# Lensing in an elastically strained space-time

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**Abstract.** Describing curved space-time as a four-dimensional manifold strained by the presence of matter or of texture defects, an additional term in the Lagrangian of space-time has to be introduced besides the Ricci scalar, accounting for the strain. The additional term produces dark matter-like effects around any given body. These effects show up both in the angular speed of freely orbiting objects and in the gravitational lensing of light. These results are obtained and discussed while treating a spherically symmetric stationary space-time configuration.

## 1. Introduction

A problem hovering above the general relativity theory (GR) from the very beginning is the nature of space-time. It cannot be a simple mathematical artifact since it interacts with matter, so in a way or another it must have physical properties on its own.

Of course we know that space-time is a curved four-dimensional manifold endowed with Lorentzian signature, and we also know that space, at the cosmic scale, appears to be expanding with a typical symmetry: the Robertson-Walker (RW) symmetry. Furthermore some pieces of evidence tell us that the expansion is accelerated [1, 2]. This acceleration is not due to matter, so commonly it is attributed to "something else" which is dubbed in various ways according to different theories, but is mostly known as *dark energy* [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. What is dark energy? There are as many answers as there are theories, but it is difficult to say that the situation is clear and that each interpretation of dark energy is physically well motivated.

A possible interpretation of the facts is given by the Strained State Theory (SST). It considers space-time as a four- dimensional deformable continuum with properties inferred by analogy and generalization from the corresponding ones of ordinary three-dimensional elastic continua. The SST theory is exposed in [14] and will be shortly reviewed in the next section. There are two relevant features with SST: a) the global symmetry of the universe is due to a defect in the texture of space-time working as defects do in material continua; b) the deformation of the manifold, due both to defects and to the presence of matter is expressed by a strain tensor which coincides with the non-trivial part of the metric tensor. To the strain tensor an *elastic* potential energy density corresponds, whose presence affects the global behavior of space-time and is responsible for the accelerated expansion. The theory has already been used to work out the luminosity/distance curve of type Ia supernovae with good results [14]; it is also consistent with the relative abundances of light elements in the universe (Big Bang Nucleosynthesis: BBN), the acoustic scale of the CMB and the primordial large scale structure (LSS) formation (these results are the object of another paper under consideration for publication). Here we shall

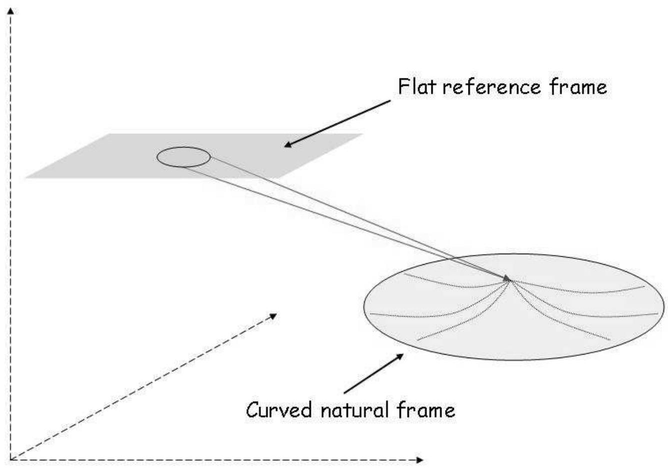


**Figure 1.** On the left we see an $N$-dimensional flat manifold (the reference manifold). On the right we have the natural $N$-dimensional manifold. The latter is curved. The curvature, in this case, is due to the presence of a defect which is indicated by the fact that a whole region of the flat manifold corresponds to a single point in the natural manifold. For the visualization everything is embedded in a flat $N + 1$-dimensional manifold.

concentrate on the implications of a strained state in the vicinity of matter in a spherically symmetric space (typical Schwarzschild problem).

## 2. Review of the SST theory

As we have written in the introduction the core idea of the Strained State Theory is that the actual space-time manifold with its global and local curvature behaves as a four-dimensional elastic continuum, so that one may think that the natural situation is obtained introducing strain in an initially flat (Minkowski) manifold. The properties of the strained manifold are expressed in terms of two parameters, which are the Lamé coefficients of space-time, $\lambda$ and $\mu$. The details of the theory may be found in [14]; here we only recollect that the global RW symmetry is assumed to be induced by a cosmic defect and that the actual metric tensor is composed of two contributions:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\epsilon_{\mu\nu}, \tag{1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor and $\epsilon_{\mu\nu}$ is the strain tensor.

The situation is schematically reproduced in fig. 1, where the effect of a defect is also shown.

The strain is obtained comparing two corresponding line elements in the reference and in the natural manifold. Of course everything most be expressed in one single coordinates system. Most often it will be the one used for the natural frame. Again a picture may help in understanding what we mean. Fig. 2 shows the situation in a simplified three-dimensional embedding with a natural manifold endowed with axial symmetry.

According to the analogy with a deformed elastic material we expect the strain to be

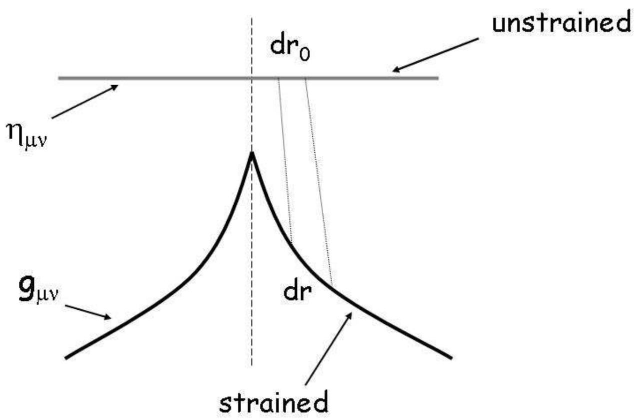


**Figure 2.** Bidimensional view of a strained axially symmetric manifold compared with its flat reference manifold. Corresponding line elements are different on the two manifolds, as well as the metric tensors. The strain tensor is proportional to the difference between the two metric tensors written with the same coordinates.

associated with an elastic deformation energy written as:

$$W_e = \frac{1}{2}\lambda\epsilon^2 + \mu\epsilon_{\alpha\beta}\epsilon^{\alpha\beta}. \tag{2}$$

Now $\epsilon = \epsilon_\alpha^\alpha$ is the trace of the strain tensor and $\lambda$ and $\mu$ are the Lamé coefficients of space-time. Here the meaning of $\lambda$ and $\mu$, notwithstanding the four dimensions and the signature, is exactly the same as in the ordinary elasticity theory in three dimensions. We assume, for simplicity, that the "elasticity" of space-time is indeed linear.

The energy in (2) allows to write a new action integral including both space-time and matter/energy in the form

$$S = \int (R + \frac{1}{2}\lambda\epsilon^2 + \mu\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} + L_{\mathrm{matter}})\sqrt{-g}d^4x. \tag{3}$$

$L_{\mathrm{matter}}$ is the Lagrangian density of ordinary matter/energy and the rest has the usual meaning. The "elastic" potential energy term belongs to geometry, i.e. space-time, even though it looks like some matter contribution. Considered from the field theoretical viewpoint this new contribution implies a "mass" associated with the gravitational interaction; usually this is said as the graviton being massive, which fact has relevant consequences when studying the propagation of gravity and, in particular, of gravitational waves [12, 15, 16, 17]. In any case our conceptual framework is entirely classical so that, properly speaking, there are no gravitons, but rather it turns out that gravity has a finite range [18, 19].

As we have seen, everything depends on the strain tensor, which in turn depends on the way events on the reference manifold are associated to their corresponding events in the natural manifold. Actually there are infinite possible ways to get a given final situation starting from a flat initial one. This apparent freedom of choice has indeed a physical meaning since our manifolds are physical. So different choices correspond to different strains and the Hamilton principle permits to identify the least strain configuration. In practice what can be seen as a gauge freedom appears as a gauge function in the line element of the flat reference written using the coordinates of the curved natural manifold.

## 3. Specific symmetries

*3.1. Robertson Walker symmetry*

This case has been treated in [14]. Here we simply quote the result for the Hubble parameter of an universe containing matter (without further distinction) and radiation. It is:

$$H = \frac{\dot{a}}{a} = c\sqrt{\frac{B}{16}}\left\{3\left(1 - \frac{(1+z)^2}{a_0^2}\right)^2 + \frac{8\kappa}{3B}(1+z)^3\left[\rho_{\mathrm{m}0} + \rho_{\mathrm{r}0}(1+z)\right]\right\}^{1/2}. \qquad (4)$$

Of course $a$ is the scale factor of the universe; dot means derivative with respect to cosmic time; $z$ is the cosmic redshift; $\rho_{\mathrm{m}0}$ is the present matter density and $\rho_{\mathrm{r}0}$ the present radiation energy density; $\kappa = 16\pi G/c^2$. The $B$ parameter is interpretable as the bulk modulus of space-time; it is a function of the Lamé coefficients:

$$B = \frac{\mu}{4}\frac{2\lambda + \mu}{\lambda + 2\mu} \qquad (5)$$

The best fit made on the luminosity/distance curve of type Ia supernovae tells us the order of magnitude of $B$, then of $\lambda$ and $\mu$. It is $B \simeq 10^{-52}\ m^{-2}$ [14]. The optimal values of the parameters from the fit are:

**Table 1.** Optimal values.

| Parameter | Value |
|---|---|
| $B$ | $(2.22 \pm 0.06) \times 10^{-52}\ m^{-2}$ |
| $\rho_{m0}$ | $(0.260 \pm 0.009) \times 10^{-26}\ kg \times m^{-3}$ |
| $B_{a0}^{-1}$ | $(0.011 \pm 0.006) \times 10^{52}\ m^2$ |

It is $B_{a0}^{-1} = 8\kappa\rho_{r0}a_0^4/9$.

## 4. The Schwarzschild symmetry

In order to treat the case of a spherically symmetric time independent configuration in space we start from the general form of a stationary spherically symmetric line element written using Schwarzschild coordinates:

$$ds^2 = f d\tau^2 - h dr^2 - r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (6)$$

where $f$ and $h$ are functions of $r$ only.

This line element is confronted with the line element of a flat Minkowskian manifold, written with the same coordinates:

$$ds^2 = d\tau^2 - \left(\frac{dw}{dr}\right)^2 dr^2 - w^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (7)$$

Now $w$ is a gauge function depending on $r$, which is there because of the multiple ways in which we may establish a correspondence between pairs of events on the two manifolds (see fig. 1).

From the comparison between the two line elements we read out the components of the strain tensor of the natural manifold. It is:

$$\begin{aligned} \varepsilon_{00} &= \tfrac{f-1}{2} \\ \varepsilon_{rr} &= -\tfrac{h-w'^2}{2} \\ \varepsilon_{\theta\theta} &= -\tfrac{r^2-w^2}{2} \\ \varepsilon_{\phi\phi} &= -\tfrac{r^2-w^2}{2}\sin^2\theta \end{aligned} \tag{8}$$

Primes denote derivatives with respect to $r$.

Once we have the strain tensor at hands, we may introduce it into the action integral (3), then applying the Hamilton principle we obtain the equations for the three unknown functions $f$, $h$ and $w$. The explicit form of the equations is highly non-linear and rather formidable, as it can be seen hereafter:

$$\begin{gathered} h-1+r\tfrac{h'}{h}-\tfrac{1}{16f^2h}\lambda r^2\left(2fh\tfrac{w^2}{r^2}-4fh-3h+fw'^2\right)\left(h-4fh+2fh\tfrac{w^2}{r^2}+fw'^2\right) \\ -\tfrac{1}{8f^2h}\mu r^2\left(2fh^2+4f^2h^2+2f^2h^2\tfrac{w^4}{r^4}-3h^2+f^2w'^4-4f^2h^2\tfrac{w^2}{r^2}-2f^2hw'^2\right)=0 \end{gathered} \tag{9}$$

$$\begin{gathered} h^2-h-\tfrac{1}{f}hrf' \\ -\tfrac{1}{16f^2}\lambda r^2\left(h-4fh+2fh\tfrac{w^2}{r^2}-3fw'^2\right)\left(h-4fh+2fh\tfrac{w^2}{r^2}+fw'^2\right) \\ -\tfrac{1}{8f^2}\mu r^2\left(h^2+4f^2h^2+2f^2h^2\tfrac{w^4}{r^4}-2fh^2-3f^2w'^4-4f^2h^2\tfrac{w^2}{r^2}+2f^2hw'^2\right)=0 \end{gathered} \tag{10}$$

$$\begin{gathered} \tfrac{\lambda}{2fh^2}w''\left(hr^2+3fr^2w'^2-4fhr^2+2fhw^2\right) \\ +\tfrac{\lambda}{h}ww'^2+\tfrac{\lambda r}{h^2}\left(\tfrac{f'}{4f}r-\tfrac{3h'}{4h}r+1\right)w'^3 \\ +\lambda\tfrac{w'}{h}\left(\left(\tfrac{1}{2}w^2-r^2-\tfrac{1}{4f}r^2\right)\tfrac{f'}{f}+\left(r^2-\tfrac{1}{2}w^2-\tfrac{1}{4f}r^2\right)\tfrac{h'}{h}+\tfrac{1}{f}r-4r\right) \\ +\lambda w\left(4-\tfrac{2}{r^2}w^2-\tfrac{1}{f}\right)+\mu\tfrac{r^2}{h^2}w''\left(3w'^2-h\right) \\ +\tfrac{\mu}{h^2}\left(2r-\tfrac{3}{2h}r^2h'+\tfrac{1}{2f}r^2f'\right)w'^3 \\ +\mu\tfrac{r}{h}\left(\tfrac{h'}{2h}r-2-\tfrac{f'}{2f}r\right)w'+2w\mu\left(1-\tfrac{w^2}{r^2}\right)=0 \end{gathered} \tag{11}$$

### *4.1. Approximate solutions*

The task of solving eq.s (9), (10), (11) is rather desperate, however we may look for approximated solutions under the hypothesis of weak enough effects. Actually from the numerical value for the Lamé coefficients inferred from subsection 3.1 we see that up to the scale of stellar systems or even galaxies, the dimensionless quantities $\lambda r^2$ and $\mu r^2$ stay well below 1. By consequence we try solutions of the type:

$$\left\{ \begin{aligned} f &\simeq f_0+f_1 \\ h &\simeq h_0+h_1 \\ w &\simeq w_0+w_1 \end{aligned} \right. \tag{12}$$

assuming that the $f_1$, $h_1$ and $w_1$ functions are proportional to $\lambda r^2$ and $\mu r^2$.

The explicit forms for $f_0$, $f_1$, etc. are then looked for recursively and perturbatively.

For the zero order of $f$ and $h$ we easily have the Schwarzschild solution:

$$h_0=\frac{1}{1-2\frac{m}{r}} \tag{13}$$

$$f_0 = \frac{1}{h_0} = 1 - 2\frac{m}{r} \tag{14}$$

Considering the fact that this solution is now referred to a deformable space-time, the central symmetry can be due to a matter distribution, in which case $m$ is a mass, as well as to a linear space-time defect[1], in which case $m$ is a parameter measuring the size and type of defect so that it can assume also negative values. Limiting our attention to the presence of an actual mass, we can also remark that in all physical situations of interest it is $m/r << 1$ even though at the scales of stellar systems or galaxies it remains many orders of magnitude bigger than $\lambda r^2$ or $\mu r^2$.

Using (13) and (14) into eq. 11 we obtain the zero order solution for $w$ in the form:

$$w_0 = \sum_{k=-1}^{\infty} a_k r^{-k} \tag{15}$$

It is $a_{-1} = 1$ and the other coefficients are all determined recursively. In practice the lowest order of $w_0$ is trivially $r$ which corresponds to the natural frame coinciding with a flat Minkowski frame.

When we go back to (9) and (10), inserting the result we have found for $w_0$, we are enabled to calculate the first order approximation for $f$ and $h$. Again we obtain a series solution whose elements, excepting the first, are all proportional to powers of $m/r$ times powers of $\lambda r^2$ and $\mu r^2$. The whole procedure is tedious but straightforward.

At the end we are left with the lowest order meaningful solution for the line element in the form:

$$\begin{array}{c} ds^2 = \left(1 - 2\frac{m}{r} + \frac{1}{8}\left(7\lambda + 2\mu\right) r^2\right) d\tau^2 - \left(\frac{1}{1-2\frac{m}{r}} + \frac{1}{4}\left(5\lambda + 2\mu\right) r^2\right) dr^2 \\ -r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \end{array} \tag{16}$$

**5. Expected effects**

Working on (16) we may deduce some effects of the strain that we would expect to see. For short let us put

$$\left\{ \begin{array}{l} \Psi = \frac{1}{8}\left(7\lambda + 2\mu\right) \\ \Lambda = \frac{1}{4}\left(5\lambda + 2\mu\right) \end{array} \right. \tag{17}$$

First we may identify various ranges of interest according to where the balance between $2m/r$ and $\Psi r^2$ or $\Lambda r^2$ is reached. Values are found in table 2:

**Table 2.** Typical ranges of the approximation in the text.

| Case | Range in metres |
|---|---|
| `Stars` | $r \sim 10^{18}$ |
| `Galaxy` | $r \sim 10^{22}$ |
| `Black hole at the galactic center` | $r \sim 10^{20}$ |

[1] Linear in four dimensions: the spherical space symmetry is around a timelike straight worldline.

Out of the full study of the geodesics of line element (16) we may single an interesting case: the circular orbits of a test particle ($r = R$). The relevant quantity we can calculate is the angular velocity which is:

$$\frac{\omega^2}{c^2} = \frac{m}{R^3} + \Psi \tag{18}$$

We see that the strain of space-time tends to increase the angular speed of a freely orbiting object by a constant amount with respect to the corresponding Keplerian orbit, provided $\Psi > 0$. This effect works in the same sense as the effect of dark matter. Of course we should keep in mind that, while going farther and farther the approximation we have made breaks down, because the dependence on the Lamé coefficients becomes much more complicated.

Another interesting phenomenology concerns the propagation of light. Let us consider the regions where the mass terms are non-negligible but comparable with the SST terms. The implicit equation of a light ray is:

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{b^2}\left(1 - \frac{b^2}{r^2} - \Psi r^2\right) - 2\frac{mr^3}{b^2}\left(1 - 2\frac{b^2}{r^2}\right) \tag{19}$$

The $b$ quantity is the geometric impact factor (the distance at which the light ray would pass from the central mass if the space-time were flat).

Again we see that the SST term acts in the same sense as the mass enhancing the curvature of the light ray.

## 6. Conclusion

We have applied the SST theory to the special and fundamental case of a space-time endowed of a spherical symmetry in space and not depending on time. The full treatment has necessarily been an approximated one, because of the complexity of the equations for the $g_{00}$ and $g_{rr}$ terms of the metric tensor and for the gauge function $w$. A perturbative treatment has been used with the small quantities $\lambda r^2$ and $\mu r^2$. The condition is satisfied for scales up to $\sim 10^{20}$ m or so. Closer to the center of symmetry we may identify two other peculiar ranges. A near region, where the mass is dominating and the zero order solution for the metric coincides with the Schwarzschild solution. An intermediate region where the mass terms are comparable with the SST terms, so that a development in powers of $m/r$ is in order. Studying the properties of the approximated metric tensor, we have found that the contribution from the strain of space-time produces a constant (at least in the intermediate region) increase of the angular velocity of a freely orbiting test particle. Furthermore we have seen that also the lensing effect about the center of symmetry is intensified. Both effects look like the ones commonly attributed to the dark matter. Quantitative evaluations do not allow for the moment to conclude that SST can completely eliminate the need for dark matter, however they go in the right direction.

The solution of the problem we have outlined in the present paper, though approximated, is a step forward in the determination of all implications of SST and follows a number of positive tests the theory has undergone, while being confronted with observation. Altogether the scenario which emerges is a physically motivated one and gives more and more confidence on the viability of SST as an interesting paradigm to describe the properties of our universe at large enough scales.

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