On the Numerical Evaluation of the Testing Integrals for the Galerkin Discretization of Surface Integral Equations

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***Abstract*—The testing integrals used to discretize surface integral equations by application of the Moment Method are usually considered as regular 'trivial' integrals to be computed.**

This paper discusses some fundamental problems relevant to the numerical evaluation of these integrals and presents several test cases.

I. Introduction

In the numerical application of the Method of Moments (MoM) to the solution of surface integral equations in the frequency domain by use of the Galerkin procedure, the element matrix is computed by evaluating double surface integrals; that is, *source* and *test* integrals.

Source integrals have been investigated by several authors (see [1-3] and reference therein); stable and efficient techniques to numerically evaluate these integrals with high/machine precision have been recently proposed [4-7].

The Galerkin procedure is typically used to weaken the singularity of the source integrals which could be problematic whenever the observation point is in the neighbourhood of the source points. The strength of the singularity depends on the used integral formulation and it can be weakened by application of appropriate mathematical theorems [8-9].

To minimize the error of the MoM matrix coefficients the source integrals must be computed with high precision, and the testing integrals should be evaluated with a precision adequate to that obtained for the source integrals.

Testing integrals can be regular, quasi-singular for test-elements located in the vicinity of the source element (Magnetic Field Integral Equation - MFIE), or singular for curved self-elements (when the test and the source element are coincident in MFIE).

The convergence properties of the testing integral depend on the size, distance, shape and reciprocal orientation of the test and source elements, in addition to the kind of the used integral equation formulation.

II. Source Integral

The source integral is the first (inner) integral to be computed in order to discretize a surface integral equation via the Galerkin procedure.

Different formulations present singularities of different strength for observation point $\boldsymbol{r}$ approaching the source point $\boldsymbol{r'}$; see Fig. 1 which, for the sake of brevity, refers to triangular elements.

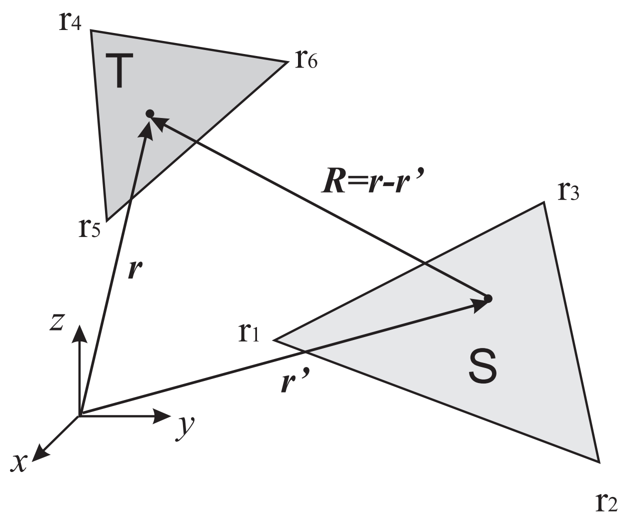

Fig. 1. Observation point $\boldsymbol{r}$ and source point $\boldsymbol{r'}$ in a triangular mesh: T is the test element, S is the source element.

The strength of the singularity $1/R^n$ (where $R=|r-r'|$) is classified in the mathematical and the mechanical engineering community in the following manner: $1/R$ is weakly singular, $1/R^2$ is strongly singular and $1/R^n$, for $n>2$, is hypersingular.

Weak singularities are always integrable; the other singularities must be considered in the general sense of the Cauchy principal value.

Let us consider the frequency-domain Electric Field Integral Equation (EFIE) for perfectly conducting bodies in free-space [8-9]. By integrating by parts and by *distributing* the differential operator onto the testing functions $\mathbf{\Lambda_m}$, the EFIE shows a weak singularity due to the presence of the free space Green function in the source integral:

$$j\omega\mu < \mathbf{\Lambda_m}, G, \mathbf{J} > + \frac{1}{j\omega\varepsilon} < \nabla\bullet\mathbf{\Lambda_m}, G, \nabla'\bullet\mathbf{J} > = < \mathbf{\Lambda_m}, \boldsymbol{E}^i > \quad (1)$$

Equation (1) represents the discretized form of the EFIE where G is the free space Green function, $\boldsymbol{E}^i$ is the incident electric field, and $\mathbf{J} = \sum \mathrm{I_n}\, \mathbf{\Lambda_n}$ (n=1..N) is the current density written in terms of basis functions $\mathbf{\Lambda_n}$ and expansion coefficients $\mathrm{I_n}$. Notice that in equation (1) the inner product $<u,v>$ and the pseudo inner product $<u,G,v>$ defined in [9] have been used:

$$< u, v > = \int uv ds; \quad < u, G, v > = \int u \int G v ds' ds \quad (2)$$

The left-hand terms of equation (1) are related to the magnetic vector and to the electric scalar potential, respectively, and define the coefficients of the MoM impedance matrix $\underline{\underline{Z}}$ of the linear system of equations:

$$\underline{\underline{Z}}\, \underline{I} = \underline{V^i} \quad (3)$$

The Magnetic Field Integral Equation (MFIE) for perfectly conducting bodies [8-9] presents strongly singular integrals due to the presence of the gradient of the free space Green's function in the source integral, see equation (4)

$$\frac{1}{2} < \mathbf{\Lambda_m}, \mathbf{J} > + < \mathbf{\Lambda_m}, n \times \int_S \mathbf{J} \times \nabla G ds' > = < \mathbf{\Lambda_m}, n \times \boldsymbol{H}^i > \quad (4)$$

where G is the free space Green function

$$G = \frac{e^{-jkR}}{4\pi R}, \quad \nabla G = -\frac{(1+kR)}{4\pi R^2}\hat{\boldsymbol{R}} \quad (5)$$

and k is the wavenumber.

The left-hand terms of (4), related to the curl of the magnetic vector potential, define the coefficients of the MoM element matrix $\underline{\underline{\beta}}$ of the linear system of equations:

$$\underline{\underline{\beta}}\, \underline{I} = \underline{I^i} \quad (6)$$

The strong singularity of the source integral in the MFIE is noticeable in particular when the test element is close or attached to the source element, or when one has to compute self-integrals (for coincident test and source elements) on curvilinear elements.

The testing integrals are not singular for weakly singular source integrals (EFIE case). Conversely, for strongly singular source integrals, the testing integrals could result to be weakly singular (MFIE case).

III. TESTING INTEGRAL

In spite of the fact that the testing integrals in EFIE formulations are not singular (see Section II), the numerical convergence of these integrals is related to the singularity of the derivatives of the integrand (the first derivative behaves as $1/R$).

The rate of convergence depends on the size, distance (both in wavelengths), shape and reciprocal orientation of the test and source elements. All these parameters influence the behaviour of the source potentials $< G, \mathbf{J} >$ (vector potential) and $< G, \nabla'\bullet\mathbf{J} >$ (scalar potential), as well as the precision of the numerical evaluation of the MoM impedance matrix (3).

For EFIE testing integrals we use the Gauss quadrature formulas proposed by Dunavant [10] for the triangular element, see Table I.

TABLE I
DUNAVANT QUADRATURE RULES

Points	1	3	4	6	7	12	13	16	19	25	27	33	37
Degree	1	2	3	4	5	6	7	8	9	10	11	12	13

These quadrature rules are almost optimal, in the sense that, for each polynomial degree, the number of points used by the rule is close or equal to the theoretical lowest possible value. Few of these rules include one or two points which are "slightly" outside the triangle; others have negative weights. Both of these occurrences are generally undesirable.

Below we list three issues that influence the precision of the testing integrals and the EFIE solution:

1) The minimum distance between the source and the test element must be considered in order to establish the *slope* of the integrand of the testing integral, *i.e.* the slope of the source potentials.

2) The size in wavelengths of the test element and its orientation with respect to the source element can influence the performance of the used quadrature since the source potentials are pseudo-oscillating functions of the observation point $\boldsymbol{r}$ (that is, with reference to Fig. 1, they contain a factor $\approx\exp(-jk|\boldsymbol{r}-\boldsymbol{r}_m|)$ with, say, $\boldsymbol{r}_m = (\boldsymbol{r}_1+\boldsymbol{r}_2+\boldsymbol{r}_3)/3$).

3) The shape of the elements is another important factor to be considered. Low-quality meshes could influence the precision of the solution because they yield to not-well conditioned impedance matrices (3). Moreover, the Dunavant rules are symmetric quadrature rules, therefore elements with small corner-angles are not optimal.

The first issue is studied in terms of the normalized-to-wavelength distance between the nearest vertices of the two triangles (see Fig. 1). The slope of the integrand (source integral) is studied by introducing interpolation formulas to establish the polynomial order of the to-be-used Dunavant formula [10], see Table I.

Interpolation formulas are also used to study the behaviour of the oscillations of the integrand for increasing size of the test element in order to establish the polynomial order of the to-be-used Dunavant formula [10] (second issue).

The third issue has been extensively studied in the literature, see for example [11]. The author of [11] proposes for a single element the quality factor

$$q = \frac{8(s-a)(s-b)(s-c)}{abc} \tag{7}$$

and for adjacent elements

$$q_{12} = 2/(1/q_1 + 1/q_2) \tag{8}$$

where *a, b, c* are the lengths of the edges of the triangular element, and *s* is the semi-perimeter, see Fig. 2. *q* factors below 0.7 are considered critical.

Other quality factors are inspired by practice, as reported in [12] where the pre-processor code considers factors such as the minimum (maximum) angle, the normalized element surface, the normalized minimum edge length.

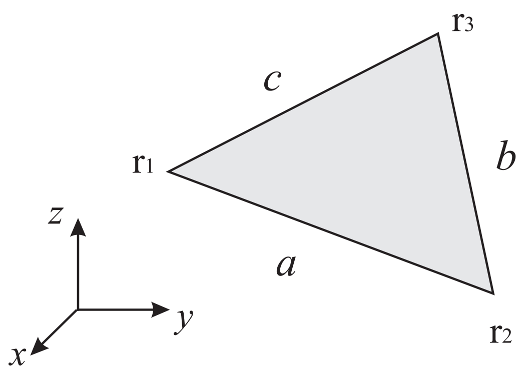

Fig. 2. Triangular element.

Several algorithms to define an optimal mesh have been studied after the algorithm presented by Delaunay in 1934 [13] to maximizes the minimum internal angle of all the triangles in the mesh.

Moreover, another parameter to be considered in selecting the quadrature rule is to compensate the loss of precision with the decay of the intensity of the source potential as a function of the distance. This property can improve the performance of the numerical code without loss of precision, provided the MoM element matrix is well conditioned.

By considering all these parameters, one can define a heuristic decision table to select the appropriate Dunavant quadrature rule. Massive tests confirm the validity of our decision algorithm.

The possible inaccuracies of the MFIE have been investigated in depth [14-17].

In our opinion, one has to consider that the testing integral in MFIE is weakly singular [16-17] because of the strongly singular behaviour of ∇G in (4). Therefore, appropriate quadrature rules to treat nearly singular and weakly singular integrals should be selected. The kind of the singularity in the testing integral is logarithmic [16-17].

Appropriate quadrature formulas can be used to treat the logarithmic singularity [18-19].

Other appropriate quadrature schemes could be of the same type of the ones used to deal with the singularity of the potential integrals in the EFIE [4-5, 20] which are based on cancellation techniques, *i.e.* on transformations with a Jacobian that vanishes at the singular point.

In the MFIE case the construction of a heuristic table useful to choose the appropriate quadrature rule depends on the quadrature formulas one wants to use [18-19], and/or on the used cancellation transformation [4-7, 20]

IV. Conclusion

In MoM applications, a precise evaluation of the testing integrals is of importance to secure high precision numerical results. The accuracy required to the testing integrals depends on several factors discussed in the paper. Decision tables to choose the appropriate quadrature rules will be presented at the conference, and validated by massive testing.

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