \mathfrak{I}_{C_r}. \tag{1}$$

Proof. From the definition of $\mathfrak{I}_{C_r}$ we get

$$\mathfrak{I}_{C_r} f_r \subseteq \mathfrak{H} = (\mathcal{B}(\mathfrak{I}) - f_r). \tag{2}$$

Conditions (1) and (2) imply

$$\mathfrak{M}^t f_r \subseteq \mathfrak{H}, \tag{3}$$

which says that f_r is inessential.
Vice versa, (3) implies the existence of syzygies whose rth components generate $\mathfrak{M}^t$, so that $\mathfrak{M}^t \subseteq \mathfrak{I}_{C_r}$. □

Remark 4.1. Proposition 4.1 can be restated reducing the problem to the affine situation. Let L be any linear form of S, regular for $S/\mathfrak{H}^{sat}$, and let $\mathfrak{I}_*, \mathfrak{H}_*, \mathfrak{I}_{C_r *}$ be the dehomogenization of $\mathfrak{I}, \mathfrak{H}, \mathfrak{I}_{C_r}$, respectively. Then f_r is inessential iff $\mathfrak{I}_{C_r *} = R$, where R is the dehomogenization of S with respect to L.

Let us now pass to problem B. It is well known that a change of a standard basis $\mathcal{B}(\mathfrak{I}) = (g_1, \ldots, g_r, \ldots, g_m)$ is equivalent to a change of its matrix $M(\mathfrak{I})$, realized by repeatedly replacing a column C_r with $C_r' = \sum_i t_{ir} C_i$, where $T = (t_{ir})$ is an invertible matrix with $t_{rr} \in K^*$ and $t_{ir} = 0$ if deg $g_i <$ deg g_r. So, Proposition 4.1 gives rise to the following corollary.

Corollary 4.1. *Let $g_r \in \mathcal{B}(\mathfrak{I})$ be the generator corresponding to the column C_r of $M(\mathfrak{I})$. Then g_r is s.i. iff the entries of every $C_r' = \sum_j t_{rj} C_j,\ t_{rr} \in K^*$, generate an ideal $\mathfrak{I}_{C_r'}$ satisfying condition (1) of Proposition 4.1.*

Let us consider again Examples 3.1 and 3.2 from this point of view. In Example 3.1, the ideals generated by the entries of the columns $C_i, i = 1 \ldots 4$, are respectively: $\mathfrak{I}_{C_1} = (z),\ \mathfrak{I}_{C_2} = (x),\ \mathfrak{I}_{C_3} = (y),\ \mathfrak{I}_{C_4} = (x, y, z)$. The only one satisfying the condition of Proposition 4.1 is $\mathfrak{I}_{C_4}$, so that the only inessential element is f.
Now, let us replace C_4 with a new column C_4', so that the fourth generator becomes essential. We have :

$${}^t C_4' = (-x + t_1 z \qquad -y + t_2 x \qquad -z + t_3 y), \quad t_i \in K.$$

The ideal $\mathfrak{I}_{C_4'}$ generated by C_4''s entries cannot contain a power of $\mathfrak{M}$ iff the linear system

$$\begin{cases} -x + t_1 z = 0 \\ -y + t_2 x = 0 \\ -z + t_3 y = 0 \end{cases}$$

has proper solutions, that is iff $t_1 t_2 t_3 = 1$. In particular, choosing $t_1 = t_2 = t_3 = 1$, we find again the basis $\mathcal{B}'(\mathfrak{I})$ already obtained with another technique.
In Example 3.2 the column corresponding to f is the fourth; it cannot be changed (apart from the multiplication by a scalar) for degree reasons. So, we find again that any standard basis has an inessential generator in degree 6. Let us observe

that, in this example, the inessential generator is the only generator of maximal degree, so that its corresponding ideal $\mathfrak{H}$ does not depend on the standard basis. In the following example the considered ideal $\mathfrak{I}$ has, in every standard basis, an inessential element of degree 11, even if 11 is not the greatest degree of its generators and another inessential element in the maximal degree in which there are two generators.

Example 4.1. Let $\mathfrak{I} \subset K[x, y, z]$ be the ideal generated by the maximal minors of the matrix

$$M(\mathfrak{I}) = \begin{pmatrix} 0 & x^5 & 0 & -y^3 & 0 \\ 0 & 0 & x^3 & z^2 & -y^2 \\ 0 & 0 & y^2 & -x & 0 \\ y^3 & 0 & z^2 & 0 & -x \end{pmatrix}.$$

We have $\mathcal{B}(\mathfrak{I}) = (g_i), i = 1 \dots 5$, where

$$g_1 = x^{10}, \qquad g_2 = y^{10}, \qquad g_3 = x^6y^5, \qquad g_4 = x^5y^7, \qquad g_5 = x^5y^5z^2 + x^9y^3.$$

The ideals generated by the entries of the columns $C_i,\ i = 1 \dots 5$, are respectively

$$\mathfrak{I}_{C_1} = (y^3), \qquad \mathfrak{I}_{C_2} = (x^5), \qquad \mathfrak{I}_{C_3} = (x^3, y^2, z^2), \qquad \mathfrak{I}_{C_4} = (y^3, z^2, x), \qquad \mathfrak{I}_{C_5} = (x, y^2).$$

This gives: $\mathfrak{I}_{C_3} \supset \mathfrak{M}^5,\ \mathfrak{I}_{C_4} \supset \mathfrak{M}^4$, while $\mathfrak{I}_{C_1}, \mathfrak{I}_{C_2}, \mathfrak{I}_{C_5}$ do not contain any power of $\mathfrak{M}$. As a consequence, g_1, g_2, g_5 are essential, while g_3 and g_4 are inessential. Perform a general basis change of $\mathfrak{I}$ by replacing only the third column C_3 with $C_3' = kC_3 + PC_4 + QC_5$, where $k \in K^*$, P and Q are linear forms. Then

$$\mathfrak{I}_{C_3'} = (Py^3, kx^3 + Pz^2 - Qy^2, ky^2 - Px, kz^2 - xQ).$$

It is immediate to verify that the system obtained by annihilating the generators of $\mathfrak{I}_{C_3'}$ has the unique solution $x = y = z = 0$, for every choice of k, P, Q. This means that the only prime ideal associated to $\mathfrak{I}_{C_3'}$ is $\mathfrak{M}$, so that g_3 is still inessential. Analogously, perform a general change of basis by replacing only the fourth column C_4 with $C_4' = k_1C_4 + k_2C_5,\ k_1 \in K^*,\ k_2 \in K$, so that

$$\mathfrak{I}_{C_4'} = (-k_1y^3, k_1z^2 - k_2y^2, -k_1x, -k_2x) = (x, y^3, k_1z^2 - k_2y^2).$$

Also in this case the generators of $\mathfrak{I}_{C_4'}$ are annihilated only by $x = y = z = 0$, so that g_4 is still inessential.

5. The general case

Now, we go back to the general case of a homogeneous ideal $\mathfrak{I} \subset S = K[y_0, \dots, y_n]$, not necessarily generated by the maximal minors of an $m \times (m + 1)$-matrix. As usual, $\mathfrak{M}$ will denote the irrelevant ideal of S.

Let $f \in \mathfrak{I}_d$ be any form that can be included in a standard basis $\mathcal{B}(\mathfrak{I})$, or, equivalently, that does not lie in $\mathfrak{I}_{d-1}S$. As we just noticed, the fact that f is essential depends on the basis $\mathcal{B}(\mathfrak{I})$. Our aim is to investigate how the *nature of f with respect to essentiality* (briefly: *the nature of f*) changes with $\mathcal{B}(\mathfrak{I})$. Some lemmas will be useful.

Lemma 5.1. *Let $\mathcal{B} = (f_1, \dots, f_{i-1}, f, f_{i+1}, \dots, f_m)$, $\mathcal{B}' = (f_1', \dots, f_{i-1}', f + h, f_{i+1}', \dots, f_m')$ be two standard bases of $\mathfrak{I}$, where $h, f_j' \in \mathfrak{H}_{(f,\ \mathcal{B})} = (f_1, \dots, \hat{f}, \dots, f_m)$. Then $f \in \mathfrak{I}_d$ is essential (resp. inessential) with respect to $\mathcal{B}$ iff $f + h$ is so with respect to $\mathcal{B}'$.*

Proof. It is enough to observe that: $\mathfrak{H}_{(f,\ \mathcal{B})} = \mathfrak{H}_{(f+h,\ \mathcal{B}')}$. As a consequence

$$(f + h)\mathfrak{M}^t \subset \mathfrak{H}_{(f+h,\ \mathcal{B}')} \Leftrightarrow f\mathfrak{M}^t \subset \mathfrak{H}_{(f,\ \mathcal{B})}. \quad \square$$

Hence a basis change acting only on $\mathcal{B}(\mathfrak{I}) - \{f\}$ does not modify f's nature, as it does not modify $\mathfrak{H}_{(f,\ \mathcal{B})}$. In particular, this situation happens for a basis change acting on elements of degree different from d.

Lemma 5.2. *The nature of $f = f_i \in \mathcal{B}(\mathfrak{I}) = (f_1, \dots, f_m)$, with respect to another basis $\mathcal{B}'(\mathfrak{I})$ containing it, is the same as with respect to a basis of the type $\widetilde{\mathcal{B}} = (f_j + a_jf)$, where $j = 1 \dots m,\ a_i = 0$ and the a_j are properly chosen.*

Proof. Let $\mathcal{B}'(\mathfrak{I}) = (f_1', \dots, f_{i-1}', f, f_{i+1}', \dots, f_m')$ be any other basis linked to $\mathcal{B}(\mathfrak{I})$ by the relation $\mathcal{B}'(\mathfrak{I}) = \mathcal{B}(\mathfrak{I})T$, where T is an invertible matrix. The submatrix T_{ii}, obtained from T by deleting the row and the column of index i, is still invertible and acts on $\mathcal{B}_1 = (f_1, \dots, \hat{f_i}, \dots, f_m)$. The basis change acting on $\mathcal{B}'(\mathfrak{I})$ with T_{ii}^{-1} produces a basis $\widetilde{\mathcal{B}}$ as described in the statement and the nature of f with respect to it is the same that it had with respect to $\mathcal{B}'(\mathfrak{I})$, thanks to Lemma 5.1. $\square$

The previous lemma gives immediately the following statement.

Proposition 5.1. *To decide the nature of $f = f_i \in \mathcal{B} = (f_1, \dots, f_m)$ when $\mathcal{B}$ is replaced by any $\mathcal{B}'$ containing it, it is enough to consider just the bases $\widetilde{\mathcal{B}}$ obtained from $\mathcal{B}$ by replacing f_j with $(f_j + a_jf)$, $j \neq i$, for all (degree-allowed) forms a_j.*

Remark. Let us observe that Corollary 4.1 can be viewed as a consequence of the previous Proposition, as the replacement of C_r with C_r' corresponds to a replacement of g_i with $g_i - t_{ri}g_r,\ i = 1 \dots m,\ i \neq r$.

Lemma 5.3. *Let c, c_1 be elements of $\mathcal{B}(\mathfrak{I})$ in the same degree d, both inessential (resp. essential), and c s.i. (resp. s.e.). Then a replacement of c with $c + \alpha c_1$, $\alpha \in K$, cannot change the nature of c_1.*

Proof. Let us consider the new basis $\mathcal{B}'(\mathfrak{I})$ in which c is replaced by $c + \alpha c_1 = c', \alpha \neq 0$. The nature of its element c_1 does not change if we replace it with $c_1 - \alpha^{-1}c' = -\alpha^{-1}c$; this shows that c_1 preserves its former nature with respect to $\mathcal{B}'(\mathfrak{I})$. $\square$

Now we point our attention to the bases having the greatest (respectively, smallest) number of essential generators in a chosen degree d; let us denote any such basis $\mathcal{B}^{(d)}(\mathfrak{I})$ (resp. $\mathcal{C}^{(d)}(\mathfrak{I})$) and $\nu_e(d)$ (resp. $\mu_e(d)$) the number of their essential entries in degree d.

Proposition 5.2. *There exist bases $\mathcal{B}_{Max}(\mathfrak{I})$ (resp. $\mathcal{B}_{min}(\mathfrak{I})$) having, in every degree d, exactly $\nu_e(d)$ (resp. $\mu_e(d)$) essential generators.*

Proof. We will prove the statement for $\mathcal{B}_{Max}$; the same reasoning can be repeated for $\mathcal{B}_{min}$. As usual, we let α be the minimal degree of an element of $\mathfrak{I}$. Let us denote by $\mathcal{B}_M^d$ a basis satisfying the required condition for every degree $\leq d$. We will prove the existence of a $\mathcal{B}_M^d$, for every d, using induction on d. For $d = \alpha$, we can chose $\mathcal{B}_M^\alpha = \mathcal{B}^{(\alpha)}$, for some choice of $\mathcal{B}^{(\alpha)}$. Now, let us suppose the existence of a $\mathcal{B}_M^d$ and produce a $\mathcal{B}_M^{d+1}$. To obtain $\mathcal{B}_M^{d+1}$ it is sufficient to replace in $\mathcal{B}^{(d+1)}$ the part of degree $\leq d$ with the analogous of the chosen $\mathcal{B}_M^d$ and modify the generators of larger degree as follows. Let us denote by ϕ_i any element of $\mathcal{B}_M^d$ of degree $> d$ and ψ_i any element of $\mathcal{B}^{(d+1)}$ of degree $>d$. We can write $\psi_i = \sum_j a_j\phi_j + \delta_i$, where $\delta_i \in \mathfrak{I}_d S$. Let us set $\psi_i' = \psi_i - \delta_i = \sum_j a_j\phi_j$. We claim that we obtain a $\mathcal{B}_M^{d+1}$ by replacing the generators of $\mathcal{B}_M^d$ of degree $\geq d+1$ with the ψ_i'. In fact, with respect to this basis, any ψ_i' has the same nature of the corresponding ψ_i with respect to $\mathcal{B}_M^{d+1}$, so that in degree $d+1$ we have the maximum number of essential elements; moreover, the elements of degree $\leq d$ have the same nature with respect to $\mathcal{B}_M^d$ and with respect to $\mathcal{B}_M^{d+1}$, as the change we made in degree $>d$ does not involve them. $\square$

Definition 5.1. Every basis satisfying the condition of Proposition 5.2 will be called *maximal* (resp. *minimal*) with respect to essentiality or, briefly, an *e-maximal basis* (resp. *e-minimal basis*). Its number of essential elements will be denoted $\nu_e(\mathfrak{I})$ (resp. $\mu_e(\mathfrak{I})$).

As a consequence of Proposition 4.1, using the standard notation $\nu(\mathfrak{I})$ to denote the cardinality of $\mathcal{B}(\mathfrak{I})$, we can state the following corollary.

Corollary 5.1. *If $\mathfrak{I} \subset K[y_0, \ldots, y_n]$, $n \geq 2$, is a perfect height 2 ideal satisfying the condition $\nu(\mathfrak{I}) \leq n+1$, then every $\mathcal{B}(\mathfrak{I})$ is an e-maximal basis (more precisely, no basis contains inessential elements).*

Proof. As every Hilbert matrix of $\mathfrak{I}$ has at most n rows, it is enough to observe that $\mathfrak{M}^h$, for every natural number h, cannot be contained in an ideal generated by at most n forms. $\square$

Remark. Thanks to Dubreil's Theorem which says that $\nu(\mathfrak{I}) \leq \alpha(\mathfrak{I})+1$, the condition of Corollary 5.1 is necessarily verified if $\alpha(\mathfrak{I}) \leq n$, that is if the minimal degree of a hypersurface containing the corresponding scheme is $\leq n$.

Now we come back to the general situation.

Proposition 5.3. *Let $\mathcal{B}(\mathfrak{I})$ and $\mathcal{B}'(\mathfrak{I})$ be two bases such that, in degree d, all their inessential (resp. essential) elements are s.i. (resp. s.e). Then the subspace of $\mathfrak{I}_d/(\mathfrak{I}_{d-1}S_1)$ generated by the inessential (resp. essential) elements of $\mathcal{B}(\mathfrak{I})$ coincides with the one generated by the inessential (resp. essential) elements of $\mathcal{B}'(\mathfrak{I})$.*

Proof. Let us consider the elements of $\mathcal{B}(\mathfrak{I})$ and $\mathcal{B}'(\mathfrak{I})$ in degree d:

$$\mathcal{B}_d(\mathfrak{I}) = (b_1, \ldots, b_h, c_1, \ldots, c_k); \qquad \mathcal{B}_d'(\mathfrak{I}) = (b_1', \ldots, b_{h'}', c_1', \ldots, c_{k'}'),$$

where b_i, b_i' are essential and c_j, c_j' are strongly inessential.

In $\mathfrak{I}_d/\mathfrak{I}_{d-1}$ we have

$$c_j' = \sum_{i=1}^{k} \gamma_i c_i + \sum_{i=1}^{h} \beta_i b_i.$$

Let us prove that $\beta_i = 0,\ i = 1 \ldots h$, so that the subspace generated by $(c_1', \ldots, c_{k'}')$ shall be inside the one generated by $(c_1, \ldots, c_k)$. As we can exchange the role of $\mathcal{B}(\mathfrak{I})$ and $\mathcal{B}'(\mathfrak{I})$, that will be enough to complete the proof. So, let us suppose $\beta_i \neq 0$ for some i and get a contradiction. In fact $\beta_i \neq 0$ implies that, in $\mathcal{B}(\mathfrak{I})$, b_i can be replaced by c_j' without changing its nature, against the hypothesis that c_j' is inessential with respect to every basis containing it.

By chance, let us observe that $h = h'$, $k = k'$.

Interchanging *inessential* and *essential* we prove the other part of the statement. $\square$

Proposition 5.3 gives immediately the following consequences.

Corollary 5.2. *Two bases whose inessential (resp. essential) elements are s.i. (resp. s.e.) must have the same number of inessential (resp. essential) elements, degree by degree.*

Theorem 5.1. *A standard basis is e-maximal (resp. e-minimal) iff its inessential (resp. essential) elements are strongly inessential (resp. strongly essential).*

Proof. We prove the statement for the e-maximal case, as the e-minimal one is analogous.

First we prove that the inessential elements of an e-maximal basis $\mathcal{B}_M$ are s.i. Let $c \in \mathcal{B}_M$ be inessential; thanks to Proposition 5.1, it is enough to check that c is still inessential with respect to the basis obtained from $\mathcal{B}_M$ by replacing each of its elements different from c, say f_i, with $f_i + a_i c = f_i'$. After such a replacement the nature of f_i' is the same as the one of f_i, so that a change of nature of c would imply the existence of a basis with one more essential element, against the maximality of $\mathcal{B}_M$.

Vice versa, let $\mathcal{B}$ be any basis whose inessential elements are s.i. We just proved that every e-maximal basis $\mathcal{B}_M$ has such a property, so that Corollary 5.2 states that $\mathcal{B}$ and $\mathcal{B}_M$ have the same number of s.i. elements. As a consequence, $\mathcal{B}$ is also e-maximal. □

Now, let us give a construction of an e-maximal (resp. e-minimal) basis.

Proposition 5.4. *Starting from any basis $\mathcal{B}(\mathfrak{I})$ it is possible to produce an e-maximal basis (resp. e-minimal basis) containing all the s.i. (resp. s.e.) elements of $\mathcal{B}(\mathfrak{I})$.*

Proof. Let us consider first the case of an e-maximal basis.

Thanks to Theorem 5.1, the aim is to produce a basis whose inessential elements are s.i. So, we start to consider the inessential, but not s.i., generators of lowest degree, following the order in which they appear in $\mathcal{B}(\mathfrak{I})$: let c_1, $\deg c_1 = d$, be the first of them. The replacement of some other elements $f_i \in \mathcal{B}(\mathfrak{I})$ with $f_i' = f_i + a_i c_1$ makes c_1 essential (Lemma 5.2), while f_i', with respect to the new basis, has the same nature as f_i (Lemma 5.1). We observe that the s.i. elements of degree d are not impacted, thanks to Lemma 5.3. Let us denote $\mathcal{B}^1(\mathfrak{I})$ the new basis at this step, in which c_1 is essential and, as a consequence, the number of inessential, but non-s.i., elements has decreased . Then we go on dealing with $\mathcal{B}^1(\mathfrak{I})$ just as we did with $\mathcal{B}(\mathfrak{I})$. After a finite number of steps, we get a basis $\mathcal{B}^u(\mathfrak{I})$ whose inessential elements are s.i.

Analogously, it is possible to produce an e-minimal basis, starting from any basis $\mathcal{B}(\mathfrak{I})$: it is enough to replace *inessential* with *essential* in the previous construction. □

Now we turn our attention to the dehomogenization $\mathfrak{I}_*$ of $\mathfrak{I}$ with respect to a generic linear form L and to the system of generators $\mathcal{B}_*$ obtained from $\mathcal{B}(\mathfrak{I})$ by dehomogenizing every form appearing in it. In general, $\mathcal{B}_*$ is not a basis for $\mathfrak{I}_*$ and our aim is to find its subsets that are bases and, among them, the ones of minimal cardinality.

The definition of inessential element can be generalized as follows.

Definition 5.2. A subset T of $\mathcal{B}(\mathfrak{I})$ is *inessential* iff $\mathfrak{I}^{sat} = (\mathcal{B}(\mathfrak{I}) - T)^{sat}$.

Let us observe that if $T = \{t\}$, then T is an inessential subset of $\mathcal{B}(\mathfrak{I})$ iff t is inessential as an element.

The following proposition, similar to Proposition 3.1, gives conditions equivalent to the one defining an inessential set $T = \{c_1, \dots, c_k\}$.

Proposition 5.5. *Let $\mathcal{B}(\mathfrak{I}) = (b_1, \dots, b_h, c_1, \dots, c_k)$, where c_i is inessential and $\deg c_i \leq \deg c_{i+1}$ for $i = 1 \dots k$. The following facts are equivalent.*

(i) *For any $i = 1 \dots k$, c_i is inessential with respect to the basis $\mathcal{B}_i = (b_1, \dots, b_h, c_1, \dots, c_i)$, generating an ideal $\mathfrak{A}_i$.*
(ii) *If $\mathfrak{A}$ is the ideal generated by $(b_1, \dots, b_h)$, then $\mathfrak{I}^{sat} = \mathfrak{A}^{sat}$.*
(iii) *For every form α_{ij}, where $\deg \alpha_{ij} = \deg c_j - \deg c_i$, $i = 1 \dots k$ and $j = (i+1) \dots k$, the element c_i is inessential with respect to the standard basis $\mathcal{B}(\alpha_{ij}) = (b_1, \dots, b_h, c_1, \dots, c_i, c_{i+1} + \alpha_{i,i+1} c_i, \dots, c_k + \alpha_{ik} c_i)$ of $\mathfrak{I}$.*
(iv) *For any $i = 1 \dots k$, c_i is inessential with respect to every standard basis of $\mathfrak{I}$ of the type $(b_1, \dots, b_h, c_1, \dots, c_i, f_{i+1}, \dots, f_k)$.*

Proof. (i) → (ii) It is enough to observe that $c_{i+1} \in \mathfrak{A}_i^{sat}$ implies $(\mathfrak{A}_{i+1})^{sat} = \mathfrak{A}_i^{sat}$.

(ii) → (i) The hypothesis implies that c_i is inessential with respect to $(b_1, \dots, b_h, c_i)$ and, as a consequence, with respect to $(b_1, \dots, b_h, c_1, \dots, c_i)$.

(i) → (iii) This is obvious, because $(b_1, \dots, b_h, c_1, \dots, c_{i-1})S \subset (b_1, \dots, b_h, c_1, \dots, c_{i-1}, c_{i+1} + \alpha_{i,i+1} c_i, \dots, c_k + \alpha_{ik} c_i)S$.

(iii) → (i) We use induction on $k - i$.

If $k - i = 0$ both conditions say that c_k is inessential with respect to $\mathcal{B}(\mathfrak{I})$.

Let us suppose the implication is true for values less than or equal to $k - (i + 1)$ and prove it for $k - i$. Induction says that $c_{i+1}, \dots, c_k \in (b_1, \dots, b_h, c_1, \dots, c_i)^{sat}$. So, it is enough to prove that $c_i \in (b_1, \dots, b_h, c_1, \dots, c_{i-1})^{sat}$ or, equivalently, that c_i is inessential with respect to $\mathcal{B}_i$. If not, according to Proposition 3.2 the essentiality of c_i with respect to $\mathcal{B}_i$ would mean that there exists a set $\{L_1, \dots, L_n, N\}$ of linear forms generating $\mathfrak{M}$ such that N is regular both for $S/(\mathfrak{A}_{i-1})^{sat}$ and for $S/(\mathfrak{A}_i)^{sat}$, and moreover

$$\forall t \in \mathbb{N}, \quad c_i N^t \notin (\mathcal{B}_{i-1}, c_i L_1, \dots, c_i L_n), \quad t \gg 0. \tag{4}$$

Hence, we will prove that it is possible to find $v_j \in K$, $j = (i + 1) \dots k$, such that

$$c_i N^t \notin (\mathcal{B}_{i-1}, c_{i+1} + v_{i+1} N^{u_{i+1}} c_i, \dots, c_k + v_k N^{u_k} c_i, c_i L_1, \dots, c_i L_n), \quad t \gg 0, \tag{5}$$

so that c_i is essential with respect to $\mathcal{B}(\alpha_{ij})$, where $\alpha_{ij} = v_j N^{u_j} c_i$, $u_j = \deg c_j - \deg c_i \geq 0$, contradicting condition (iii).

To this aim, it is enough to prove that the v_j's can be chosen to realize the inclusion $((c_j + v_j N^{u_j} c_i)S)_t \subseteq (\mathcal{B}_{i-1}, c_i L_1, \dots, c_i L_n)$, $t \gg 0$, $j > i$, or equivalently

$$(c_j + v_j N^{u_j} c_i) L_w^t \in (\mathcal{B}_{i-1}, c_i L_1, \dots, c_i L_n), \quad t \gg 0, \ j > i, \ w = 1 \dots n \tag{6}$$

and

$$(c_j + v_j N^{u_j} c_i)N^t \in (\mathcal{B}_{i-1}, c_iL_1, \dots, c_iL_n), \quad t \gg 0,\ j > i. \tag{7}$$

Now, (6) is equivalent to : $c_jL_w^t \in (\mathcal{B}_{i-1}, c_iL_1, \dots, c_iL_n, c_iN) = \mathcal{B}_i,\ t \gg 0,\ w = 1 \dots n$. But we already observed that induction implies $c_j \in (b_1, \dots, b_h, c_1, \dots, c_i)^{sat},\ j > i$, hence (6) is true.

Let us consider (7). Using induction, we have: $\exists t_0,\ c_jN^{t_0} = h_1 + c_iP + \alpha c_iN^{t_1}$, where $h_1 \in \mathfrak{A}_{i-1},\ P \in (L_1, \dots, L_n),\ \alpha \in K$ and $t_1 = t_0 + (\deg c_j - \deg c_i)$. So, to realize (7), it is enough to choose $v_j = -\alpha$.

(iii) $\leftrightarrow$ (iv) It is enough to use Lemma 5.2. $\square$

An immediate consequence of the previous proposition and of Theorem 5.1 is the following corollary.

Corollary 5.3. *If $\mathcal{B}(\mathfrak{I})$ is an e-maximal basis, then the set of all its inessential elements is an inessential set.*

In the special case of perfect height 2 ideals, Proposition 4.1, Corollary 4.1 and Proposition 5.5 give rise to the following proposition.

Proposition 5.6. *Let $\mathfrak{M}(\mathfrak{I})$ be a Hilbert matrix with respect to the basis $\mathcal{B}(\mathfrak{I})$. The subset T, $|T| = s$, of $\mathcal{B}(\mathfrak{I})$ corresponding to the columns $C_{i_1}, \dots, C_{i_j}, \dots, C_{i_s}$, $i_1 < i_2 < \cdots < i_s$, of $\mathfrak{M}(\mathfrak{I})$ is inessential iff, $\forall i_j \in (i_1, \dots, i_s)$, the following condition is satisfied:*

(*) *For every choice of the forms t_{i_j,i_h}, where $h \geq j$ and $\deg\ t_{i_j,i_h} = \deg\ g_h - \deg\ g_j$, the entries of*

$$C'_{i_j} = \sum_{h \geq j} t_{i_j,i_h} C_{i_h}, \quad t_{i_j,i_j} = 1$$

generate an ideal $\mathfrak{I}_{C'_{i_j}}$ containing some power of the irrelevant ideal $\mathfrak{M}$.

Definition 5.3. A subset $T \subset \mathcal{B}(\mathfrak{I})$ is called a *maximal inessential set* of $\mathcal{B}(\mathfrak{I})$ iff it is inessential and it is not properly included in any other inessential set of $\mathcal{B}(\mathfrak{I})$.

Remark 5.1. An inessential set of maximal cardinality will be clearly a maximal inessential set, but, in general, the converse is not true.

The interest of maximal inessential sets lies in the following statement.

Proposition 5.7. *Let $\mathcal{B}(\mathfrak{I}) = (b_1, \dots, b_h, c_1, \dots, c_k)$, where $T = (c_1, \dots, c_k)$ is a maximal inessential set. Then, in any dehomogenization $\mathfrak{I}_*$ of $\mathfrak{I}$ with respect to a linear form L, the set $\mathcal{B}_L(\mathfrak{I}_*) = ((b_1)_*, \dots, (b_h)_*)$ is a set of generators of $\mathfrak{I}_*$. Moreover, in the K-space $\mathfrak{M}_1$, the subset of the linear forms L such that $\mathcal{B}_L(\mathfrak{I}_*)$ is not a basis of $\mathfrak{I}_*$ is a finite union of proper linear subspaces (briefly, we can say that $\mathcal{B}_L(\mathfrak{I}_*)$ is generically a basis).*

Proof. The first assertion comes immediately from the definition of inessential set. In fact, $c_i \in (b_1, \dots, b_h)^{sat}$ means that

$$\forall L \in \mathfrak{M}_1, \quad (\exists t)\ c_iL^t \in (b_1, \dots, b_h).$$

As a consequence, $(c_i)_* \in ((b_1)_*, \dots, (b_h)_*)$, in the dehomogenization with respect to L.

The second part of the statement can be proved by observing that $(b_i)_* \in ((b_1)_*, \dots, (b_h)_*)$ is equivalent to $b_iL^t \in (b_1, \dots, \hat{b}_i, \dots, b_h)$, for some t. As

$$b_iL_j^{t_j} \in (b_1, \dots, \hat{b}_i, \dots, b_h), \quad j = 1, 2 \Rightarrow b_i(L_1 + L_2)^{2sup(t_1,t_2)} \in (b_1, \dots, \hat{b}_i, \dots, b_h),$$

the set of all the linear forms L for which $\mathcal{B}_L(\mathfrak{I}_*)$ is not a basis is a finite union of linear subspaces V_i of $\mathfrak{M}_1$. The equality $V_i = \mathfrak{M}_1$ cannot hold, as it would imply $b_i \in (b_1, \dots, \hat{b}_i, \dots, b_h)^{sat}$, against the hypothesis on the maximality of T. $\square$

As a consequence of Proposition 5.7, we can say that a subset of a standard basis $\mathcal{B}(\mathfrak{I})$ gives rise to a minimal basis for the dehomogenization of $\mathfrak{I}$ with respect to a generic linear form iff it is the complement, in $\mathcal{B}(\mathfrak{I})$, of an inessential subset with maximal cardinality.

Lemma 5.4. *Let $\mathcal{B}(\mathfrak{I}) = (b_1, \dots, b_h, c_1, \dots, c_k)$, where $T = (c_1, \dots, c_k)$ is an inessential set. In a basis $\mathcal{B}'(\mathfrak{I}) = (b'_1, \dots, b'_h, c'_1, \dots, c'_k)$, where $(b'_1, \dots, b'_h)S = (b_1, \dots, b_h)S = \mathfrak{L}$, the subset $T' = (c'_1, \dots, c'_k)$ is still inessential.*

Proof. The hypothesis says that $T \subset \mathfrak{L}^{sat}$. As a consequence, the inclusion $T' \subset \mathfrak{L}^{sat}$ also holds. $\square$

The following proposition says that we can find an inessential set of maximal cardinality among the inessential sets of the e-minimal bases.

Proposition 5.8. *Let $\mathcal{B}_m(\mathfrak{I})$ be the e-minimal basis produced for $\mathcal{B}(\mathfrak{I})$ according to Proposition 5.4. For every inessential subset $V \subset \mathcal{B}(\mathfrak{I})$, there exists an inessential subset $V' \subset \mathcal{B}_m(\mathfrak{I})$ with the same cardinality as V.*

Proof. $\mathcal{B}_m(\mathfrak{I})$ is obtained from $\mathcal{B}(\mathfrak{I})$ by replacing every element $f \in \mathcal{B}(\mathfrak{I})$ with $f' = f + \sum a_jf_j$, where the f_j's are elements outside V. As a consequence, V is replaced by V' so that V and V' have the same cardinality. We can now apply Lemma 5.4 where $T = V$. $\square$

From the previous proposition it is immediate to get the following corollary.

Corollary 5.4. *The maximal cardinality of the inessential subsets of $\mathcal{B}_m(\mathfrak{I})$ is not less than the one of the inessential subsets of $\mathcal{B}(\mathfrak{I})$.*

Now we point our attention to the bases whose inessential elements form an inessential set.

Definition 5.4. A basis $\mathcal{B}(\mathfrak{I})$ whose essential elements generate an ideal $\mathfrak{E}$ such that $\mathfrak{E}^{sat} = \mathfrak{I}^{sat}$ is called an *essential basis* (briefly: *E-basis*).

Let us produce an ideal with an *E-basis* containing exactly two inessential elements, both in maximal degree, each of which is *s.i.*

Example 5.1. Let $\mathfrak{I} \subset K[x, y, z]$ be the ideal generated by the maximal minors of the matrix

$$M(\mathfrak{I}) = \begin{pmatrix} z^2 & 0 & 0 & y & x \\ 0 & x^2+y^2 & 0 & 0 & y \\ 0 & 0 & 0 & x & z \\ 0 & 0 & x^2-y^2 & z & 0 \end{pmatrix}.$$

Any linear combination of the last two columns produces a new column generating $\mathfrak{M}$; this means that the two generators of maximal degree are strongly inessential. More precisely, $\mathfrak{I} = (g_1, \dots, g_5)$, where: $g_1 = (x^4 - y^4)(x^2 - yz)$, $g_2 = xyz^2(x^2 - y^2)$, $g_3 = z^4(x^2 + y^2)$, $g_4 = z^3(x^4 - y^4)$, $g_5 = xz^2(x^4 - y^4)$ and $\mathfrak{L} = (g_1, g_2, g_3)$ is such that $\mathfrak{L}^{sat} = \mathfrak{I} = \mathfrak{I}^{sat}$, as can be seen with a direct computation; however, the last assertion is a consequence of Definition 5.2 and Proposition 5.6.

Now, our aim is to produce, starting from any standard basis $\mathcal{B}(\mathfrak{I})$, an E-basis $\mathcal{B}_E(\mathfrak{I})$, containing all the essential elements of $\mathcal{B}(\mathfrak{I})$.

Lemma 5.5. *Let $\mathcal{B}(\mathfrak{I}) = (b_1, \dots, b_h, c_1, \dots, c_k)$ be any standard basis for $\mathfrak{I}$, where b_i is essential for $i = 1 \dots h$ and c_i is inessential for $i = 1 \dots k$. There exists a maximal inessential set $V \subset \mathcal{B}(\mathfrak{I})$ such that*

$$c_i \notin V \Rightarrow \{c_i, V \cap (c_{i+1}, \dots, c_k)\} \text{ is not inessential.} \tag{8}$$

Proof. We define V, step by step, by means of the following conditions:

(i) $c_k \in V$;
(ii) $c_i \in V$ for $i < k \Leftrightarrow c_i \in (\mathcal{B}(\mathfrak{I}) - \{c_i, V \cap (c_{i+1}, \dots, c_k)\})^{sat}$.

Condition (8) coincides with (ii) and the inessentiality of V is an immediate consequence of the definition of inessential set. Moreover, condition (8) implies the maximality of V. $\square$

Proposition 5.9. *Starting from any basis $\mathcal{B}(\mathfrak{I})$, it is possible to produce an E-basis $\mathcal{B}_E(\mathfrak{I})$, whose set of essential elements includes the ones of $\mathcal{B}(\mathfrak{I})$.*

Proof. Let us focus our attention on the maximal inessential set $V = (v_1, \dots, v_r)$ defined in Lemma 5.5. Condition (8) says that every inessential element c_i outside V becomes essential by a replacement of v_j with $v'_j = v_j + \sum a_{ij}c_i$, where the a_{ij} are properly chosen step by step (see Proposition 5.5). Moreover, Lemma 5.4 assures that the subset $V' = (v'_j)$, $j = 1 \dots r$ is inessential in $\mathcal{B}_E(\mathfrak{I})$. As a consequence, $\mathcal{B}_E(\mathfrak{I})$ turns out to be an E-basis. $\square$

Remark 5.2. Let us observe that the maximal inessential set $V \subset \mathcal{B}(\mathfrak{I})$ satisfying condition (8) is not, in general, of maximal cardinality among the inessential sets of $\mathcal{B}(\mathfrak{I})$. However, it may happen that, in another basis $\mathcal{B}'(\mathfrak{I})$, there exists an inessential set of maximal cardinality satisfying condition (8). To understand this situation, let us consider the following example.

Example 5.2. We consider the ideal $\mathfrak{I} \subset S = K[x, y, z]$, generated by the maximal minors of the matrix

$$M = \begin{pmatrix} yz & 0 & 0 & x^2 & x \\ 0 & x^2 & y^2 & 0 & y \\ 0 & 0 & x^2 & z^2 & z \\ 0 & 0 & z & y & 0 \end{pmatrix}.$$

$\mathfrak{I}$ is a perfect height 2 ideal with M as a matrix of syzygies. In fact its generators $g_1, \dots, g_5$, corresponding to the columns $C_1, \dots, C_5$, are respectively:

$$g_1 = x^3(-xz^2 - x^2y + z^3), \quad g_2 = y^2z(y^2z - x^2y + z^3),$$
$$g_3 = -x^2y^2z^2, \quad g_4 = x^2yz^3, \quad g_5 = x^2yz(x^2y - z^3)$$

and they do not have a common factor.

Thanks to Proposition 4.1, we see that, with respect to the basis $\mathcal{B}(\mathfrak{I}) = (g_1, \dots, g_5)$, g_1, g_2 are essential, while g_3, g_4, g_5 are inessential. More precisely, by means of Proposition 5.6, we can verify that $T_1 = \{g_3, g_4\}$ and $T_2 = \{g_5\}$ are maximal inessential sets. Moreover, $\mathfrak{I}$ cannot have an inessential set with three elements; otherwise, $\mathfrak{I}$ would be the saturation of an ideal $\mathfrak{L}$ generated by two elements, but such a $\mathfrak{L}$ is already saturated. So, T_1 is an inessential set of maximal cardinality that does not satisfy condition (8), while T_2 is the maximal inessential set of $\mathcal{B}(\mathfrak{I})$ satisfying condition (8).

Concretely, the situation is as follows. In the generic dehomogenization $\mathfrak{I}_*$ of $\mathfrak{I}$ with respect to a regular linear form, $\mathcal{B}(\mathfrak{I})$ gives rise to two different bases of $\mathfrak{I}_*$:

$$\mathcal{B}_1(\mathfrak{I}_*) = ((g_1)_*, (g_2)_*, (g_5)_*), \qquad \mathcal{B}_2(\mathfrak{I}_*) = ((g_1)_*, (g_2)_*, (g_3)_*, (g_4)_*).$$

In S, $\mathcal{B}_1 = (g_1, g_2, g_5)$ generates an ideal $\mathfrak{I}_1$, defining the same projective scheme as $\mathfrak{I}$, whose saturation is $\mathfrak{I}$. In $\mathfrak{I}_1$, the basis $\mathcal{B}_1$ consists of essential elements; however, in $\mathcal{B}(\mathfrak{I})$ g_5 is not essential.

Similarly, $\mathcal{B}_2 = (g_1, g_2, g_3, g_4)$ generates an ideal $\mathfrak{I}_2$, defining the same projective scheme as $\mathfrak{I}$, that is such that $(\mathfrak{I}_2)^{sat} = \mathfrak{I}$. The entries of $\mathcal{B}_2$ are essential and they remain essential in $\mathcal{B}_E(\mathfrak{I}) = (g_1, g_2, g_3, g_4, g_5 + yg_3 - xg_4)$, which turns out to be an E-basis; unfortunately, $\mathcal{B}_2(\mathfrak{I}_*)$ is not of minimal cardinality.

Now, let us consider the new basis $\mathcal{B}'(\mathfrak{I}) = (g'_1, \ldots, g'_5)$, where

$$g'_1 = g_1, \qquad g'_2 = g_2 + g_4, \qquad g'_3 = g_3 + g_4, \qquad g'_4 = g_4, \qquad g'_5 = g_5,$$

corresponding to the following matrix of syzygies

$$M'(\mathfrak{I}) = \begin{pmatrix} yz & 0 & 0 & x^2 & x \\ 0 & x^2 & y^2 & -x^2 - y^2 & y \\ 0 & 0 & x^2 & z^2 - x^2 & z \\ 0 & 0 & z & y - z & 0 \end{pmatrix}.$$

Proposition 5.6 says that $\{g'_4, g'_5\}$ is an inessential set of $\mathcal{B}'(\mathfrak{I})$, which turns out to be of maximal cardinality. Moreover, $\{g'_4, g'_5\}$, as a subset of $\mathcal{B}'(\mathfrak{I})$, satisfies condition (8). So, $\mathcal{B}'(\mathfrak{I})$ is an E-basis, whose essential elements give rise to a basis of $\mathfrak{I}_*$ of minimal cardinality.

References

[1] G. Beccari, C. Massaza, Separating sequences of 0-dimensional schemes, Matematiche 61 (2006) 37–68.
[2] G. Campanella, Standard bases of perfect homogeneous polynomial ideals of height 2, J. Algebra 101 (1986) 47–60.
[3] E. Davis, A. Geramita, F. Maroscia, Perfect homogeneous ideals: Dubreil's theorem revisited, Bull. Sci. Math. 108 (1984) 143–185.
[4] W. Fulton, Algebraic curves, An Introduction to Algebraic Geometry, W.A., New York, 1974.
[5] R. Hartshorne, Algebraic Geometry, Springer-Verlag, New York, 1977.
[6] M. Kreuzer, L. Robbiano, Computational Commutative Algebra 2, Springer, 2005.
[7] F.S. Macaulay, Some properties of enumeration in the theory of modular systems, Proc. Lond. Math. Soc. 26 (1927) 531–555.
[8] D.G. Northcott, Finite Free Resolutions, Cambridge University Press, Cambridge, 1976.
[9] L. Robbiano, An introduction to the theory of Hilbert functions, Queen's Papers in Pure and Appl. Math. 85C (1990) 1–32.
[10] R.P. Stanley, Hilbert functions of greaded algebras, Adv. Math. 28 (1978) 57–83.
[11] O. Zariski, P. Samuel, Commutative Algebra, Vol.II, Van Nostrand Company, New York, 1963.