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The Period of a free-swinging pendulum in adiabatic and non-adiabatic gravitational potential variations

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Abstract
An experiment aimed at the determination of the Newtonian constant of gravitation G is in progress at the Politecnico di Torino. This experiment is based on measurement of the period of a simple pendulum, when its gravitational potential is forced to cycle by moving two source masses between two positions, respectively, near and far from the oscillation plane. In previous papers various geometrical and physical effects were analysed, including internal, environmental and technical ones. Most effects are made irrelevant at the desired level of uncertainty (10^{-4}) by the common mode rejection provided by the differential measurement scheme. The remaining effects were shown to impose reasonable constraints of symmetry and geometrical accuracy, and are quickly reviewed here. In this paper the energy exchange between the oscillating mass and the gravitational potential is analysed, and its effect on the period is evaluated. It is shown that the uncertainty contribution of the latter does not constitute a problem for the desired 10^{-4} accuracy of G determination.

List of frequently used symbols

a	semi-distance between the source masses (figure 1)
a_N	value of a when the source masses are near the oscillating plane
a_F	value of a when the source masses are far from the oscillating plane
g	local acceleration of gravity
G	Newtonian constant of gravitation
I	moment of inertia of the pendulum ($I = mL^2$ when m is a point mass)
L	radius of gyration of the pendulum (figure 1)
m	oscillating mass
M	source masses
T_0	asymptotic period of the pendulum ($T_0 = 2\pi\sqrt{I/mgL}$)
V_M	volume of the source masses
θ	instantaneous swing angle
θ_0	swing amplitude of the pendulum
ρ_M	density of the source masses

1. Introduction

In the experiment working at the Politecnico di Torino a ball lens, suspended by two fibres, is excited to swing freely in a pendulum mode. During the ringdown, two source masses are periodically moved back and forth between a ‘near’ and a ‘far’ position, along a direction orthogonal to the oscillating plane, in order to modify the gravitational potential. By differential measurement of the pendulum period, it is possible to determine how much it changes due to such potential variations. Assuming the Newtonian model, a value of G can then be derived from measurements of mass and geometry. Figure 1 shows a schematic of the pendulum with the source masses [1].

The idea to use such a simple pendulum to determine the Newtonian constant of gravitation G is based on three considerations. One is the fact that the oscillating mass falls out of the equations, obliterating the need to measure it very accurately, and fostering miniaturization. The second is that such a pendulum can be realized with a high degree of ideality, as confirmed by the obtained Q-factor, minimizing in this way

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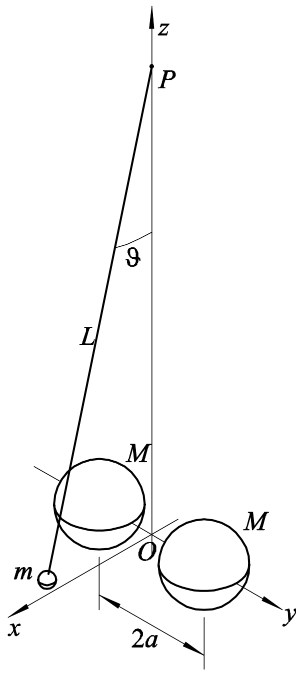

Figure 1. Oscillating mass m and source masses M layout.

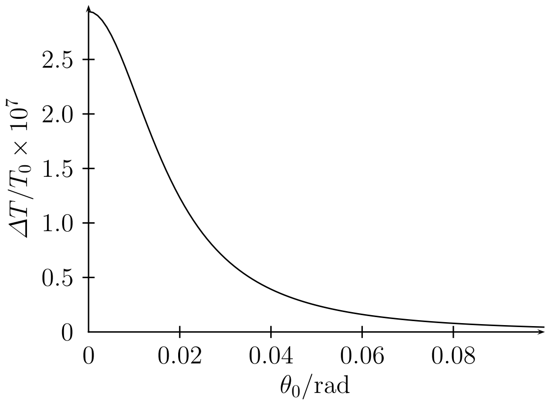

Figure 2. Fractional period shift caused by source masses displacement.

all systematics and pushing the uncertainty down to resolution limitations. The third is that, contrary to intuition, in measuring periods by timing zero crossing instants, much can be gained in resolution by minimizing the period duration, as flicker noise of voltage comparators improves and zero crossing speed increases at a given oscillation amplitude. Moreover, shorter periods reduce the sensed environmental noise, which has a peak in the low tenths of a hertz. All three accounts indicate an advantage for a pendulum over a Cavendish type torsion apparatus.

A quick review of systematics is given in the next section. It is argued that an accuracy of 10^{-4} on G appears obtainable as a result of already analysed contributions. In this paper, the question is considered whether the fashion in which source masses are moved may have an effect on G evaluation at the target level of accuracy through the energy that it may inject in the pendulum.

In fact, in all experiments based on swing time measurements, whether in a simple or a torsion pendulum, non-linearities, implied in the nature of the experiment or caused by non-idealities of some components, yield dependence of the natural frequency on the swing amplitude. An accurate analysis is then necessary in order to evaluate the systematic effect caused by amplitude variations. The latter in fact perturbs the swing period of interest and could represent a significant contribution to the final accuracy. The problem is analysed here first under the assumption of constant swing angle (which means no effect on the period), and then considering the total energy stored in the experiment as an invariant (adiabatic assumption). Moreover, three intermediate approaches, based, respectively, on an approximated numerical evaluation, on direct solution of a simplified version of the pendulum equation of motion and on the theory of adiabatic invariants, are also presented. In all methods, because of the high Q-factor, the effect on period due to the natural ringdown during a source masses cycle is neglected, for its fractional contribution is in the order of 10^{-14} on the period (smaller than 10^{-6} for G).

Since the period uncertainty evaluation, calculated by all approaches, yields a negligible contribution for the desired final accuracy, the solution based on constant swing angle represents the simplest method to reach the experiment aim, and, therefore, is used to estimate G from measured periods.

2. Experimental setup

The experiment is based on a 1 m long pendulum in vacuum; its natural frequency is about 0.5 Hz and the Q-factor is approximately 10^6. The oscillating mass is a 5 mm diameter ball lens, suspended by two 12 µm diameter Kevlar® fibres; the period is measured by an optical system, which senses the lens crossing of its low point placed on its resting direction.

Two 30 mm diameter gold spheres act as source masses, periodically moved, by two motorized stages, between a near position ($a = a_N = 18.7$ mm) and a far position ($a = a_F = 118.7$ mm). Both size and travel of the source masses are optimized in order to maximize signal within the existing constraints [2, 3]. With these parameters, the expected fractional period shift $\Delta T/T_0$, caused by source masses displacement, is shown in figure 2.

In previous papers [1–3] an extensive evaluation of accuracy contributions from various sources was discussed. For most of them the differential scheme shifts the burden on the stability of some physical quantity over the experiment's repetition period, which is set to 300 s. Among these are the effects which produce a variation in the length of the suspension fibres, such as fibre thermal expansion and oscillation driven variable tension and other medium and long term geometry variations. Since the adopted criteria of elaboration of the experimental data include linear drift removal through double differencing, only quadratic drift is relevant for accuracy in these cases. The most worrisome contribution of this type might be the fibre thermal expansion, which could have a quadratic term in relation to ambient temperature transients. However, feasible millikelvin temperature stabilization of the whole experimental apparatus takes this contribution below the 10^{-4} level for G.

Accuracy is instead relevant for effects which are synchronous with the repetition rate, such as geometry covariant electrostatic effects and positioning or density uniformity of the source masses. The former were eliminated altogether by interposing two electrostatic shields between the moving source masses and the swinging mass (which is then moving along the trench formed by the two shields). The latter must be guaranteed by construction. Positioning of the source masses must be known better than 3×10^{-5} of their distance in the near position for the desired 10^{-4} accuracy on G, since the signal depends on the cube of such distance (see equation (2)), and this is probably the most stringent mechanical accuracy requirement as it amounts to an uncertainty of $2\,\mu$m on the distance of the source masses. As for density uniformity, the net effect is to alter the effective distance of the source masses; therefore the same criteria must be applied. However, noble metals are well suited materials for the target uniformity and this is the reason why pure gold was used.

3. Mathematical model

It can be shown that the Lagrangian of the pendulum with two uniform spherical source masses placed symmetrically on either side of the swing plane, their centres of mass aligned horizontally across the centre of the oscillating mass at rest, is given by

$$\mathcal{L}(\theta, \dot{\theta}; a) = \frac{1}{2} I \dot{\theta}^2 - 2mgL[1 + 2\epsilon(a)h(\theta, a)] \sin^2 \frac{\theta}{2}, \quad (1)$$

with

$$\epsilon(a) = \frac{GML}{ga^3}, \quad (2)$$

$$h(\theta, a) = \frac{2}{\sqrt{\xi(\theta, a)} + \xi(\theta, a)}, \quad (3)$$

$$\xi(\theta, a) = 1 + \frac{4 \sin^2 \theta/2}{(a/L)^2}. \quad (4)$$

Since a changes periodically at the repetition rate $T_R \gg T_0$ in a quasi-trapezoidal way between near and far position, all the functions described above are actually time dependent.

From equation (1) it is possible to derive the equation of motion of the pendulum, which can be solved analytically after appropriate linearization. This approach is presented in section 7.

An alternative way to obtain the model of the pendulum is the following. The energy of the system is

$$E(a) = \frac{1}{2} I \left[\dot{\theta}^2 + 4\omega_0^2 [1 + 2\epsilon(a)h(\theta, a)] \sin^2 \frac{\theta}{2} \right], \quad (5)$$

where $\omega_0 = 2\pi/T_0 = \sqrt{mgL/I}$.

Equation (5) yields the differential equation

$$\dot{\theta} = \pm \left\{ \frac{2E(a)}{I} - 4\omega_0^2 [1 + 2\epsilon(a)h(\theta, a)] \sin^2 \frac{\theta}{2} \right\}^{1/2}, \quad (6)$$

which gives a calculable expression for the period of the pendulum, if suitable assumptions are made about the total energy stored in the pendulum and the energy exchange between the pendulum and the varying gravitational potential. Calculations are carried out here with four different approaches: the constant swing amplitude assumption (section 4), the adiabatic assumption, in which the stored energy is constant (section 5), the discrete approach (section 6) and the adiabatic invariant approach (section 8).

4. Constant swing amplitude

The simplest assumption consists in considering the swing amplitude θ_0 as an invariant. This represents the case in which the source masses are instantaneously shifted at exactly the time when $\theta(t) = \theta_0$.

From (5) with $\dot{\theta} = 0$, the energy stored in the system is then

$$E(a) = 2\omega_0^2 I [1 + 2\epsilon(a)h(\theta_0, a)] \sin^2 \frac{\theta_0}{2}. \quad (7)$$

In this case, the period of the pendulum can be written, by integrating equation (6), as

$$T(\theta_0, a) = \frac{T_0}{\pi} \int_0^{\theta_0} \left(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right)^{-1/2} (1 + 2\epsilon(a) w(\theta, \theta_0, a))^{-1/2} \, d\theta, \quad (8)$$

with

$$w(\theta, \theta_0, a) = \frac{2}{\xi(\theta_0, a)\sqrt{\xi(\theta, a)} + \xi(\theta, a)\sqrt{\xi(\theta_0, a)}}. \quad (9)$$

Since $|\epsilon(a) w(\theta, \theta_0, a)| \ll 1$ for $\theta \in [-\theta_0, \theta_0]$, equation (8) can be replaced by its first-order approximation, obtaining

$$T(\theta_0, a) = T_0[\alpha(\theta_0) - \epsilon(a)\beta(\theta_0, a)], \quad (10)$$

where

$$\alpha(\theta_0) = \frac{1}{\pi} \int_0^{\theta_0} \left(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right)^{-1/2} d\theta \quad (11)$$

is the usual elliptic integral, which describes the period dependence on swing amplitude for an unperturbed simple pendulum, and $\epsilon(a)\beta(\theta_0, a)$ is the perturbation due to the source masses, where

$$\beta(\theta_0, a) = \frac{1}{\pi} \int_0^{\theta_0} \left(\sin^2 \frac{\theta_0}{2} - \sin^2 \frac{\theta}{2} \right)^{-1/2} w(\theta, \theta_0, a) \, d\theta. \quad (12)$$

As equation (11) points out, considering θ_0 as an invariant (i.e. no amplitude modulation of the pendulum), implies no effect on period except for the one introduced by the varying gravitational potential and described by the perturbation term.

5. Constant energy assumption

In the alternative adiabatic assumption, the initial energy stored in the pendulum

$$E_0 = 2mgL\sin^2\left(\frac{\theta_i}{2}\right) \tag{13}$$

is an invariant, and θ_i is set in order to satisfy the initial energy value. This assumption is ideally represented by the instantaneous shifting of the source masses at exactly the time when $\theta(t) = 0$.

By integrating equation (6) with assumption (13), the period is

$$T(\theta_i, a) = \frac{T_0}{2\pi}\int_{-\theta_0(a)}^{\theta_0(a)}\left[\sin^2\frac{\theta_i}{2} - (1 + 2\epsilon(a)h(\theta, a))\sin^2\frac{\theta}{2}\right]^{-1/2} d\theta, \tag{14}$$

where $\theta_0(a)$ is the actual swing amplitude, which can be determined by solving the equation

$$\sin^2\frac{\theta_i}{2} - (1 + 2\epsilon(a)h(\theta_0, a))\sin^2\frac{\theta_0}{2} = 0. \tag{15}$$

This yields two solutions for equation (15), i.e. θ_{0N}, θ_{0F}, which can be combined to obtain an increment $\Delta\theta$ of the swing angle caused by the instantaneous perturbation of the gravitational potential. Then, the swing amplitude change induced by the displacement of source masses affects additionally the period of the pendulum. An evaluation of this effect is then necessary to guarantee the desired accuracy on the determination of G.

Since it is possible to demonstrate that equation (14) meets equation (8) under the substitution $\theta_0 = \theta_0(a)$, the effect on the period can be evaluated by using approximation (10), in which the dependence on varying swing angle is expressed by its Taylor's series. Considering only its linear term, as $\theta_0 \approx \theta_i$, yields

$$T(\theta_i, a) = T_0\left[\alpha(\theta_i) + (\theta_0 - \theta_i)\left.\frac{\partial\alpha}{\partial\theta_0}\right|_{\theta_0=\theta_i} - \epsilon(a)\left(\beta(\theta_i, a) + (\theta_0 - \theta_i)\left.\frac{\partial\beta}{\partial\theta_0}\right|_{\theta_0=\theta_i}\right)\right]. \tag{16}$$

Evaluating equation (16) with $\theta_0 = \theta_{0N}$ and $\theta_0 = \theta_{0F}$ and neglecting the contribution $\Delta\theta\epsilon(a)\partial\beta/\partial\theta_0$, first-order approximation of the effect on period is given by

$$\Delta T_{CE} = \frac{T_0}{8}\theta_i\Delta\theta = \frac{T_0}{8}\theta_i(\theta_{0N} - \theta_{0F}). \tag{17}$$

An evaluation of the fractional effect $\Delta T_{CE}/T_0$ for $\theta_i \in [0, 0.1]$ rad is shown in figure 3. For all practical values of θ_i, the fractional effect on period is lower than 2.5×10^{-11}, which allows one to neglect the adiabatic contribution if the desired accuracy level for the evaluation of G is not better than 10^{-4}.

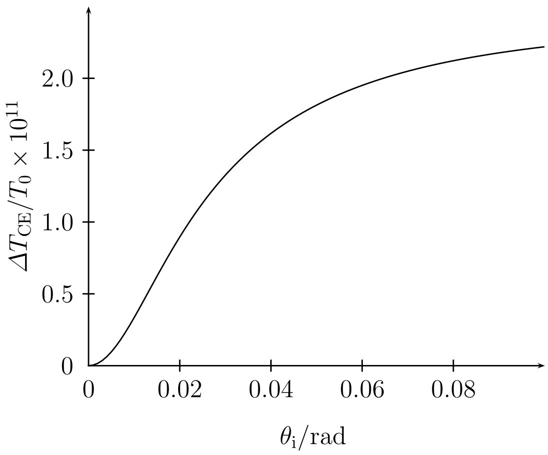

Figure 3. Fractional effect of adiabaticity on period under constant energy assumption.

6. Discrete approach

As the source masses are not instantaneously moved, the constant energy approach can be refined by considering a discrete-time evolution of the pendulum dynamics and the masses shifting. The angle θ_0 can be considered a generic function of θ_i and a:

$$\theta_0(t) = f(\theta_i, a(t)). \tag{18}$$

An infinitesimal increment of (18) is given by

$$\delta\theta_0(t) = \frac{\partial f}{\partial a}\frac{\partial a}{\partial t}\delta t, \tag{19}$$

where $\partial a/\partial t$ represents the source masses velocity. An infinitesimal contribution to the period change caused by the moving masses can be described, as in equation (17), by

$$dT_{DA} = \frac{T_0}{8}\theta_0(t)\frac{\partial\theta_0}{\partial a}\frac{\partial a}{\partial t}\delta t, \tag{20}$$

and then

$$\Delta T_{DA} = \frac{T_0}{8}\int_0^{\tau_M}\theta_0(t)\frac{\partial\theta_0}{\partial a}\frac{\partial a}{\partial t}\,dt, \tag{21}$$

where τ_M is the travel time of source masses.

Figure 4 shows results of the numerical evaluation versus travel time, considering three swing amplitudes, i.e. $\theta_i = 0.1$ rad, $\theta_i = 0.05$ rad, $\theta_i = 0.01$ rad. When τ_M approaches 0, the effect is the same as the instantaneous shifting, shown in figure 3.

7. Analytical approach

With the Lagrangian approach, the effect of the varying gravitational potential on the pendulum dynamics can be obtained by solving the equation of motion of the pendulum. This can be derived from equation (1), obtaining

$$\ddot{\theta} + \omega_0^2\left\{4\epsilon(a)\frac{\partial h(\theta, a)}{\partial\theta}\sin^2\frac{\theta}{2} + \sin\theta[1 + 2\epsilon(a)h(\theta, a)]\right\} = 0. \tag{22}$$

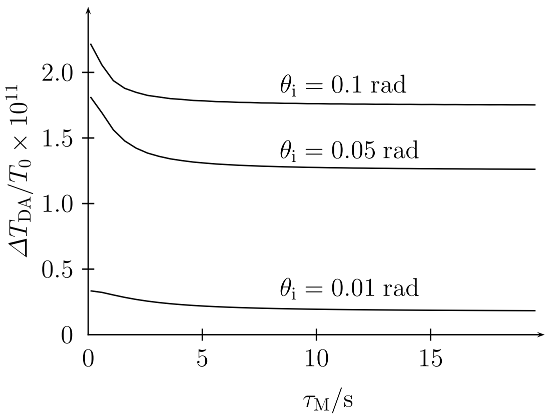

Figure 4. Discrete approach evaluation of the fractional effect of adiabaticity on period.

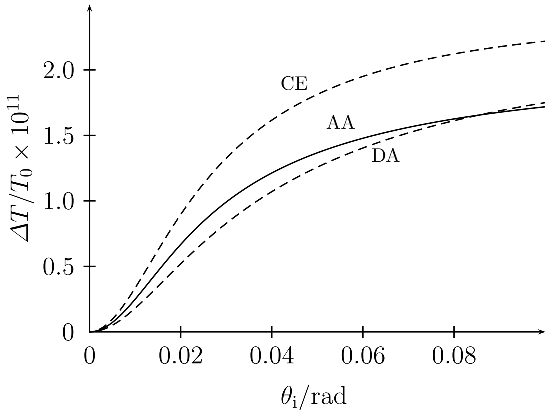

Figure 5. A comparison between the fractional effect evaluated by the constant energy approach (dashed upper curve, CE), the discrete approach (dashed lower curve, DA) and the analytical approach (continuous curve, AA).

Since the first term in braces is $o(\theta)$ and $h(\theta, a) = O(1)$, the last equation can be simplified by considering its first-order approximation:

$$\ddot{\theta} + \omega_0^2[1 + 2\epsilon(a)]\theta = 0. \tag{23}$$

By considering a periodic source masses shifting, equation (23) can be rewritten in the form of Hill's equation

$$\ddot{\theta} + [\omega_0^2 + 2q\chi(t)]\theta = 0, \tag{24}$$

where q is a constant and $\chi(t)$ is a periodic function [4].

The simplest version of Hill's equation is the Mathieu equation ($\chi(t) = \cos(t)$). Imposing a cosine-type source masses effect, the term $\epsilon(a)$ of equation (23) can be rewritten in the form

$$\epsilon(a) = \epsilon_{\mathrm{M}} + \epsilon_{\mathrm{D}}\cos(\omega_{\mathrm{M}}t), \tag{25}$$

with $\omega_{\mathrm{M}} = 2\pi/\tau_{\mathrm{M}}$, and

$$\epsilon_{\mathrm{M}} = \frac{\epsilon_{\mathrm{F}} + \epsilon_{\mathrm{N}}}{2}, \qquad \epsilon_{\mathrm{D}} = \frac{\epsilon_{\mathrm{F}} - \epsilon_{\mathrm{N}}}{2}, \tag{26}$$

where $\epsilon_{\mathrm{N}} = \epsilon(a_{\mathrm{N}})$ and $\epsilon_{\mathrm{F}} = \epsilon(a_{\mathrm{F}})$.

The equation of motion is then

$$\ddot{\theta} + \omega_0^2[1 + 2\epsilon_{\mathrm{M}} + 2\epsilon_{\mathrm{D}}\cos(\omega_{\mathrm{M}}t)]\theta = 0, \tag{27}$$

A solution of equation (27), since $\epsilon_{\mathrm{M}} \ll 1$ and $\epsilon_{\mathrm{D}} \ll 1$, arises from Riccati's equation

$$\ddot{\theta} + \omega_0^2\rho^2(t)\theta = 0, \tag{28}$$

with $\rho^2(t) = 1 + 2\epsilon_{\mathrm{M}} + 2\epsilon_{\mathrm{D}}\cos(\omega_{\mathrm{M}}t)$, which yields

$$\begin{aligned}\theta(t) &= \frac{B}{\sqrt{\rho(t)}}\cos\left[\omega_0\left(\int_0^t \rho(\tau)\,\mathrm{d}\tau\right)\right] \\ &\approx B\left(1 - \frac{1}{2}\left(\epsilon_{\mathrm{M}} + \epsilon_{\mathrm{D}}\cos(\omega_{\mathrm{M}}t)\right)\right) \\ &\quad\times \cos\left[\omega_0(1 + \epsilon_{\mathrm{M}})t + \frac{\omega_0\epsilon_{\mathrm{D}}}{\omega_{\mathrm{M}}}\sin(\omega_{\mathrm{M}}t)\right],\end{aligned}$$

where B is set in order to satisfy initial conditions.

In (29) it is possible to identify an amplitude modulation coefficient, which is responsible for the adiabatic effect, and a frequency modulation term, already derived in the other approaches. An evaluation of equation (29) with $\theta_0 = 0.1$ rad shows a period shift effect, caused by amplitude modulation, of 30 ps (1.5×10^{-11} fractional), according to the results deduced from figure 4.

A more general model can be derived from equation (27) by substituting the cosine-type law with the Fourier series expansion of a generic periodic function $\chi(t)$, and then applying the superposition rule. For example, imposing a square wave type source masses position cycle, the equations that have to be solved are

$$\ddot{\theta} + \omega_0^2\left[1 + 2\epsilon_{\mathrm{M}} + 2\epsilon_{\mathrm{D}}\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos((2k+1)\omega_{\mathrm{M}}t)\right]\theta = 0. \tag{29}$$

Evaluated solutions of equation (29), with $\theta_0 = 0.1$ rad, yield a period shift of 32 ps, which confirms the discrete approach, as it amounts to a relative error of 1.6×10^{-11}.

A comparison between the fractional effect on period $\Delta T/T_0$ evaluated by the three approaches is shown in figure 5.

8. Adiabatic invariant approach

Consider a system with one degree of freedom, parametrized by λ, and let q and p be the generalized coordinate and momentum, respectively, and Λ be an integration path over the complete range of variation of the coordinate during the period; then, according to the theory of adiabatic invariants, the integral

$$J = \frac{1}{2\pi}\oint_{\Lambda} p\,\mathrm{d}q \tag{30}$$

is an invariant for slow variations of λ [5].

Since, in this experiment, the source masses shifting is 20 times slower than the period of the pendulum, the assumption to consider J as an invariant is applicable [6].

From equations (5) and (13), the adiabatic integral is given by

$$J = \frac{2I\omega_0}{\pi}\int_{-\theta_0}^{\theta_0}\left[\sin^2\frac{\theta_\mathrm{i}}{2} - (1+2\epsilon(a)h(\theta,a))\sin^2\frac{\theta}{2}\right]^{1/2}\mathrm{d}\theta. \tag{31}$$

An approximation of the integral (31) is represented by the model of a harmonic oscillator with varying natural frequency, described by

$$J \approx \frac{2}{\pi}\int_0^{\theta_0}\sqrt{2IE_{\mathrm{N,F}} - I^2\omega_0^2(1+2\epsilon_{\mathrm{N,F}})\theta^2}\,\mathrm{d}\theta = \frac{E_{\mathrm{N,F}}}{\omega_0(1+2\epsilon_{\mathrm{N,F}})}, \tag{32}$$

where $E_{\mathrm{N,F}}$ are the values of energy evaluated when $a(t) = a_\mathrm{N}$ and $a(t) = a_\mathrm{F}$ respectively, and $\epsilon_{\mathrm{N,F}}$ are defined in section 7.

By evaluating the energy variation ΔE due to the switching between near and far source masses position (i.e. replacing ϵ_N by ϵ_F), it is possible to calculate the imposed amplitude modulation and then determine the effect on the swing period. Calculated values confirm the results obtained by the analytical approach.

9. Conclusion

In this paper, the effect on the period of a simple pendulum produced by forced variations of gravitational conditions under adiabatic and non-adiabatic assumptions is discussed. This is caused by non-linearities, which perturb the isochronism, and link the period to the swing amplitude. Four approaches are considered, namely, a worst case, a discrete approximation, an analytical analysis and an adiabatic invariant approach. The evaluated contribution shows a negligible effect on the pendulum period, if the desired relative accuracy is not better than 10^{-11} on the period or 10^{-4} on G.

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