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Capacity scaling of large wireless networks with heterogeneous clusters / Martina, Valentina; Garetto, Michele; Leonardi, Emilio. - In: PERFORMANCE EVALUATION. - ISSN 0166-5316. - STAMPA. - 67:(2010), pp. 1203-1218.  
[10.1016/j.peva.2010.08.008]

*Availability:*

This version is available at: 11583/2375701 since:

*Publisher:*

Elsevier

*Published*

DOI:10.1016/j.peva.2010.08.008

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# Capacity scaling of large wireless networks with heterogeneous clusters

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## Abstract

**We analyze the capacity scaling laws of wireless networks where the spatial distribution of nodes over the network area exhibits a high degree of clustering. In particular we consider the presence of heterogeneous clusters, both in size and in population, which are common in many real systems. We completely characterize the scaling exponent of the resulting network capacity by providing upper and lower bounds which differ at most by a poly-logarithmic factor in the number of nodes.**

## I. INTRODUCTION AND RELATED WORK

The fundamental problem of determining the asymptotic capacity of large ad hoc wireless networks has received significant interest in the past several years, starting from the seminal work of Gupta and Kumar [1]. A variety of results are currently available under different system assumptions related to the interference model (i.e., protocol or physical model), channel fading, scaling of the network area, constraints on the power/transmission range, and shape of the power attenuation function (see [2] for a survey of results).

One critical aspect that can affect the applicability of existing results to real network scenarios is the way in which nodes are assumed to be distributed over the area, since network topology can strongly affect the overall system performance.

In [1] Gupta and Kumar have shown that the per-node throughput is upper bounded by  $1/\sqrt{n}$  under arbitrary nodes placement. Later on, Franceschetti et al. [3] have proven, using percolation theory results, that the above upper bound is actually achievable (under the physical interference model) when nodes are distributed according to a Homogeneous Poisson Point (HPP) process over the network area. Hence the case in which nodes are distributed according to a HPP process is optimal in terms of capacity.

The natural question, which has received little attention so far, is whether  $1/\sqrt{n}$  is actually achievable in more general network topologies which cannot be adequately represented by a HPP process. Indeed, most of the topologies generated by natural growing processes (such as urban or sub-urban settlements) are characterized by large inhomogeneities in the nodes spatial distribution, since preferential attachment phenomena produce high degree of clustering [4].

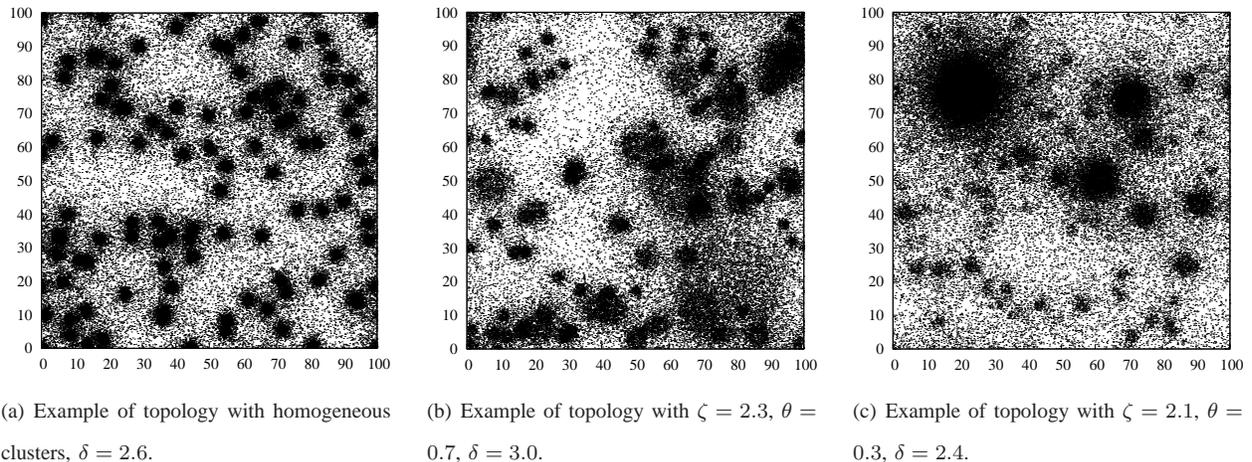


Fig. 1. Examples of topologies with  $n = 100,000$  nodes belonging to 100 different clusters ( $\nu = 0.4$ ), distributed over the square  $100 \times 100$  ( $\alpha = 0.4$ ).

In our previous work [5], [6], we have derived both lower and upper bounds to the capacity of wireless networks with inhomogeneous node density. In particular, we have considered the case of several identical clusters, in which the node density decays from the cluster centre with a power law of exponent  $\delta$  (one example is reported in Figure 1(a)). For this class of topologies the per-node throughput can be significantly smaller than  $1/\sqrt{n}$ , and turns out to be intrinsically related to the node density of the least populated areas.

In this paper we move one step forward toward the capacity analysis of realistic inhomogeneous networks, by considering a much richer class of point processes generating heterogeneous clusters (both in size and population), which are usually found in real networks. In particular we consider power laws for both the cluster size and the cluster population, which naturally appear in many growing systems [7]. To this extent we generalize the approach of [5], [6], developing a methodology that permits to characterize the capacity of rather complex and heterogeneous topologies such as those shown in Figures 1(b) and 1(c).

Prior to our work, only a few papers have considered the scaling behavior of network in which nodes are not uniformly distributed. In [8] the authors consider  $n$  nodes distributed over  $\sqrt{n}$  lines, or clustered around  $\sqrt{n}$  neighborhoods. However, both cases lead to topologies which do not contain significant inhomogeneities in the node density, thus the resulting capacity is similar to that derived by Gupta and Kumar.

In [9] the authors consider a system which contains many circular clusters with uniform node density within them, surrounded by a sea of nodes with much lower node density. The only quantity that scales with  $n$  is the network size. Below a critical network size, the per-node throughput is limited by the amount of data that a cluster can exchange with the sea of nodes, whereas above the critical size the per-node throughput is limited by the capacity of the sea of nodes. In contrast to [9], we consider a much more general class of clustered topologies, which requires also different techniques to compute the resulting network capacity.

## II. SYSTEM ASSUMPTIONS AND NOTATION

### A. Network Topology

We consider networks composed of a random number  $N$  of nodes (being  $\mathbb{E}[N] = n$ ) distributed over a square region  $\mathcal{O}$  of edge  $L$ . To avoid border effects, we consider wrap-around conditions at the network edges (i.e., the network area is assumed to be the surface of a two-dimensional Torus). The network physical extension  $L$  is allowed to scale with the average number of nodes, since this is expected to occur in many growing systems. Throughout this work we will assume that<sup>1</sup>  $L = \Theta(n^\alpha)$ , with  $\alpha \in [0, 1/2]$ , which permits to model all intermediate systems in between the two extreme cases usually referred to in the literature as *dense network* ( $\alpha = 0$ ) and *extended network* ( $\alpha = 1/2$ ).

Nodes are grouped into a random number  $M$  of clusters, with  $\mathbb{E}[M] = m$ . Each cluster has a centre denoted by  $c_j$ , for  $j = 1 \dots M$ . Cluster centres are placed on  $\mathcal{O}$  according to a HPP process of intensity  $\phi_c = m/L^2$ . We allow the average number of clusters  $m$  to scale with  $n$  as well, according to the law  $m = \Theta(n^\nu)$ , with  $\nu \in (0, 1]$ .

Each cluster generates an Inhomogeneous Poisson Point (IPP) process of nodes around the cluster centre  $c_j$ , whose local intensity at point  $\xi$  is denoted by  $\psi_j(\xi)$ . Hence the number of nodes belonging to cluster  $j$  is itself a random variable, whose mean  $q_j$  is given by the integral over  $\mathcal{O}$  of the local intensity  $\psi_j(\xi)$ , which is assumed to be rotationally invariant around the cluster centre. Therefore we can express  $\psi_j(\xi)$  in the form

$$\psi_j(\xi) = q_j k_j(\xi, c_j) = q_j k_j(\|\xi - c_j\|)$$

where  $\|\xi - c_j\|$  denotes the distance<sup>2</sup> of point  $\xi$  from cluster center  $c_j$ , and  $k_j(\cdot)$  is a kernel function whose integral over  $\mathcal{O}$  is equal to 1.

We remark that our model for heterogeneous clusters can be regarded as a special case of generalized shot-noise Cox processes [10]. The overall node process over the network domain  $\mathcal{O}$  is a random field obtained by the superposition of the individual point processes generated by the clusters. Given the positions of clusters' centres  $\mathbf{c} = \{c_j\}_{j=1}^M$  and the mean clusters populations  $\mathbf{q} = \{q_j\}_{j=1}^M$ , the conditional local intensity at  $\xi$  of the overall point process is

$$\Phi(\xi|\mathbf{c}, \mathbf{q}) = \sum_j q_j k_j(\xi, c_j)$$

Notice that  $\Phi(\xi|\mathbf{c}, \mathbf{q})$  is a standard (inhomogeneous) Poisson point process.

We first describe the distribution of the  $q_j$ 's, and then specify the associated kernel functions  $k_j(\cdot)$ 's.

<sup>1</sup>Given two functions  $f(n) \geq 0$  and  $g(n) \geq 0$ :  $f(n) = o(g(n))$  means  $\lim_{n \rightarrow \infty} f(n)/g(n) = 0$ ;  $f(n) = O(g(n))$  means  $\limsup_{n \rightarrow \infty} f(n)/g(n) = c < \infty$ ;  $f(n) = \omega(g(n))$  is equivalent to  $g(n) = o(f(n))$ ;  $f(n) = \Omega(g(n))$  is equivalent to  $g(n) = O(f(n))$ ;  $f(n) = \Theta(g(n))$  means  $f(n) = O(g(n))$  and  $g(n) = O(f(n))$ ; at last  $f(n) \sim g(n)$  means  $\lim_{n \rightarrow \infty} f(n)/g(n) = 1$ .

<sup>2</sup>Given any two points  $X_1 = (x_1, y_1) \in \mathcal{O}$  and  $X_2 = (x_2, y_2) \in \mathcal{O}$  we define their distance as  $d(X_1, X_2) = \min_{u, v \in \{-L, 0, L\}} \sqrt{(x_1 + u - x_2)^2 + (y_1 + v - y_2)^2}$

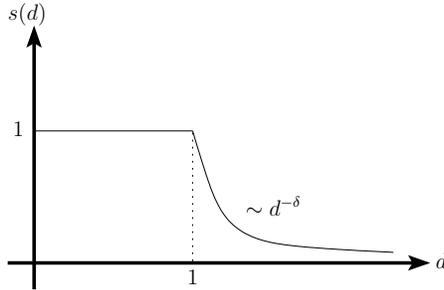


Fig. 2. Function  $s(d)$ , determining the shape of kernels  $k_j(\cdot)$ .

To model the presence of a few large clusters together with many small clusters, we consider that the cluster population size is distributed according to a power law, assigning to each  $q_j$  a discrete random value  $q \in \mathbb{N}$  according to the Zipf's law

$$f_\zeta(q) = G q^{-\zeta} \quad q \in \{q_{\min}, \dots, q_{\max}\}, \quad (1)$$

where  $G$  is a normalization constant and  $\zeta > 2$ . To guarantee that the average number of nodes in the network is  $\Theta(n)$ , it is necessary to select  $q_{\min} = \Theta(n^{1-\nu})$ . We instead assume that  $q_{\max} = \Theta(n^{1-\beta})$ , where  $0 \leq \beta \leq \nu$  is a free parameter that will be better specified later.

Once the population size of each cluster has been assigned, the shape of kernel function  $k_j(\cdot)$ , which dictates how nodes belonging to the cluster are distributed around the cluster centre, must reflect the fact that bigger clusters are expected to occupy a larger network region than smaller clusters. At the same time, we want some nodes to stay arbitrarily far from their cluster centre, filling those regions in between the clusters. At last, we want  $k_j(\cdot)$  to be a summable, non-increasing, bounded and continuous function whose integral over the entire network area is equal to 1.

To satisfy all requirements above, we start with the function  $s(d)$  reported in Figure 2, which can be expressed as

$$s(d) = \mathbb{I}_{(d < 1)} + d^{-\delta} \cdot \mathbb{I}_{(d \geq 1)}.$$

Then, for every  $j$ , we define a parameter  $r_j$  that we call cluster radius of cluster  $j$ . Kernel function  $k_j(\cdot)$  is finally obtained rescaling and normalizing  $s(d)$  over the network area  $\mathcal{O}$ :

$$k_j(\xi, c_j) = \frac{s(\|\xi - c_j\|/r_j)}{\int_{\mathcal{O}} s(\|\xi' - c_j\|/r_j) d\xi'}, \quad (2)$$

where  $\int_{\mathcal{O}} s(\|\xi' - c_j\|/r_j) d\xi' = \Theta(r_j^2)$  for any  $\delta > 2$ . By so doing, the node density of cluster  $j$  is constant within a disc of radius  $r_j$  centered at  $c_j$ , and decays as a power law of exponent  $\delta$  outside the disc.

For greater flexibility, we let the cluster radius to depend on  $q_j$  according to  $r_j = (q_j/q_{\min})^\theta$ , where  $\theta \geq 0$  is one additional parameter of our model which allows to model different levels of node concentration around the

Symbol	Definition
$n$	average number of nodes
$L$	edge length of the network area
$\alpha$	growth exponent of $L$ : $L = \Theta(n^\alpha)$ , $\alpha \in [0, 1/2]$
$m$	average number of clusters
$\nu$	growth exponent of $m$ : $m = \Theta(n^\nu)$ , $0 < \nu \leq 1$
$\zeta$	exponent of Zipf's distribution of cluster population size, $\zeta > 2$
$\theta$	exponent of cluster radius, $\theta > 0$
$\beta$	exponent of maximum cluster population size, $q_{\max} = \Theta(n^{1-\beta})$
$\delta$	power-law decay of node density, $\delta > 2$

TABLE I  
SYSTEM PARAMETERS

cluster centre. However we must be careful that the radius of the biggest clusters, having population size  $q_{\max}$ , does not exceed in order sense the edge  $L = \Theta(n^\alpha)$  of the network area. This is satisfied when  $\beta \geq \nu - \alpha/\theta$ , which combined with previous constraints on  $\beta$  leads to the following range of feasible values for  $\beta$ :

$$\max\{0, \nu - \alpha/\theta\} \leq \beta \leq \nu. \quad (3)$$

Notice that we have normalized to 1 the radius of clusters having minimum population size  $q_{\min}$ . This is not restrictive, since one can play with the network edge  $L = n^\alpha$  to account for different values of the minimum cluster radius.

Table I summarizes all parameters of our model, which allow us to obtain a wide range of network topologies. Figures 1(b) and 1(c) provide two example topologies containing on average  $n = 100,000$  nodes. In both cases there are (on average) 100 clusters distributed over a torus surface of edge length 100 (i.e.,  $\alpha = \nu = 0.4$ ). The topology of Figure 1(b) is characterized by  $\zeta = 2.3$ ,  $\theta = 0.7$  and  $\delta = 3$ , resulting in quite large cluster radiuses, and rapidly decaying node density outside the cluster discs. In the topology of Figure 1(c) the distribution of cluster populations is more skewed ( $\zeta = 2.1$ ), but cluster discs are smaller ( $\theta = 0.3$ ) while the node density decays more slowly outside them ( $\delta = 2.4$ ). In both cases we have chosen the smallest possible value for  $\beta$ , equal to 0.

### B. Communication Model

We assume that time is divided into slots of equal duration, and that in each slot an optimal scheduling policy enables a set of transmitter-receiver pairs to communicate over point-to-point wireless links which are modeled as Gaussian channels of unit bandwidth. We assume that interference among simultaneous transmissions is described by the following version of the *generalized physical model*, according to which the rate achievable by node  $i$  transmitting to node  $j$  in a given time slot is limited to

$$R_{ij} = \min\{R_0, \log_2(1 + \text{SINR}_j)\}$$

where  $R_0$  is the maximum rate attainable over a link due to physical limitations of transmitters/receivers (maximum data speed of I/O devices, finite set of possible modulation schemes, etc) and  $\text{SINR}_j$  is the signal to interference

$e_\Lambda$	condition
$\frac{1}{2}$	$\alpha - \frac{\nu}{2} < 0$
$\frac{1}{2} - (\alpha - \frac{\nu}{2})(\frac{\delta}{2} - 1)$	$\alpha - \frac{\nu}{2} \geq 0 \wedge \alpha - \frac{\nu}{2} + (\nu - \beta)(\frac{\zeta'}{2} - \theta) \geq 0 \wedge \theta < \frac{\zeta'\delta - 2}{2(\delta - 2)}$
$\frac{1}{2} - (\alpha - \frac{\nu}{2})(\frac{\delta}{2} - 1) + \frac{\nu - \beta}{2} [(\delta - 2)\theta - (\frac{\zeta'\delta}{2} - 1)]$	$\alpha - \frac{\nu}{2} \geq 0 \wedge \alpha - \frac{\nu}{2} + (\nu - \beta)(\frac{\zeta'}{2} - \theta) \geq 0 \wedge \theta \geq \frac{\zeta'\delta - 2}{2(\delta - 2)}$
$\frac{1}{2} - \frac{\nu_0}{2}(\zeta' - 1)$	$\alpha - \frac{\nu}{2} \geq 0 \wedge \alpha - \frac{\nu}{2} + (\nu - \beta)(\frac{\zeta'}{2} - \theta) < 0 \wedge \theta \geq \frac{\zeta'\delta - 2}{2(\delta - 2)}$
$\frac{1}{2} - \min \left\{ (\alpha - \frac{\nu}{2})(\frac{\delta}{2} - 1), \frac{\nu_0}{2}(\zeta' - 1) \right\}$	$\alpha - \frac{\nu}{2} \geq 0 \wedge \alpha - \frac{\nu}{2} + (\nu - \beta)(\frac{\zeta'}{2} - \theta) < 0 \wedge \theta < \frac{\zeta'\delta - 2}{2(\delta - 2)}$

TABLE II

SCALING EXPONENT OF NETWORK CAPACITY.  $\zeta' = \zeta - 1$ .  $\nu_0 = \frac{2\alpha - \nu}{2\theta - \zeta'}$

and noise ratio at receiver  $j$ :

$$\text{SINR}_j = \frac{P_i \ell_{ij}}{N_0 + \sum_{k \in \Delta, k \neq i} P_k \ell_{kj}}$$

Here,  $\Delta$  is the set of nodes which are enabled to transmit in the given slot,  $P_i$  is the power emitted by node  $i$ ,  $\ell_{ij}$  is the power attenuation between  $i$  and  $j$ , and  $N_0$  is the ambient noise power. The power attenuation is assumed to be a deterministic function of the distance  $d_{ij}$  between  $i$  and  $j$ , according to  $\ell_{ij} = d_{ij}^{-\gamma}$ , with  $\gamma > 2$ . We assume that nodes can employ different transmitting powers, according to an optimal strategy of power assignment to simultaneous transmissions.

### C. Traffic Model

Similarly to previous work we focus on *permutation traffic patterns*, i.e., traffic patterns according to which every node is source and destination of a single data flow at rate  $\lambda$ . Sources and destinations of data flows are randomly matched, establishing  $N$  end-to-end flows in the network. Our goal is to maximize the common rate  $\lambda$  concurrently achievable by all flows, or, equivalently, to maximize the network capacity, defined as  $\Lambda = N\lambda$ .

## III. SUMMARY OF RESULTS

Similarly to previous work, we characterize the scaling law of the network capacity  $\Lambda$  by the scaling exponent  $e_\Lambda$ , defined as,

$$e_\Lambda = \lim_{n \rightarrow \infty} \frac{\log \Lambda(n)}{\log n}$$

The scaling exponent allows to ignore all poly-logarithmic factors, i.e., factors which are  $O(\log n)^k$ , for any finite  $k$ . Since our lower and upper bounds differ at most by a poly-logarithmic factor, the corresponding scaling exponents match, so we can claim that our characterization of the network capacity in terms of the scaling exponent is exact. Results are reported in Table II as function of the system parameters  $\alpha, \nu, \delta, \zeta, \theta, \beta$ . To simplify the expressions, we have used  $\zeta' = \zeta - 1$  and  $\nu_0 = \frac{2\alpha - \nu}{2\theta - \zeta'}$ .

Whenever  $\alpha - \nu/2 < 0$  we get the maximum possible exponent  $e(\Lambda) = \frac{1}{2}$ , equivalent to a system in which nodes are distributed according to a HPP process; otherwise for  $\alpha - \nu/2 > 0$  the network capacity is in general reduced

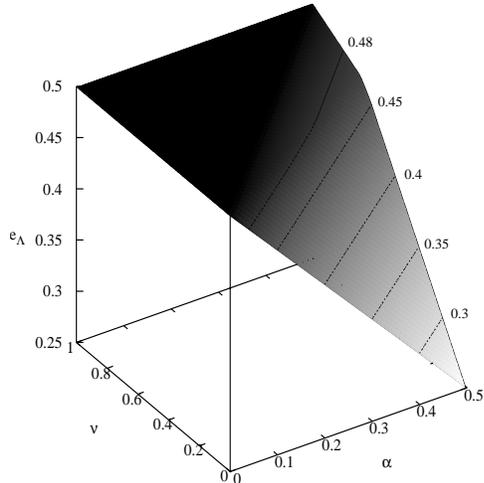


Fig. 3. Scaling exponent of  $\Lambda$  as function of  $\alpha$  and  $\nu$ , for fixed  $\zeta = 2.1$ ,  $\theta = 0.8$ ,  $\delta = 3$ .

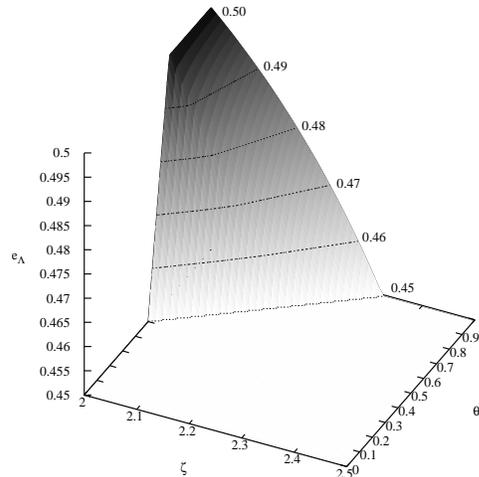


Fig. 4. Scaling exponent of  $\Lambda$  as function of  $\zeta$  and  $\theta$ , for fixed  $\alpha = 0.4$ ,  $\nu = 0.6$ ,  $\delta = 3$ .

for effect of nodes inhomogeneities. The behavior of the capacity is rather complex, coming to depend on all of the system parameters ( $\alpha$ ,  $\delta$ ,  $\nu$ ,  $\beta$ ,  $\zeta$ ,  $\theta$ ). In general we can observe that the system capacity is a non-increasing function of  $\alpha$ ,  $\delta$ ,  $\beta$  and  $\zeta$ , whereas it is a non-decreasing function of  $\nu$  and  $\theta$ .

To have a more immediate feeling of how the capacity depends on the the system parameters, Figure 3 reports a 3D plot of  $e(\Lambda)$  as function of  $\alpha$  and  $\nu$ , for fixed  $\zeta = 2.1$ ,  $\theta = 0.8$ ,  $\delta = 3$ ; while Figure 4 reports a 3D plot of  $e(\Lambda)$  as function of  $\zeta$  and  $\theta$ , for fixed  $\alpha = 0.4$ ,  $\nu = 0.6$ ,  $\delta = 3$ . In all cases we have set the smallest  $\beta = \max\{0, \nu - \alpha/\theta\}$  (see (3)). In particular, Figure 4 highlights the important role that parameters driving the distribution of cluster population size can have on the overall system capacity. In particular, by increasing the spreading of the nodes belonging to the same cluster over the network area, i.e., for  $\zeta \rightarrow 2$  and  $\theta \rightarrow 1$ , we can achieve the same capacity exponent  $e(\Lambda) \rightarrow \frac{1}{2}$  as if the nodes were homogeneously distributed.

#### IV. PRELIMINARIES

We introduce some basic properties and existing results that are needed in the analysis presented in the next section. The first lemma is a standard concentration result about HPP processes.

**Lemma 1:** Consider an average number  $m$  of points distributed over  $\mathcal{O}$  according to an HPP of intensity  $\phi$ . Let  $\mathcal{A} = \{A_k\}$  be a regular tessellation of  $\mathcal{O}$  (or any sub-region of  $\mathcal{O}$ ), whose tiles  $A_k$  have a surface  $|A_k| > 16 \frac{\log m}{\phi}$ ,  $\forall k$ . Let  $U(A_k)$  be the number of points falling in  $A_k$ . Then, uniformly over the tessellation, for every  $k$  it holds  $\frac{\phi|A_k|}{2} < \inf_k U(A_k) \leq \sup_k U(A_k) < 2\phi|A_k|$ .

We do not repeat the proof of this lemma, which is based on a standard application of the Chernoff bound (see [11]).

**Corollary 1:** As immediate consequence of Lemma 1, if we consider a regular tessellation  $\mathcal{A} = \{A_k\}$  in which

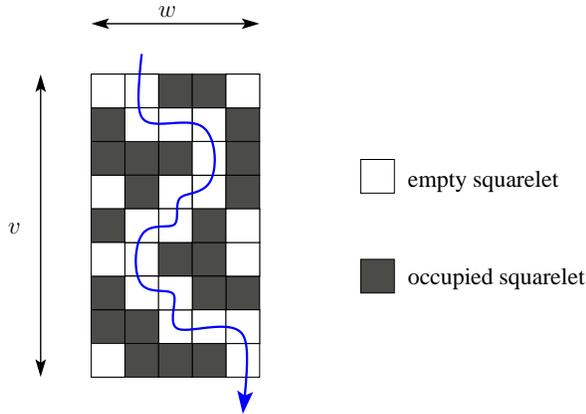


Fig. 5. Example of site percolation. There exists a path of empty squarelets crossing the rectangular grid  $w \times v$  from the top edge to the bottom edge.

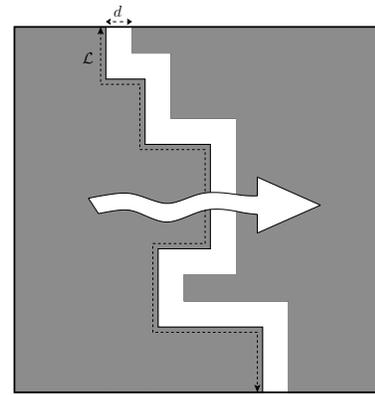


Fig. 6. Network flow passing through an empty corridor of length  $\mathcal{L}$  and width  $d$ .

$|A_k| = O(\log m/\phi)$ ,  $\forall k$ , then uniformly over the tessellation  $U(A_k) = O(\log m)$ .

We will need the following result from percolation theory (see [12]):

**Lemma 2:** Consider a rectangular grid of squarelets (as in Figure 5), having  $v$  squarelets on the (long) vertical edge and  $w$  squarelets on the (short) horizontal edge. Squarelets are independently marked as empty with probability  $p$ , and occupied with probability  $1 - p$ . Two squarelets are adjacent if they have a common edge. Let  $p_c^s \sim 0.59$  be the critical probability of independent site percolation on the square lattice. Then, if  $w = \Omega(\log v)$ , for any  $p > p_c^s$  and as  $v \rightarrow \infty$ , there exists w.h.p. a vertical crossing path of empty adjacent squarelets, comprising  $\Theta(v)$  (empty) squarelets.

The following property, established in [6], allows to upper bound the maximum amount of data that can be transferred across a network cut, under the same communication model adopted in this paper.

**Lemma 3:** Suppose there exists a corridor of width  $d$  and length  $\mathcal{L}$  dividing the network area in two parts (see Figure 6), and which does not contain any node. Then the amount of data that can be transferred from one part to the other across the corridor is  $O(\mathcal{L}/d)$ .

In particular, we can obtain an upper bound to the aggregate network capacity by considering a cut dividing area  $\mathcal{O}$  in two parts of area  $\Theta(L^2)$ , since w.h.p. there are  $\Theta(n)$  flows established across it. Lemma 3 suggests that to obtain the tightest possible bound we need to find an empty corridor crossing the network area from the top edge to the bottom edge, having minimum length  $\mathcal{L}$  and maximum width  $d$ . Lemma 3 has been already used in [6] to derive an upper bound to the capacity achievable in the case of identical clusters. In this paper, beside extending the analysis to heterogeneous clusters (both in size and population), we will adopt a different technique to identify the optimal corridor to which we can apply Lemma 3, which also improves upon the bound derived in [6] for identical clusters (actually, we believe that the technique presented here provides the best possible upper bound for

our class of network topologies, including the special case of homogeneous clusters).

The following lower bound has been instead obtained in [5] in the case of identical clusters.

**Lemma 4:** Consider the clustered point process described in Section II-A, in which the population size is the same for all clusters, i.e.,  $q_j = q$ , ( $j = 1 \dots M$ ). Let  $\underline{\Phi} = \inf_{\mathcal{O}} \Phi(\xi)$  be the minimum node density in the network. Then it is possible to find a scheduling-routing scheme providing an aggregate capacity  $\Lambda = \Theta\left(\max\{L\sqrt{\underline{\Phi}}, \sqrt{m}\}\right)$  being  $\underline{\Phi} = \Omega(n^{1-\nu-\delta(\alpha-\nu/2)})$ .

The basic idea underlying the scheduling-routing mentioned in the above lemma is to extract from the overall point process a set of nodes  $\mathbf{X}_0$  distributed according to a HPP process, and use this set as the main transport infrastructure of the network, whose capacity can be computed using well-known results [3]. If the minimum node density  $\underline{\Phi}$  is not too low, a standard thinning technique can be applied to extract from the overall point process a subset of nodes distributed according to a HPP of intensity  $\underline{\Phi}$ . Alternatively, one can select just one node per cluster, and obtain with the selected nodes a HPP process of intensity  $m/L^2$  (this alternative leads to the second term in the max function that appears in lemma 4). The main difficulty is then to show that the rest of nodes can communicate with the nodes of the main infrastructure at a per-flow rate higher than that sustainable over the main infrastructure (i.e., the network throughput is not throttled by communications between nodes in  $\mathbf{X}_0$  and nodes not belonging to  $\mathbf{X}_0$ ). The interested reader is referred to [5] for the details.

## V. ANALYSIS FOR FINITE NUMBER OF CLASSES

In the following we analyze a simplified scenario for the cluster population size: the techniques developed for this case will come in handy later on in Section VI, in which we analyze the case of cluster population distributed according to a Zipf's distribution. In particular, in this section we consider the case of non-homogeneous cluster population size, where only a finite number of values can be taken. We assume that there exist a finite number  $H$  of classes, representing the possible value that the cluster population size can take. For every  $h \in \{0, \dots, H-1\}$ , the population size value corresponding to class  $h$  is assumed to be

$$q_h = \Theta(q_{\min} n^{h\mu}), \quad (4)$$

where  $q_{\min} = \Theta(n^{1-\nu})$  is the minimum population size, and  $\mu > 0$  is a parameter that specifies how the populations of the different classes are spaced apart. Note that  $q_h = \Theta(n^{1-\nu+h\mu})$ .

Every cluster is assigned to one of these class independently and identically with respect to other clusters. In particular, every cluster  $j$  is assigned a random mark  $h_j$  taking values in  $\{0, H-1\}$ , according to the distribution

$$p_h = Pr(h_j = h) = G' q_h^{-\zeta'}, \quad (5)$$

where  $G'$  is a normalization factor, and  $\zeta' > 1$ . We have  $p_h = \Theta(n^{-h\mu\zeta'})$  for any  $\zeta' > 1$ . Notice that our simplified model with finite number of classes can be used as an approximation of the original Zipf's distribution. This is accomplished by slicing the domain of the original Zipf's distribution into a finite set of intervals  $I_h =$

$[q_{\min} n^{h\mu}, q_{\min} n^{(h+1)\mu})$ , for all  $0 \leq h < H$ , where  $\mu > 0$ , and assuming that all clusters within an interval have the same size. The approximation becomes better and better as we increase the number of classes (i.e., letting  $\mu$  tend to zero).

The average number of clusters assigned to class  $h$  is  $m_h = m p_h = \Theta(n^{\nu-h\mu\zeta'})$ . The average number of nodes belonging to cluster assigned to class  $h$  is  $n_h = m_h q_h = \Theta(n^{1-h\mu(\zeta'-1)})$ .

Having defined the cluster population size in the case of a finite number of classes, we need to specify the kernel function  $k_h(\cdot)$  that characterizes the IPP generated by each cluster of class  $h$ : it can be obtained from the cluster population size  $q_h$  in exactly the same way as described in Section II. Following the rationale outlined there, the radius of class  $h$  is set to  $r_h = \Theta(n^{h\mu\theta})$ ,  $\theta > 0$ .

Note that the centres of clusters belonging to class  $h$  are distributed over the network area according to a HPP process of intensity

$$\phi_c(h) = \phi_c p_h. \quad (6)$$

We also introduce  $d_c(h) = \sqrt{1/\phi_c(h)} = \Theta(n^{\alpha-\nu/2+h\mu\zeta'/2})$ , which is the typical distance between clusters belonging to class  $h$ . More precisely,  $d_c(h)$  is the edge of the squarelet in which we expect to find, on average, one cluster centre belonging to class  $h$ . Quantities  $d_c(h)$ 's are fundamental in our analysis, as explained in the next section.

#### A. Asymptotic analysis of the local node density

The first step of our analysis is the characterization of the asymptotic node density over the network area. Let  $\mathbf{h} = \{h_j\}_{j=1}^M$  the collection of marks assigned to clusters, and  $\mathbf{c}$  the position of the clusters' centers. In the case of a finite number of classes, we can express the conditional local intensity of nodes at point  $\xi$ , given the sets  $\mathbf{h}$  and  $\mathbf{c}$ :

$$\Phi(\xi|\mathbf{c}, \mathbf{h}) = \sum_{h=0}^{H-1} \sum_{j:h_j=h} q_h k_h(\xi, c_j) = \sum_{h=0}^{H-1} \phi_c(h)(\xi|\mathbf{c}, \mathbf{h})$$

where  $\phi_c(h)(\xi|\mathbf{c}, \mathbf{h}) = \sum_{j:h_j=h} q_h k_h(\xi, c_j)$  is the contribution of clusters of class  $h$ . To simplify the notation, in the following we will write  $\Phi(\xi)$  and  $\phi_c(h)(\xi)$  instead of  $\Phi(\xi|\mathbf{c}, \mathbf{h})$  and  $\phi_c(h)(\xi|\mathbf{c}, \mathbf{h})$ , respectively.

We introduce the following quantities,  $\bar{\Phi} = \sup_{\mathcal{O}} \Phi(\xi)$  and  $\underline{\Phi} = \inf_{\mathcal{O}} \Phi(\xi)$ , denoting, respectively, the supremum and the infimum of  $\Phi(\xi)$  over  $\mathcal{O}$ . Similarly, for each class  $h$ , we define  $\bar{\Phi}_h = \sup_{\mathcal{O}} \phi_c(h)(\xi)$  and  $\underline{\Phi}_h = \inf_{\mathcal{O}} \phi_c(h)(\xi)$ , which are the supremum and the infimum of  $\phi_c(h)(\xi)$  over  $\mathcal{O}$ .<sup>3</sup> Recall that the above quantities are random variables depending on the positions  $\mathbf{c}$  of the cluster centres and their marking  $\mathbf{h}$ . Note that  $\underline{\Phi} \geq \sum_h \underline{\Phi}_h$  and  $\bar{\Phi} \leq \sum_h \bar{\Phi}_h$ .

<sup>3</sup>In the following, with slight abuse of terminology we will refer to  $\bar{\Phi}$  ( $\bar{\Phi}_h$ ) and  $\underline{\Phi}$  ( $\underline{\Phi}_h$ ), respectively as the maximum and the minimum of  $\Phi(\xi)$  ( $\phi_c(h)(\xi)$ ) over  $\mathcal{O}$ .

The following theorem characterizes the extreme values of  $\phi_c(h)(\xi)$ , for each  $h$ .

**Theorem 1:** Let  $\eta_h = \frac{d_c(h)\sqrt{\log m_h}}{r_h}$ . If  $\eta_h = o(1)$ , then it is possible to find two positive constants  $g_h, G_h$ , with  $g_h < G_h$ , such that  $\forall \xi_0 \in \mathcal{O}$ ,

$$g_h \frac{n_h}{L^2} < \phi_c(h)(\xi_0) < G_h \frac{n_h}{L^2} \quad \text{w.h.p.} \quad (7)$$

Hence in this case  $\underline{\Phi}_h = \Theta(\overline{\Phi}_h) = \Theta(n_h/L^2)$ . Instead, when  $\eta_h = \Omega(1)$  it results, w.h.p.,  $\underline{\Phi}_h = o(\overline{\Phi}_h)$ . Moreover:  $\overline{\Phi}_h = O(q_h \log m_h)$  and  $\underline{\Phi}_h = \Omega(q_h \log m_h s(\eta_h)/r_h^2)$ .

For the proof of Theorem 1, see Appendix A. Here, we provide an intuitive interpretation of the main result of the theorem. When  $\eta_h = o(1)$  the typical distance between clusters belonging to class  $h$ , i.e.,  $d_c(h)$ , becomes in order sense smaller than the cluster radius  $r_h$ . As consequence the density of nodes belonging to class  $h$  tends to become uniformly constant over the entire network domain. We say in this case that class  $h$  is in the *cluster-dense* regime.

On the contrary, when  $\eta_h = \Omega(1)$  the typical distance between neighboring class- $h$  clusters is larger (in order sense) than the class- $h$  cluster radius. Hence the density of nodes belonging to class  $h$  is no longer uniformly distributed (in order sense), i.e.,  $\underline{\Phi}_h = o(\overline{\Phi}_h)$ . We say in this case that class  $h$  is in the *cluster-sparse* regime.

Since for the same values of the system parameters the various classes can be in different regimes (either *dense* or *sparse*), we distinguish the following four cases:

- *full cluster-dense* regime, when all classes are in the *cluster-dense* regime. This case occurs when  $\alpha - \nu/2 + h\mu(\zeta/2 - \theta) < 0$  for any  $h$ , which requires that:
  - (i)  $\alpha - \nu/2 < 0$  if  $\theta \geq \zeta'/2$ .
  - (ii)  $\alpha - \nu/2 + (H-1)\mu(\zeta'/2 - \theta) < 0$  if  $\theta < \zeta'/2$ .
- *full cluster-sparse* regime, when all classes are in the *cluster-sparse* regime. This case occurs when  $\alpha - \nu/2 + h\mu(\zeta'/2 - \theta) > 0$  for any  $h$ , which requires that:
  - (i)  $\alpha - \nu/2 \geq 0$  if  $\theta \leq \zeta'/2$ .
  - (ii)  $\alpha - \nu/2 + (H-1)\mu(\zeta'/2 - \theta) \geq 0$  if  $\theta > \zeta'/2$ .
- *$\tilde{h}$ -sparse* regime, when classes  $0 \dots \tilde{h}$  are in the *cluster sparse* regime, and classes  $\tilde{h} + 1 \dots (H-1)$  are in the *cluster-dense* regime. This case can occur only when  $\theta > \zeta'/2$ , and requires the existence of  $\tilde{h} \in \{0, \dots, (H-1)\}$  such that
  - (i)  $\alpha - \nu/2 + \tilde{h}\mu(\zeta'/2 - \theta) \geq 0$ .
  - (ii)  $\alpha - \nu/2 + (\tilde{h} + 1)\mu(\zeta'/2 - \theta) < 0$ .
- *$\tilde{h}$ -dense* regime, when classes  $0 \dots \tilde{h}$  are in the *cluster dense* regime, and classes  $\tilde{h} + 1 \dots (H-1)$  are in the *cluster-sparse* regime. This case can occur only when  $\theta < \zeta'/2$ , and requires the existence of  $\tilde{h} \in \{0, \dots, (H-1)\}$  such that

- (i)  $\alpha - \nu/2 + \tilde{h}\mu(\zeta'/2 - \theta) < 0.$
- (ii)  $\alpha - \nu/2 + (\tilde{h} + 1)\mu(\zeta'/2 - \theta) \geq 0.$

Notice that the value of  $\theta$  with respect to that of  $\zeta'/2$  is critical. When  $\theta > \zeta'/2$  the more ‘uniformly dense’ clusters are the biggest ones. On the contrary, when  $\theta < \zeta'/2$  the more ‘uniformly dense’ clusters are the smallest ones.

### B. Capacity Upper bound

To compute an upper bound to the network capacity we are going to apply Lemma 3, finding an empty corridor that divides the area in two parts each having area  $\Theta(L^2)$ . Recall that the optimal corridor should have minimum length  $\mathcal{L}$  and maximum width  $d$ . We observe that  $\mathcal{L}$  cannot be smaller than  $L$ . To maximize  $d$ , the corridor must traverse those network regions where the node density is minimum. In particular, we need to identify a connected region traversing the network area from top to bottom, and staying as far as possible from cluster centres, especially from the biggest ones (i.e., those having mark  $h = H - 1$ ), which produce a large node density in their proximity. Intuitively, the optimal corridor should stay at a distance from clusters of class  $h$  which increases with  $h$ .

We first focus on the *full cluster-sparse* regime, in which clusters of any class are well separated from each other, allowing to find an empty crossing path from the top to the bottom edge of the network area, which does not contain any cluster centre. After analyzing this case, it will be clear how we can handle the concurrent presence of some classes (in the extreme case, all classes) in the *cluster-dense* regime.

First, we build a sequence of nested corridors  $\mathcal{P}_0 \subset \mathcal{P}_1 \dots \subset \mathcal{P}_h \subset \dots \mathcal{P}_{H-1}$ , satisfying the property that, for each  $h \in \{0, \dots, H-1\}$ , corridor  $\mathcal{P}_h$  does not contain any cluster centre belonging to class  $h$ . Then, within corridor  $\mathcal{P}_0$ , we look for a final corridor  $\mathcal{P}_s$  free of nodes, to which we can eventually apply Lemma 3. The existence of the sequence of nested corridors  $\mathcal{P}_0 \subset \dots \subset \mathcal{P}_{H-1}$  is guaranteed by the following theorem:

**Theorem 2:** In the *full cluster-sparse regime*, it is possible to find a sequence of nested corridors  $\mathcal{P}_0 \subset \dots \subset \mathcal{P}_h \subset \dots \mathcal{P}_{H-1} \subset \mathcal{O}$ , such that the width of corridor  $\mathcal{P}_h$  ( $h = 0, \dots, H-1$ ) is  $D_h = \Theta(1/\sqrt{\phi_c(h)})$ .

*Proof:* Our construction starts with the biggest clusters (i.e., those of class  $h = H - 1$ ), which are more sparse, and thus permit to find the largest initial corridor  $\mathcal{P}_{H-1}$ . We consider a vertical slice of the network area of width  $\Theta(L)$  and height  $L$ , and divide it into a regular grid of squarelets of edge  $D_{H-1}$ , chosen in such a way that the probability that no cluster center of class  $H - 1$  falls within one of them is larger than  $p_c^s$ , the critical probability of site percolation in square lattice. This requires that  $e^{-\phi_{H-1} D_{H-1}^2} > p_c^s$  (recall that  $\phi_c(h)$  is the intensity of the HPP of cluster centres of class  $h$ , defined in (6)), which is satisfied (in order sense) when  $D_{H-1} = \Theta(1/\sqrt{\phi_{H-1}})$ . The horizontal and the vertical number of squarelets in the slice are of the same order of magnitude, hence we can apply Lemma 2 and establish the existence of a corridor  $\mathcal{P}_{H-1}$  of width  $D_{H-1}$  and physical length  $\Theta(L)$  which does not contain any cluster centre of class  $H - 1$ .

Once we have found corridor  $\mathcal{P}_{H-1}$ , we can sequentially find all of the other nested corridors

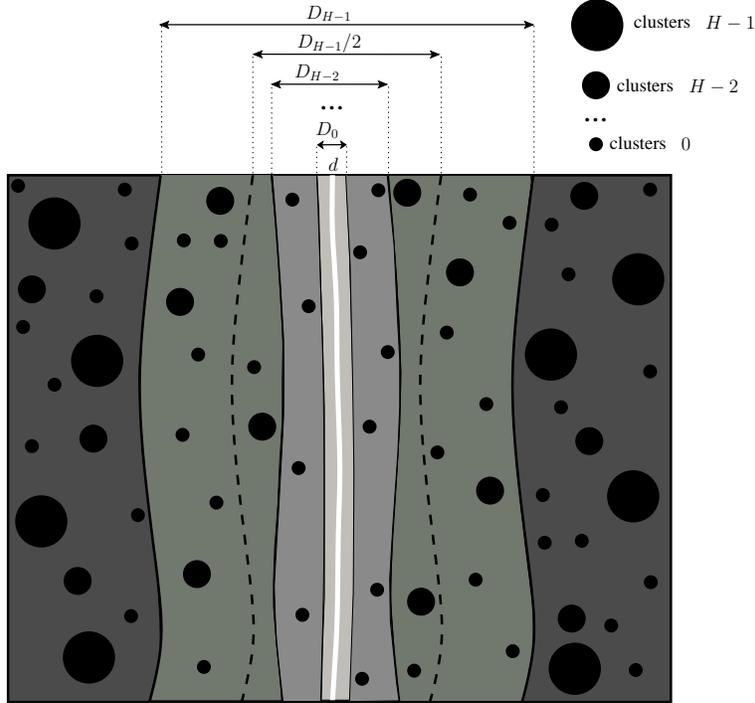


Fig. 7. Construction of the sequence of nested corridors

$\mathcal{P}_{H-2}, \mathcal{P}_{H-3}, \dots, \mathcal{P}_0$  using the following iterative construction. We consider the generic corridor  $\mathcal{P}_h$ , with  $h > 0$ , and denote by  $\mathcal{P}'_h \subset \mathcal{P}_h$  the central part of  $\mathcal{P}_h$ , having width  $D_h/2$ . Inside  $\mathcal{P}'_h$ , we look for the inner corridor  $\mathcal{P}_{h-1}$ , which must not contain any cluster centre belonging to class  $h-1$ . Figure 7 provides a graphical illustration of our approach.

We divide  $\mathcal{P}'_h$  into a regular grid of squarelets of edge  $D_{h-1}$ , chosen in such a way that the probability that no cluster center of class  $h-1$  falls within one of them is larger than  $p_c^s$ . This requires that  $e^{-\phi_{h-1} D_{h-1}^2} > p_c^s$  which is satisfied (in order sense) when  $D_{h-1} = \Theta(1/\sqrt{\phi_{h-1}})$ . Let  $w_{h-1}$  and  $v_{h-1}$  be, respectively, the horizontal and vertical number of squarelets of edge  $D_{h-1}$  that we can put within  $\mathcal{P}'_h$ . We have  $w_{h-1} = \Theta(D_h/D_{h-1}) = \Theta(n^{\mu\zeta'/2})$ , which does not depend on  $h$ . Moreover,  $v_{h-1} = o(n)$ , since  $v_{h-1} = L/D_{h-1}$  where  $D_{h-1} = \Theta(L/\sqrt{m_{h-1}})$ , with  $m_{h-1} = O(n)$ . Hence, condition  $w_{h-1} = \Omega(\log v_{h-1})$  is verified and we can apply Lemma 2 to establish the existence of corridor  $\mathcal{P}_{h-1} \subset \mathcal{P}'_h \subset \mathcal{P}_h$ . Iterating sequentially this step from  $h = H-1$  down to  $h = 1$ , we obtain the desired sequence of nested corridors. ■

At last, we need to establish the existence of a path  $\mathcal{P}_s \subset \mathcal{P}_0$  containing no nodes. We again consider the central part  $\mathcal{P}'_0$ , having width  $D_0/2$ , of  $\mathcal{P}_0$ , and look for corridor  $\mathcal{P}_s$  only within  $\mathcal{P}'_0$ . This time the problem is more difficult, because the point process of individual nodes within  $\mathcal{P}'_0$  is no longer a HPP process, hence we cannot apply exactly the same technique adopted above for the other corridors.

A loose upper bound could be obtained assuming (as a worst case) that the node process within  $\mathcal{P}'_0$  is a HPP

process of intensity uniformly equal to the maximum intensity attained by the point process within  $\mathcal{P}'_0$ . One could then build a regular grid of squarelets dimensioned in accordance to the above maximum intensity, and apply again Lemma 2. This approach has been followed in [6] in the case of homogeneous clusters. Here we propose a novel methodology which allows to obtain an improved bound (which we believe to be in order sense tight, i.e., leading to an estimate of the network capacity matching, in order sense, the actual network capacity).

Recall from Section V-A that, given the set  $\mathbf{h}$  of marks assigned to clusters and their locations  $\mathbf{c}$ , the point process generated by clusters of class  $h$ , and the overall node process generated by all clusters, are standard inhomogeneous Poisson point processes whose (conditional) intensities are denoted by  $\phi_c(h)(\xi)$  and  $\Phi(\xi)$ , respectively.

We introduce the following definition of the mean node density within a generic (Lebesgue-measurable) domain  $\mathcal{B}$ :

$$\mathcal{E}_{\mathcal{B}}[\Phi(\xi)] \triangleq \frac{\int_{\mathcal{B}} \Phi(\xi) d\xi}{\int_{\mathcal{B}} d\xi} \quad (8)$$

In the hypothetical case in which the intensity of the point process within  $\mathcal{P}'_0$  were uniformly equal to its mean  $\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]$ , we could build a regular grid of squarelets of edge  $z_x = \Theta(\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]^{-1/2})$ , and apply Lemma 2 to establish the existence of a corridor  $\mathcal{P}_s \subset \mathcal{P}'_0$  having width  $z_x$ . Clearly, this hypothetical corridor would provide an improved upper bound to the capacity (using Lemma 3), because its width is larger than the one that we obtain assuming that the node process within  $\mathcal{P}'_0$  has intensity uniformly equal to its maximum value within  $\mathcal{P}'_0$ .

Now, even if the point process within  $\mathcal{P}'_0$  is not a HPP process of intensity  $\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]$ , the following theorem allows to establish the existence of a corridor having width equal to  $z_x$  as defined above.

**Theorem 3:** Let  $\mathcal{P}_0$  be the innermost corridor found according to the construction in Theorem 2. Let  $\mathcal{P}'_0 \subset \mathcal{P}_0$  be a corridor having the same length and half the width of  $\mathcal{P}_0$ . Then, it is possible to find a corridor  $\mathcal{P}_s \subset \mathcal{P}'_0$  empty of nodes, having length  $\Theta(L)$ , and width  $z_x = \Theta(\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]^{-1/2})$ .

*Proof:* We consider for simplicity the case in which path  $\mathcal{P}'_0$  has a rectangular shape. However, the same approach can be applied to a general path, dividing  $\mathcal{P}'_0$  into a sequence of partially overlapped rectangles. The basic idea is to construct an irregular tessellation of  $\mathcal{P}'_0$  in which the sizes of the tiles are locally adapted to the intensity of point process  $\Phi(\xi)$ . We consider tiles of rectangular shape, in which the horizontal edge  $z_x$  is the same for all tiles, while the vertical edge  $z_y(\xi)$  can vary, being adapted to the local intensity  $\Phi(\xi)$ . Notice that we force all tiles in the same row to have the same vertical dimension. This choice does not affect the tightness of our improved bound, because  $\Phi(\xi)$  does not change significantly in the horizontal direction (actually,  $\Phi(\xi)$  is of the same order of magnitude over any horizontal line within  $\mathcal{P}'_0$ ). Figure 8 provides a graphical illustration of our approach.

Let  $N_A$  be the total number of tiles of the tessellation, and  $\mathcal{A}_k$  denote the generic tile. As already said, we set the horizontal edge of all tiles equal to  $z_x = \mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]$ . Let  $p = Pr(\mathcal{A}_k \text{ free of nodes}) = e^{-\int_{\mathcal{A}_k} \Phi(\xi) d\xi}$ . We dimension the vertical edge  $z_y(\xi)$  in such a way that  $p > p_c^s$  over all tiles belonging to the same row. By so doing, we can map our irregular tessellation into a bidimensional lattice homologous to the one in Figure 5. Thus we left

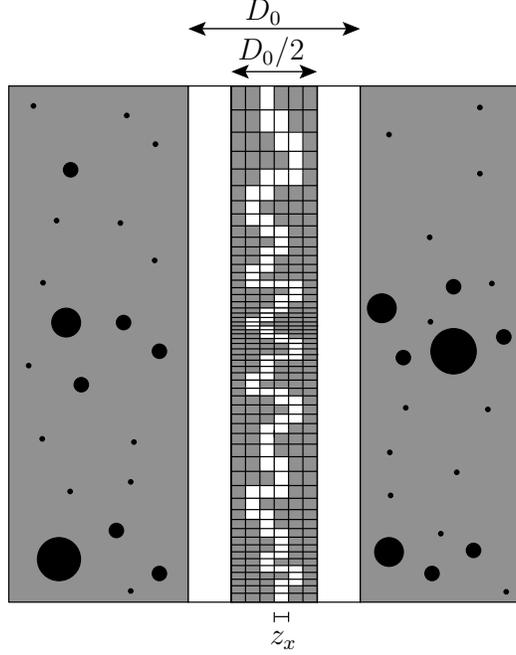


Fig. 8. Novel approach to identify an empty path of width  $z_x$  which does not contain any node, nested in the innermost path  $\mathcal{P}'_0$  of width  $D_0$ . The vertical edge of the tiles is adapted to the local intensity of the point process.

unchanged the underlying discrete geometry over which we can apply Lemma 2, provided that the number of tiles  $N_y$  along the vertical direction and the number of tiles  $N_x$  along the horizontal direction satisfy  $N_x = \Omega(\log N_y)$ . Since by hypotheses all tiles are dimensioned in such a way that  $\int_{\mathcal{A}_k} \Phi(\xi) d\xi > -\log p_c^s$ , we can assume that for some constant  $\epsilon$

$$N_A < \frac{\int_{\mathcal{P}'_0} \Phi(\xi) d\xi}{-\log \epsilon p_c^s} \quad (9)$$

Considering that  $N_A = N_x \times N_y$ , and that  $z_x = \mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]$  we have

$$N_y = N_A/N_x \stackrel{(9)}{=} O\left(\frac{LD_0 \mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]}{N_x}\right) \stackrel{N_x = D_0/2z_x}{=} O(L/z_x).$$

Since  $D_0 = \Omega(\log L)$ , we can indeed apply Lemma 2 and establish the existence of an empty corridor in the underlying lattice, having width  $z_x$  and comprising  $\Theta(N_y)$  tiles. Since by construction the average vertical size of the tiles is  $\bar{z}_y = L/N_y$ , we conclude that the empty corridor has average length  $\Theta(L)$ .

The following theorem characterizes the asymptotic behavior of the mean node density within  $\mathcal{P}'_0$ .

**Theorem 4:** The mean node density within  $\mathcal{P}'_0$  is  $\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)] = O\left(\sum_h q_h \frac{D_h^{-\delta}}{r_h^{2-\delta}}\right)$ .

For the proof of Theorem 4 see Appendix B.

From Theorem 4 we derive a lower bound for  $z_x = \Omega\left(\sum_h q_h \frac{D_h^{-\delta}}{r_h^{2-\delta}}\right)^{-1/2}$ , on which we can apply Lemma 3 and

obtain our final upper bound to the network capacity  $\Lambda = O(L/z_x)$ .

Our approach can be easily extended to the case in which some (or all) classes are in the *cluster-dense* regime. Indeed, the contribution of these classes to the overall density of the node process is almost uniform over the network area, being  $\Phi_h = \Theta(\bar{\Phi}_h)$  for any class  $h$  in the *cluster-dense* regime (see Section V-A). Hence these classes are ignored in the construction of the nested corridors  $\mathcal{P}_h$ , which is to be done only for classes in the *cluster-sparse* regime. The contribution of classes in the *cluster-dense* regime to the overall node density must instead be taken into account when we look for the final corridor  $\mathcal{P}_s$  containing no nodes.

In the *full cluster-dense* regime, being  $\Phi(\xi) = \Theta(\frac{n}{L^2})$  uniformly over the whole domain  $\mathcal{O}$ , the maximal width of a corridor containing no nodes is  $z_x = \Theta(L/\sqrt{n})$ , i.e., it is equal (in order sense) to the typical distance between neighboring nodes in a uniformly dense network.

In the  $\tilde{h}$ -*sparse* regime, the mean density of nodes within  $\mathcal{P}'_0$  can be evaluated (in order sense) as

$$\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)] = \Theta \left( \sum_{h \leq \tilde{h}} q_h \frac{D_h^{-\delta}}{r_h^{2-\delta}} + \sum_{h > \tilde{h}} \frac{n_h}{L^2} \right)$$

and we can set  $z_x = \Theta(\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)]^{-1/2})$ .

In the  $\tilde{h}$ -*dense* regime, the smallest clusters in the sparse regime belong to class  $\tilde{h} + 1$ , hence we look for the final corridor free of nodes within  $\mathcal{P}'_{\tilde{h}+1}$ , in which

$$\mathcal{E}_{\mathcal{P}'_{\tilde{h}+1}}[\Phi(\xi)] = \Theta \left( \sum_{h \leq \tilde{h}} \frac{n_h}{L^2} + \sum_{h > \tilde{h}} q_h \frac{D_h^{-\delta}}{r_h^{2-\delta}} \right)$$

Then we can set  $z_x = \Theta(\mathcal{E}_{\mathcal{P}'_{\tilde{h}+1}}[\Phi(\xi)]^{-1/2})$ .

In all cases, the final upper bound to the network capacity is  $\Lambda = O(L/z_x)$ . Notice that in the *full cluster-dense* and in the  $\tilde{h}$ -*dense* regimes we recover the well known result that  $\Lambda = O(\sqrt{n})$  [1].

### C. Capacity Lower Bound

Lower bounds to the network capacity are obtained by evaluating the performance of constructive scheduling-routing schemes specifically tailored to the topologies generated by our model. In particular, we generalize the scheduling-routing scheme developed in [5] for the case of homogeneous clusters, whose performance is given by Lemma 4. The basic idea is still to extract from the overall node process  $\mathbf{X}$  a set of nodes  $\mathbf{X}_0$  distributed according to a HPP process, and use such nodes as the main transport infrastructure through which data are transferred across the network area. Similarly to the case of homogeneous clusters,  $\mathbf{X}_0$  is either obtained by extracting a set of nodes with intensity equal to  $\Phi$ , or it is formed by just one node per cluster, if this provides a richer set of nodes (i.e., if  $\phi_c > \Phi$ ). Then the main challenge is to show that the aggregate throughput is ultimately given by the capacity

of the main infrastructure, i.e., that communications between set  $\mathbf{X}_0$  and the rest of the nodes do not throttle the capacity available over the main infrastructure.

**Theorem 5:** Consider the case of heterogeneous clusters belonging to a finite number of classes, as specified earlier in this section. Then it is possible to find a scheduling-routing scheme providing an aggregate capacity  $\Lambda = \Theta(\max\{L\sqrt{\underline{\Phi}}, \sqrt{m}\})$ .

*Proof:* In the *full cluster-dense* regime we have  $\underline{\Phi} = \Theta(\bar{\Phi})$ , hence we can exploit a general result (see [5], Theorem 2) that assures that in this case we always get a network capacity  $\Lambda = \Theta(L\underline{\Phi}) = \Theta(\sqrt{n})$ . In the other regimes, there are some classes (in the extreme case, all classes) in the *cluster sparse* regime. In this case, the simplest approach is to separately consider the nodes of each class  $h$  (together with the nodes in  $\mathbf{X}_0$ ), as if they were the only nodes present in the network, and to devote to each class a constant fraction of time, during which we schedule only transmission between nodes belonging to class  $h$  or to  $\mathbf{X}_0$ . In more detail, we introduce a scheduling super-frame given by the succession of  $H + 1$  frames  $0, 1, \dots, H$  of equal duration. During frame  $h$ , with  $h = 0, 1, \dots, H - 1$ , we consider only the nodes belonging to class  $h$  and to the main transport infrastructure. This frame is used to make nodes belonging to class  $h$  exchange traffic with nodes belonging to  $\mathbf{X}_0$ . The last frame  $h = H$  is instead devoted entirely to the main transport infrastructure, and it is used to transfer data of all classes over large distances across the network area. Notice that communication among nodes belonging to different classes occur only using nodes of  $\mathbf{X}_0$  as intermediate relays. Since the number of classes  $H$  is supposed to be finite, this strategy, although suboptimal, achieves in order sense the same performance of a network consisting only of the main transport infrastructure, since the loss introduced by the scheduling super-frame is  $1/H = \Theta(1)$ . It remains to show that, during the generic frame  $h$ , nodes belonging to class  $h$  can exchange traffic with nodes in  $\mathbf{X}_0$  without throttling down the per-node throughput. However, for this we can simply adapt the scheduling strategy developed for the case of homogeneous clusters. More in detail, for each class  $h$ , we separately consider the sub-region  $\mathcal{O}'_h$  of the network area in which  $\Phi_h = O(\underline{\Phi})$  and the sub-region  $\mathcal{O}''_h$  in which  $\Phi_h = \omega(\underline{\Phi})$ . Notice that  $\mathcal{O}''_h$  can be empty, if  $q_h \log m_h = O(\underline{\Phi})$ .

The two sub-regions above can be again considered in isolation, since we can assign to each sub-region (when both are non empty) half of the frame devoted to class  $h$  without affecting the overall performance in order sense. Nodes belonging to  $\mathcal{O}'_h$  can directly communicate with nodes in  $\mathbf{X}_0$  using single-hop transmissions, in the same way adopted for the *full cluster-dense* regime (see [5], Theorem 2). Nodes belonging to  $\mathcal{O}''_h$  must adopt, instead, the hierarchical multi-hop scheme described in [5], which allows to spread out the traffic generated by the ‘peaks’ of nodes belonging to class  $h$  over the ground-level infrastructure  $\mathbf{X}_0$ . The only difference is that in this case the ground-level infrastructure can be above the one given by  $\underline{\Phi}_h$ , i.e., the density of the main infrastructure could be higher than the minimum node density generated by class  $h$ . This situation makes even easier the traffic spreading procedure described in [5], reducing the number of hops required to reach the main infrastructure. We conclude that the proposed strategy allow to achieve the same capacity of a network in which nodes are distributed according

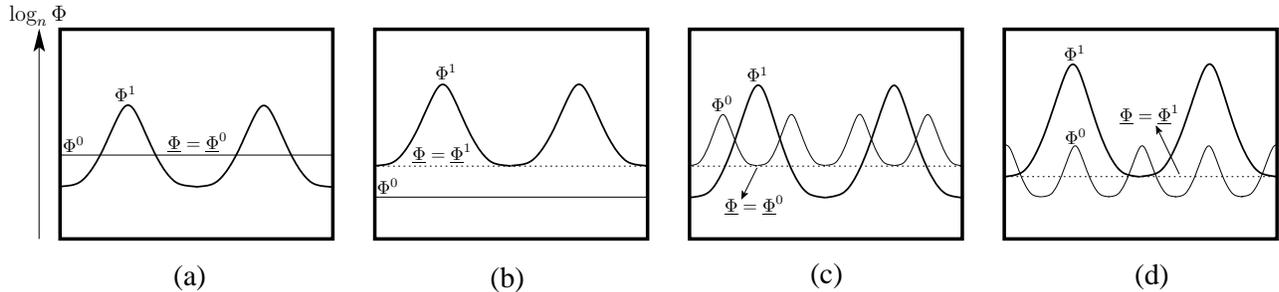


Fig. 9. Illustration of the different cases that can occur with two classes of clusters. For simplicity, we have considered the case of a uni-dimensional network.

to a HPP of intensity  $\max\{\phi_c, \underline{\Phi}\}$ , which is  $\Lambda = \Theta(\max\{L\sqrt{\underline{\Phi}}, \sqrt{m}\})$ .  $\blacksquare$

Figure 9 provides a graphical illustration of the different cases that can occur with  $H = 2$  classes of clusters, assuming  $\theta < \zeta'/2$ . It refers to the case in which there are many small clusters belonging to class 0 and a few large clusters belonging to class 1. Cases (a) and (b) in Figure 9 provide an example of mixed regime (more precisely, a  $0$ -dense regime, according to the definitions in Section V-A), whereas cases (c) and (d) in Figure 9 correspond to the *full cluster-sparse* regime. The minimum network density is determined by clusters of class 0 in cases (a) and (c), and by clusters of class 1 in cases (b) and (d). The extension to a generic number of cluster classes is straightforward.

Using the lower bound of  $\underline{\Phi}_h$  given in Theorem 1, it turns out that the network capacity achievable by our schemes exactly matches the corresponding upper bound when the overall capacity is dominated by the contribution of classes in the *cluster dense regime*, while it differs only by a poly-log factor when capacity is determined by classes in the *cluster sparse regime*. In this case, the poly-log gap between lower bounds and upper bounds is entirely due to the lower bound, since the proposed scheduling routing schemes do not always achieve optimal throughput. We are confident that employing more sophisticated techniques a constructive lower bound that exactly matches the corresponding upper bound can be found; however we leave this issue for future investigations.

## VI. ZIPF'S DISTRIBUTION OF CLUSTER POPULATION SIZE

We are now ready to extend the analysis to the case in which clusters' populations are distributed according to a Zipf's distribution of exponent  $\zeta$ . The basic idea is to reduce the analysis of this case to that of a system with finite number of cluster classes. This can be done by slicing the domain of the original Zipf's distribution into intervals  $I_h = [q_{\min} n^{h\mu}, q_{\min} n^{(h+1)\mu})$ , for all  $0 \leq h < H$ , where  $\mu > 0$ , and assuming that all clusters within one interval belong to the same class. Since our analysis for finite number of classes requires that all clusters belonging to the same class are homogeneous, an approximation is needed at this point, as we have to assign the same nominal population size  $q_h$  to all clusters in  $I_h$ .

Considering that the network capacity is intimately related to the minimum node density over the area, to

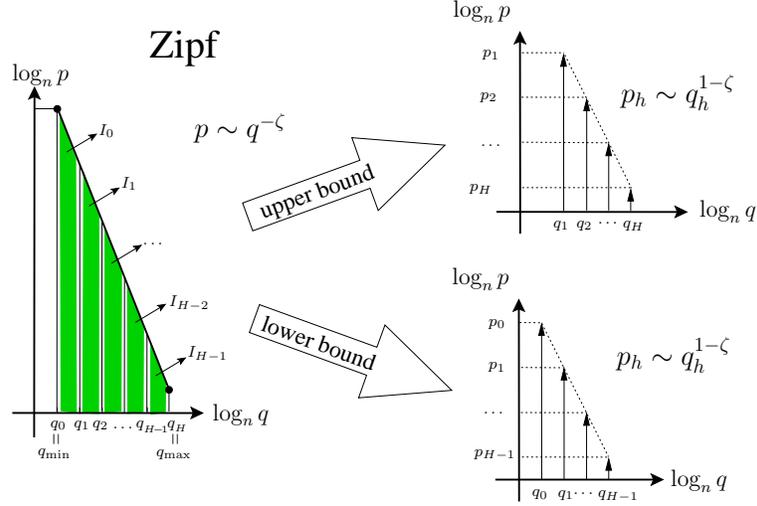


Fig. 10. Slicing of the Zipf's distribution of cluster population

obtain an upper/lower bound to the capacity in the Zipf case we need to assure that the above approximation provides a corresponding upper/lower bound to the resulting node density. This is easily accomplished by setting  $q_h = \sup_{q \in I_h} = q_{\min} n^{(h+1)\mu}$  when we wish to upper bound the system capacity, and  $q_h = \inf_{q \in I_h} = q_{\min} n^{h\mu}$  when we wish to lower bound the capacity (see Figure 10).

In this way, by employing the techniques developed in the previous section, we can obtain for any  $\mu > 0$  both an upper bound  $\bar{\Lambda}(\mu)$  and a lower bound  $\underline{\Lambda}(\mu)$  to the network capacity of the original Zipf case.

Note that, by construction, the fraction of clusters falling in class  $h$ , for both lower and upper bounds, is

$$p_h = G \sum_{q_{\min} n^{h\mu}}^{q_{\min} n^{(h+1)\mu} - 1} q^{-\zeta} \approx G \int_{q_{\min} n^{h\mu}}^{q_{\min} n^{(h+1)\mu}} q^{-\zeta} dq = G' q_{\min} n^{h\mu(1-\zeta)} [1 + o(1)] = G'' q_h^{1-\zeta} \quad (10)$$

(expressed in terms of the  $q_h$  to be used for the lower bound). Hence a Zipf's distribution of exponent  $\zeta$  is mapped into a model with finite number of classes in which the exponent is  $\zeta' = \zeta - 1$  (see (5)). For this reason in this paper we have always assumed that  $\zeta > 2$  in the Zipf's distribution, whereas  $\zeta' > 1$  in the case of  $H$  classes.

Now, considering that upper and lower bounds become tighter and tighter as  $\mu$  is reduced, we obtain, for any  $n$ , the best bounds by letting  $\mu \rightarrow 0$ :

$$\underline{\Lambda} = \lim_{\mu \rightarrow 0} \underline{\Lambda}(\mu) \leq \Lambda \leq \lim_{\mu \rightarrow 0} \bar{\Lambda}(\mu) = \bar{\Lambda}$$

Since our upper/lower bounds for the case of finite number of classes are asymptotically tight except for poly-log terms, we conclude that our analysis allows to obtain the scaling exponent  $e(\Lambda)$  of the system, as reported in Table II. Indeed,  $e(\underline{\Lambda}) = e(\bar{\Lambda})$ , i.e., lower bound  $\underline{\Lambda}$  differs at most by a poly-log term from the upper bound  $\bar{\Lambda}$  also in the Zipf case.

## VII. CONCLUSIONS

In this paper we have proposed a methodology to upper and lower bound the asymptotic capacity of a static wireless networks with heterogeneous clusters. We have first analyzed the case in which there are  $H$  classes of homogeneous clusters, and then generalized the approach to the more complex case in which the cluster population size is distributed according to a Zipf's distribution. In both cases the obtained upper and lower bounds have been shown to be tight except for poly-log terms. Our results suggest that cluster heterogeneity can have in some cases a significant impact on the achievable network capacity.

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## APPENDIX A

### PROOF OF THEOREM 1

The main steps of the proof are: i) the domain  $\mathcal{O}$  is divided into squarelets; ii) the local intensity at  $\xi_0$  is expressed as sum of contributions, each due to cluster centres located in the same squarelet; iii) applying Lemma 1, every contribution is bounded w.h.p. (both from below and from above); iv) the upper (lower) bound is shown to converge w.h.p. to some value for  $n \rightarrow \infty$ .

Consider a generic point  $\xi_0 \in \mathcal{O}$  and a class  $h$ . Let  $\mathcal{A}^h = \{A_k^h\}$  denote a regular square tessellation of  $\mathcal{O}$ , such that each squarelet  $A_k^h$  has area  $|A_k^h| = 16 \eta_h^2$ . Let  $\underline{d}_{0k}^h$  and  $\bar{d}_{0k}^h$  be, respectively, the inferior and the superior of the distances between points  $\xi \in A_k^h$  and  $\xi_0$ , i.e.,  $\underline{d}_{0k}^h = \inf_{\xi \in A_k^h} \|\xi - \xi_0\|$  and  $\bar{d}_{0k}^h = \sup_{\xi \in A_k^h} \|\xi - \xi_0\|$ ; at last, let

$\underline{U}(A_k^h)$  and  $\overline{U}(A_k^h)$  be, respectively, a lower bound and an upper bound to the number of cluster centers of type  $h$  falling in  $A_k^h$ . We recall that, by definition:  $\phi_c(h)(\xi_0) = \sum_{j:h_j=h} q_h k_h(c_j, \xi_0)$ , and  $k_h(c_j, \xi_0)$  has the same form as (2). It results:

$$\sum_k \frac{q_h}{r_h^2} s(\underline{d}_{0k}^h/r_h) \underline{U}(A_k^h) < \underline{\phi}_c(h) \leq \phi_c(h)(\xi_0) \leq \overline{\phi}_c(h) < \sum_k \frac{q_h}{r_h^2} s(\underline{d}_{0k}^h/r_h) \overline{U}(A_k^h). \quad (11)$$

Applying Lemma 1 we have that, w.h.p., uniformly over  $k$ ,  $\underline{U}(A_k^h) \geq (m_h/2L^2)|A_k^h|$  and  $\overline{U}(A_k^h) \leq (2m_h/L^2)|A_k^h|$ . Moreover, if we introduce the variable  $\underline{D}_{0k}^h = \underline{d}_{0k}^h/r_h$  (and analogously  $\overline{D}_{0k}^h$ ), we observe that i)  $\sum_k q_h s(\overline{D}_{0k}^h) \frac{|A_k^h|}{r_h^2}$  and  $\sum_k q_h s(\underline{D}_{0k}^h) \frac{|A_k^h|}{r_h^2}$  can be interpreted, respectively, as lower Riemann sum and upper Riemann sum of  $\int_0^\infty q_h D \cdot s(D) dD$ ; ii) since  $\eta_h(m) = o(1)$ , the mesh size of the partitions associated to Riemann sums vanishes to 0 as  $n \rightarrow \infty$ . As consequence:

$$\sum_k q_h s(\overline{D}_{0k}^h) \frac{|A_k^h|}{r_h^2} \sim \sum_k q_h s(\underline{D}_{0k}^h) \frac{|A_k^h|}{r_h^2} \sim q_h \int_0^\infty D \cdot s(D) dD = q_h = \frac{n_h}{m_h}$$

and we conclude that:

$$\frac{n_h}{2L^2} = q_h \frac{m_h}{2L^2} < \phi_c(h)(\xi_0) < q_h \frac{2m_h}{L^2} = \frac{2n_h}{L^2}$$

Thus (7) is verified for any  $0 < g \leq 1/2$  and  $G \geq 2$ .

On the other hand, when  $\eta_h = \Omega(1)$ , the sums in (11) provide, respectively, an upper bound and a lower bound to the local intensity. It turns out:  $\underline{\phi}_c(h) > \sum_k q_h s(\underline{D}_{0k}^h) \frac{\overline{U}(A_k^h)}{r_h^2} = \Theta(q_h \log m_h)$  and  $\overline{\phi}_c(h) < \sum_k q_h s(\overline{D}_{0k}^h) \frac{\underline{U}(A_k^h)}{r_h^2} = \Theta(q_h \log m_h s(D_c^h \sqrt{\log m_h}))$ . ■

## APPENDIX B

### PROOF OF THEOREM 4

We first observe that our construction of nested corridors guarantees that any cluster centre belonging to class  $h$  ( $h = 0 \dots H - 1$ ) stay at a distance at least  $D_h/4$  from any point belonging to corridor  $\mathcal{P}'_0$ . To simplify the geometry, we suppose that  $\mathcal{P}'_0$  has a perfect rectangular shape; however we emphasize that our arguments can be extended to the more general case. We have:

$$\begin{aligned} \int_{\mathcal{P}'_0} \Phi(\xi) d\xi &= \int_0^{D_0/2} \int_0^L \Phi(\xi_x, \xi_y) d\xi_y d\xi_x = \int_0^{D_0/2} \int_0^L \sum_j q(h_j) k_{h_j}(c_{jx}, c_{jy}, \xi_x, \xi_y) d\xi_y d\xi_x = \\ &= \sum_h \sum_{j:h_j=h} \int_0^{D_0/2} \left( \int_0^L q_h k_h(c_{jx}, c_{jy}, \xi_x, \xi_y) d\xi_y \right) d\xi_x \quad (12) \end{aligned}$$

In the last member of (12) we can observe that the quantity inside the brackets is constant with respect to the vertical component of the cluster's center position  $c_{jy}$ . Thus we can write:

$$\int_{\mathcal{P}'_0} \Phi(\xi) d\xi = \sum_h \sum_{j:h_j=h} F_h(d_{jx})$$

where  $d_{jx} = \inf_{\xi \in \mathcal{P}'_0} |c_{jx} - \xi_x|$  is the horizontal component of the distance between the cluster's center and points in  $\mathcal{P}'_0$ , and  $F_h(d_{jx}) = \int_0^{D_0/2} (\int_0^L q_h k_h(c_{jx}, c_{jy}, \xi_x, \xi_y) d\xi_y)$

Now we evaluate the contribution to the node density of a specific class  $h$ :

$$\int_{\mathcal{P}'_0} \phi_c(h)(\xi) d\xi = \sum_{j:h_j=h} F_h(d_{jx}).$$

First, we divide the whole domain  $\mathcal{O} \setminus \mathcal{P}_h$  into stripes  $\mathcal{P}_h^k$  parallel to  $\mathcal{P}_h$  of dimensions  $D_h \times L$  (i.e., congruent with  $\mathcal{P}_h$ ); then we upper-bound the contribution of clusters with center in every stripe by lower-bounding the horizontal component of the distance between the cluster's centres and points of  $\mathcal{P}'_0$ .

To simplify the notation we restrict ourselves to considering only clusters centres placed in the right half of the network area with respect to the cut  $\mathcal{P}'_0$  (the same can be done for clusters on the left half). The contribution of class  $h$  is:

$$\int_{\mathcal{P}'_0} \phi_c(h)(\xi) d\xi = \sum_{j:h_j=h} F_h(d_{jx}) \leq \sum_k N_h^k F_h(\underline{d}_x^k)$$

where  $N_h^k$  is the number of cluster's centres of class  $h$  falling within the  $k$ -th stripe  $\mathcal{P}_h^k$ , and  $\underline{d}_x^k = D_h/4 + kD_h$  is by construction the minimal distance between the  $k$ -th stripe and  $\mathcal{P}'_0$ . Applying corollary 1 we can conclude that w.h.p., uniformly over  $k$ ,  $N_h^k = \Theta(\phi_c(h)LD_h)$ . Thus, summing over all classes, we obtain:

$$\int_{\mathcal{P}'_0} \Phi(\xi) d\xi = O\left(\sum_h \phi_c(h)D_hL \sum_k F_h(D_h/4 + kD_h)\right)$$

After some calculations, it turns out that  $F_h(D_h/4 + kD_h) = \Theta(D_0D_h \frac{q_h}{r_h^2} s((D_h/4 + kD_h)/r_h))$ . Then it is easy to verify that  $\sum_h \sum_k F(D_h/4 + kD_h) = \Theta(\sum_h D_0D_h \frac{q_h}{r_h^2} s(D_h/r_h))$ . At last, recalling that by construction  $D_h^2 = 1/\phi_c(h)$  we have:

$$\int_{\mathcal{P}'_0} \Phi(\xi) d\xi = O\left(\sum_h LD_0 \frac{q_h}{r_h^2} s\left(\frac{D_h}{r_h}\right)\right)$$

Thus we obtain

$$\mathcal{E}_{\mathcal{P}'_0}[\Phi(\xi)] = \frac{\int_{\mathcal{P}'_0} \phi(\xi) d\xi}{\int_{\mathcal{P}'_0} d\xi} = O\left(\sum_h q_h \frac{D_h^{-\delta}}{r_h^{2-\delta}}\right)$$

■