Helix submanifolds of euclidean spaces

Helix submanifolds of euclidean spaces

Antonio J. Di Scala* and Gabriel Ruiz-Hernández †

Abstract

A submanifold of $\mathbb{R}^n$ whose tangent space makes constant angle with a fixed direction d is called a helix. In the first part of the paper we study helix hypersurfaces. We give a local description of how these hypersurfaces are constructed. As an application we construct (non flat) minimal helices hypersurfaces in $\mathbb{R}^n$ for $n > 3$. In the second part we give a characterization of helix submanifolds related to the solutions of the so called eikonal differential equation. As a corollary we give necessary and sufficient conditions for a manifold M to be immersed as an helix in some Euclidean space. In the third part of this paper we study r-helices submanifolds. That is to say submanifolds such that its tangent space makes a constant angle with r linearly independent directions.

Mathematics Subject Classification(2000): $53B25, 53C40$.

Keywords: helix submanifold, eikonal function, shadow boundary, constant angle submanifolds.

1 Introduction

Recently, M. Ghomi solved in [6] the shadow problem formulated by H. Wente. He used the concept of shadow boundary (or horizon) in his work. Also J. Choe used the same concept to study the stability index of complete minimal

*The first author is supported by the Project M.I.U.R. "Riemann Metrics and Differenziable Manifolds" and by G.N.S.A.G.A. of I.N.d.A.M., Italy.

†The second author is partially supported by CNPq.

surfaces in $\mathbb{R}^3$ (see [2]). As the second author observed in [7, pag. 2] shadow boundaries are naturally related to helix submanifolds i.e. submanifolds whose tangent space makes constant angle with a fixed direction d. Helix surfaces has also been studied in non flat ambient spaces (see for example [4, 5]). An interesting motivation for the study of helix hypersurfaces comes also from the physics of interfaces of liquid cristals (see [8] for details).

The plan of this article is as follows. Section 2, contains a local characterization of all helix hypersurfaces of $\mathbb{R}^n$, see Theorem 2.4 and its converse Theorem 2.7. As an application we show, in Subsection 2.3, how to construct non flat minimal helices of $\mathbb{R}^n$ for $n \geq 4$. Thus, we solve a question posed in [7, pag. 8, Question 3.1]. In Section 3, we give a description of helix submanifolds by using solutions of the eikonal differential equation (see Definition 3.2). Namely, we prove Theorem 3.3 which gives a way to construct helix submanifolds by using an eikonal function and its converse Theorem 3.4. Corollary 3.6 gives the necessary and sufficient conditions for a manifold M to be immersed as an helix in some Euclidean space. In the last Section 4 we introduce the concept of r-helix submanifold and prove some results about them.

We are going to use basic definitions and results from Submanifold Geometry, e.g. shape operators, structure equations, flat normal bundle, etc. These well-known objects, definitions and facts can be found in the books [1] or [3].

2 Helix Hypersurfaces

Let $d \in \mathbb{R}^n$ be any direction (i.e. a unitary vector) and let $V \subset \mathbb{R}^n$ be a linear subspace. The *angle* θ between d and V is the angle between the vectors d and $\pi_V(d)$, where $\pi_V : \mathbb{R}^n \longrightarrow V$ is the orthogonal projection onto V. That is to say, $cos(\theta) := \langle d, \pi_V(d) \rangle$.
Here is the definition of a helix hypersurface of a Euclidean space.

Definition 2.1 Given a hypersurface $M \subset \mathbb{R}^n$ and an unitary vector $d \neq 0$ in $\mathbb{R}^n$, we say that M is a *helix w.r. to* d if for each $q \in M$ the angle between d and T_qM is constant.

Let $M \subset \mathbb{R}^n$ be a hypersurface and let $\xi : M \to \nu(M)$ be a unit normal vector field, where $\nu(M)$ denotes the normal bundle. Then the above definition is equivalent to the fact that $\langle d, \xi \rangle$ is a constant function along M. If M is a helix of angle θ we will denote it by M_θ.

Remark 2.2 There are two special cases of helix hypersurfaces $M \subset \mathbb{R}^n$: When the angle of the helix is $\theta = \pi/2$ and the helix is connected, it lies in a hyperplane orthogonal to d. So in this case the helix should be an open subset of a hyperplane.
When the angle is $\theta = 0$, the direction d is tangent to the helix M_θ. Since

d is constant, M_θ is ruled by parallel straight line segments. So in this case the helix splits (locally) as trivial factor times a hypersurface of $\mathbb{R}^{n-1}$.

Remark 2.3 A helix hypersurface whose tangent space makes constant angle with a transversal direction (i.e. $\theta \neq 0$) should be orientable. Let us observe that there are nonorientable helix hypersurfaces with $\theta = 0$: Define $M := M' \times \mathbb{R}$ where M' is a nonorientable hypersurface in $\mathbb{R}^{n-1}$. Then M is a nonorientable helix in $\mathbb{R}^n$ with respect to a direction along the $\mathbb{R}$ factor.
Also a helix hypersurface has zero Gauss-Kronecker curvature: the shape operator is singular, i.e. $\det(A) = 0$ (see Remark 2.6).

2.1 Construction

Here is a method to construct helix hypersurfaces $M_\theta \subset \mathbb{R}^n$. Actually, in the next subsection we will show that our method gives (locally) all the helix hypersurfaces.

Let $H \subset \mathbb{R}^{n-1}$ be an orientable hypersurface in $\mathbb{R}^{n-1}$ and let η be an unitary normal vector field of H.
Without loss of generality we can assume that d is the vector $(0, \ldots, 0, 1) \in \mathbb{R}^n$. We can immerse H in $\mathbb{R}^n$ in a canonical way. That is, $H \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$.
As smooth manifold M_θ will be diffeomorphic to $M = H \times \mathbb{R}$, i.e. we add one dimension to the manifold H. In the immersion we will describe, the image of H will be orthogonal to the direction d and the image of the $\mathbb{R}$ factor in $H \times \mathbb{R}$ will make constant angle with the vector d.

We are ready to describe the immersion of M in $\mathbb{R}^n$. We explain this in the following way:

First we define the vector field $\mathrm{T}(x) := \sin(\theta)\, \eta(x) + \cos(\theta) d$, where $x \in H$ (recall that η is normal to H). So T is a vector field defined along the submanifold H. Since H is a submanifold of codimension two in $\mathbb{R}^n$ with flat normal space, T is normal parallel along H.

The immersion $f_\theta : M \to \mathbb{R}^n$ is as follows:

$$f_\theta(x, s) := x + s\, \mathrm{T}(p).$$

For $-\epsilon < s < \epsilon$ enough small, f_θ is an immersion. See Figure 1.

Theorem 2.4 *The immersed submanifold $M_\theta := f_\theta(M)$ is a helix.*

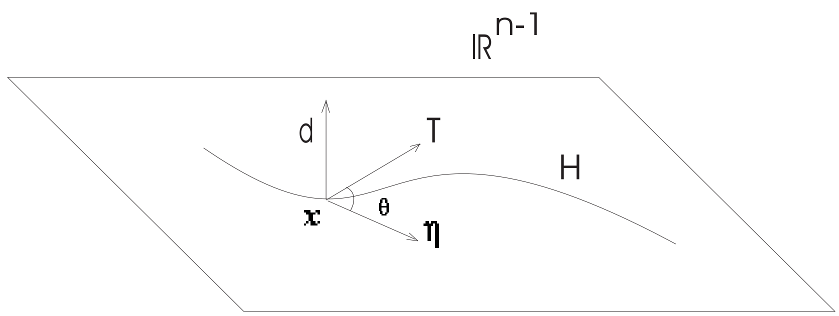
Proof. We will compute an unitary normal vector field ξ of the image $M_\theta = f_\theta(M)$ and verify that its angle with d is constant and equal to θ. In order

to do this we are going to describe the tangent space of $M_\theta = f_\theta(M)$ at each point $f_\theta(x,s)$.
We should observe that f_θ is the identity when we restrict it to the points $(x,0)$, i.e. $f_{\theta|H} = Id_H$. Let us identify H with $f_\theta(H \times \{0\})$ and M with its image M_θ.
Then, at a point (x,s) the tangent space $T_{(x,s)}M_\theta$ is given by:

$$T_{(x,s)}M_\theta := T_xH \oplus \mathbb{R}\,\mathrm{T}(x)\ .$$

Affirmation: $\xi(x) := -\cos(\theta)\,\eta(x) + \sin(\theta)d$ is an unitary normal vector field of M_θ. We can verify this by as follows: for each $x \in H$, $\xi(x) \in T_xH^{\perp}$, because η and d are orthogonal to H. Finally,
$\langle \xi(x), \mathrm{T}(x)\rangle = \langle -\cos(\theta)\ \eta(x) + \sin(\theta)d,\ \sin(\theta)\ \eta(x) + \cos(\theta)d\rangle = 0$.
Since η is orthogonal to d, $\langle \xi, d\rangle = \sin(\theta)$. Now it is clear that M_θ is a helix of angle θ. $\square$

Figure 1: Helix M_θ



2.2 Reconstruction

In this subsection we will show that every helix $M_\theta \subset \mathbb{R}^n$ can be obtained, by the construction described in the previous subsection. Namely, starting with a hypersurface $H \subset \mathbb{R}^{n-1}$.

Let ξ and T be unitary vector fields tangent and normal to M_θ respectively, so that $d = \cos(\theta)\,\mathrm{T}(p) + \sin(\theta)\xi(p)$.

Lemma 2.5 *The integral curves of* T *are straight lines of* $\mathbb{R}^n$.

Proof. Let us denote by D the standard covariant derivative in $\mathbb{R}^n$ and by ∇ the induced covariant derivative in M_θ. First let us observe that, T will be well defined if we assume that $\cos(\theta) \neq 0$, otherwise M_θ would be an open subset of a hyperplane and T would not be well defined. **Let us assume that** $\cos(\theta) \neq 0$**.**

Let $A^{\xi}(X) := -D_X \xi$ be the shape operator of the helix M_θ and let $\alpha(X,Y)$ be its second fundamental form, i.e. $\langle A^{\xi}(X), Y\rangle = \langle \alpha(X,Y), \xi\rangle$. Let X be an arbitrary vector field on M_θ. Taking the covariant derivative in each part of the equation $d = \cos(\theta)\,\mathrm{T} + \sin(\theta)\xi$ with respect to X, we obtain:

$$0 = \cos(\theta) D_X \,\mathrm{T} + \sin(\theta) D_X \xi \ ,$$

taking the normal component and the tangential one (in M_θ) we get:

$$0 = \cos(\theta)(\nabla_X \,\mathrm{T} + \alpha(X, \mathrm{T})) - \sin(\theta) A^{\xi}(X) \ .$$

From this, we have:

$$0 = \cos(\theta)\nabla_X \,\mathrm{T} - \sin(\theta) A^{\xi}(X) \, ,$$

$$0 = \cos(\theta)\alpha(X, \mathrm{T}).$$

So $\alpha(X, \mathrm{T}) = 0$ for each $X \in T_pM$.
Let us apply this latter condition to prove that $A^{\xi}(\mathrm{T}) = -D_T\xi = 0$: it follows from the relations $\langle D_T \xi, X\rangle = -\langle A^{\xi}(\mathrm{T}), X\rangle = -\langle \xi, \alpha(T, X)\rangle = 0$ and finally $\langle D_T \xi, \xi\rangle = 0$. Therefore, $0 = \cos(\theta)\nabla_T \,\mathrm{T} - \sin(\theta) A^{\xi}(T) = \cos(\theta)\nabla_T \,\mathrm{T}$.
Hence we deduce the next two conditions:

$$\alpha(\mathrm{T}, \mathrm{T}) = 0 \ \text{ and } \nabla_{\mathrm{T}} \,\mathrm{T} = 0 \ .$$

To prove that the integral curves of T are straight lines is enough verify that $D_{\mathrm{T}}\mathrm{T} = 0$. But

$$D_{\mathrm{T}} \,\mathrm{T} = \nabla_{\mathrm{T}} \,\mathrm{T} + \alpha(\mathrm{T}, \mathrm{T}) = 0 \ ,$$

this proves the lemma. □

Remark 2.6 The condition $A^{\xi}(\mathrm{T}) = 0$ means that the shape operator of any helix hypersurface M_θ is singular: $\det(A^{\xi}) = 0$.

Theorem 2.7 *Each helix hypersurface M_θ is locally obtained by the construction given in Theorem 2.4.*

Proof. Without loss of generality we can assume that $d = (0, \dots, 0, 1)$. We can assume that $\theta \neq \{0, \ \pi/2\}$ (see Remark 2.2). Otherwise we are done.
So, d is transversal and non orthogonal to M_θ. Hence, each hyperplane Q orthogonal to d and so that $Q \cap M_\theta \neq \emptyset$ is transversal to M_θ.
We are ready for the local reconstruction.

Let T the tangential component of d along M_θ as in Lemma 2.5, i.e. T is an unitary vector field on M_θ. By the same Lemma 2.5, we know that the integral curves of T are straight line segments.

Let $p \in M_\theta$ and let Q_p be a hyperplane through p and orthogonal to d. We define the submanifold of $H := Q_p \cap M_\theta$ of M_θ. Is clear that for each point x of H passes a straight line segment of M_θ with direction $\mathrm{T}(x)$. Hence, there is an open neighborhood U of p in M_θ so that any point in U can be written as $x + sT(x)$, where $x \in H \cap U$ and $-\epsilon < s < \epsilon$ for some $\epsilon > 0$ enough small. $\square$

2.3 Minimal helices

Notice that a helix surface in $\mathbb{R}^3$ with zero mean curvature should be an open subset of a plane. Indeed, since it has zero Gauss curvature (see Remark 2.3) it follows that the shape operator vanishes. In general this is not true for a minimal (i.e. with zero mean curvature) helix hypersurface M_θ of dimension greater than two. We are going to use our classification of helix hypersurfaces to classify the minimal helix hypersurfaces.

Given $p \in M_\theta$, Q_p will denote the hyperplane through p and orthogonal to d. As before we can assume that $\theta \neq \{0,\ \pi/2\}$ (see again Remark 2.2). Otherwise we are done. Hence, d is transversal and nonorthogonal to M_θ. So, $H := Q_p \cap M_\theta$ is a submanifold.

Theorem 2.8 *A helix hypersurface M_θ is minimal if and only if every $H := Q_p \cap M_\theta$ is minimal in $\mathbb{R}^n$.*

Proof. The idea is to observe that using the shape operator $A^\xi : TM_\theta \longrightarrow TM_\theta$ of M_θ, we can calculate the shape operator (or the Weingarten operators) of H.
Let ∇ be as before, i.e. the covariant derivative of M_θ induced by the Euclidean standard Levi-Civita connection. Similarly, let $\nabla^{H^\perp}$ be the normal connection induced on the normal bundle of H as submanifold of $\mathbb{R}^n$.

The normal space of H is flat, has dimension two and is generated by ξ and T in each point $x \in H$.
Let A^ξ_H be the shape operator of H (as submanifold of $\mathbb{R}^n$) in the direccion ξ.
The Weingarten formula of $H \subset \mathbb{R}^n$ is $D_X\xi = -A^\xi_H(X) + {\nabla^{H}}^{\perp}_X \xi$, for every tangent vector field on H. Since H has flat normal space and $A^\xi(X) = -D_X\xi$, $A^\xi(X) = A^\xi_H(X)$.

If M_θ is minimal then

$$trace(A^\xi) = \sum_{j=1}^{n-1} \langle A^\xi(X_j), X_j\rangle = \sum_{j=1}^{n-1} \langle \alpha(X_j, X_j), \xi\rangle = 0,$$

where $X_1, \ldots, X_{n-1}$ is any local orthonormal basis of TM_θ. Since T is an unitary tangent vector field of M_θ we can assume that $X_{n-1} = \mathrm{T}$. Morover, will be convenient to take the vector fields $X_1, \ldots, X_{n-2}$ orthogonal to direction d. So, they will be tangent to each $H = Q_p \cap M_\theta$. So using the latter computation plus the relation $A^\xi(\mathrm{T}) = 0$ we obtain,

$$trace(A^\xi_H) = \sum_{j=1}^{n-2} \langle A^\xi_H(X_j), X_j \rangle = \sum_{j=1}^{n-2} \langle A^\xi(X_j), X_j \rangle + \langle A^\xi(\mathrm{T}), \mathrm{T} \rangle =$$

$$= trace(A^\xi) = 0 \ .$$

But this means that H is minimal in $\mathbb{R}^n$. The same computation show that if every H is minimal in the Euclidean ambient then M_θ is minimal too. $\square$

3 Eikonal functions and higher codimensional helices

Here is the definition of a helix submanifold of the Euclidean space.

Definition 3.1 Given an Euclidean submanifold of arbitrary codimension $M \subset \mathbb{R}^n$ and an unitary vector $d \neq 0$ in $\mathbb{R}^n$, we say that M is a *helix submanifold w.r. to* d if for each $q \in M$ the angle between d and T_qM is constant.

Notice that if the angle between d and T_qM is zero i.e. $d \in TM$ then M splits as an extrinsic product. So we are going to assume along this section that the angle between d and T_qM is not zero.

The term "helix" comes from the helix curves in $\mathbb{R}^3$. Naturally, a helix curve in $\mathbb{R}^3$ has codimension 2, then it is interesting try to extend its construction to $\mathbb{R}^n$. In order to do that, we need the following definition.

Definition 3.2 *Let* (M, g) *be a Riemannian manifold, where* g *is the metric. Let* $f : M \to \mathbb{R}$ *be a function and let* ∇f *be its gradiant i.e.* $df(X) = g(\nabla f, X)$. *We say that* f *is Eikonal* if it satisfies:

$$\|\nabla f\| = cte \ .$$

The following results show us the relationship between eikonal functions and helices.

Theorem 3.3 *Let $i: M \to \mathbb{R}^n$ be a submanifold and let $f: M \to \mathbb{R}$ be an eikonal function, where M has the induced metric by $\mathbb{R}^n$, i.e. the metric of the image $i(M) \subset \mathbb{R}^n$. Then $\phi(M)$ is a helix, where $\phi: M \to \mathbb{R}^n \times \mathbb{R}$ is the immersion given by*

$$\phi(p) := (i(p), f(p)) \ .$$

The direction is $d = (0,1)$ and the angle θ between d and $\nu(M)$ (normal space) is determined by the equality

$$\cos(\theta) = \frac{-1}{\sqrt{1 + \|\nabla f\|^2}}.$$

Proof. Let $\eta_1, \eta_2, \cdots, \eta_k$ be a frame (i.e. k-orthonormal vector fields), around a point $p \in M$, of the normal space of the immersion $i: M \to \mathbb{R}^n$. We can use these vector fields to construct other frame around p of the immersion $\phi: M \to \mathbb{R}^{n+1}$. The normal vector fields $\tau_i := (\eta_i, 0)$ for $1 \leq i \leq k$ and $E := \frac{(\nabla f, -1)}{\sqrt{1+\|\nabla f\|^2}}$, define the frame: First, let us observe that the vectors τ_i and E are orthogonals to $T\phi(M)$, moreover $\langle \tau_i, \tau_j \rangle = \langle \eta_i, \eta_j \rangle = \delta_{ij}$ (η_i is a frame). Since $\nabla f \in TM$, $\langle E, \tau_i \rangle = \langle \nabla f, \eta_i \rangle = 0$.
To see that $\phi(M)$ is a helix with respect the direction $d = (0,1)$, we shall verify that the projection of d into the normal space has constant length. Using the latter frame we can see that the projection of d into its normal component is: $\langle d, E \rangle E$. So its length is constant if and only if $\langle d, E \rangle$ is constant. But $\langle d, E \rangle = \cos(\theta) = \frac{-1}{\sqrt{1+\|\nabla f\|^2}}$. □

Let us remark that if the eikonal function is nonconstant, the induced metric on M by the helix immersion ϕ is different from the corresponding metric induced by the immersion i. The following result shows that is true the converse of Theorem 3.3 (cf. [8, Sect.2, eq. (10)]).

Theorem 3.4 *Let $\phi: M \to \mathbb{R}^n \times \mathbb{R}$ be a helix submanifold with $d = (0,1)$ and angle different from zero. Let us assume (without lost of generality) that $(0,0) \in \phi(M)$. Let $\phi(p) = (i(p), f(p))$ be the $\mathbb{R}^n \times \mathbb{R}$ expression of ϕ, where $f(p) = \langle \phi(p), d \rangle$. Then, f is eikonal with respect to the metric induced by the immersion $i: M \to \mathbb{R}^n$ on M.*

Proof. The proof follows the same line as the previous proof. Namely, as above, introduce the normal vector fields $\tau_i := (\eta_i, 0)$ for $1 \leq i \leq k$ and $E := \frac{(\nabla f, -1)}{\sqrt{1+\|\nabla f\|^2}}$, where ∇f is w.r. to the metric induced by the immersion $i: M \to \mathbb{R}^n$ (Notice that $i: M \to \mathbb{R}^n$ is an immersion since the angle between d and TM is different from zero). So the normal vectors fields E and $\tau_i := (\eta_i, 0)$ for $1 \leq i \leq k$ are a frame around $p \in M$ of the the normal bundle of the immersion $\phi: M \to \mathbb{R}^n \times \mathbb{R}$. To see that f is eikonal with respect to the metric induced by the immersion $i: M \to \mathbb{R}^n$ on M we use that the projection of d into the normal space to $\phi(M)$ has constant length. Thus, by using the latter frame we can see that the projection of d into its normal component is: $\langle d, E\rangle E$. So its length is constant if and only if $\langle d, E\rangle$ is constant. But $\langle d, E\rangle = \cos(\theta) = \frac{-1}{\sqrt{1+\|\nabla f\|^2}}$. It follows that $\|\nabla f\|$ is constant. $\square$

Here we have an application of the above results.

Lemma 3.5 *Let M be a differentiable manifold. Then M can be immersed as an helix submanifold with angle $\theta \neq 0$ w.r. to a direction d of some Euclidean space if and only if M admits a Riemannian metric g and an eikonal function $f : M \to \mathbb{R}$ w.r. to g such that:*

$$\cos(\theta) = \frac{-1}{\sqrt{1+\|\nabla f\|^2}}$$

Proof. Notice that Theorem 3.4 implies that the existence condition of the Riemannian metric g and the eikonal function f is necessary. Reciprocally, if the Riemannian metric g and the eikonal function f are given on M we can use the well-known embedding Theorem of Nash to get an isometric embedding $i : M \to \mathbb{R}^n$ such that $g = i^* g_0$, where g_0 is the flat metric of $\mathbb{R}^n$. Then we can use Theorem 3.3 to conclude that M can be immersed as an helix submanifold of $\mathbb{R}^{n+1}$. $\square$
Let us observe that any differentiable manifold can be embedded as a helix Euclidean submanifold with angle $\theta = \pi/2$ and if it is compact then the angle $\pi/2$ is necessary. A helix Euclidean submanifold with $\theta = 0$ splits (locally) necessarily with a trivial factor $\mathbb{R}$ (see Remark 2.2 for the case of hypersurfaces). So, the interesting case is when $\theta \neq 0$ or $\pi/2$.
It is possible to show that a differentiable manifold M admits a Riemannian metric g and an eikonal function $f : M \to \mathbb{R}$ w.r. to g if and only if M is non compact. Indeed, it is known that M is non compact if and only if there exists a smooth function f without critical points. Then it is not difficult to see that f is eikonal w.r. to an adequate Riemannian metric g. Thus, we get the following corollary.

Corollary 3.6 *Let M be a differentiable manifold. Then M can be immersed as an helix submanifold with angle $\theta \neq 0$ or $\pi/2$ w.r. to a direction d of some Euclidean space if and only if M is non compact.*

4 r-Helix submanifolds

It can happens that a given submanifold $M \subset \mathbb{R}^n$ is a helix w.r. to two or more independent directions. For an hypersuface M notice that if M is an helix w.r. to d and d' then M is also a helix w.r. to any direction in the linear span of d and d'. This gives a motivation for the following definition.

Definition 4.1 *A submanifold* $M \subset \mathbb{R}^n$ *is a* r*-helix if there exist a linear subspace* $\mathcal{H} \subset \mathbb{R}^n$ *of dimension* $r = dim(\mathcal{H})$ *such that* M *is a helix w.r. to any direction* $d \in \mathcal{H}$*. The subspace* $\mathcal{H}$ *will be called the subspace of helix directions .*

Here is a characterization of a r-helix in terms of the projectors $\pi_p : \mathbb{R}^n \to T_pM$.

Proposition 4.2 *The submanifold* $M \subset \mathbb{R}^n$ *is a* r*-helix if and only if there exist a* r*-dimensional subspace* $\mathcal{H} \subset \mathbb{R}^n$*, such that :*

$$\|\pi_p(v)\|$$

does not depends of $p \in M$ *for all* $v \in \mathcal{H}$*.*

Proof. It is enough to prove the proposition when $dim(\mathcal{H}) = 1$. So assume that M is a helix w.r. to $d \in \mathcal{H}$. Then we can decompose $d = \cos(\theta)\,\mathrm{T} + \sin(\theta)\xi$, where $\mathrm{T} \in T_pM$ and $\xi \in \nu_pM$. So $\pi_p(d) = \cos(\theta)\,\mathrm{T}$ and therefore $\|\pi_p(d)\| = \cos(\theta)^2 = c$. Reciprocally, if $\|\pi_p(d)\|$ is constant then $d = \cos(\theta)\,\mathrm{T} + \sin(\theta)\xi$, where $\cos(\theta)$ should be constant. Hence M is a helix w.r. to d. □

Remark 4.3 *Notice that if a submanifold* $M \subset \mathbb{R}^n$ *is a helix w.r. to the directions* d_1 *and* d_2 *then* M *is not necessarily a* 2*-helix.*

Let now M be a r-helix and let $(d_1, d_2, \cdots, d_r)$ be a base of $\mathcal{H}$. We can decompose each vector d_j in its tangent and normal components:

$$d_j = \cos(\theta_j)\,\mathrm{T}_j(p) + \sin(\theta_j)\xi_j \ ,$$

where each θ_j is constant. Taking derivative with respect to $X \in TM$ we obtain:

$$0 = \cos(\theta_j) D_X\,\mathrm{T}_j(p) + \sin(\theta_j) D_X \xi_j =$$

$$0 = \cos(\theta_j)(\nabla_X\,\mathrm{T}_j(p) + \alpha(X, \mathrm{T}_j(p))) + \sin(\theta_j)(\nabla^{\perp}_X \xi_j - A^{\xi_j}(X)).$$

Consequently:

$$\begin{aligned} 0 &= \cos(\theta_j)\nabla_X\,\mathrm{T}_j(p) - \sin(\theta_j)A^{\xi_j}(X) \text{ and} \qquad (1)\\ 0 &= \cos(\theta_j)\alpha(X, \mathrm{T}_j(p)) + \sin(\theta_j)\nabla^{\perp}_X \xi_j. \qquad (2)\end{aligned}$$

Definition 4.4 The *nullity space* of the second fundamental form α (of $M \subset \mathbb{R}^n$) consist of $\{X \in TM \mid \alpha(X, Y) = 0, \text{ for every } Y \in TM\}$.

Definition 4.5 The *second normal space* of $M \subset \mathbb{R}^n$ consist of the normal vectors, $\xi \in \nu(M)$, such that the shape operator in its direction is zero, i.e. $A^{\xi} = 0$.

As a consequence of the equations (1) and (2), we have the following results.

Lemma 4.6 *The vector field* T_j *is in the nullity of* α *if and only if* ξ_j *is parallel with respect to the normal connection. In this case the shape operators* A^{ξ_j} *and the vector fields* T_j *commute between them. Moreover, the integral curves of* T_i *are geodesics of* M *.*

Proof. Equation (2) shows that ξ_j is parallel if and only if $\alpha(X, T_j) = 0$, for every vector field X on M. This proves the next affirmation: T_j is in the nullity space of α is equivalent to the parallelism of ξ_j.
For the second part we will use the *Ricci equation*, namely: Let X, Y be vector fields on M and let ξ be a normal vector field of M,

$$R^{\perp}(X,Y)\xi = \alpha(X, A^{\xi}(Y)) - \alpha(A^{\xi}(X), Y),$$

where $R^{\perp}(X,Y)\xi = \nabla^{\perp}_X\nabla^{\perp}_Y\xi - \nabla^{\perp}_Y\nabla^{\perp}_X\xi - \nabla^{\perp}_{[X,Y]}\xi$.
If η is another normal vector field of M, the Ricci equation can be rewritten as $\langle R^{\perp}(X,Y)\xi, \eta\rangle = \langle A^{\xi}(A^{\eta}(X)), Y\rangle - (A^{\eta}A^{\xi}(X)), Y\rangle$. By hyphotesis, the normal vector field ξ_j is parallel with respect to the normal connection: $\nabla^{\perp}_X\xi_j = 0$, which implies that $R^{\perp}(X,Y)\xi_j = 0$.
So $A^{\xi_k}(A^{\xi_j}(X)) = A^{\xi_j}(A^{\xi_k}(X))$.
Finally, let us verify that the vector fields T_j commute between them. By equation (1), (2) and the parallelism of the normal vector fields ξ_j, we have that for each vector field X on M,

$$\langle \nabla_{\mathrm{T}_j}\mathrm{T}_i, X\rangle = \langle \tan(\theta_i)A^{\xi_i}(\mathrm{T}_j), X\rangle = \tan(\theta_i)\langle\alpha(\mathrm{T}_j, X), \xi_i\rangle = 0.$$

This proves that $\nabla_{\mathrm{T}_j}\mathrm{T}_i = 0$, in particular the integral curves of each T_i are geodesics of M. Now we just apply the equation $[\mathrm{T}_i, \mathrm{T}_j] = \nabla_{\mathrm{T}_i}\mathrm{T}_j - \nabla_{\mathrm{T}_j}\mathrm{T}_i = 0$. □

Lemma 4.7 *Each* ξ_j *is in the second normal space of* $M \subset \mathbb{R}^n$ *if and only if* T_j *is parallel in* M *.*

Proof. By definition T_j is parallel in M if $\nabla_X \mathrm{T}_j = 0$, for every vector field X on M. From equation (1), we deduce that T_j is parallel in M if and only if $A^{\xi}_j = 0$. □

It is natural to try of generalize the classification of helix hypersurfaces given in Theorem 2.7 into submanifolds of higher codimension, i.e. r-helices. A first step is towards such a classification is the following problem:

Problem. Classify r-helices so that the normal components, ξ_j, of the directions d_j satisfy:

$$\nabla^{\perp}\xi_j = 0 \ ,$$

i.e. every ξ_j is parallel with respect to the normal connection.

4.1 r-helices with $r > k$ where k is the codimension.

Let $M \subset \mathbb{R}^n$ be a submanifold of $\mathbb{R}^n$ and let $k = \dim(\nu(M)) = n - \dim(M)$ be the codimension.

Theorem 4.8 *Let $M \subset \mathbb{R}^n$ be a r-helix with $r > k$. Then M is a Riemannian product submanifold in $\mathbb{R}^n$, i.e. $M = I \times N \subset \mathbb{R} \times \mathbb{R}^{n-1}$ where $I \subset \mathbb{R}$ is an open interval and $N \subset \mathbb{R}^{n-1}$ is a $r-1$-helix.*

Proof. Let us assume that M is a helix with respect to directions d_i, for every $1 \leq i \leq r$. Let $D \subset \mathbb{R}^n$ be the vector subspace generated by directions d_i's. Let us observe that M is a 1-helix with respect to any unitary vector in D. Without loss of generality, we can assume that the origin $0 \in \mathbb{R}^n$ is in M. Hence the hypothesis implies that $dim(D) + dim(T_0M) > n$. Therefore $dim(D \bigcap T_0M) > 0$. Let $v \in D \bigcap T_0M$. So v is parallel in $\mathbb{R}^n$ and tangent to M. This concludes the proof. □

This latter Theorem has the following corollary.

Corollary 4.9 *Let M^n be a $(n-1)$-helix of $\mathbb{R}^{n+1}$, where $n > 1$. Then M is flat, i.e. curvature tensor is zero. In fact, a $(n-1)$-helix of $\mathbb{R}^{n+1}$ is (locally) a product of an open subset of $\mathbb{R}^{n-2}$ and a helix surface in $\mathbb{R}^3$*

Proof. By induction over n and using the Theorem 4.8, we can see that M^n is the aforementioned Riemannian product. To conclude the proof, we should observe that for $n = 2$ the Gauss curvature of a helix in $\mathbb{R}^3$ is zero. □

Acknowledgements The second author is indebted to Antonio J. Di Scala for his invitation to participate in this work and his help throughout it.

References

[1] Berndt, J.; Console S. and Olmos C.: *Submanifolds and holonomy,* Chapman & Hall/CRC , Research Notes in Mathematics 434 (2003).

[2] Choe, J.: *Index, vision number and stability of complete minimal surfaces,* Arch. Rational Mech. Anal. 109 (1990), no. 3, 195–212.

[3] Dajczer, M.: *Submanifolds and isometric immersions,* Mathematics Lecture Series 13, Publish or Perish, Inc. Houston, Texas (1999).

[4] Dillen, F. and Munteanu, M.I: *Constant Angle Surfaces in $\mathbb{H}^2 \times \mathbb{R}$,* arXiv:0705.3744 (2007).

[5] Dillen, F.; Fastenakels, J.; Van der Veken, J. and Vrancken, L.: *Constant Angle Surfaces in $S^2 \times \mathbb{R}$,* to appear in Monatsh. Math.

[6] Ghomi, M.: *Shadows and convexity of surfaces,* Ann. of Math. (2) 155 (2002), no. 1, 281–293.

[7] Ruiz-Hernandez, G.: *Helix, shadow boundary and minimal submanifolds,* arXiv:0706.1524 (2007).

[8] Cermelli, P. and Di Scala, A.J.: *Constant-angle surfaces in liquid cristals,* Philosophical Magazine vol. 87, pp. 1871-1888 (2007).

Dipartimento di Matematica,
Politecnico di Torino,
Corso Duca degli Abruzzi 24, 10129 Torino, Italy
`antonio.discala@polito.it`

IMPA, Rio de Janeiro, Brasil and
Facultad de Matemáticas, UAG,
Carlos E. Adame 54, 39650 Guerrero, México
`gruiz@cimat.mx`