Response Variability of High-Speed Interconnects via Hermite Polynomial Chaos

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Abstract

This paper focuses on the stochastic analysis of dynamical circuits via the Hermite Polynomial Chaos theory. The proposed approach facilitates the inclusion of external uncertainties, like tolerances or process variations, in the circuit analysis. The mechanics of the method amounts to expanding the output variables into a sum of a limited number of orthogonal basis functions and generating an extended matrix for the modified nodal analysis. The advocated method, while providing accurate results, turns out to be more efficient than the classical Monte Carlo technique in determining the circuit response sensitivity to parameters variability. A realistic application example involving a coupled-line structure concludes the paper.

1 Introduction

The availability of simulation techniques for the stochastic analysis of high-speed digital links in the early design phase is highly desirable. Even if a circuit equivalent of the link is available, the effects of the possible uncertainties in the circuit parameters due to the manufacturing process or the temperature variation need to be taken into account for the realistic prediction of the system performance.

Within this framework, the typical resource allowing to collect some quantitative information on the statistical behavior of the circuit response is based on the application of the brute-force Monte Carlo (MC) method, or possible complementary methods based on the optimal selection of the subset of model parameters in the whole design space [1]. Theses methods, however, are computationally expensive, thus preventing their application to the analysis of complex realistic structures.

Recently, an effective solution that overcomes the previous limitation has been proposed. It focuses on the polynomial chaos theory [2, 3, 4] and on the representation of the stochastic solution of a dynamical circuit in terms of orthogonal polynomials [5, 6]. This idea allows to readily extend the basic tools for the circuit analysis like the modified nodal analysis (MNA) and to describe the stochastic behavior of the circuit responses with arbitrary distribution by means of an analytical formula.

In this paper, the proposed approach is illustrated by means of a simple example and its strength is shown on a realistic interconnect structure.

2 Hermite polynomial chaos

This Section provides a quick overview of the mathematical background allowing to understand the application of the proposed method to the stochastic analysis of a dynamical circuit.

The idea underlying the polynomial chaos resides in the representation of a stochastic process via the sum of orthogonal basis functions. Within this framework, a *generic* stochastic process Y can be approximated by means of the following truncated series

$$Y(\epsilon) = \sum_{k=0}^{P} Y_k \cdot \phi_k(\epsilon) \tag{1}$$

where $\{\phi_k\}$ are Hermite polynomials expressed in terms of the standard Gaussian variable ϵ with zero mean and unit variance and $\{Y_k\}$ are the linear coefficients of the expansion. As an example, the first three polynomial terms are $\phi_0 = 1, \phi_1 = \epsilon, \phi_2 = (\epsilon^2 - 1)$ where ϕ_0 accounts for the deterministic behavior of Y.

For a given process, approximation (1) is defined by the number of terms P (limited within the range $2 \div 5$ for practical applications) and by the expansion coefficients that are computed via the projection of Y onto the orthogonal components ϕ_0, ϕ_1, ... The orthogonal relation of the Hermite polynomials takes the form

$$< \phi_k, \phi_j > = < \phi_k^2 > \delta_{kj} \tag{2}$$

where δ_{kj} is the Kronecker delta and $< \cdot, \cdot >$ denotes the inner product in the Hilbert space of the variable ϵ defined by

$$< \phi_k, \phi_j > = \int_{-\infty}^{+\infty} \phi_k(\epsilon)\phi_j(\epsilon) \exp(-\epsilon^2/2)/(\sqrt{2\pi}) d\epsilon. \tag{3}$$

Readers are referred to [2, 3, 4] and references therein for a comprehensive and formal discussion of polynomial chaos, including the generalization of (1) to multiple random variables.

3 Stochastic analysis of a dynamical circuit

This section discusses the modification of the basic MNA tool allowing for the analysis of a dynamical circuit that includes the effects of the statistical variation of circuit parameters via the polynomial chaos theory.

For the sake of simplicity, the discussion is based on the simple RC circuit shown in Fig. 1. In this example, both the conductance and the capacitance are assumed to be Gaussian random variables defined by

$$\left\{ \begin{array}{lcl} G & = & G_0 + G_1\epsilon \\ C & = & C_0 + C_1\epsilon \end{array} \right. \tag{4}$$

where ϵ is a the standard normal distribution with zero mean and unit variance ($G_0 = 1$ S, $C_0 = 1$ F, $G_1 = 1/5$ S, $C_1 = 1/10$ F; scaled values are used for demonstration purposes).

If needed, equation (4) can be suitably modified to account for non-gaussian stochastic distributions via (1) as well as for the dependence of the circuit parameters to multiple random variables.

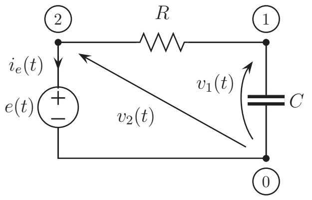

Figure 1: Tutorial example for the illustration of stochastic analysis application to dynamical circuits.

Frequency-domain analysis. For the deterministic case, the MNA equation in the Laplace domain for the example of Fig. 1 is

$$(\mathbf{G} + s\mathbf{C})\mathbf{X}(s) = \mathbf{U}(s) \tag{5}$$

where $\mathbf{X} = [V_1(s)\ V_2(s)\ I_e(s)]^T$, $\mathbf{X} = [E(s)\ 0\ 0]^T$ and

$$\mathbf{G} = \begin{bmatrix} G & -G & 0 \\ -G & G & 1 \\ 0 & 1 & 0 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} C & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

The inclusion of stochastic variations (4) and the expansion of the unknown variables of $\mathbf{X}$ in terms of second-order Hermite polynomials, leads to a modified version of (5)

$$\begin{aligned}(\mathbf{G}_0 + \mathbf{G}_1\epsilon + s\mathbf{C}_0 + s\mathbf{C}_1\epsilon)(\mathbf{X}_0(s)+ \\ +\mathbf{X}_1(s)\epsilon + \mathbf{X}_2(s)(\epsilon^2 - 1)) \ = \ \mathbf{U}(s)\end{aligned} \tag{7}$$

where the interpretation of the new matrices is straightforward.

Projection of (7) on the first three Hermite polynomials leads to the following augmented system, where the random variable ϵ does not appear,

$$\left(\begin{bmatrix} \mathbf{G}_0 & \mathbf{G}_1 & 0 \\ \mathbf{G}_1 & \mathbf{G}_0 & 2\mathbf{G}_1 \\ 0 & 2\mathbf{G}_1 & 2\mathbf{G}_0 \end{bmatrix} + s \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_1 & \mathbf{0} \\ \mathbf{C}_1 & \mathbf{C}_0 & 2\mathbf{C}_1 \\ \mathbf{0}_1 & 2\mathbf{C}_1 & 2\mathbf{C}_0 \end{bmatrix} \right) \begin{bmatrix} \mathbf{X}_0 \\ \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{U} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{8}$$

The above equation belongs to the same class of (5) and can be solved in the frequency domain via direct matrix inversion, thus effectively providing the expansion coefficients of the stochastic approximation of the circuit of Fig. 1.

It is worth noticing that the proposed method involves the solution of the augmented MNA problem (8), which is $(P+1)$ times larger than the deterministic system (5). However, for small values of P (as typically occurs in practice) the additional overhead due to matrix inversion is much less than the time required to run a large number of MC simulations. What is more important, the solution of (8) via matrix inversion allows to compute the quantitative information on the spreading of the circuit responses, and the related probability density function (PDF).

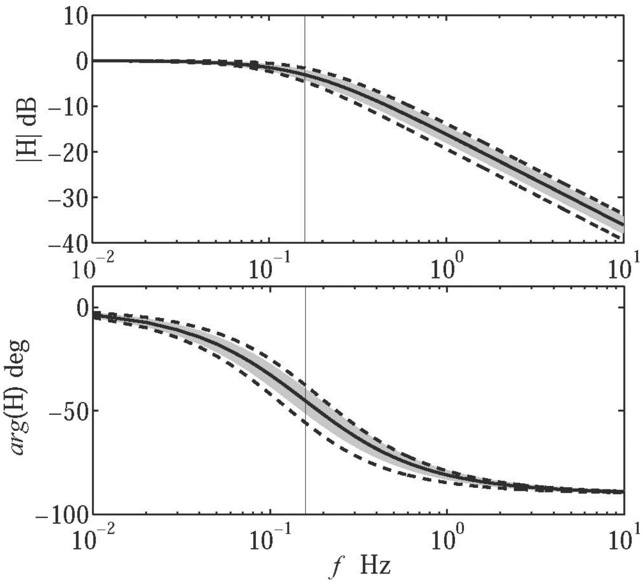

Figure 2: Bode plots (magnitude and phase) of the transfer function $H(s) = V_1(s)/E(s)$ for the example of Fig. 1. Solid black line: deterministic response; dashed lines: 3σ tolerance limit of the second order polynomial chaos expansion (9); solid gray lines: 100 responses obtained by means of the MC method.

As an example, Fig. 2 shows the Bode plot (magnitude and phase) of the transfer function $H(s) = V_1(s)/E(s)$ where

$$V_1(s) = V_{1,0}(s) + V_{1,1}(s)\epsilon + V_{1,2}(s)(\epsilon^2 - 1) \tag{9}$$

is the second order polynomial chaos expansion of the unknown voltage V_1, according to (7); the coefficients $V_{1,0}$, $V_{1,1}$ and $V_{1,2}$ are obtained from the solution of (8). The dashed curves of Fig. 2 represent the numerically-computed $\pm 3\sigma$ interval of the transfer function. For comparison, the deterministic response with nominal values of the circuit elements is reported in Fig. 2 as a solid line; also, a limited set of MC simulations (100, in order not to clutter the figure) are plotted as gray lines. Clearly, the dashed curves of Fig. 2 provide only a qualitative information of the spread of responses due to parameters uncertainty. A better quantitative prediction is only possible from the knowledge of the actual PDF of network responses. This fact can be clearly appreciated in Figure 3, by comparing the PDF of $|V_1(j\omega)|$ computed for ω=1 rad/s over a very large number of MC simulations, and the distribution obtained form the analytical expansion (9). The good agreement between the two curves and, in particular, the accuracy in predicting the left tail of the reference distribution, confirms the potential of the proposed method. In addition, for this simple example, it is also clear that a polynomial chaos expansion with three terms seems already accurate enough to capture the dominant statistical information of the system response.

Time-domain analysis. In order to compute the transient response of a dynamical circuit that includes the stochastic variation of parameters, two possible approaches are available. The simplest approach is the direct conversion of equation (8) in

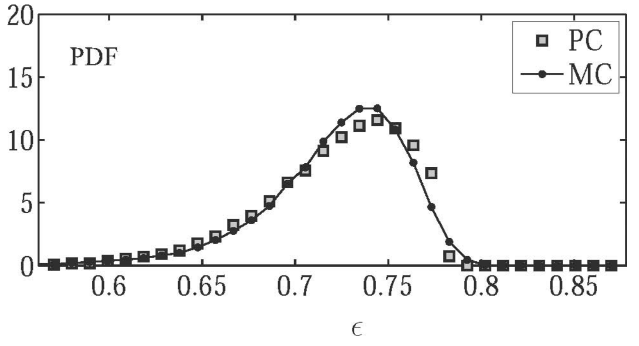

Figure 3: Probability density function of $|V_1(j\omega)|$, $\omega = 1$ rad/s. The two distributions correspond to the reference curve obtained via 40000 MC simulations and via the second order polynomial chaos expansion (9) (PC).

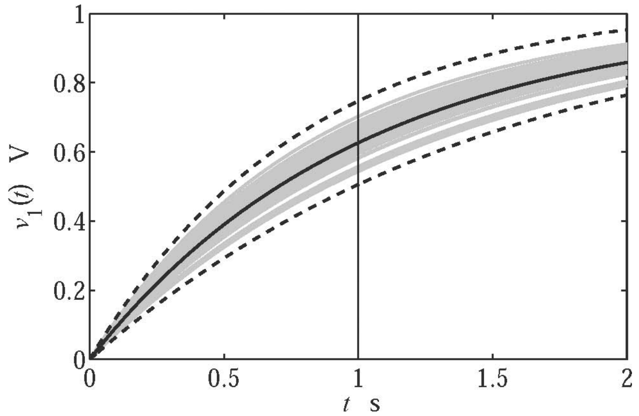

Figure 4: Transient response of the voltage $v_1(t)$ for the circuit of Fig. 1. Solid black line: deterministic response; dashed lines: 3σ tolerance limit of the second order polynomial chaos expansion; solid gray lines: 100 responses obtained with the MC method.

time domain and the application of well established integration techniques for the solution of differential-algebraic equations (*e.g.,* the Matlab® function `ode23t`). It goes without saying that the applicability of this approach is limited to linear circuits. On the other hand, the recent literature proposes an effective solution consisting in the following steps [5]:

(i) convert the continuous-time constitutive relations of the circuit elements into a discrete-time form via the trapezoidal integration rule (for instance, the capacitor equation $i(t) = Cdv/dt$ becomes $i(k) = (2C/T)v(k) - [(2C/T)v(k-1) + i(k-1)]$, being $t = kT$ and T the time step);

(ii) recast the discrete-time constitutive relations by using the polynomial chaos expansion of the voltage and current sequences, the description of the variations of circuit parameters and the inner product defined by (3);

(ii) write the classical MNA formulation, including possible nonlinear components, by using the above-modified constitutive relations.

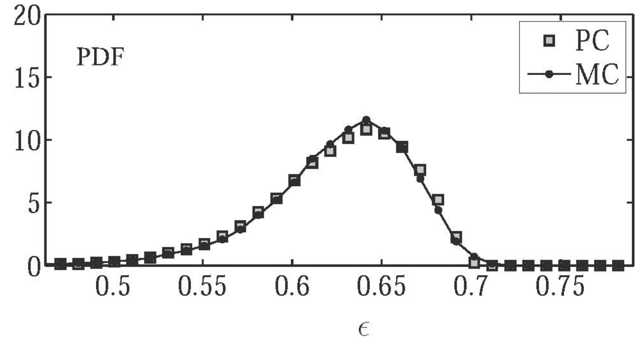

Figure 5: Probability density function of $v_1(t)$, $t = 1$ s. Of the two distributions, the one marked MC refers to 40000 MC simulations, and the one marked PC refers to the transient voltage response obtained via second order polynomial chaos expansion.

As already done for the frequency-domain case, the RC circuit of Fig. 1 is used as a simple test case for the first application of the time-domain stochastic analysis. Figures 4 and 5 show the response of the transient current $v_1(t)$ and the PDF of the voltage signal computed for $t = 1$ s, thus confirming the feasibility of the proposed method.

For completeness, Table 1 collects the standard deviation values of voltage $v_1(t)$, for $t = 1$ s, computed for an increasing number of responses obtained via MC simulations. The error of the computed standard deviation, reported in the third column of Table 1 indicates that a large number of simulations is required to obtain a good estimation with a small error. On the other hand, the second order polynomial chaos description of voltage $v_1(t)$ produces a standard deviation value of 4.0194e-2, that is a very close to the value attainable with a very large number of MC simulations.

Table 1: Standard deviation σ of the voltage $v_1(t)$, t=1 s and its relative error computed via an increasing number N of MC simulations (the reference value for the error determination is assumed to be 4.02e-2, obtained with 40,000 MC runs).

N	σ	error
20	3.2821e-2 V	18 %
200	4.1062e-2 V	2 %
2,000	4.1065e-2 V	2 %
20,000	4.02e-2 V	0 %

4 Application

In this Section, the proposed method is applied to the simulation of the realistic interconnect structure shown in Fig. 6. The two receivers in Fig. 6 are represented by the shunt connection of a $500\,\Omega$ resistor and a 10 pF capacitance, and the drivers by the series connection of an ideal voltage source and a $30\,\Omega$ resistor. The coupled line is a 5 cm-long PCB microstrip structure modeled as a lossless LC line, whose nominal parameters are given in [7]. In this example, the elements of the per-unit-length inductance and capacitance matrices are assumed to be Gaussian random variables with a standard deviation corresponding to 5% of their nominal value.

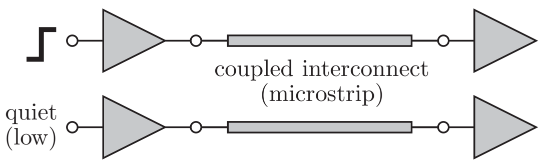

Figure 6: Application test case.

Figure 7 shows the predicted deterministic far-end transient voltage responses on the active and quiet lines, the $\pm 3\sigma$ interval, and the responses of 100 MC simulations. The curves confirm the results obtained with the simple RC circuit. In addition, Fig. 8 shows the PDF of the crosstalk voltage response for t=2 ns computed via MC simulations and by means of the second order polynomial chaos expansion. The good matching in Fig. 8 confirms that low-order polynomial chaos expansions are sufficient to capture the non-Gaussian distribution of the statistical responses of this class of circuits.

5 Conclusions

This paper concentrates on the application of the Polynomial Chaos theory to the stochastic analysis of interconnect structures. The proposed method is based on the description of the circuit elements, randomly dependent on a parameter (*e.g.,* physical dimension or process variation), in terms of Hermite polynomials. The use of such orthogonal basis functions, combined with standard methods for the analysis of dynamical circuits (as the MNA method in this paper), allows us to devise a powerful procedure for the simulation of circuits both in frequency and time domain with the inclusion of a quantitative prediction of the parameters variability on the circuit responses. The strengths of the approach has been demonstrated on a realistic interconnect structure, for which the variability of crosstalk consequent to parameters indetermination is shown.

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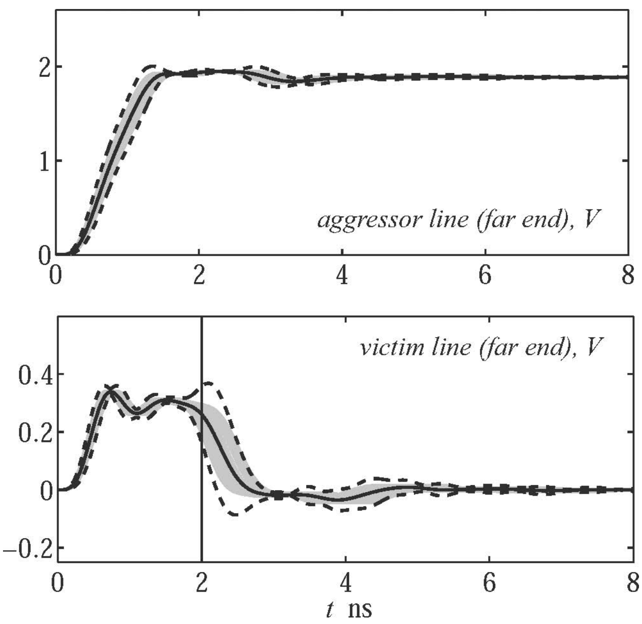

Figure 7: Far-end transient voltage responses on active (top panel) and quiet (bottom panel) lines of the structure of Fig. 6. Solid black line: deterministic response; dashed lines: 3σ tolerance limit of the second order polynomial chaos expansion; solid gray lines: 100 responses obtained with the MC method.

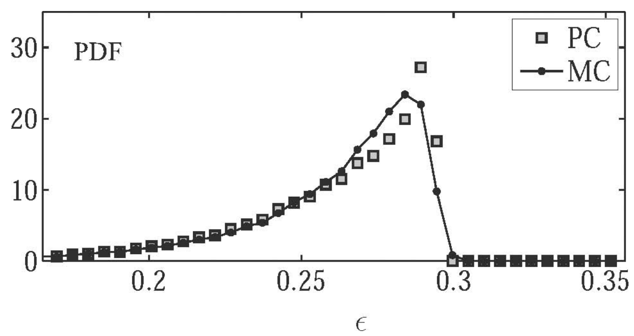

Figure 8: Probability density function of the crosstalk response of Fig. 7 (bottom panel) for $t = 2$ ns. Of the two distributions, the one marked MC refers to 40000 MC simulations, and the one marked PC refers to the transient voltage response obtained via second order polynomial chaos expansion.