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The capacitated transshipment location problem under uncertainty: a computational study

(Article begins on next page)

20 May 2024

THE CAPACITATED TRANSSHIPMENT LOCATION PROBLEM UNDER UNCERTAINTY: A COMPUTATIONAL STUDY

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ABSTRACT

We consider the stochastic capacitated transshipment problem for freight transportation where an optimal location of the transshipment facilities, which minimizes the total cost, must be found. The total cost is given by the sum of the total fixed cost plus the expected minimum total flow cost, when the throughput costs of the facilities are random variables with unknown probability distribution. By applying the asymptotic approximation method derived from the Extreme Value Theory, a deterministic non-linear model, which belongs to a wide class of Entropy maximizing models, is then obtained. The computational results show a very good performance of this deterministic model when compared to the stochastic one.

INTRODUCTION

Multi-Echelon Distribution Systems are quite common for distributing freight in supply chains and complex transportation systems. In these systems, the freight is not directly delivered from origins to destinations, but it is consolidated into intermediate depots.

Recently Multi-Echelon Distribution Systems have been investigated in City Logistics for their flexibility in freight delivery and the opportunity to integrate environmental-friendly vehicles into urban logistics systems (Crainic *et al.* 2009; Perboli *et al.* 2008). One of the main issues associated to Multi-Echelon Distribution Systems is the location of intermediate depots, which highly affects the total transportation cost. While the transportation cost can be easily calculated for the links outside the urban area, for urban links this task becomes quite difficult. In fact, the transportation cost between intermediate depots and customers, which are usually located in the urban area, highly changes day-by-day and is affected by several parameters, including the hour of the day.

Thus, whenever medium term decisions must be taken (e.g. the location of intermediate depots, which is a decision that must be valid for several years), the transportation cost between intermediate depots and customers cannot be considered deterministic, but intrinsically stochastic. Then the transportation cost becomes a stochastic variable, whose probability distribution is unknown. In fact, any assumption on the shape of this probability distribution would not be realistic, being affected by many and incommensurable parameters.

In this paper we consider the Capacitated Transshipment Location Problem under Uncertainty (CTLPU), a variant of the classical transshipment problem where the costs between intermediate depots and customers are stochastic with unknown probability distribution. More in details, given a set of origins with given supply, a set of destinations with given demand, a set of potential depot locations with deterministic fixed costs of location, upper capacity constraints for the depots, and random generalized transportation costs from origins to destinations through the depots, the CTLPU consists in finding a depot location which minimizes the total cost, given by the sum of the deterministic total fixed cost plus the expected minimum total flow cost, subject to supply, demand, and upper depot capacity constraints. Each generalized transportation cost is a random variable given by the sum of a deterministic transportation cost from an origin to a destination through an intermediate depot plus a random term, with unknown probability distribution, which represents the throughput cost of the intermediate depot.

In the huge literature on the Transshipment Facility Location Problem there are just a few papers where stochasticity is considered, but this stochasticity mainly concerns the arc capacity or the customer demand, while random costs are generally ignored. Glockner and Nemhauser (2000) consider a dynamic network flow problem where arc capacities are random variables and derive a multistage stochastic linear program. In Zhou and Liu (2003) a capacitated location-allocation problem with stochastic demands is originally formulated as expected value model, chance-constrained programming and dependent-chance programming according to different criteria. In Klose and Drexl (2005) the authors present a review of some contributions to the current state-of-the-art on facility location problems. Also probabilistic models are presented where some of the input data of the location models are subject to uncertainty. A more recent review covering stochastic and some non-linear facility location problems is due to Snyder (2006). In Zhao and Sen (2006) the authors formulate the multi-location transshipment as a two-stage stochastic program with recourse, where the demand is stochastic. In Shakeel and Gupta (2008) the authors analyze a stochastic fractional transshipment problem with uncertain demands and prohibited routes, which is solved by reformulating the stochastic transshipment problem into an equivalent deterministic

transportation problem. In Unnikrishnan *et al.* (2009) a two-stage linear program with recourse formulation is developed to determine the optimal storage capacity to be installed on transshipment nodes by shippers in a dynamic shipper carrier network under stochastic demand. In the first stage, the shipper decides the optimal capacity to be installed on transshipment nodes, while in the second stage, the shipper chooses a routing strategy based on the realized demand.

Just a few papers concerning location problems with stochastic costs are available in literature. Among them, Snyder *et al.* (2007) consider a scenario-based stochastic version of a joint location-inventory model that minimizes the expected cost of locating depots, allowing costs, lead times, demand, and some other parameters to be stochastic. In Tadei *et al.* (2009c) the authors consider a stochastic *p*-median problem where the cost for using a depot is a random variable, with unknown probability distribution. In Tadei *et al.* (2009a) the CTLPU, where only upper capacity constraints for the depots are considered, is introduced. The problem is modeled as a stochastic program, and a deterministic approximation of it, named CTLPD, is given. Although the authors showed that CTLPD is quite satisfactory when the random costs follow a Gumbel distribution, no tests have been made with other probability distributions.

In this paper, we present a deep analysis of the impact of different probability distributions on the optimum of CTLPU, as well as on the number of depots located. In particular, we analyze the performance of the deterministic approximation CTLPD by comparing its results with those obtained from CTLPU under three different probability distributions: Gumbel, Laplace, and Uniform.

THE STOCHASTIC PROBLEM AND ITS DETERMINISTIC APPROXIMATION

Let be:
- : set of origins
- : set of destinations
- : set of potential transshipment locations
- L_k : set of throughput operation scenarios at transshipment facility $k \in K$
- P_i : supply at origin $i \in I$
- Q_j : demand at destination $j \in J$
- U_k : throughput capacity of transshipment facility $k \in K$
- f_k : fixed cost of locating a transshipment facility $k \in K$
- y_k : binary variable which takes value 1 if transshipment facility $k \in K$ is located, 0 otherwise
- c_{ij}^k : unit transportation cost from origin $i \in I$ to destination $j \in J$ through transshipment facility $k \in K$
- θ_{kl} : unit throughput cost of transshipment facility $k \in K$ in throughput operation scenario $l \in L_k$
- s_{ij}^k : flow from origin $i \in I$ to destination $j \in J$ through transshipment facility $k \in K$.

Let us assume

i.   the system is balanced, i.e. $\sum_{i \in I} P_i = \sum_{j \in J} Q_j = T$

ii.   the unit throughput costs θ_{kl} are independent and identically distributed (i.i.d.) random variables with a common and unknown probability distribution

$$\Pr\{\theta_{kl} \geq x\} = F(x) \tag{1}$$

Assumption i. is a standard one and it is straightforward to balance the system if necessary. Assumption ii. is justified by the fact that the unit throughput costs usually vary among transshipment facilities and inside each of them in a random way and are quite difficult to be measured. Thus they become random variables with unknown probability distribution. Moreover, these random variables are independent each other and there is no reason to consider different shapes for their unknown probability distributions (Leonardi 1983; Leonardi and Tadei 1984).

Let $r_{ij}^{kl}(\theta)$ be the stochastic generalized unit transportation cost from origin i to destination j through transshipment facility k in throughput operation scenario l given by

$$r_{ij}^{kl}(\theta) = c_{ij}^k + \theta_{kl}, \quad i \in I, j \in J, k \in K, l \in L_k \tag{2}$$

with unknown probability distribution

$$\Pr\{r_{ij}^{kl}(\theta) \geq x\} = \Pr\{c_{ij}^k + \theta_{kl} \geq x\} = \Pr\{\theta_{kl} \geq x - c_{ij}^k\} = F(x - c_{ij}^k) \tag{3}$$

Let us define

$$\bar{\theta}_k = \min_{l \in L_k} \theta_{kl}, \quad k \in K \tag{4}$$

with unknown probability distribution

$$H(x) = \Pr\{\bar{\theta}_k \geq x\} \tag{5}$$

As $\bar{\theta}_k \geq x \Leftrightarrow \theta_{kl} \geq x, l \in L_k$ and θ_{kl} are independent, using (1) one gets

$$H(x) = \Pr\{\bar{\theta}_k \geq x\} = \prod_{l \in L_k} \Pr\{\theta_{kl} \geq x\} = \prod_{l \in L_k} F(x) = [F(x)]^{n_k} \tag{6}$$

where $n_k = |L_k|$ is the number of the different throughput operation scenarios at the transshipment facility k.

The stochastic generalized unit transportation cost from origin i to destination j through transshipment facility k is the minimum among the costs for the different throughput operation scenarios at facility k and becomes

$$\bar{r}_{ij}^k(\theta) = \min_{l \in L_k} r_{ij}^{kl}(\theta) = c_{ij}^k + \min_{l \in L_k} \theta_{kl} = c_{ij}^k + \bar{\theta}_k, \quad i \in I, j \in J, k \in K \tag{7}$$

The CTLPU may be formulated as follows

$$\min_y \sum_{k \in K} f_k y_k + E_\theta \left[\min_s \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} \bar{r}_{ij}^k(\theta) s_{ij}^k \right] \tag{8}$$

subject to

$$\sum_{j \in J} \sum_{k \in K} s_{ij}^k = P_i, \quad i \in I \tag{9}$$

$$\sum_{i \in I} \sum_{k \in K} s_{ij}^k = Q_j, \quad j \in J \tag{10}$$

$$\sum_{i \in I} \sum_{j \in J} s_{ij}^k \le U_k y_k, \quad k \in K \tag{11}$$

$$s_{ij}^k \ge 0, i \in I, j \in J, k \in K \tag{12}$$

$$y_k \in \{0, 1\}, k \in K \tag{13}$$

where E_θ denotes the expected value with respect to θ; the objective function (8) expresses the minimization of the total cost given by the sum of the minimum total fixed cost plus the expected minimum total flow cost; constraints (9) and (10) ensure that supply at each origin i and demand at each destination j are satisfied; constraints (11) ensure the capacity restriction at each transshipment facility k; (12) are the non-negativity constraints, and (13) are the integrality constraints.

It is shown (see Tadei *et al.* 2009a) that, by applying the asymptotic approximation method derived from the Extreme Value Theory (Galambos 1978), the deterministic approximation of the Capacitated Transshipment Location Problem under Uncertainty, named CTLPD, becomes

$$\min_y \sum_{k \in K} f_k y_k + \max_s \left[-\frac{1}{\beta} \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \ln s_{ij}^k - \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} s_{ij}^k \left(c_{ij}^k - \frac{1}{\beta} \right) \right] \tag{14}$$

subject to

$$\sum_{j \in J} \sum_{k \in K} s_{ij}^k = P_i, \quad i \in I \tag{15}$$

$$\sum_{i \in I} \sum_{k \in K} s_{ij}^k = Q_j, \quad j \in J \tag{16}$$

$$\sum_{i \in I} \sum_{j \in J} s_{ij}^k \le U_k y_k, \quad k \in K \tag{17}$$

$$s_{ij}^k \ge 0, i \in I, j \in J, k \in K \tag{18}$$

$$y_k \in \{0, 1\}, k \in K \tag{19}$$

which is a mixed-integer deterministic non-linear model in the unknowns y_k and s_{ij}^k.

We observe that the non-linearity affects only the objective function through the Entropy term

$-\dfrac{1}{\beta}\displaystyle\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} s_{ij}^k \ln s_{ij}^k$, while all the constraints are linear.

It is interesting to note that when $\beta \to +\infty$ problem (14)-(19) turns into the classical Capacitated Transshipment Location Problem (CTLP). In fact, the Entropy term in the objective function disappears and only the linear classical total transportation cost does remain. This is also coherent with the well-known property of the multinomial Logit model which states that this model collapses into a classical minimum transportation cost choice model as $\beta \to +\infty$ (Domencich and McFadden, 1975).

We also observe that, provided $\displaystyle\max_s -\dfrac{1}{\beta}\sum_{i\in I}\sum_{j\in J}\sum_{k\in K} s_{ij}^k \ln s_{ij}^k \geq 0$, the optimum of the classical Capacitated Transshipment Location Problem is a Lower Bound for the Capacitated Transshipment Location Problem under Uncertainty.

COMPUTATIONAL RESULTS

In this section we compare the Capacitated Transshipment Location Problem under Uncertainty CTLPU, given by (8)-(13), with its deterministic approximation CTLPD, given by (14)-(19).

This section is organized as follows. As no instance for CTLPU is available in literature, new instances are generated and introduced next (see subsection Instance generation). The setting of a commercial stochastic solver to solve CTLPU as well as the identification of an appropriate non-linear solver to solve CTLPD are given in subsection "Stochastic solver setting and non-linear solver identification". A detailed comparison between CTLPU under different probability distributions and its deterministic approximation CTLPD is discussed in subsection "Comparison between CTLPU under different probability distributions and CTLPD". Finally, in order to show the effect of the Entropy term in CTLPD, in subsection "Comparison between CTLPD and the classical CTLP" we compare CTLPD with the classical CTLP, and we evaluate the speed of convergence of CTLPD to the classical CTLP, when the value of parameter β does increase.

Instance generation

We consider a subset of the test classes given in Keskin and Uster (2007), where the authors generate 4 classes of 20 instances each. Here, due to the much higher computational effort required to solve the stochastic and the non-linear problems, we consider only the first class from Keskin and Uster (2007) and generate 10 instances instead of 20, using uniform distribution with corresponding ranges according to the following criteria:

- number of depots $|I|$ is drawn from $U[2, 3]$
- number of customers $|J|$ is drawn from $U[30, 40]$

- number of possible locations for the transshipments $|K|$ is drawn from $U[10, 20]$
- supply P_i is drawn from $U[900, 1000]$
- demand Q_j is drawn from $U\left[1, \sum_{i\in I} P_i/|J|\right]$. If necessary, the demand of the last customer is adjusted so that the total demand is equal to the total supply
- capacity U_k is drawn from $U[0.5avU, 3avU]$, where $avU = \sum_{i\in I} P_i/|K|$
- unit transportation cost c_{ij}^k is drawn from $U[1, 10]$
- fixed cost $f_k = TCU_k/(|I||J|)$, where TC is the total unit transportation cost over all the possible arcs
- random cost $\bar{\theta}_k$ is generated using three different probability distributions, Gumbel, Laplace, and Uniform, as follows (the cumulative distribution functions are considered):
  - Gumbel: $\exp(-e^{-\beta x})$ (with mode equal to 0). The parameter β is set to 0.1, which ensures to have a mean of the Gumbel distribution (≈ 5.7) quite close to the mean of the distribution used to obtain the deterministic unit costs c_{ij}^k. In this way, the random costs $\bar{\theta}_k$ have the same order of magnitude of the deterministic unit costs c_{ij}^k
  - Laplace:

$$
\begin{cases}
0.5\exp\left(\dfrac{x-\mu}{b}\right) & \text{if } x < \mu \\[2mm]
1-0.5\exp\left(-\dfrac{x-\mu}{b}\right) & \text{if } x \geq \mu
\end{cases}
$$

  with mean equal to μ. The parameters of the distribution are set such that the mean of the Laplace distribution is the same of the Gumbel one
  - Uniform:

$$
\begin{cases}
0 & \text{if } x < a \\[1mm]
\dfrac{x-a}{b-a} & \text{if } a \leq x < b \\[1mm]
1 & \text{if } x \geq b
\end{cases}
$$

  - The costs are generated in the range $[a, b] = [1, 10]$, such that the mean of the Uniform distribution is quite close to the Gumbel one.

The random unit generalized transportation costs $\bar{r}_{ij}^k$ in (8) are computed by (7). If some of them become negative, they are set to 1.

Stochastic solver setting and non-linear solver identification

As stated above, we compare CTLPU with its deterministic approximation CTLPD.
The solution of CTLPU is generated by implementing the stochastic model in XPress-SP, i.e. the stochastic programming module provided by XPress (see Dormer *et al.* 2005).

The tests are performed by generating an appropriate number of scenarios for each instance. In order to tune this number, we start with 50 scenarios and increase them by step 50. Then we solve each instance 10 times, reinitializing every time the pseudo-random generator of the stochastic components with a different seed, and compute the standard deviation and the mean of the optima over the 10 runs. The appropriate number of scenarios is then fixed to the smallest value ensuring for each instance a maximum ratio between the standard deviation and the mean less than 0.5% (see Kaut and Wallace 2007). According to our tests, this value is fixed to 100 scenarios, which show a maximum ratio between standard deviation and mean equal to 0.17%.

In order to solve the deterministic approximation CTLPD we consider the most efficient and effective state-of-the-art non-linear solvers: BonMIN, MinLP, KNITRO, LINGO and FilMINT. In order to have uniformity in input, output and computational results, we use the NEOS infrastructure (Czyzyk *et al.* 1998) to make the tests, giving to the solvers a time limit of 1000 seconds per instance.
According to our results, BonMIN and KNITRO outperform the other solvers, obtaining the best solutions on the overall set of instances. By comparing each other these two solvers, BonMIN is 10 times faster than KNITRO, which also shows some memory problems when running large instances (with more than 50000 arcs). For these reasons, we select BonMIN (release 1.1) (Bonami *et al.* 2008; Bonami and Lee 2009) to solve CTLPD within a time limit of 1000 seconds. The parameters are set to their default values, which show a satisfactory behavior both in accuracy and computational effort.

Comparison between CTLPU under different probability distributions and CTLPD

In the following we analyze the performance of the deterministic approximation CTLPD, by comparing its results with those obtained from CTLPU under the three different probability distributions. All the tests are done on a Pentium Q6600 2.4GHz Machine with 2 Gb of RAM. The parameter β in the CTLPD objective function (14) is set to 0.1 as the corresponding β value in the Gumbel distribution for CTLPU.
A comparison between the optimum of the deterministic approximation CTLPD and that of the stochastic model CTLPU under the three different probability distributions is presented in Table 1.
The table columns have the following meaning:

- Column 1: instance number
- Column 2: optimum of CTLPD
- Columns 3-5: optimum of CTLPU using Gumbel, Laplace, and Uniform distributions, respectively
- Columns 6-8: percentage gap between the stochastic optimum using Gumbel, Laplace, and Uniform distributions, respectively, and the deterministic approximation one. The gap is computed as $100(S-D)/S$, where S is the optimum of CTLPU and D the optimum of CTLPD.

Table 1 reports the results for each instance, as well as their mean in the last row.
The optima of the stochastic problem and its deterministic approximation are quite close together, with a mean gap around 2%. The gap is lower for the Gumbel and Laplace distributions than the Uniform one, which is coherent with the assumption done for the probability distribution used to derive CTLPD (i.e. it acts as an exponential function in its left

tail). About the negative value for some gap values present in Table 1, we remind that when the random cost $\overline{\theta}_k$ is generated and added to the deterministic unit cost in (7), if the resulting $\overline{r}_{ij}^k$ is negative it is set to 1. This implies a slight change in the distribution functions, producing such negative gap values.

The results of Table 1, even if satisfactory, are not sufficient to qualify the performance of CTLPD. In fact, besides the optimum, another important comparison concerns the optimal solution of the two models. This is considered in Tables 2 and 3.

Instances	Objective function				Gap		
	Det	Stoch			Gumbel	Laplace	Uniform
		Gumbel	Laplace	Uniform			
1	142713	137460	139664	134570	3.82%	2.18%	6.05%
2	209429	207013	209238	202495	1.17%	0.09%	3.42%
3	150860	144510	145031	147152	4.39%	4.02%	2.52%
4	167359	164393	165939	161654	1.80%	0.86%	3.53%
5	157160	151061	152683	148561	4.04%	2.93%	5.79%
6	211108	210291	210567	213969	0.39%	0.26%	-1.34%
7	244105	243214	245251	239280	0.37%	-0.47%	2.02%
8	248086	243645	245213	249019	1.82%	1.17%	-0.37%
9	247005	243887	246621	239930	1.28%	0.16%	2.95%
10	188291	181987	184353	185853	3.46%	2.14%	1.31%
Mean	**196612**	**192746**	**194456**	**192248**	**2.25%**	**1.43%**	**2.93%**

Table 1 - Comparison between the optimum of the deterministic approximation CTLPD and that of the stochastic model CTLPU under the three different probability distributions

Table 2 compares the optimal solutions of CTLPD with those of CTLPU, in terms of open facilities. The table columns have the following meaning:

- Column 1: instance number
- Columns 2-5: number of open facilities in CTLPD and CTLPU under Gumbel, Laplace, and Uniform distributions, respectively
- Columns 6, 8, 10: number of open facilities which are equal in the optimal solution of CTLPD and CTLPU under the three different distributions
- Columns 7, 9, 11: percentage of open facilities which are equal in the optimal solution of CTLPD and CTLPU under the three different distributions.

Instances	Number of open facilities				Common open facilities					
	Det	Stoch			Gumbel		Laplace		Uniform	
		Gumbel	Laplace	Uniform	Number	%	Number	%	Number	%
1	8	6	8	8	4	67%	8	100%	8	100%
2	9	7	7	8	5	71%	5	71%	6	75%
3	12	10	10	10	8	80%	8	80%	7	70%
4	9	6	7	8	4	67%	3	43%	6	75%
5	9	8	8	8	5	63%	6	75%	6	75%
6	13	12	12	12	9	75%	10	83%	10	83%
7	9	7	7	7	5	71%	3	43%	5	71%
8	9	8	9	8	7	88%	9	100%	6	75%
9	9	8	9	8	6	75%	6	67%	5	63%
10	13	12	11	11	11	92%	9	82%	9	82%
Mean	**10**	**8**	**9**	**9**	**6**	**75%**	**7**	**74%**	**7**	**77%**

Table 2 - Comparison of the open facilities in the optimal solutions of the deterministic approximation CTLPD and the stochastic model CTLPU under different probability distributions

Table 3 reports the results for each instance, as well as their mean in the last row.
The main conclusion we can draw from these results is that the optimal solution of the stochastic model under the three different distributions is quite similar to that of its deterministic approximation, since on average 75% of the open facilities are in common. Let us consider instance 1 with Laplace and Uniform distributions, for which the open facilities in the optimal solution are exactly the same of those of the deterministic approximation. Nevertheless, the gap between the two optima is 2.18% and 6.05% (see Table 1), respectively. We should then conclude that this gap is due to a different flow distribution in the two optimal solutions.
In order to verify this conclusion we consider the optimum of CTLPU when the open facilities are those of the CTLPD optimal solution and we compare this optimum with the original CTLPU optimum in Table 3.
The table columns have the following meaning:

- Column 1: instance number
- Column 2-4: percentage gap between the stochastic optimum when the facilities are compulsory opened as in the CTLPD optimal solution and the stochastic optimum when CTLPU can decide which facilities must be opened.

Instances	Gumbel	Laplace	Uniform
1	0.30%	0.00%	0.00%
2	0.22%	0.16%	0.36%
3	0.19%	0.47%	0.34%
4	0.73%	0.08%	0.29%
5	1.31%	0.40%	0.77%
6	0.57%	0.81%	0.03%
7	0.36%	0.45%	0.06%
8	0.27%	0.40%	0.79%
9	0.53%	0.26%	0.15%
10	0.38%	0.45%	0.84%
Mean	**0.49%**	**0.35%**	**0.38%**

Table 3 - Performance of the optimal solution of the deterministic approximation CTLPD when used as optimal solution of the stochastic model CTLPU

According to these results, the gap between the optimum of CTLPU obtained with given open facilities and the original one is on average less than 0.5 for all the three distributions. This implies that the optimal decisions taken by CTLPD and CTLPU in terms of open facilities are equivalent (i.e. they generate almost the same optimum) and that the gap between the CTLPD optimum and the CTLPU one when the open facilities are different is mainly due to a different flow distribution in the two optimal solutions.

Comparison between CTLPD and the classical CTLP

The discussion of the computational results ends by showing the behavior of the Entropy term of CTLPD in (14).

In Table 4 the contribution given by the Entropy term to the optimum of CTLPD is presented. The table compares the optimum of CTLPD with that of the classical CTLP, which differs from the former by the Entropy term.
The table columns have the following meaning:

- Column 1: instance number
- Column 2: percentage gap between the optimum of CTLPD and that of the classical CTLP
- Column 3: number of open facilities which are equal in the two optimal solutions
- Column 4: percentage of open facilities which are equal in the two optimal solutions.

Table 4 reports the results for each instance, as well as their mean in the last row.
According to the results, even if a large part of the open facilities are common to the two optimal solutions, the gap between the two optima is relevant, showing an important role played by the Entropy term in CTLPD.
We remind that when $\beta \to +\infty$ the coefficient of the Entropy term tends to 0 and CTLPD turns into the classical CTLP.
The last test we perform is devoted to show the speed of convergence of CTLPD to the classical CTLP, while the value of parameter β increases. Figure 1 reports the mean gap between the optimum of CTLPD and that of the classical CTLP while β varies.
According to Figure 1, the gap is almost zero for β equal to 5, so a very fast convergence is guaranteed.

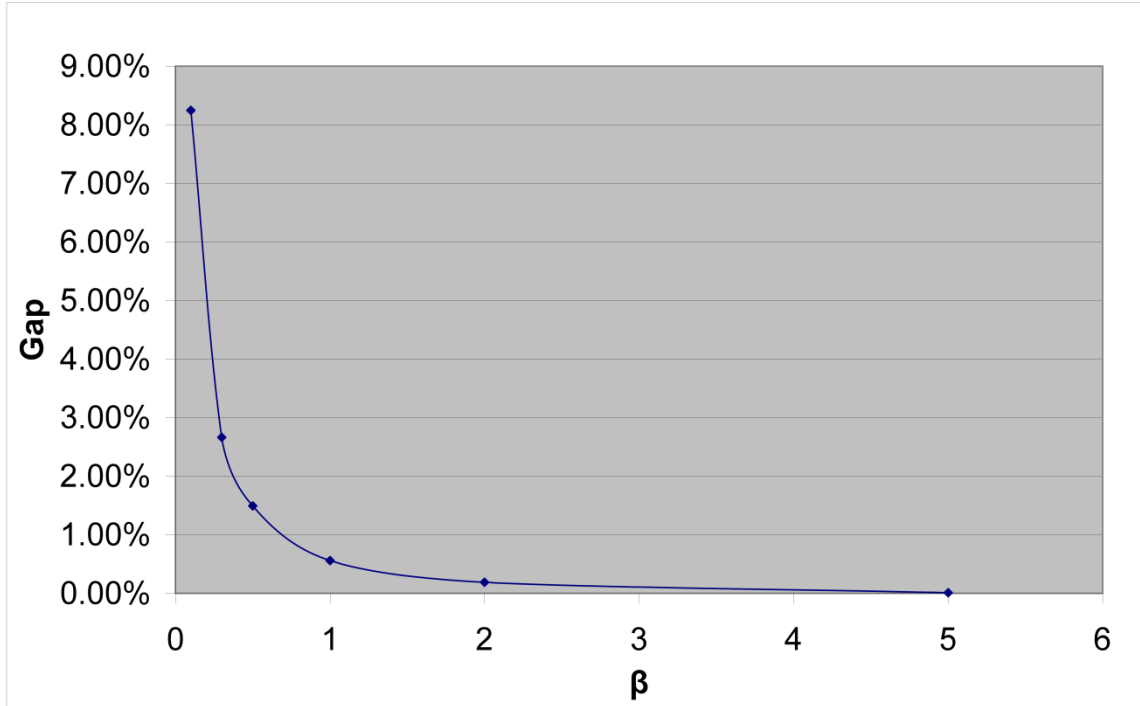| Instances	Gap	Comparison	
		Common open facilities	Common open facilities (%)
1	14.68%	5	71%
2	11.89%	6	75%
3	6.48%	8	73%
4	10.59%	5	71%
5	13.66%	6	75%
6	3.85%	10	77%
7	8.93%	9	100%
8	8.87%	6	75%
9	10.24%	5	71%
10	3.89%	11	92%
Mean	9.31%	7	78%

Table 4 - Contribution of the Entropy term in CTLPD

Figure 1 - Convergence of the deterministic approximation CTLPD to the classical CTLP as $\beta \to +\infty$

CONCLUSION

In this paper the Capacitated Transshipment Location Problem under Uncertainty CTLPU, which is a stochastic program, has been approximated to an equivalent non-linear deterministic Capacitated Transshipment Location Problem CTLPD, which belongs to a wide class of Entropy maximizing models. The performance of CTLPD is pretty good. In fact, the mean gap between the two optima is around 2%.

The gap is smaller for the Gumbel and Laplace distributions (2.25% and 1.43%, respectively) than the Uniform one (2.93%) This is coherent with the assumption done for the probability distribution used to derive CTLPD, i.e. the costs act as an exponential function in their left tail.

Also when we consider the number of located depots which are in common between the solutions of the stochastic model and the deterministic one, the results are satisfactory. In fact the behavior of the three probability distributions is quite similar to that of the deterministic approximation, since on average 75% of open depots are in common. Moreover, additional tests prove that, even when the depots opened by CTLPU and CTLPD are exactly the same, there is still a small gap between the costs due to some different freight flows.

The role of the Entropy term in CTLPD, weighted by the non-negative parameter β set to 0.1, is particularly relevant. We observe that this role is highly affected by the value of the parameter β. In fact, when this parameter increases, the contribution of the Entropy term to the optimum rapidly decreases and for $\beta = 5$ CTLPD collapses into the classical linear Capacitated Transshipment Location Problem. Both CTLPU and CTLPD have been exactly solved for small size instances (up to 3 origins, 20 potential transshipment locations and 40 destinations) in 1000 seconds by means of existing solvers. Larger instances will likely deserve some heuristic approaches (Tadei 2009b).

REFERENCES

Bonami, P., L. T. Biegler, A. R. Conn, G. Cornuéjols, I. E. Grossmann, C. D. Laird, J. Lee, A. Lodi, F. Margot, N. Sawaya and A. Wächter (2008). An algorithmic framework for convex mixed integer non-linear programs. *Discrete Optimization.* **5**, 186–204.

Bonami, P. and J. Lee (2009). BONMIN 1.1 Users' Manual. *COIN-OR.*

Crainic, T.G., N. Ricciardi and G. Storchi (2009). Models for evaluating and planning city logistics systems. *Transportation Science.* **43(4)**, 432–454.

Czyzyk, J., M. Mesnier and J. Moré (1998). The NEOS server. *IEEE Journal on Computational Science and Engineering.* **5**, 68–75.

Daskin, M. S., C. R. Coullard and Z. M. Shen (2002). An inventory-location model: formulation, solution algorithm and computational results. *Annals of Operations Research.* **5**, 227–246.

Domencich, T. and D. McFadden (1975). *Urban travel dynamics: a behavioral analysis.* North Holland.

Dormer, A., A. Vazacopoulos, N. Verma and H. Tipi (2005). Modeling & Solving Stochastic Programming Problems in Supply Chain Management Using Xpress-SP. In: *Supply Chain Optimization* (Geunes, J., Pardalos, P. M. eds.), pp. 307-354. Springer-Verlag.

Galambos, J. (1978). *The asymptotic theory of extreme order statistics.* John Wiley.

Glockner, G.D. and G. L. Nemhauser (2000). A dynamic network flow problem with uncertain arc capacities: Formulation and problem structure. *Operations Research.* **48**, 233–242.

Kaut, M. and S. W. Wallace (2007). Evaluation of scenario generation methods for stochastic programming. *Pacific Journal of Optimization.* **3**, 257–271.

Keskin, B.B. and H. Uster (2007). A scatter search-based heuristic to locate capacitated transshipment points. *Computers and Operations Research.* **34**, 3112–3125.

Klose, A. and A. Drexl (2005). Facility location models for distribution system design. *European Journal of Operational Research.* **162**, 4–29.

Leonardi, G. (1983). The use of random utility theory in building location-allocation models. In: *Locational analysis of public facilities* (Thisse, J. F., H. G. Zoller eds.), pp. 357–383. North Holland.

Leonardi, G. and R. Tadei (1984). Random utility demand models and service location. *Regional Science and Urban Economics.* **14**, 399–431.

Perboli, G., R. Tadei and D. Vigo (2008). The Two-Echelon Capacitated Vehicle Routing Problem: models and math-based heuristics. *CIRRELT-2008-55*, CIRRELT, Montreal, Canada, and *Transportation Science,* forthcoming.

Ricciardi, N., R. Tadei and A. Grosso (2002). Optimal facility location with random throughput costs. *Computers and Operations Research.* **29**, 593–607.

Shakeel, J. and S. N. Gupta (2008). Capacitated stochastic transshipment problem with prohibited routes. *Int. J. Agric. Stat. Sci.* **4**, 127–142.

Snyder, L.V. (2006). Facility location under uncertainty: a review. *IIE Transactions.* **18**, 547–564.

Snyder, L.V., M. S. Daskin and C. Teo (2007). The stochastic location model with risk pooling. *European Journal of Operational Research.* **179**, 1221–1238.

Tadei, R., N. Ricciardi and G. Perboli (2009a). The capacitated transshipment location problem under uncertainty. *CIRRELT-2009-38*, CIRRELT, Montreal, Canada. Submitted.

Tadei, R., N. Ricciardi and G. Perboli (2009b). A Lagrangian heuristic for the capacitated transshipment location problem under uncertainty. *CIRRELT-2009-56*, CIRRELT, Montreal, Canada.

Tadei, R., N. Ricciardi and G. Perboli (2009c). The stochastic p-median problem with unknown cost probability distribution. *Operations Research Letters.* **37**, 135–141.

Unnikrishnan, A., V. Valsaraj and S. T. Waller (2009). Stochastic and dynamic shipper carrier network design problem. *Networks and Spatial Economics.* **DOI: 10.1007/s11067-008-9094-3**.

Zhao, L. and S. Sen (2006). A comparison of sample-path-based simulation-optimization and stochastic decomposition for multi-location transshipment problems. In: *Proceedings of the 2006 Winter Simulation Conference* (Perrone, L.F., F. P. Wieland, J. Liu, B. G. Lawson, D. M. Nicol, R. M. Fujimoto eds.), pp. 238–245.

Zhou, J. and B. Liu (2003). New stochastic models for capacitated location allocation problem. *Computers & Industrial Engineering.* **45**, 111–125.