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Analysis of laminated shells by a sinusoidal shear deformation theory and radial basis functions collocation, accounting for through-the-thickness deformations

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A B S T R A C T

In this paper, the static and free vibration analysis of laminated shells is performed by radial basis functions collocation, according to a sinusoidal shear deformation theory (SSDT). The SSDT theory accounts for through-the-thickness deformation, by considering a sinusoidal evolution of all displacements with the thickness coordinate. The equations of motion and the boundary conditions are obtained by the Carrera's Unified Formulation, and further interpolated by collocation with radial basis functions.

1. Introduction

The efficient load-carrying capabilities of shell structures make them very useful in a variety of engineering applications. The continuous development of new structural materials leads to ever increasingly complex structural designs that require careful analysis. Although analytical techniques are very important, the use of numerical methods to solve shell mathematical models of complex structures has become an essential ingredient in the design process.

The most common mathematical models used to describe shell structures may be classified in two classes according to different physical assumptions: the Koiter model [1], based on the Kirchhoff hypothesis and the Naghdi model [2], based on the Reissner–Mindlin assumptions that take into account the transverse shear deformation. In this paper, the Unified Formulation (UF) by Carrera [3–8] is proposed to derive the equations of motion and boundary conditions. This formulation contains a large variety of 2D models that differ in the order of used expansion in thickness direction and in the manner the variables are modelled along the thickness. In particular, the UF is here applied to analyse laminated shells by radial basis functions (RBF) collocation, according to a sinus-based shear deformation theory that accounts for through-the-thickness

deformations. This theory is an expansion of the developments by Touratier [9–11], and Vidal and Polit [12]. It allows a sinusoidal variation of all displacement components along the thickness and it is more convenient than the classical Taylor polynomials because the sine function can be expressed by means of Taylor expansion. Moreover, the derivative of displacements in the deformations does not reduce the order of the approximating functions. The choice of the sine function can be justified from the three-dimensional point of view, using the work of Cheng [13]. As it can be seen in Polit [14], a sine term appears in the solution of the shear equation (see Eq. (7) in [2]). Therefore, the kinematics proposed can be seen as an approximation of the exact three-dimensional solution. Furthermore, the sine function has an infinite radius of convergence and its Taylor expansion includes not only the third-order terms but all the odd terms.

The most common numerical procedure for the analysis of the shells is the finite element method [15–19]. It is known that the phenomenon of numerical locking may arise from hidden constraints that are not well represented in the finite element approximation and, in scientific literature, it is possible to find many methods to overcome this problem [20–25]. The present paper, that performs the bending and free vibration analysis of laminated shells by collocation with radial basis functions, avoids the locking phenomenon. A radial basis function, $\phi(\|x - x_j\|)$ is a spline that depends on the Euclidian distance between distinct data centers $x_{j,j} = 1, 2, \dots, N \in \mathbb{R}^n$, also called nodal or collocation points. The

so-called unsymmetrical Kansa method, introduced by Kansa [26], is employed. The use of radial basis function for the analysis of structures and materials has been previously studied by numerous authors [27–41]. The authors have recently applied the RBF collocation to the static deformations of composite beams and plates [42–44].

In this paper it is investigated for the first time how the Unified Formulation can be combined with radial basis functions to the analysis of thin and thick laminated shells, using the SSDT, allowing for through-the-thickness deformations. The quality of the present method in predicting static deformations and free vibrations of thin and thick laminated shells is compared and discussed with other methods in some numerical examples.

2. Applying the Unified Formulation to SSDT

The Unified Formulation proposed by Carrera, also known as CUF, is a powerful framework for the analysis of beams, plates and shells. This formulation has been applied in several finite element analysis, either using the Principle of Virtual Displacements, or by using the Reissner's Mixed Variational theorem. The stiffness matrix components, the external force terms or the inertia terms can be obtained directly with this UF, irrespective of the shear deformation theory being considered.

In this section, it is shown how to obtain the fundamental nuclei, which allows the derivation of the equations of motion and boundary conditions according to CUF, in weak form for the finite element analysis and in strong form for the present RBF collocation.

3. A sinus shear deformation theory

The present sinus shear deformation theory involves the following expansion of displacements

$$\begin{aligned} u &= u_0 + z u_1 + \sin\left(\frac{\pi z}{h}\right) u_3; & v &= v_0 + z v_1 + \sin\left(\frac{\pi z}{h}\right) v_3; & w & \\ &= w_0 + z w_1 + \sin\left(\frac{\pi z}{h}\right) w_3 \end{aligned} \quad (1)$$

where u_0 , v_0 and w_0 are translations of a point at the middle-surface of the plate, and u_1 , v_1 , u_3 , v_3 denote rotations. This theory is an expansion of early developments by Touratier [9–11], and Vidal and Polit [12]. It considers a sinusoidal variation of all displacements u , v , w , allowing for through-the-thickness deformations. Extension to shells becomes evident in the next sections.

3.1. Governing equations and boundary conditions in the framework of Unified Formulation

Shells are bi-dimensional structures in which one dimension (in general the thickness in z direction) is negligible with respect to

the other two in-plane dimensions. Geometry and the reference system are indicated in Fig. 1. The square of an infinitesimal linear segment in the layer, the associated infinitesimal area and volume are given by:

$$\begin{aligned} ds_k^2 &= H_\alpha^k d\alpha^2 + H_\beta^k d\beta^2 + H_z^k dz^2 \\ d\Omega_k &= H_\alpha^k H_\beta^k d\alpha d\beta \\ dV &= H_\alpha^k H_\beta^k H_z^k d\alpha d\beta dz \end{aligned} \quad (2)$$

where the metric coefficients are:

$$H_\alpha^k = A^k (1 + z/R_\alpha^k), \quad H_\beta^k = B^k (1 + z/R_\beta^k), \quad H_z^k = 1 \quad (3)$$

k denotes the k -layer of the multilayered shell; R_α^k and R_β^k are the principal radii of curvature along the coordinates α and β , respectively. A^k and B^k are the coefficients of the first fundamental form of Ω_k (Γ_k is the Ω_k boundary). In this work, the attention has been restricted to shells with constant radii of curvature (cylindrical, spherical, toroidal geometries) for which $A^k = B^k = 1$.

Although one can use the UF for one-layer, isotropic shell, a multi-layered shell with N_l layers is considered. The Principle of Virtual Displacements (PVD) for the pure-mechanical case reads:

$$\sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \left\{ \delta \epsilon_{pG}^k T \sigma_{pC}^k + \delta \epsilon_{nC}^k T \sigma_{nC}^k \right\} d\Omega_k dz = \sum_{k=1}^{N_l} \delta L_e^k \quad (4)$$

where Ω_k and A_k are the integration domains in plane (α, β) and z direction, respectively. Here, k indicates the layer and T the transpose of a vector, and δL_e^k is the external work for the k th layer. G means geometrical relations and C constitutive equations.

The steps to obtain the governing equations are:

- Substitution of the geometrical relations (subscript G).
- Substitution of the appropriate constitutive equations (subscript C).
- Introduction of the Unified Formulation.

Stresses and strains are separated into in-plane and normal components, denoted, respectively, by the subscripts p and n . The mechanical strains in the k th layer can be related to the displacement field $\mathbf{u}^k = \{u_\alpha^k, u_\beta^k, u_z^k\}$ via the geometrical relations:

$$\begin{aligned} \epsilon_{pG}^k &= [\epsilon_{\alpha\alpha}^k, \epsilon_{\beta\beta}^k, \epsilon_{\alpha\beta}^k]^T = (\mathbf{D}_p^k + \mathbf{A}_p^k) \mathbf{u}^k, & \epsilon_{nC}^k &= [\epsilon_{zz}^k, \epsilon_{\beta z}^k, \epsilon_{z\beta}^k]^T \\ &= (\mathbf{D}_{n\Omega}^k + \mathbf{D}_{nz}^k - \mathbf{A}_n^k) \mathbf{u}^k \end{aligned} \quad (5)$$

The explicit form of the introduced arrays follows:

$$\mathbf{D}_p^k = \begin{bmatrix} \frac{\partial_\alpha}{H_\alpha^k} & 0 & 0 \\ 0 & \frac{\partial_\beta}{H_\beta^k} & 0 \\ \frac{\partial_\beta}{H_\beta^k} & \frac{\partial_\alpha}{H_\alpha^k} & 0 \end{bmatrix}, \quad \mathbf{D}_{n\Omega}^k = \begin{bmatrix} 0 & 0 & \frac{\partial_z}{H_z^k} \\ 0 & 0 & \frac{\partial_\beta}{H_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{nz}^k = \begin{bmatrix} \partial_z & 0 & 0 \\ 0 & \partial_z & 0 \\ 0 & 0 & \partial_z \end{bmatrix}, \quad (6)$$

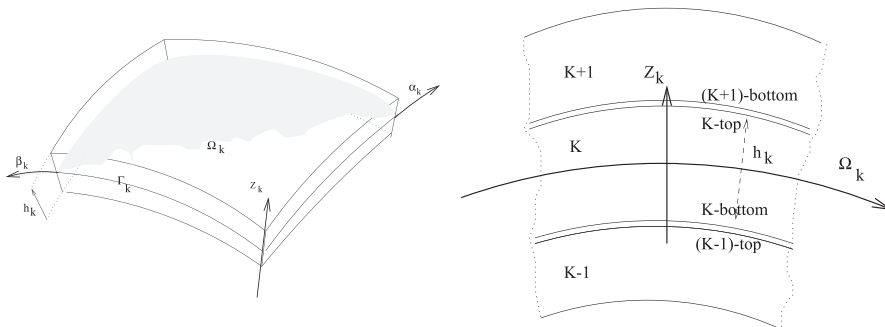


Fig. 1. Geometry and notations for a multilayered shell (doubly curved).

$$\mathbf{A}_p^k = \begin{bmatrix} 0 & 0 & \frac{1}{H_x^k R_x^k} \\ 0 & 0 & \frac{1}{H_\beta^k R_\beta^k} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{A}_n^k = \begin{bmatrix} \frac{1}{H_x^k R_x^k} & 0 & 0 \\ 0 & \frac{1}{H_\beta^k R_\beta^k} & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (7)$$

The 3D constitutive equations are given as:

$$\begin{aligned} \sigma_{pC}^k &= \mathbf{C}_{pp}^k \epsilon_{pG}^k + \mathbf{C}_{pn}^k \epsilon_{nG}^k \\ \sigma_{nC}^k &= \mathbf{C}_{np}^k \epsilon_{pG}^k + \mathbf{C}_{nn}^k \epsilon_{nG}^k \end{aligned} \quad (8)$$

with

$$\begin{aligned} \mathbf{C}_{pp}^k &= \begin{bmatrix} C_{11}^k & C_{12}^k & C_{16}^k \\ C_{12}^k & C_{22}^k & C_{26}^k \\ C_{16}^k & C_{26}^k & C_{66}^k \end{bmatrix}, \quad \mathbf{C}_{pn}^k = \begin{bmatrix} 0 & 0 & C_{13}^k \\ 0 & 0 & C_{23}^k \\ 0 & 0 & C_{36}^k \end{bmatrix}, \\ \mathbf{C}_{np}^k &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ C_{13}^k & C_{23}^k & C_{36}^k \end{bmatrix}, \quad \mathbf{C}_{nn}^k = \begin{bmatrix} C_{55}^k & C_{45}^k & 0 \\ C_{45}^k & C_{44}^k & 0 \\ 0 & 0 & C_{33}^k \end{bmatrix} \end{aligned} \quad (9)$$

According to the Unified Formulation by Carrera, the three displacement components u_x , u_β and u_z and their relative variations can be modelled as:

$$\begin{aligned} (u_x, u_\beta, u_z) &= F_\tau(u_{x\tau}, u_{\beta\tau}, u_{z\tau}), \quad (\delta u_x, \delta u_\beta, \delta u_z) \\ &= F_s(\delta u_{xs}, \delta u_{\beta s}, \delta u_{zs}) \end{aligned} \quad (10)$$

where the F_τ are functions of the thickness coordinate z and τ is a sum index. Taylor expansions from first- up to fourth-order are employed: $F_0 = z^0 = 1$, $F_1 = z^1 = z$, ..., $F_N = z^N$, ..., $F_4 = z^4$ if an Equivalent Single Layer (ESL) model is used. For ESL approach, one means that the displacements are assumed for the whole laminate if a multi-layer structure is considered.

Resorting to the displacement field in Eq. (10), the vectors $F_\tau = [1 \quad z \quad \sin(\frac{z}{h})]$ are chosen for the displacements u , v , w . Then, all the terms of the equations of motion are obtained by integrating through the thickness direction.

It is interesting to note that under this combination of the Unified Formulation and RBF collocation, the collocation code depends only on the choice of F_τ , in order to solve this type of problems. A MATLAB code has been designed that, just by changing F_τ , can analyse static deformations, and free vibrations for any type of C^0 shear deformation theory. An obvious advantage of the present methodology is that the tedious derivation of the equations of motion and boundary conditions for a particular shear deformation theory is no longer an issue, as this MATLAB code does all that work for us.

In Fig. 2, it is shown the assembling procedures of the stiffness matrix on layer k for the ESL approach.

Substituting the geometrical relations, the constitutive equations and the Unified Formulation into the variational statement PVD, for the k th layer, one has:

$$\begin{aligned} \sum_{k=1}^{N_l} \left\{ \int_{\Omega_k} \int_{A_k} \left\{ ((\mathbf{D}_p + \mathbf{A}_p) \delta \mathbf{u}^k)^T (\mathbf{C}_{pp}^k (\mathbf{D}_p + \mathbf{A}_p) \mathbf{u}^k + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) \mathbf{u}^k) \right. \right. \\ \left. \left. + ((\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) \delta \mathbf{u}^k)^T (\mathbf{C}_{np}^k (\mathbf{D}_p + \mathbf{A}_p) \mathbf{u}^k \right. \right. \\ \left. \left. + \mathbf{C}_{nn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) \mathbf{u}^k) \right\} d\Omega_k dz_k \right\} = \sum_{k=1}^{N_l} \delta L_c^k \end{aligned} \quad (11)$$

At this point, the formula of integration by parts is applied:

$$\begin{aligned} \int_{\Omega_k} ((\mathbf{D}_\Omega) \delta \mathbf{a}^k)^T \mathbf{a}^k d\Omega_k &= - \int_{\Omega_k} \delta \mathbf{a}^{kT} ((\mathbf{D}_\Omega^T) \mathbf{a}^k) d\Omega_k \\ &+ \int_{\Gamma_k} \delta \mathbf{a}^{kT} ((\mathbf{I}_\Omega) \mathbf{a}^k) d\Gamma_k \end{aligned} \quad (12)$$

where \mathbf{I}_Ω matrix is obtained applying the *Gradient theorem*:

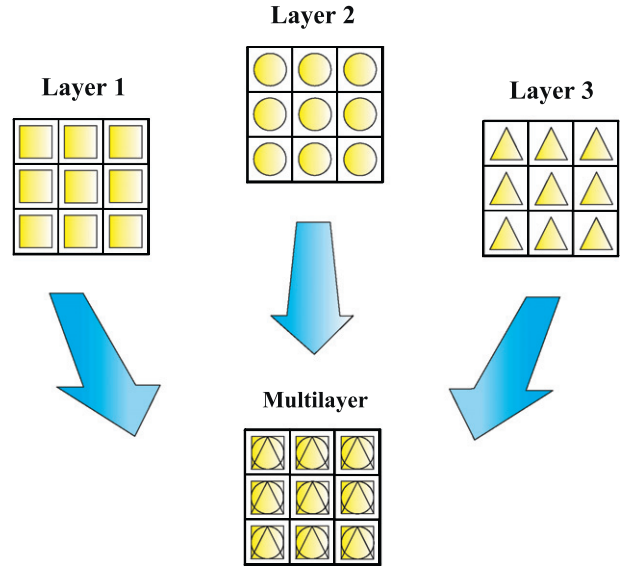


Fig. 2. Assembling procedure for ESL approach.

$$\int_{\Omega} \frac{\partial \psi}{\partial X_i} dv = \oint_{\Gamma} n_i \psi ds \quad (13)$$

being n_i the components of the normal \hat{n} to the boundary along the direction i . After integration by parts and the substitution of CUF, the governing equations and boundary conditions for the shell in the mechanical case are obtained:

$$\begin{aligned} \sum_{k=1}^{N_l} \left\{ \int_{\Omega_k} \int_{A_k} \left\{ \delta \mathbf{u}_s^{kT} [(-\mathbf{D}_p + \mathbf{A}_p)^T F_s (\mathbf{C}_{pp}^k (\mathbf{D}_p + \mathbf{A}_p) F_\tau \mathbf{u}_\tau^k \right. \right. \\ \left. \left. + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) F_\tau \mathbf{u}_\tau^k) \right] \right. \\ \left. + \delta \mathbf{u}_s^{kT} [(-\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n)^T F_s (\mathbf{C}_{np}^k (\mathbf{D}_p + \mathbf{A}_p) F_\tau \mathbf{u}_\tau^k \right. \\ \left. + \mathbf{C}_{nn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) F_\tau \mathbf{u}_\tau^k) \right] \right\} d\Omega_k dz_k \\ + \sum_{k=1}^{N_l} \left\{ \int_{\Gamma_k} \int_{A_k} \left\{ \delta \mathbf{u}_s^{kT} [F_s (\mathbf{C}_{pp}^k (\mathbf{D}_p + \mathbf{A}_p) F_\tau \mathbf{u}_\tau^k \right. \right. \\ \left. \left. + \mathbf{C}_{pn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) F_\tau \mathbf{u}_\tau^k) \right] + \delta \mathbf{u}_s^{kT} [F_s (\mathbf{C}_{np}^k (\mathbf{D}_p - \mathbf{A}_p) F_\tau \mathbf{u}_\tau^k \right. \\ \left. + \mathbf{C}_{nn}^k (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n) F_\tau \mathbf{u}_\tau^k) \right] \right\} d\Gamma_k dz_k \right\} = \sum_{k=1}^{N_l} \left\{ \int_{\Omega_k} \delta \mathbf{u}_s^{kT} F_s \mathbf{p}_u^k \right\} \end{aligned} \quad (14)$$

where \mathbf{I}_p^k and \mathbf{I}_{np}^k depend on the boundary geometry:

$$\mathbf{I}_p = \begin{bmatrix} \frac{n_x}{H_z} & 0 & 0 \\ 0 & \frac{n_\beta}{H_\beta} & 0 \\ \frac{n_\beta}{H_\beta} & \frac{n_x}{H_x} & 0 \end{bmatrix}; \quad \mathbf{I}_{np} = \begin{bmatrix} 0 & 0 & \frac{n_x}{H_x} \\ 0 & 0 & \frac{n_\beta}{H_\beta} \\ 0 & 0 & 0 \end{bmatrix} \quad (15)$$

The normal to the boundary of domain Ω is:

$$\hat{n} = \begin{bmatrix} n_x \\ n_\beta \end{bmatrix} = \begin{bmatrix} \cos(\varphi_\alpha) \\ \cos(\varphi_\beta) \end{bmatrix} \quad (16)$$

where φ_α and φ_β are the angles between the normal \hat{n} and the direction α and β , respectively.

The governing equations for a multi-layered shell subjected to mechanical loadings are:

$$\delta \mathbf{u}_s^{kT} : \mathbf{K}_{uu}^{k\tau s} \mathbf{u}_\tau^k = \mathbf{P}_u^k \quad (17)$$

where the fundamental nucleus $\mathbf{K}_{uu}^{k\tau s}$ is obtained as:

$$\begin{aligned} \mathbf{K}_{uu}^{kts} = & \int_{A_k} \left[[-\mathbf{D}_p + \mathbf{A}_p]^T \mathbf{C}_{pp}^k [\mathbf{D}_p + \mathbf{A}_p] + [-\mathbf{D}_p + \mathbf{A}_p]^T \mathbf{C}_{pn}^k [\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n] \right. \\ & + [-\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n]^T \mathbf{C}_{np}^k [\mathbf{D}_p + \mathbf{A}_p] \\ & \left. + [-\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n]^T \mathbf{C}_{nn}^k [\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n] \right] F_\tau F_s H_\alpha^k H_\beta^k dz \quad (18) \end{aligned}$$

and the corresponding Neumann-type boundary conditions on Γ_k are:

$$\mathbf{\Pi}_d^{kts} \mathbf{u}_c^k = \mathbf{\Pi}_d^{kts} \bar{\mathbf{u}}_c^k \quad (19)$$

where

$$\begin{aligned} \mathbf{\Pi}_d^{kts} = & \int_{A_k} \left[\mathbf{I}_p^T \mathbf{C}_{pp}^k [\mathbf{D}_p + \mathbf{A}_p^\tau] + \mathbf{I}_p^T \mathbf{C}_{pn}^k [\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n^\tau] + \mathbf{I}_{np}^T \mathbf{C}_{np}^k [\mathbf{D}_p + \mathbf{A}_p^\tau] \right. \\ & \left. + \mathbf{I}_{np}^T \mathbf{C}_{nn}^k [\mathbf{D}_{n\Omega} + \mathbf{D}_{nz} - \mathbf{A}_n^\tau] \right] F_\tau F_s H_\alpha^k H_\beta^k dz \quad (20) \end{aligned}$$

and \mathbf{P}_{ut}^k are variationally consistent loads with applied pressure.

3.2. Fundamental nuclei

The following integrals are introduced to perform the explicit form of fundamental nuclei:

$$\begin{aligned} \left(J_\alpha^{kts}, J_\beta^{kts}, J_\gamma^{kts}, J_\delta^{kts}, J_\epsilon^{kts}, J_\zeta^{kts} \right) &= \int_{A_k} F_\tau F_s \left(1, H_\alpha, H_\beta, \frac{H_\alpha}{H_\beta}, \frac{H_\beta}{H_\alpha}, H_\alpha H_\beta \right) dz \\ \left(J_{\alpha\beta}^{kts}, J_{\alpha\gamma}^{kts}, J_{\alpha\delta}^{kts}, J_{\alpha\epsilon}^{kts}, J_{\alpha\zeta}^{kts} \right) &= \int_{A_k} \frac{\partial F_\tau}{\partial z} F_s \left(1, H_\alpha, H_\beta, \frac{H_\alpha}{H_\beta}, \frac{H_\beta}{H_\alpha}, H_\alpha H_\beta \right) dz \\ \left(J_{\beta\alpha}^{kts}, J_{\beta\gamma}^{kts}, J_{\beta\delta}^{kts}, J_{\beta\epsilon}^{kts}, J_{\beta\zeta}^{kts} \right) &= \int_{A_k} F_\tau \frac{\partial F_s}{\partial z} \left(1, H_\alpha, H_\beta, \frac{H_\alpha}{H_\beta}, \frac{H_\beta}{H_\alpha}, H_\alpha H_\beta \right) dz \\ \left(J_{\gamma\alpha}^{kts}, J_{\gamma\beta}^{kts}, J_{\gamma\delta}^{kts}, J_{\gamma\epsilon}^{kts}, J_{\gamma\zeta}^{kts} \right) &= \int_{A_k} \frac{\partial F_\tau}{\partial z} \frac{\partial F_s}{\partial z} \\ & \left(1, H_\alpha, H_\beta, \frac{H_\alpha}{H_\beta}, \frac{H_\beta}{H_\alpha}, H_\alpha H_\beta \right) dz \quad (21) \end{aligned}$$

The fundamental nuclei \mathbf{K}_{uu}^{kts} is reported for doubly curved shells (radii of curvature in both α and β directions (see Fig. 1)):

$$\begin{aligned} \left(\mathbf{K}_{uu}^{kts} \right)_{11} &= -C_{11}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\alpha^\tau - C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{66}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \partial_\beta^\tau \\ & + C_{55}^k \left(J_{\alpha\beta}^{kts} - \frac{1}{R_{\alpha k}} J_\beta^{kts} - \frac{1}{R_{\alpha k}} J_\beta^{kts} + \frac{1}{R_{\alpha k}^2} J_{\beta/\alpha}^{kts} \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{12} &= -C_{12}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{26}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \partial_\beta^\tau - C_{66}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau \partial_\beta^\tau \\ & + C_{45}^k \left(J_{\alpha\beta}^{kts} - \frac{1}{R_{\beta k}} J_\alpha^{kts} - \frac{1}{R_{\beta k}} J_\beta^{kts} + \frac{1}{R_{\beta k}} \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{13} &= -C_{11}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau - C_{12}^k \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau - C_{13}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau - C_{16}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau \\ & - C_{26}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\beta^\tau - C_{36}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + C_{45}^k \left(J_\alpha^{kts} \partial_\beta^\tau - \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau \right) \\ & + C_{55}^k \left(J_\beta^{kts} \partial_\alpha^\tau - \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{21} &= -C_{12}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{26}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \partial_\beta^\tau - C_{66}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau \partial_\beta^\tau \\ & + C_{45}^k \left(J_{\alpha\beta}^{kts} - \frac{1}{R_{\beta k}} J_\alpha^{kts} - \frac{1}{R_{\beta k}} J_\beta^{kts} + \frac{1}{R_{\beta k}} \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{22} &= -C_{22}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \partial_\beta^\tau - C_{26}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \partial_\beta^\tau - C_{26}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{66}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\alpha^\tau \\ & + C_{44}^k \left(J_{\alpha\beta}^{kts} - \frac{1}{R_{\beta k}} J_\alpha^{kts} - \frac{1}{R_{\beta k}} J_\beta^{kts} + \frac{1}{R_{\beta k}^2} J_{\beta/\alpha}^{kts} \right) \end{aligned}$$

$$\begin{aligned} \left(\mathbf{K}_{uu}^{kts} \right)_{23} &= -C_{12}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau - C_{22}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\beta^\tau - C_{23}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau - C_{16}^k \\ & \times \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau - C_{26}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\alpha^\tau - C_{36}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau \\ & + C_{45}^k \left(J_\beta^{kts} \partial_\alpha^\tau - \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \right) + C_{44}^k \left(J_\alpha^{kts} \partial_\beta^\tau - \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{31} &= C_{11}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + C_{12}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\alpha^\tau + C_{13}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau + C_{16}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau \\ & + C_{26}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\beta^\tau + C_{36}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau - C_{45}^k \left(J_\alpha^{kts} \partial_\beta^\tau - \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau \right) \\ & - C_{55}^k \left(J_\beta^{kts} \partial_\alpha^\tau - \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{32} &= C_{12}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau + C_{22}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\beta^\tau + C_{23}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + C_{16}^k \frac{1}{R_{\alpha k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \\ & + C_{26}^k \frac{1}{R_{\beta k}} J_{\alpha/\beta}^{kts} \partial_\alpha^\tau + C_{36}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau - C_{45}^k \left(J_\beta^{kts} \partial_\alpha^\tau - \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \right) \\ & - C_{44}^k \left(J_\alpha^{kts} \partial_\beta^\tau - \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} \partial_\beta^\tau \right) \\ \left(\mathbf{K}_{uu}^{kts} \right)_{33} &= C_{11}^k \frac{1}{R_{\alpha k}^2} J_{\beta/\alpha}^{kts} + C_{22}^k \frac{1}{R_{\beta k}^2} J_{\alpha/\beta}^{kts} + C_{33}^k J_{\alpha/\beta}^{kts} + 2C_{12}^k \frac{1}{R_{\alpha k}} \\ & \times \frac{1}{R_{\beta k}} J_{\beta/\alpha}^{kts} + C_{13}^k \frac{1}{R_{\alpha k}} \left(J_\beta^{kts} + J_\beta^{kts} \right) + C_{23}^k \\ & \times \frac{1}{R_{\beta k}} \left(J_\alpha^{kts} + J_\alpha^{kts} \right) - C_{44}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \partial_\beta^\tau - C_{55}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau \partial_\alpha^\tau \\ & - C_{45}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau \partial_\beta^\tau - C_{45}^k J_{\beta/\alpha}^{kts} \partial_\beta^\tau \partial_\alpha^\tau \quad (22) \end{aligned}$$

The application of boundary conditions makes use of the fundamental nuclei $\mathbf{\Pi}_d$ in the form:

$$\begin{aligned} \left(\mathbf{\Pi}_{uu}^{kts} \right)_{11} &= n_\alpha C_{11}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\beta C_{66}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + n_\beta C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\alpha C_{16}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{12} &= n_\alpha C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\beta C_{26}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + n_\alpha C_{12}^k J_{\beta/\alpha}^{kts} \partial_\beta^\tau + n_\beta C_{66}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{13} &= n_\alpha \frac{1}{R_{\alpha k}} C_{11}^k J_{\beta/\alpha}^{kts} + n_\alpha \frac{1}{R_{\beta k}} C_{12}^k J_{\beta/\alpha}^{kts} + n_\alpha C_{13}^k J_{\beta/\alpha}^{kts} + n_\beta \frac{1}{R_{\alpha k}} C_{16}^k J_{\beta/\alpha}^{kts} \\ & + n_\beta \frac{1}{R_{\beta k}} C_{26}^k J_{\alpha/\beta}^{kts} + n_\beta C_{36}^k J_{\alpha/\beta}^{kts} \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{21} &= n_\alpha C_{16}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\beta C_{26}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + n_\beta C_{12}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\alpha C_{66}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{22} &= n_\alpha C_{66}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\beta C_{22}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + n_\beta C_{26}^k J_{\alpha/\beta}^{kts} \partial_\alpha^\tau + n_\alpha C_{26}^k J_{\beta/\alpha}^{kts} \partial_\beta^\tau \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{23} &= n_\alpha \frac{1}{R_{\alpha k}} C_{16}^k J_{\beta/\alpha}^{kts} + n_\alpha \frac{1}{R_{\beta k}} C_{26}^k J_{\alpha/\beta}^{kts} + n_\alpha C_{36}^k J_{\alpha/\beta}^{kts} + n_\beta \frac{1}{R_{\alpha k}} C_{12}^k J_{\beta/\alpha}^{kts} \\ & + n_\beta \frac{1}{R_{\beta k}} C_{22}^k J_{\alpha/\beta}^{kts} + n_\beta C_{23}^k J_{\alpha/\beta}^{kts} \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{31} &= -n_\alpha \frac{1}{R_{\alpha k}} C_{55}^k J_{\beta/\alpha}^{kts} + n_\alpha C_{55}^k J_{\beta/\alpha}^{kts} - n_\beta \frac{1}{R_{\alpha k}} C_{45}^k J_{\beta/\alpha}^{kts} + n_\beta C_{45}^k J_{\beta/\alpha}^{kts} \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{32} &= -n_\alpha \frac{1}{R_{\beta k}} C_{45}^k J_{\alpha/\beta}^{kts} + n_\alpha C_{45}^k J_{\alpha/\beta}^{kts} - n_\beta \frac{1}{R_{\beta k}} C_{44}^k J_{\alpha/\beta}^{kts} + n_\beta C_{44}^k J_{\alpha/\beta}^{kts} \\ \left(\mathbf{\Pi}_{uu}^{kts} \right)_{33} &= n_\alpha C_{55}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\beta C_{44}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau + n_\beta C_{45}^k J_{\beta/\alpha}^{kts} \partial_\alpha^\tau + n_\alpha C_{45}^k J_{\alpha/\beta}^{kts} \partial_\beta^\tau \quad (23) \end{aligned}$$

One can note that all the equations written for the shell degenerate in those for the plate when $\frac{1}{R_{\alpha k}} = \frac{1}{R_{\beta k}} = 0$. In practice, the radii of curvature are set to 10^9 .

3.3. Dynamic governing equations

The PVD for the dynamic case is expressed as:

$$\begin{aligned} & \sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \left\{ \delta \epsilon_{pG}^k \sigma_{pC}^k + \delta \epsilon_{nG}^k \sigma_{nC}^k \right\} d\Omega_k dz \\ & = \sum_{k=1}^{N_l} \int_{\Omega_k} \int_{A_k} \rho^k \delta \mathbf{u}^{kT} \ddot{\mathbf{u}}^k d\Omega_k dz + \sum_{k=1}^{N_l} \delta L_e^k \end{aligned} \quad (24)$$

where ρ^k is the mass density of the k th layer and double dots denote acceleration.

By substituting the geometrical relations, the constitutive equations and the Unified Formulation, one obtains the following governing equations:

$$\delta \mathbf{u}_s^{kT} : \mathbf{K}_{uu}^{k\tau s} \mathbf{u}_\tau^k = \mathbf{M}^{k\tau s} \ddot{\mathbf{u}}_\tau^k + \mathbf{P}_{u\tau}^k \quad (25)$$

In the case of free vibrations one has:

$$\delta \mathbf{u}_s^{kT} : \mathbf{K}_{uu}^{k\tau s} \mathbf{u}_\tau^k = \mathbf{M}^{k\tau s} \ddot{\mathbf{u}}_\tau^k \quad (26)$$

where $\mathbf{M}^{k\tau s}$ is the fundamental nucleus for the inertial term. The explicit form of that is:

$$\begin{aligned} \mathbf{M}_{11}^{k\tau s} &= \rho^k J_{\alpha\beta}^{k\tau s} \\ \mathbf{M}_{12}^{k\tau s} &= 0 \\ \mathbf{M}_{13}^{k\tau s} &= 0 \\ \mathbf{M}_{21}^{k\tau s} &= 0 \\ \mathbf{M}_{22}^{k\tau s} &= \rho^k J_{\alpha\beta}^{k\tau s} \\ \mathbf{M}_{23}^{k\tau s} &= 0 \\ \mathbf{M}_{31}^{k\tau s} &= 0 \\ \mathbf{M}_{32}^{k\tau s} &= 0 \\ \mathbf{M}_{33}^{k\tau s} &= \rho^k J_{\alpha\beta}^{k\tau s} \end{aligned} \quad (27)$$

where the meaning of the integral $J_{\alpha\beta}^{k\tau s}$ has been illustrated in Eq. (21). The geometrical and mechanical boundary conditions are the same of the static case. Because the static case is only considered, the mass terms will be neglected.

4. The radial basis function method

4.1. The static problem

Radial basis functions (RBF) approximations are mesh-free numerical schemes that can exploit accurate representations of the boundary, are easy to implement and can be spectrally accurate. In this section the formulation of a global unsymmetrical collocation RBF-based method to compute elliptic operators is presented.

Consider a linear elliptic partial differential operator L and a bounded region Ω in \mathbb{R}^n with some boundary $\partial\Omega$. In the static problems, the displacements (\mathbf{u}) are computed from the global system of equations

$$\mathcal{L}\mathbf{u} = \mathbf{f} \text{ in } \Omega \quad (28)$$

$$\mathcal{L}_B\mathbf{u} = \mathbf{g} \text{ on } \partial\Omega \quad (29)$$

where \mathcal{L} , \mathcal{L}_B are linear operators in the domain and on the boundary, respectively. The right-hand side of (28) and (29) represent the external forces applied on the plate or shell and the boundary conditions applied along the perimeter of the plate or shell, respectively. The PDE (Partial Differential Equation) problem defined in (28) and (29) will be replaced by a finite problem, defined by an algebraic system of equations, after the radial basis expansions.

4.2. The eigenproblem

The eigenproblem looks for eigenvalues (λ) and eigenvectors (\mathbf{u}) that satisfy

$$\mathcal{L}\mathbf{u} + \lambda\mathbf{u} = 0 \text{ in } \Omega \quad (30)$$

$$\mathcal{L}_B\mathbf{u} = 0 \text{ on } \partial\Omega \quad (31)$$

As in the static problem, the eigenproblem defined in (30) and (31) is replaced by a finite-dimensional eigenvalue problem, based on RBF approximations.

4.3. Radial basis functions approximations

The radial basis function (ϕ) approximation of a function (\mathbf{u}) is given by

$$\tilde{\mathbf{u}}(\mathbf{x}) = \sum_{i=1}^N \alpha_i \phi(\|\mathbf{x} - \mathbf{y}_i\|_2), \quad \mathbf{x} \in \mathbb{R}^n \quad (32)$$

where \mathbf{y}_i , $i = 1, \dots, N$ is a finite set of distinct points (centers) in \mathbb{R}^n . The most common RBFs are

$$\text{Cubic : } \phi(r) = r^3$$

$$\text{Thin plate splines : } \phi(r) = r^2 \log(r)$$

$$\text{Wendland functions : } \phi(r) = (1-r)_+^m p(r)$$

$$\text{Gaussian : } \phi(r) = e^{-(cr)^2}$$

$$\text{Multiquadrics : } \phi(r) = \sqrt{c^2 + r^2}$$

$$\text{Inverse multiquadrics : } \phi(r) = (c^2 + r^2)^{-1/2}$$

where the Euclidian distance r is real and non-negative and c is a positive shape parameter. Hardy [45] introduced multiquadrics in the analysis of scattered geographical data. In the 1990s Kansa [26] used multiquadrics for the solution of partial differential equations. Considering N distinct interpolations, and knowing $u(x_j)$, $j = 1, 2, \dots, N$, one finds α_i by the solution of a $N \times N$ linear system

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{u} \quad (33)$$

where $\mathbf{A} = [\phi(\|\mathbf{x} - \mathbf{y}_i\|_2)]_{N \times N}$, $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \dots, \alpha_N]^T$ and $\mathbf{u} = [u(x_1), u(x_2), \dots, u(x_N)]^T$.

4.4. Solution of the static problem

The solution of a static problem by radial basis functions considers N_l nodes in the domain and N_B nodes on the boundary, with a total number of nodes $N = N_l + N_B$. One can denote the sampling points by $\mathbf{x}_i \in \Omega$, $i = 1, \dots, N_l$ and $\mathbf{x}_i \in \partial\Omega$, $i = N_l + 1, \dots, N$. At the points in the domain, the following system of equations is solved

$$\sum_{i=1}^N \alpha_i \mathcal{L}\phi(\|\mathbf{x} - \mathbf{y}_i\|_2) = \mathbf{f}(\mathbf{x}_j), \quad j = 1, 2, \dots, N_l \quad (34)$$

or

$$\mathcal{L}^l \boldsymbol{\alpha} = \mathbf{F} \quad (35)$$

where

$$\mathcal{L}^l = [\mathcal{L}\phi(\|\mathbf{x} - \mathbf{y}_i\|_2)]_{N_l \times N} \quad (36)$$

At the points on the boundary, the following boundary conditions are imposed

$$\sum_{i=1}^N \alpha_i \mathcal{L}_B\phi(\|\mathbf{x} - \mathbf{y}_i\|_2) = \mathbf{g}(\mathbf{x}_j), \quad j = N_l + 1, \dots, N \quad (37)$$

or

$$\mathbf{B}\boldsymbol{\alpha} = \mathbf{G} \quad (38)$$

where

$$\mathbf{B} = \mathcal{L}_B \phi [(\|x_{N_l+1} - y_j\|_2)]_{N_B \times N}$$

Therefore, one can write a finite-dimensional static problem as

$$\begin{bmatrix} \mathcal{L}^I \\ \mathbf{B} \end{bmatrix} \boldsymbol{\alpha} = \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \quad (39)$$

By inverting the system (39), one obtains the vector $\boldsymbol{\alpha}$. Then, the solution \mathbf{u} is calculated using the interpolation Eq. (32).

4.5. Solution of the eigenproblem

Consider N_I nodes in the interior of the domain and N_B nodes on the boundary, with $N = N_I + N_B$. The interpolation points are denoted by $x_i \in \Omega$, $i = 1, \dots, N_I$ and $x_i \in \partial\Omega$, $i = N_I + 1, \dots, N$. At the points in the domain, the following eigenproblem is defined

$$\sum_{i=1}^N \alpha_i \mathcal{L} \phi(\|x - y_i\|_2) = \lambda \tilde{\mathbf{u}}(x_j), j = 1, 2, \dots, N_I \quad (40)$$

or

$$\mathcal{L}^I \boldsymbol{\alpha} = \lambda \tilde{\mathbf{u}}^I \quad (41)$$

where

$$\mathcal{L}^I = [\mathcal{L} \phi(\|x - y_i\|_2)]_{N_I \times N_I} \quad (42)$$

At the points on the boundary, the imposed boundary conditions are

$$\sum_{i=1}^N \alpha_i \mathcal{L}_B \phi(\|x - y_i\|_2) = 0, \quad j = N_I + 1, \dots, N \quad (43)$$

or

$$\mathbf{B} \boldsymbol{\alpha} = \mathbf{0} \quad (44)$$

Eqs. (41) and (44) can now be solved as a generalized eigenvalue problem

$$\begin{bmatrix} \mathcal{L}^I \\ \mathbf{B} \end{bmatrix} \boldsymbol{\alpha} = \lambda \begin{bmatrix} \mathbf{A}^I \\ \mathbf{0} \end{bmatrix} \boldsymbol{\alpha} \quad (45)$$

where

$$\mathbf{A}^I = \phi [(\|x_{N_I} - y_j\|_2)]_{N_I \times N_I}$$

4.6. Discretization of the equations of motion and boundary conditions

The radial basis collocation method follows a simple implementation procedure. Taking Eq. (39), one computes

$$\boldsymbol{\alpha} = \begin{bmatrix} \mathbf{L}^I \\ \mathbf{B} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{F} \\ \mathbf{G} \end{bmatrix} \quad (46)$$

This $\boldsymbol{\alpha}$ vector is then used to obtain solution $\tilde{\mathbf{u}}$, by using (32). If derivatives of $\tilde{\mathbf{u}}$ are needed, such derivatives are computed as

$$\frac{\partial \tilde{\mathbf{u}}}{\partial x} = \sum_{j=1}^N \alpha_j \frac{\partial \phi_j}{\partial x} \quad (47)$$

$$\frac{\partial^2 \tilde{\mathbf{u}}}{\partial x^2} = \sum_{j=1}^N \alpha_j \frac{\partial^2 \phi_j}{\partial x^2}, \text{ etc.} \quad (48)$$

In the present collocation approach, one needs to impose essential and natural boundary conditions. Consider, for example, the condi-

tion $w = 0$, on a simply supported or clamped edge. The conditions are enforced by interpolating as

$$w = 0 \rightarrow \sum_{j=1}^N \alpha_j^W \phi_j = 0 \quad (49)$$

Other boundary conditions are interpolated in a similar way.

4.7. Free vibrations problems

For free vibration problems, the external forces are set to zero, and an harmonic solution is assumed for the displacements $u_0, u_1, v_0, v_1, \dots$

$$u_0 = U_0(x, y) e^{i\omega t}; \quad u_1 = U_1(x, y) e^{i\omega t}; \quad u_3 = U_3(x, y) e^{i\omega t} \quad (50)$$

$$v_0 = V_0(x, y) e^{i\omega t}; \quad v_1 = V_1(x, y) e^{i\omega t}; \quad v_3 = V_3(x, y) e^{i\omega t} \quad (51)$$

$$w_0 = W_0(x, y) e^{i\omega t}; \quad w_1 = W_1(x, y) e^{i\omega t}; \quad w_3 = W_3(x, y) e^{i\omega t} \quad (52)$$

where ω is the frequency of natural vibration. Substituting the harmonic expansion into Eq. (45) in terms of the amplitudes $U_0, U_1, U_3, V_0, V_1, V_3, W_0, W_1, W_3$, one can obtain the natural frequencies and vibration modes for the plate or shell problem, by solving the eigenproblem

$$[\mathcal{L} - \omega^2 \mathcal{G}] \mathbf{X} = \mathbf{0} \quad (53)$$

where \mathcal{L} collects all stiffness terms and \mathcal{G} collects all terms related to the inertial terms. In (53) \mathbf{X} are the modes of vibration associated with the natural frequencies defined as ω .

5. Numerical examples

All numerical examples consider a Chebyshev grid (see Fig. 3) and a Wendland function, defined as

$$\phi(r) = (1 - cr)_+^8 (32(cr)^3 + 25(cr)^2 + 8cr + 1) \quad (54)$$

where the shape parameter (c) was obtained by an optimization procedure, as detailed in Ferreira and Fasshauer [46].

5.1. Spherical shell in bending

A laminated composite spherical shell is here considered, of side a and thickness h , composed of layers oriented at $[0^\circ/90^\circ/0^\circ]$

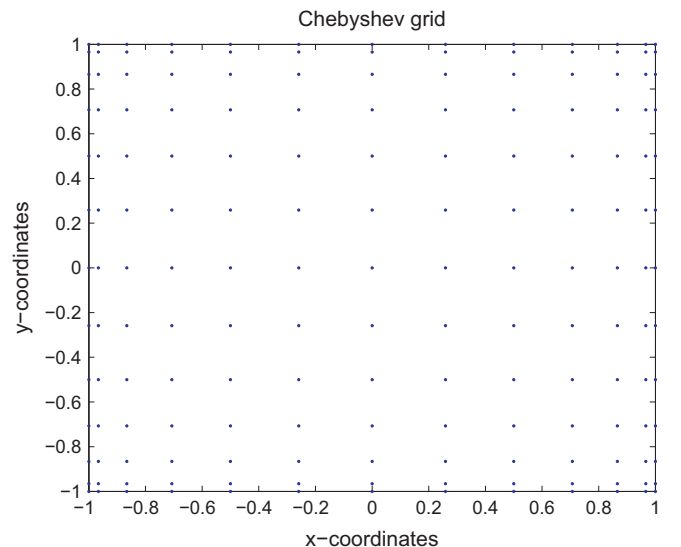


Fig. 3. A sketch of a Chebyshev grid for 13×13 points.

Table 1

Non-dimensional central deflection, $\bar{w} = w \frac{10^2 E_2 h^3}{P_0 a^4}$ variation with various number of grid points per unit length, N for different R/a ratios, for $R_1 = R_2$.

| | a/h | Method | R/a | | | | | |
|-----------------|-------|-------------------|--------|--------|--------|--------|--------|--------|
| | | | 5 | 10 | 20 | 50 | 100 | 10^9 |
| [0°/90°/0°] | 10 | Present (13 × 13) | 6.6874 | 6.9044 | 6.9615 | 6.9781 | 6.9806 | 6.9816 |
| | 10 | Present (17 × 17) | 6.6879 | 6.9047 | 6.9618 | 6.9784 | 6.9809 | 6.9819 |
| | 10 | Present (21 × 21) | 6.6880 | 6.9048 | 6.9618 | 6.9784 | 6.9809 | 6.9820 |
| | 10 | HSDT [47] | 6.7688 | 7.0325 | 7.1016 | 7.1212 | 7.1240 | 7.125 |
| | 10 | FSDT [47] | 6.4253 | 6.6247 | 6.6756 | 6.6902 | 6.6923 | 6.6939 |
| | 100 | Present (13 × 13) | 1.0244 | 2.3651 | 3.5151 | 4.0692 | 4.1629 | 4.1951 |
| | 100 | Present (17 × 17) | 1.0249 | 2.3661 | 3.5165 | 4.0707 | 4.1644 | 4.1966 |
| | 100 | Present (21 × 21) | 1.0250 | 2.3662 | 3.5167 | 4.0709 | 4.1646 | 4.1966 |
| | 100 | HSDT [47] | 1.0321 | 2.4099 | 3.617 | 4.2071 | 4.3074 | 4.3420 |
| | 100 | FSDT [47] | 1.0337 | 2.4109 | 3.6150 | 4.2027 | 4.3026 | 4.3370 |
| [0°/90°/90°/0°] | 10 | Present (13 × 13) | 6.7199 | 6.9418 | 7.0004 | 7.0174 | 7.0201 | 7.0211 |
| | 10 | Present (17 × 17) | 6.7204 | 6.9423 | 7.0007 | 7.0178 | 7.0204 | 7.0214 |
| | 10 | Present (21 × 21) | 6.7205 | 6.9423 | 7.0008 | 7.0178 | 7.0204 | 7.0215 |
| | 10 | HSDT [47] | 6.7865 | 7.0536 | 7.1237 | 7.1436 | 7.1464 | 7.1474 |
| | 10 | FSDT [47] | 6.3623 | 6.5595 | 6.6099 | 6.6244 | 6.6264 | 6.6280 |
| | 100 | Present (13 × 13) | 1.0190 | 2.3581 | 3.5119 | 4.0694 | 4.1638 | 4.1962 |
| | 100 | Present (17 × 17) | 1.0195 | 2.3591 | 3.5132 | 4.0708 | 4.1653 | 4.1978 |
| | 100 | Present (21 × 21) | 1.0195 | 2.3592 | 3.5134 | 4.0711 | 4.1655 | 4.1980 |
| | 100 | HSDT [47] | 1.0264 | 2.4024 | 3.6133 | 4.2071 | 4.3082 | 4.3430 |
| | 100 | FSDT [47] | 1.0279 | 2.4030 | 3.6104 | 4.2015 | 4.3021 | 4.3368 |

Table 2

Nondimensionalized fundamental frequencies of cross-ply laminated spherical shells, $\bar{\omega} = \omega \frac{a}{h} \sqrt{\rho/E_2}$, laminate ([0°/90°/90°/0°]).

| a/h | Method | R/a | | | | | |
|-------|-------------------|-------------------|---------|---------|---------|---------|---------|
| | | 5 | 10 | 20 | 50 | 100 | 10^9 |
| 10 | Present (13 × 13) | 12.0999 | 11.9378 | 11.8967 | 11.8851 | 11.8835 | 11.8829 |
| | Present (17 × 17) | 12.0995 | 11.9375 | 11.8964 | 11.8849 | 11.8832 | 11.8827 |
| | Present (21 × 21) | 12.0994 | 11.9375 | 11.8964 | 11.8849 | 11.8832 | 11.8827 |
| | HSDT [47] | 12.040 | 11.840 | 11.790 | 11.780 | 11.780 | 11.780 |
| | 100 | Present (13 × 13) | 31.2175 | 20.5753 | 16.8713 | 15.6760 | 15.4977 |
| 100 | Present (17 × 17) | 31.2076 | 20.5690 | 16.8663 | 15.6714 | 15.4931 | 15.4333 |
| | Present (21 × 21) | 31.2063 | 20.5683 | 16.8658 | 15.6711 | 15.4929 | 15.4331 |
| | HSDT [47] | 31.100 | 20.380 | 16.630 | 15.420 | 15.230 | 15.170 |

Table 3

Nondimensionalized fundamental frequencies of cross-ply laminated spherical shells, $\bar{\omega} = \omega \frac{a}{h} \sqrt{\rho/E_2}$, laminate ([0°/90°/0°]).

| a/h | Method | R/a | | | | | |
|-------|-------------------|-------------------|---------|---------|---------|---------|---------|
| | | 5 | 10 | 20 | 50 | 100 | 10^9 |
| 10 | Present (13 × 13) | 12.1258 | 11.9661 | 11.9256 | 11.9142 | 11.9126 | 11.9120 |
| | Present (17 × 17) | 12.1254 | 11.9658 | 11.9253 | 11.9140 | 11.9123 | 11.9112 |
| | Present (21 × 21) | 12.1253 | 11.9658 | 11.9253 | 11.9140 | 11.9123 | 11.9112 |
| | HSDT [47] | 12.060 | 11.860 | 11.810 | 11.790 | 11.790 | 11.790 |
| | 100 | Present (13 × 13) | 31.1360 | 20.5441 | 16.8634 | 15.6764 | 15.4993 |
| 100 | Present (17 × 17) | 31.1262 | 20.5388 | 16.8584 | 15.6718 | 15.4948 | 15.4353 |
| | Present (21 × 21) | 31.1249 | 20.5381 | 16.8579 | 15.6714 | 15.4944 | 15.4349 |
| | HSDT [47] | 31.020 | 20.350 | 16.620 | 15.420 | 15.240 | 15.170 |

and [0°/90°/90°/0°]. The shell is subjected to a sinusoidal vertical pressure of the form

$$p_z = P \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi y}{a}\right)$$

with the origin of the coordinate system located at the lower left corner on the midplane and P the maximum load (at center of shell).

The orthotropic material properties for each layer are given by

$$E_1 = 25.0E_2, \quad G_{12} = G_{13} = 0.5E_2, \quad G_{23} = 0.2E_2, \quad \nu_{12} = 0.25$$

The in-plane displacements, the transverse displacements, the normal stresses and the in-plane and transverse shear stresses are presented in normalized form as

$$\begin{aligned} \bar{w} &= \frac{10^3 w_{(a/2, a/2, 0)} h^3 E_2}{P a^4}, \quad \bar{\sigma}_{xx} = \frac{\sigma_{xx(a/2, a/2, h/2)} h^2}{P a^2}, \quad \bar{\sigma}_{yy} \\ &= \frac{\sigma_{yy(a/2, a/2, h/4)} h^2}{P a^2}, \quad \bar{\tau}_{xz} = \frac{\tau_{xz(0, a/2, 0)} h}{P a}, \quad \bar{\tau}_{xy} = \frac{\tau_{xy(0, 0, h/2)} h^2}{P a^2} \end{aligned}$$

The shell is simply supported on all edges.

In Table 1, the static deflections for the present shell model are compared with results of Reddy shell formulation using first-order and third-order shear deformation theories [47]. Nodal grids with 13 × 13, 17 × 17, and 21 × 21 points are considered. Various values of R/a and two values of a/h (10 and 100) are taken for the analysis. Results are in good agreement for various a/h ratios with the higher-order results of Reddy [47].

Table 4Nondimensionalized fundamental frequencies of cross-ply cylindrical shells, $\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho/E_2}$.

| R/a | Method | [0/90/0] | | [0/90/90/0] | |
|-------|-------------------|-------------|------------|-------------|------------|
| | | $a/h = 100$ | $a/h = 10$ | $a/h = 100$ | $a/h = 10$ |
| 5 | Present (13 × 13) | 20.4988 | 11.9234 | 20.5310 | 11.9010 |
| | Present (17 × 17) | 20.4871 | 11.9230 | 20.5214 | 11.9007 |
| | Present (21 × 21) | 20.4856 | 11.9230 | 20.5202 | 11.9007 |
| | FSDT [47] | 20.332 | 12.207 | 20.361 | 12.267 |
| | HSDT [47] | 20.330 | 11.850 | 20.360 | 11.830 |
| 10 | Present (13 × 13) | 16.8448 | 11.9149 | 16.8598 | 11.8874 |
| | Present (17 × 17) | 16.8448 | 11.9146 | 16.8538 | 11.8872 |
| | Present (21 × 21) | 16.8441 | 11.9146 | 16.8531 | 11.8872 |
| | FSDT [47] | 16.625 | 12.173 | 16.634 | 12.236 |
| | HSDT [47] | 16.620 | 11.800 | 16.630 | 11.790 |
| 20 | Present (13 × 13) | 15.8048 | 11.9127 | 15.8055 | 11.8840 |
| | Present (17 × 17) | 15.7998 | 11.9125 | 15.8006 | 11.8838 |
| | Present (21 × 21) | 15.7993 | 11.9125 | 15.8001 | 11.8838 |
| | FSDT [47] | 15.556 | 12.166 | 15.559 | 12.230 |
| | HSDT [47] | 15.55 | 11.79 | 15.55 | 11.78 |
| 50 | Present (13 × 13) | 15.4988 | 11.9121 | 15.4972 | 11.8831 |
| | Present (17 × 17) | 15.4942 | 11.9119 | 15.4926 | 11.8829 |
| | Present (21 × 21) | 15.4938 | 11.9119 | 15.4922 | 11.8829 |
| | FSDT [47] | 15.244 | 12.163 | 15.245 | 12.228 |
| | HSDT [47] | 15.24 | 11.79 | 15.23 | 11.78 |
| 100 | Present (13 × 13) | 15.4546 | 11.9120 | 15.4527 | 11.8830 |
| | Present (17 × 17) | 15.4501 | 11.9118 | 15.4481 | 11.8827 |
| | Present (21 × 21) | 15.4497 | 11.9118 | 15.4477 | 11.8827 |
| | FSDT [47] | 15.198 | 12.163 | 15.199 | 12.227 |
| | HSDT [47] | 15.19 | 11.79 | 15.19 | 11.78 |
| Plate | Present (13 × 13) | 15.4398 | 11.9120 | 15.4378 | 11.8829 |
| | Present (17 × 17) | 15.4353 | 11.9118 | 15.4333 | 11.8827 |
| | Present (21 × 21) | 15.4349 | 11.9118 | 15.4328 | 11.9927 |
| | FSDT [47] | 15.183 | 12.162 | 15.184 | 12226 |
| | HSDT [47] | 15.170 | 11.790 | 15.170 | 11.780 |

5.2. Free vibration of spherical and cylindrical laminated shells

Nodal grids with 13×13 , 17×17 , and 21×21 points are considered. In Tables 2 and 3 the nondimensionalized natural frequencies from the present SSDT theory for various cross-ply spherical shells are compared with analytical solutions by Reddy and Liu [47], who considered both the First-order Shear Deformation Theory (FSDT) and the High-order Shear Deformation Theory (HSDT). The first-order theory overpredicts the fundamental natural frequencies of symmetric thick shells and symmetric shallow thin shells. The present radial basis function method is compared with analytical results by Reddy [47] and shows excellent agreement.

Table 4 contain nondimensionalized natural frequencies obtained using the present SSDT theory for cross-ply cylindrical shells with lamination schemes [0/90/0], [0/90/90/0]. Present results are compared with analytical solutions by Reddy and Liu [47] who considered both the first-order (FSDT) and the third-order (HSDT) theories. The present radial basis function method is compared with analytical results by Reddy [47] and shows excellent agreement.

In Fig. 4, the first four vibrational modes of cross-ply laminated spherical shells, $\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho/E_2}$, are illustrated for a laminate ([0°/90°/90°/0°]), using a grid of 13×13 points, for $a/h = 100$, $R/a = 10$. The modes of vibration are quite stable.

6. Concluding remarks

In this paper a sinusoidal shear deformation theory was implemented for the first time for laminated orthotropic elastic shells through a multiquadrics discretization of equations of motion and boundary conditions. The multiquadric radial basis function method for the solution of shell bending and free vibration prob-

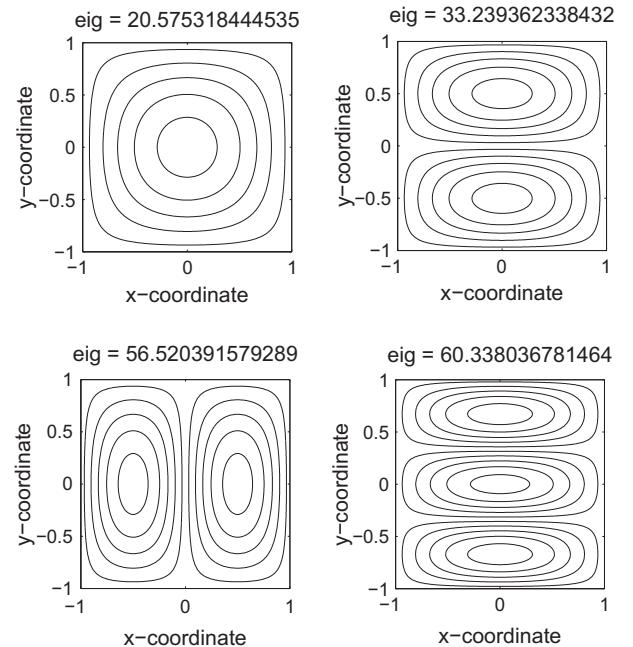


Fig. 4. First four vibrational modes of cross-ply laminated spherical shells, $\bar{\omega} = \omega \frac{a^2}{h} \sqrt{\rho/E_2}$, laminate ([0°/90°/90°/0°]) grid 13×13 points, $a/h = 100$, $R/a = 10$.

lems was presented. Results for static deformations and natural frequencies were obtained and compared with other sources. This meshless approach demonstrated that is very successful in the static deformations and free vibration analysis of laminated compos-

ite shells. Advantages of radial basis functions are absence of mesh, ease of discretization of boundary conditions and equations of equilibrium or motion and very easy coding. The static displacements and the natural frequencies obtained from present method are shown to be in excellent agreement with analytical solutions.

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