Cramér-Rao, Bound for Hybrid Peer-to-Peer Positioning

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I. Introduction

Novel cooperative positioning methods have been proposed to operate in GPS-challenged environments. However, such cooperative schemes can be also used in combination with GPS, so as to improve positioning accuracy in cases where GPS measurements are available (i) intermittently; or (ii) from a limited number of satellites; or (iii) are strongly affected by noise or multi-path. "Hybrid cooperative positioning" schemes can thus be designed to fuse information from peers and from GPS satellites. This contribution provides a theoretical characterization of achievable performance of hybrid cooperative positioning, by expressing the Cramér-Rao lower bound (CRLB) for the aforementioned scenario. Our results extend [**?**], by including the unknown clock bias, and [**?**], by taking into account satellites in addition to terrestrial devices.

II. Problem Formulation

Given a heterogeneous network (Fig. **??**), comprising satellites with known clock bias and known position, anchor nodes with known position but unknown clock bias, and agents with unknown clock bias and unknown position. Let $\mathcal{M}$ be the set of agents, $\mathcal{S}$ the set of satellites, $\mathcal{A}$ the set of anchors; denote by $\mathcal{S}_m$ the set of satellites that agent m can see, by $\mathcal{A}_m$ the set of anchors that agent m can communicate with, and by $\mathcal{M}_m$ the set of peers it can communicate with. The position of a satellite $s \in \mathcal{S}$, an anchor $a \in \mathcal{A}$, and an agent $m \in \mathcal{M}$, are indicated respectively by $\mathbf{x}_s$, $\mathbf{x}_a$, $\mathbf{x}_m$. Our focus will be on 2-dimensional positioning, from which the extension to 3-dimensional case is straightforward. The variable b_m represents the clock bias of agent m, expressed in distance units.

Two types of measurements are available to agent m: $r_{n\to m}$ is the measured distance between agent m and node $n \in \mathcal{A}_m \cup \mathcal{M}_m$, with $r_{n\to m} = \|\mathbf{x}_n - \mathbf{x}_m\| + v_{n\to m}$, where $v_{n\to m}$ is measurement noise; $\rho_{s\to m}$ is a pseudorange measurement between node m and satellite $s \in \mathcal{S}_m$, $\rho_{s\to m} = \|\mathbf{x}_s - \mathbf{x}_m\| + b_m + v_{s\to m}$, where $v_{s\to m}$ is measurement noise.We assume that: (i) all measurement noise is zero-mean Gaussian; (ii) for peer-to-peer measurements, the link variance is symmetric: $\sigma^2_{n\to m} = \sigma^2_{m\to n}$.

Our goal is to compute the CRLB of the deterministic unknown $[\mathbf{X}, \mathbf{b}]$, where $\mathbf{X} = \{\mathbf{x}_{m\in\mathcal{M}}\}$ and $\mathbf{b} = \{b_{m\in\mathcal{M}}\}$, as a function of the (range and pseudorange) measurement noise variances $\sigma^2_{a\to m}$, $\sigma^2_{n\to m}$, $\sigma^2_{s\to m}$, and of the network geometry.

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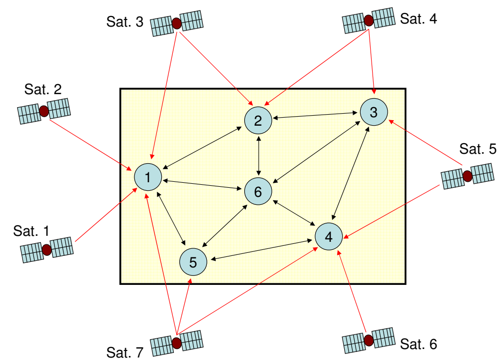

Figure 1. Example network topology. Agents' positions (in m): 1: [-5 1]; 2: [0 3]; 3: 5 4]; 4: [3 -2]; 5: [-3 -4]; 6: [0 0]. Satellites' positions (in m): 1: [-50 -10]; 2: [-50 20]; 3: [-20 50; 4: [12 50; 5: [50 0]; 6: [10 -50]; 7: [-10 -50]. Measurement noise: $\sigma_{s\to m} = 3$m $\forall m \in \mathcal{M}, s \in \mathcal{S}_m$; $\sigma_{n\to m} = 0.10$m $\forall m \in \mathcal{M}, n \in \mathcal{M}_m$.

III. Fisher Information Matrix

The CRLB of any unbiased estimator of $[\mathbf{X}, \mathbf{b}]$ is obtained by inverting the corresponding Fisher information matrix (FIM). Let $\mathbf{F}$ the FIM for our hybrid scenario. We will first compute the FIM under a non-cooperative setting, and then extend this result to the cooperative case.

A. Non-cooperative Case

We focus on a single agent, say m. Then the log-likelihood function of its measurements with respect to anchors and satellites is

$$\begin{aligned}
&\log p\left(\{r_{a\to m}\}_{a\in\mathcal{A}_m}, \{\rho_{s\to m}\}_{s\in\mathcal{S}_m} | \mathbf{x}_m, b_m\right) \\
&= \sum_{a\in\mathcal{A}_m} \log p\left(r_{a\to m} | \mathbf{x}_m\right) + \sum_{s\in\mathcal{S}_m} \log p\left(\rho_{s\to m} | \mathbf{x}_m, b_m\right) \\
&\doteq \Lambda_m\left(\mathbf{x}_m, b_m\right).
\end{aligned}$$

The Fisher information matrix is given by

$$\mathbf{F}_m = -\mathbb{E}\left\{ H_m\left(\Lambda_m\left(\mathbf{x}_m, b_m\right)\right)\right\},$$

where the expectation is with respect to the measurements, and $H_m(\cdot)$ is the Hessian operator containing the second-order

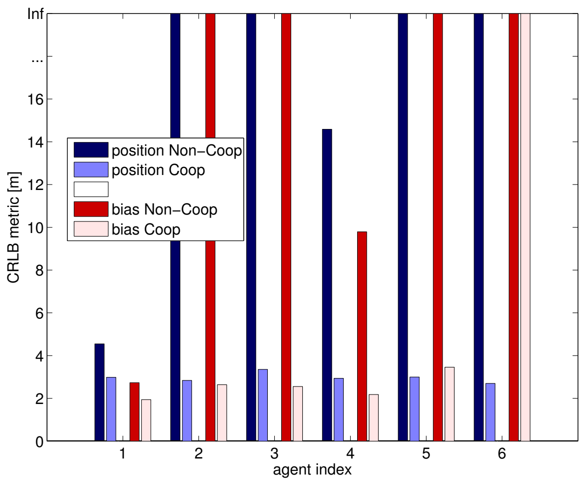

Figure 2. Comparison of position- and bias-CRLB: cooperative vs. non-cooperative setting.

partial derivatives with respect to each element of $[\mathbf{x}_m, b_m]$. In case of 2D positioning $\mathbf{F}$ is a 3 by 3 matrix:

$$\mathbf{F}_m = \begin{bmatrix} \mathbf{F}_{\mathbf{x}_m} & \mathbf{f}_{\mathbf{x}_m,b_m} \\ \mathbf{f}^T_{\mathbf{x}_m,b_m} & F_{b_m} \end{bmatrix} \succeq 0, \quad (1)$$

where

$$\begin{aligned} \mathbf{F}_{\mathbf{x}_m} &= \sum_{a\in\mathcal{A}_m} \frac{1}{\sigma^2_{a\to m}} \mathbf{q}_{am}\mathbf{q}^T_{am} + \sum_{s\in\mathcal{S}_m} \frac{1}{\sigma^2_{s\to m}} \mathbf{q}_{sm}\mathbf{q}^T_{sm} \\ F_{b_m} &= \sum_{s\in\mathcal{S}_m} \frac{1}{\sigma^2_{s\to m}} \\ \mathbf{f}_{\mathbf{x}_m,b_m} &= \sum_{s\in\mathcal{S}_m} -\frac{1}{\sigma^2_{s\to m}} \mathbf{q}_{sm}, \end{aligned}$$

in which $\mathbf{q}_{im} = \frac{\mathbf{x}_i - \mathbf{x}_m}{\|\mathbf{x}_i - \mathbf{x}_m\|}$ is a unit-length column vector between $\mathbf{x}_m$ and $\mathbf{x}_i$. Considering all M agents, the global non-cooperative FIM is a block-diagonal matrix as

$$\mathbf{F}_{\text{non-coop}} = \begin{bmatrix} \mathbf{F}_1 & & \\ & \ddots & \\ & & \mathbf{F}_M \end{bmatrix}. \quad (2)$$

B. Cooperative Case

The log-likelihood function is now

$$\begin{aligned} &\log p\left(\{r_{n\to m}\}_{n\in\mathcal{A}_m\cup\mathcal{M}_m}, \{\rho_{s\to m}\}_{s\in\mathcal{S}_m} \left| \mathbf{x}_m, b_m \right.\right) \\ &= \sum_{m\in\mathcal{M}} \Lambda_m(\mathbf{x}_m, b_m) + \underbrace{\sum_{m\in\mathcal{M}} \sum_{n\in\mathcal{M}_m} \log p\left(r_{n\to m} \left| \mathbf{x}_m \right.\right)}_{\doteq \Lambda_{\text{coop}}(\mathbf{X})}. \end{aligned}$$

The Fisher information matrix is of the form

$$\mathbf{F} = \mathbf{F}_{\text{non-coop}} + \mathbf{F}_{\text{coop}} \quad (3)$$

and has dimension $3M \times 3M$, for M agents. The first term $\mathbf{F}_{\text{non-coop}}$, representing the non-cooperative contribution, is given in (**??**), while the cooperative part $\mathbf{F}_{\text{coop}}$ can be expressed as

$$\mathbf{F}_{\text{coop}} = -\mathbb{E}\left\{H(\Lambda_{\text{coop}}(\mathbf{X}))\right\},$$

where $H(\cdot)$ is again the Hessian operator. Notice first of all that $\Lambda_{\text{coop}}(\mathbf{X})$ does not depend on the bias. Under the hypothesis of Gaussian measurement noise in peer-to-peer communication, the result is a block matrix of the form

$$\mathbf{F}_{\text{coop}} = \begin{bmatrix} \mathbf{F}'_1 & \mathbf{0} & \mathbf{K}_{12} & \mathbf{0} & \ldots & \mathbf{K}_{1M} & \mathbf{0} \\ \mathbf{0}^T & 0 & \mathbf{0}^T & 0 & & \mathbf{0}^T & 0 \\ \mathbf{K}_{21} & \mathbf{0} & \mathbf{F}'_2 & \mathbf{0} & & & \\ \mathbf{0}^T & 0 & \mathbf{0}^T & 0 & & & \\ \vdots & & & & \ddots & & \\ \mathbf{K}_{M1} & \mathbf{0} & & & & \mathbf{F}'_M & \mathbf{0} \\ \mathbf{0}^T & 0 & & & & \mathbf{0}^T & 0 \end{bmatrix} \succeq 0, \quad (4)$$

where

$$\begin{aligned} \mathbf{F}'_m &= \sum_{n\in\mathcal{M}_m} \frac{1}{\sigma^2_{n\to m}} \mathbf{q}_{nm}\mathbf{q}^T_{nm} \\ \mathbf{K}_{mn} &= \begin{cases} -\frac{1}{\sigma^2_{n\to m}} \mathbf{q}_{nm}\mathbf{q}^T_{nm}, & \text{if } n \in \mathcal{M}_m \\ \mathbf{0} & \text{otherwise.} \end{cases} \end{aligned}$$

The above results allow to compute $\mathbf{F}$ for a given network configuration and, by inverting (**??**), to express the CRLB.

IV. Numerical Results

We consider a representative indoor GPS-challenged scenario with 6 agents, no anchors and 7 satellites, where only agents close to windows can see satellites (Fig. **??**). Observe that agents 1 and 4 see enough satellites for positioning without cooperation, while agent 6 sees no satellites at all. In Fig. **??**, we show for a non-cooperative and cooperative setting, (expressed in meters) $\sqrt{\text{Tr}(\mathbf{J}_{\mathbf{x}_m})}$ and $\sqrt{J_{b_m}}$, where $\mathbf{J}_{\mathbf{x}_m}$ and J_{b_m} are, respectively, the m-th position- and bias-related blocks of $\mathbf{J}$, that is the CRLB matrix obtained by inversion of $\mathbf{F}_{\text{non-coop}}$ (**??**) or $\mathbf{F}$ (**??**) after removing rows and columns corresponding to non-estimable variables[1]. As expected, cooperation allows a significant performance improvement in positioning accuracy: without cooperation, only node 1 and node 4 have a finite CRLB; in the cooperative case, the CRLB is finite for all nodes, with lower values for nodes 1 and 4 compared to the previous ones. Notice that at least one satellite connection is necessary for estimating the bias. This explains why the bias-CRLB of node 6 is still infinite even with cooperation, while it is finite for node 5 thanks to its connection with satellite 7.

V. Conclusion and Further Work

The results derived in this paper give insight into the gains in hybrid cooperative positioning, node placement strategies, and network blind spots. We plan a detailed investigation of hybrid networks for realistic operational scenarios.

[1]Non-estimable variables are: positions and biases, for nodes whose total number of connections is less than 3; biases, for nodes connected to no satellites. These variables generate matrix singularities, hence CRLB $\to \infty$.

References

[1] Larsson, E. G, "Cramér-Rao Bound Analysis of Distributed Positioning in Sensor Networks," *IEEE Signal Processing Letters*, 2004, vol. 11, part 3, pp. 334-337

[2] Y. Shen, H. Wymeersch, and M. Z. Win, "Fundamental Limits of Wideband Cooperative Localization via Fisher Information", *IEEE Wireless Communications and Networking Conference (WCNC)*, pp.3951-3955, March 2007