Submitted to Physical Review E

Generalized Maxwell state and H-theorem for the lattice Boltzmann method

Pietro Asinari[1] and Ilya V. Karlin[2,3]

[1]*Department of Energetics, Politecnico di Torino, Corso Duca degli Abruzzi 24, Torino, Italy*
[2]*Institute of Energy Technology, ETH Zurich, 8092 Zurich, Switzerland*
[3]*School of Engineering Sciences, University of Southampton, SO17 1BJ Southampton, UK*

(Dated: February 17, 2009)

Generalized Maxwell distribution function is derived analytically for the lattice Boltzmann (LB) method. All the previously introduced equilibria for LB are found as special cases of the generalized Maxwellian. The generalized Maxwellian is used to derive an new class of multiple-relaxation-time LB models and prove the H–theorem for them.

PACS numbers: 47.11.-j, 05.20.Dd

A branch of kinetic theory - the lattice Boltzmann (LB) method - has recently met with a remarkable success as a powerful alternative for solving the hydrodynamic Navier–Stokes equations, with applications ranging from large Reynolds number flows to flows at a micron scale, porous media, and multiphase flows see, e. g., [1–3] and references therein. The LB method solves a fully discrete kinetic equation for populations $f_\alpha(\boldsymbol{x}, t)$, designed in a way that it reproduces the Navier–Stokes equations in the hydrodynamic limit in D dimensions. Populations correspond to discrete velocities $\boldsymbol{v}_\alpha$ for $\alpha = 0, 1, ..., Q-1$, which fit into a regular spatial lattice with the nodes $\boldsymbol{x}$. This enables a simple and highly efficient algorithm based on (a) nodal relaxation and (b) streaming along the links of the regular spatial lattice. On the other hand, numerical stability of the LB method remains a critical issue [2]. Recalling the role played by the Boltzmann's H–theorem in enforcing macroscopic evolutionary constraints (the second law of thermodynamics), pertinent entropy functions have been proposed [4–8]. The full connection of LB to kinetic theory was established by the discrete-velocity analog of the Maxwellian (see Eq. (2) below).

Admittedly, however, that other heuristic methods were proposed recently to enhance stability of LB. The rationale behind one of them, the multiple–relaxation–time (MRT) [9–11], is sound: Since the incompressible flow is the only concern, the bulk viscosity arising in the quasi-compressible LB scheme can be viewed as a free parameter and tuned in order to enhance stability. However, in spite of popularity of the MRT method, to date, it cannot be considered as a consistent kinetic theory but rather a numerical trick where tuning of parameters is based on experience rather than on physics.

In this paper, we present a new consideration of the LB models, and derive a crucial result: the closed-form generalized equilibrium (see Eq. (3) below). The generalized equilibrium is the analog of the anisotropic Gaussian, and is a long-needed relevant distribution in the LB method. This finding further allows us to introduce an innovative class of entropy-based MRT LB models which enjoy both the H–theorem and the additional free–tunable parameter for controlling the bulk viscosity, where the range is dictated by the entropy.

For the sake of presentation and without any loss of generality, we consider the popular nine-velocity model, the so-called D2Q9 lattice, of which the discrete velocities are: $\boldsymbol{v}_0 = (0, 0)$, $\boldsymbol{v}_i = (\pm c, 0)$ and $(0, \pm c)$ for $i = 1$–4, and $\boldsymbol{v}_i = (\pm c, \pm c)$, for $i = 5$–8 [12] where c is the lattice spacing. Recall that the D2Q9 lattice derives from the three–point Gauss–Hermite formula [13], with the following weights of the quadrature $w(-1) = 1/6$, $w(0) = 2/3$ and $w(+1) = 1/6$. Let us arrange in the list v_x all the components of the lattice velocities along the x–axis and similarly in the list v_y. Analogously let us arrange in the list f all the populations f_α. Algebraic operations for the lists are always assumed component-wise. The sum of all the elements of the list p is denoted by $\langle p \rangle = \sum_{i=0}^{Q-1} p_i$. The dimensionless density ρ, the flow velocity $\boldsymbol{u}$ and the second–order moment (pressure tensor) $\boldsymbol{\Pi}$ are defined by $\rho = \langle f \rangle$, $\rho u_i = \langle v_i f \rangle$ and $\rho \Pi_{ij} = \langle v_i\, v_j f \rangle$ respectively.

On the lattice under consideration, the convex entropy function (H–function) is defined as [5]

$$H(f) = \big\langle f\, \ln\left(f/W\right)\big\rangle, \tag{1}$$

where $W = w(v_x)\, w(v_y)$. The H–function minimization problem is considered in the sequel. It is well known [5] that the equilibrium population list f_M is defined as the solution of the minimization problem $f_M = \min_{f \in \mathsf{P}_M} H(f)$, where P_M is the set of functions such that $\mathsf{P}_M = \big\{ f > 0 : \langle f \rangle = \rho,\ \langle \boldsymbol{v} f \rangle = \rho \boldsymbol{u} \big\}$. In other words, minimization of the H–function (1) under the constraints of mass and momentum conservation yields [6]

$$f_M = \rho \prod_{\alpha = x,y} w(v_\alpha)\left(2 - \varphi(u_\alpha/c)\right)\left(\frac{2(u_\alpha/c) + \varphi(u_\alpha/c)}{1 - (u_\alpha/c)}\right)^{v_\alpha/c}, \tag{2}$$

where $\varphi(z) = \sqrt{3z^2 + 1}$. A remarkable feature of the equilibrium (2) which it shares with the ordinary Maxwellian is that it is a product of one-dimensional equilibria. In order to ensure the positivity of f_M, the low Mach number limit must be considered, i.e. $|u_\alpha| < c$.

In this paper, we derive a novel constrained equilibrium, or quasi–equilibrium [14], by requiring, in addition, that the diagonal components of the pressure tensor $\boldsymbol{\Pi}$

have some prescribed values. Hence let us introduce a different minimization problem. The quasi–equilibrium population list f_G is defined as the solution of the minimization problem $f_G = \min_{f\in \mathsf{P}_G} H(f)$, where $\mathsf{P}_G \subset \mathsf{P}_M$ is the set of functions such that $\mathsf{P}_G = \big\{f > 0 : \langle f\rangle = \rho,\ \langle \boldsymbol{v} f\rangle = \rho \boldsymbol{u},\ \langle v_\alpha^2 f\rangle = \rho \Pi_{\alpha\alpha}\big\}$. In other words, minimization of the H–function (1) under the constraints of mass and momentum conservation and prescribed diagonal components of the pressure tensor yields

$$f_G = \rho \prod_{\alpha=x,y} w(v_\alpha)\, \frac{3\,(c^2-\Pi_{\alpha\alpha})}{2\,c^2} \left(\sqrt{\frac{\Pi_{\alpha\alpha}+c\,u_\alpha}{\Pi_{\alpha\alpha}-c\,u_\alpha}}\right)^{v_\alpha/c} \left(\frac{2\sqrt{\Pi_{\alpha\alpha}^2-c^2\,u_\alpha^2}}{c^2-\Pi_{\alpha\alpha}}\right)^{v_\alpha^2/c^2} . \tag{3}$$

To ease notation, we use $\Pi = (\Pi_{xx}, \Pi_{yy})$ for a generic point on the two-dimensional plane of parameters. In order to ensure the positivity of f_G, it is required that $\Pi \in \Omega$ where $\Omega = \{\Pi : c\,|u_x| < \Pi_{xx} < c^2,\ c\,|u_y| < \Pi_{yy} < c^2\}$ is a convex rectangular in the plane of parameters for each velocity $\boldsymbol{u}$ (see Fig. 1).

The generalized Maxwellian (3) is the central result of this paper, and is the key to the derivations below. It is interesting to note that, while the equilibrium (2) is analogous to the ordinary Maxwellian (spherically symmetric Gaussian $f_M \sim \exp\{-m(\boldsymbol{v}-\boldsymbol{u})^2/2k_B\Theta_0\}$, shifted from the origin by the amount of mean flow velocity $\boldsymbol{u}$, and with the width proportional to the fixed temperature $\Theta_0 = c^2/3$), the quasi-equilibrium (3) resembles the anisotropic Gaussian, $f_G \sim \exp\{-(1/2)(\boldsymbol{v}-\boldsymbol{u})\cdot \boldsymbol{\Pi}^{-1}\cdot(\boldsymbol{v}-\boldsymbol{u})\}$. The latter generalized Maxwellian corresponds to the ellipsoidal symmetry, and is among the only few analytic results on the relevant distribution functions in the classical kinetic theory [15]. It is revealing that also in the LB realm the analog of the generalized Maxwellian has a nice closed form (3). The physical sense of (3) is that it distinguishes the relaxation of the diagonal components of the pressure tensor (and hence also of the trace of this tensor) among other non-conserved moments, and hence one expects a control over the dynamics of the trace which is responsible for the bulk viscosity (see below). Moreover, it is possible to evaluate explicitly the H–function in the generalized Maxwell states (3), $H_G = H\big(f_G\big)$, the result is elegantly written

$$H_G = \rho \ln \rho + \rho \sum_{\alpha=x,\,y}\ \sum_{k=-,\,0,\,+} w_k\, a_k(\Pi_{\alpha\alpha}) \ln\big(a_k(\Pi_{\alpha\alpha})\big), \tag{4}$$

where $w_\pm = w(\pm 1)$, $w_0 = w(0)$, $a_\pm(\Pi_{\alpha\alpha}) = 3\,(\Pi_{\alpha\alpha} \pm c\,u_\alpha)/c^2$ and $a_0(\Pi_{\alpha\alpha}) = 3\,(c^2-\Pi_{\alpha\alpha})/(2\,c^2)$ (see Fig. 1).

Finally, with the help of f_G (3), let us derive a constrained equilibrium f_C which brings the H-function to a minimum among all the population lists with a fixed trace of the pressure tensor $T(\boldsymbol{\Pi}) = \Pi_{xx} + \Pi_{yy}$. In terms of the parameter set Ω, this is equivalent to require that the point $C = (\Pi^C_{xx}, \Pi^C_{yy})$ belongs to a line segment $\mathsf{L}_T = \big\{\Pi \in \Omega : \Pi_{xx}+\Pi_{yy} = T\big\}$, and the constrained equilibrium C is that minimizing the function H_G (4) on L_T (see Fig. 1). Since the restriction of a convex function to a line is also convex, the solution to the latter problem exists and is found by $[(\partial H_G/\partial \Pi_{xx}) - (\partial H_G/\partial \Pi_{yy})]_{(\Pi_{xx},\Pi_{xx})\in \mathsf{L}_T} = 0$, which yields a cubic equation in terms of the normal stress dif-

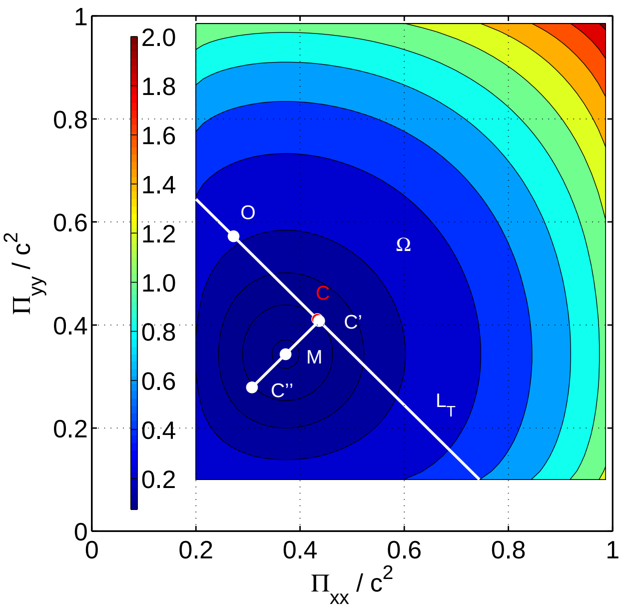

FIG. 1: (Color online) Contour plot of the entropy H_G (4) at $\rho = 1$, $u_x = -0.2$ and $u_y = 0.1$ ($c = 1$). Rectangular domain is the positivity domain Ω. M is the image of the Maxwellian (2). O is the image of a generic non-equilibrium state while C is the image of the constrained equilibrium (7) (minimum of H_G on the line L_T). C' is the low Mach number approximation of C, while the line segment connecting C' and C'' represents admissible generalized equilibria $E(\omega)$ (16) with $E(1) = C'$ and $E(\omega^*) = C''$ at $\omega^* = -1$.

ference $N = \Pi_{xx}^C - \Pi_{yy}^C$,

$$
\begin{aligned}
N^3 + a\,N^2 + b\,N + d = 0, \\
a = -\frac{1}{2}\,(u_x^2 - u_y^2),\ b = (2\,c^2 - T)\,(T - u^2), \\
d = -\frac{1}{2}\,(u_x^2 - u_y^2)\,(2\,c^2 - T)^2.
\end{aligned}
\tag{5}
$$

Let us define $p = -a^2/3 + b$, $q = 2\,a^3/27 - a\,b/3 + d$ and $\Delta = (q/2)^2 + (p/3)^3$. As long as $\Delta \geq 0$, which is well satisfied in the low Mach number limit, the Cardano formula implies

$$
\Pi_{xx}^C = \frac{T}{2} + \frac{1}{2}\left(r - \frac{p}{3\,r} - \frac{a}{3}\right),\ r = \sqrt[3]{-\frac{q}{2} + \sqrt{\Delta}}, \tag{6}
$$

while $\Pi_{yy}^C = T - \Pi_{xx}^C$. Thus, substituting (6) into (3), we find the constrained equilibrium

$$
f_C = f_G(\rho, \boldsymbol{u}, \Pi_{xx}^C(\boldsymbol{u}, T), \Pi_{yy}^C(\boldsymbol{u}, T)). \tag{7}
$$

Before proceeding any further, we mention that the generalized Maxwellian (3) is consistent with and extends the previously known results:

(i) The point of global minimum of the function H_G (4) on Ω is found from $(\partial H_G/\partial \Pi_{\alpha\alpha}) = 0$. The corresponding solution $M = (\Pi_{xx}^M, \Pi_{xx}^M)$, where $\Pi_{\alpha\alpha}^M = -c^2/3 + (2c^2/3)\sqrt{1 + 3(u_\alpha/c)^2}$, recovers the equilibrium f_M (2) upon substitution into (3): $f_M = f_G(\rho, \boldsymbol{u}, \Pi_{xx}^M(\boldsymbol{u}), \Pi_{yy}^M(\boldsymbol{u}))$.

(ii) In Ref. [7], a different LB equilibrium f_Θ was introduced as the entropy minimization problem under fixed density, momentum and energy. That equilibrium was evaluated exactly only for vanishing velocity in [7] while a series expansion was used for $\boldsymbol{u} \neq 0$. The new result reported above solves the problem of Ref. [7] *exactly* for any velocity: Substituting $T = 2\Theta + u^2$ (two-dimensional ideal gas equation of state, with Θ the temperature) into (7), it is simply $f_\Theta(\rho, \boldsymbol{u}, \Theta) = f_G(\rho, \boldsymbol{u}, \Pi_{xx}^C(\boldsymbol{u}, 2\Theta + u^2), \Pi_{yy}^C(\boldsymbol{u}, 2\Theta + u^2))$. Expanding the exact solution Π_{xx}^C (6) in terms of the velocity u yields the approximate solution consistent with Ref. [7], namely

$$
\Pi_{xx}^C = \Theta + \left(\frac{\Theta + 1}{4\Theta}\right) u_x^2 + \left(\frac{3\Theta - 1}{4\Theta}\right) u_y^2 + O(u^4). \tag{8}
$$

(iii) In Ref. [16], a so-called guided equilibrium $\tilde{f}_\Theta$ was introduced in order to derive LB method for compressible flows. That equilibrium is recovered by simply assuming the Maxwell-Boltzmann form of the diagonal components, $\Pi_{xx} = \Theta + u_x^2$ and $\Pi_{yy} = \Theta + u_y^2$, in (3): $\tilde{f}_\Theta(\rho, \boldsymbol{u}, \Theta) = f_G(\rho, \boldsymbol{u}, \Theta + u_x^2, \Theta + u_y^2)$.

Thus, the generalized Maxwellian (3) and its implication, the constrained equilibrium (7), unifies *all* the equilibria introduced previously on the D2Q9 lattice.

Armed with the constrained equilibrium, we now proceed with the derivation of the kinetic equation. By means of the usual equilibrium M and the newly found constrained equilibrium C, let us define the generalized equilibrium $E(\omega) = \left(\Pi_{xx}^E(\omega), \Pi_{yy}^E(\omega)\right)$ as a linear interpolation between the points M and C on the Π-plane

$$
E(\omega) = (1 - \omega)\,M + \omega\,C, \tag{9}
$$

where ω is a free parameter, and its admissible range will be defined next. Thus, the generalized equilibrium list is defined as

$$
f_{GE}(\omega) = f_G(\rho, \boldsymbol{u}, \Pi_{xx}^E(\omega), \Pi_{yy}^E(\omega)). \tag{10}
$$

Considering kinetic equation of the form, $\partial_t f + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} f = J(f)$, let us define the following collision operator

$$
J(f) = \lambda\left[f_{GE}(\omega) - f\,\right], \tag{11}
$$

where $\lambda > 0$ is a parameter, ruling the relaxation toward the generalized equilibrium. In the continuum limit, λ is related to the kinematic viscosity. While Eq. (11) reminds the popular Bhatnagar-Gross-Krook (BGK) model [17], the collision integral (11) depends on two parameters, λ and ω. In view of the analogy of f_G with the anisotropic Gaussian, this is somewhat similar to the so-called ellipsoidal statistical model [17]. However, in our case, the leading order of the macroscopic equations recovered in the continuum limit does not depend on ω, which is a tunable parameter for enhancing the stability of the corresponding LB scheme. Collision operator (11) conserves mass and momentum, i.e. $\langle J(f)\rangle = 0$ and $\langle \boldsymbol{v} J(f)\rangle = 0$. Note that, at $\omega = 0$, (11) reduces to the BGK LB model of Ref. [5], while at $\omega = 1$ it becomes the so-called consistent LB model with energy conservation [7] (see Remark (ii) above).

The key for proving the H-theorem for the kinetic equation is to establish the non-positivity of the entropy production σ due to the relaxation term (11), where

$$
\sigma = \left\langle\, \ln(f/W)\, J(f)\, \right\rangle. \tag{12}
$$

Clearly, if $f = f_M$, then $C = M$ and $\Pi^E(\omega) = \Pi^M$ for any ω. From Remark (i), it follows that entropy production annihilates at the equilibrium, $\sigma(f_M) = 0$. In the general case, we have

$$
\frac{\sigma}{\lambda} \leq H_{GE}(\omega) - H(f) \leq H_{GE}(\omega) - H_G(\Pi), \tag{13}
$$

where $H_{GE}(\omega) = H_G\big(\Pi_{xx}^E(\omega), \Pi_{yy}^E(\omega)\big)$. The first inequality is due to the convexity of the H-function, while the second holds because $f_G(\Pi_{xx}, \Pi_{yy})$, by definition, minimizes H among all the population lists with the moments (Π_{xx}, Π_{yy}). Recalling that $\boldsymbol{\Pi}(f_{GE}(1))$ and $\boldsymbol{\Pi}(f_G(\Pi_{xx}, \Pi_{yy}))$ have the same trace and taking into account the definition of the point C, inequality (13) can be rewritten

$$
\begin{aligned}
\frac{\sigma}{\lambda} &\leq H_{GE}(\omega) - H_{GE}(1) + H_{GE}(1) - H_G(\Pi) \\
&\leq H_{GE}(\omega) - H_{GE}(1).
\end{aligned}
\tag{14}
$$

What remains is to estimate the range of ω such that $H_{GE}(\omega) \leq H_{GE}(1)$. Clearly, since $M = E(0)$ is the absolute minimum of H_G, and because $H_{GE}(\omega)$ is a convex function (a restriction of a convex function to a line), the right hand side of Eq. (14) is non-positive if $0 \le \omega < 1$. This proves non-positivity of the entropy production in the interval $0 \le \omega < 1$. In order to extend the proof to $\omega < 0$, let us consider the entropy estimate [5] (see also [18]):

$$H_{GE}\left(\omega^*\right) = H_{GE}\left(1\right). \tag{15}$$

Thanks to the convexity of $H_{GE}\left(\omega\right)$, the non-trivial solution $\omega^* < 0$ to this equation is unique when it exists. In the opposite case, we take $\omega^* < 0$ from the condition, $E(\omega^*) \in \partial\,\Omega$, where $\partial\,\Omega$ is the boundary of the positivity domain Ω. In both cases, for $\omega^* \le \omega \le 0$, it holds $H_{GE}\left(\omega\right) \le H_{GE}\left(1\right)$. Thus, if ω takes values in the interval $\omega^* \le \omega < 1$, the entropy production is non-positive, $\sigma \le 0$, which proves the existence of the H–theorem for the proposed model. Note that, from the entropy estimate, it follows that ω^*, in general, depends on the state f. However, Eq. (15) drastically simplifies at low Mach numbers which we consider next.

In the case of diffusion scaling [19, 20], i.e. the flow regime with Kn $\sim$ Ma $\sim u/c \ll 1$, where Kn is the Knudsen number and Ma is the Mach number, equation (8) simplifies to $\Pi_{xx}^C = (T/2) + (\Pi_{xx}^M - \Pi_{yy}^M)/2 + O(u^4)$ and similar to Π_{yy}^C. Introducing these results in Eqs. (9) allows one to recast the definition of the generalized equilibrium, namely

$$\Pi_{\alpha\alpha}^E(\omega) = \Pi_{\alpha\alpha}^M + \omega\,\frac{T - T_M}{2} + O(u^4). \tag{16}$$

Using (16) in the definition of the collision operator (11) and neglecting all the terms in the higher moments which are two order of magnitude (with regards to u) smaller than the corresponding terms required to recover incompressible Navier–Stokes equations [20], the collision operator can be simplified to

$$J'(f) = A\left(f_M - f\right), \tag{17}$$

where $A = \lambda\, B^{-1} \Lambda\, B$ and 9×9 matrices B and Λ are

$$\Lambda = \mathrm{diag}\left([0,0,0], \begin{bmatrix} r_+ & r_- \\ r_- & r_+ \end{bmatrix}, [1,1,1,1]\right), \tag{18}$$
$$B = \left[1,\, v_x,\, v_y,\, v_x^2,\, v_y^2,\, v_x v_y,\, v_x^2 v_y,\, v_x v_y^2,\, v_x^2 v_y^2\right]^T,$$

with $r_\pm = (r \pm 1)/2$ and $r = 1 - \omega$. Operator J' is a MRT collisional operator with collision matrix A (characterized by two relaxation frequencies λ and $\delta = r\lambda$). It is possible to prove by means of the asymptotic analysis [21] that, in the continuum limit, J' leads to the kinematic (shear) viscosity and the second (bulk) viscosity coefficients given respectively by

$$\nu = \frac{c^2}{3\lambda}, \qquad \xi = \frac{c^2}{3\delta}. \tag{19}$$

Finally, for low Mach numbers, the entropy H_{GE} can be estimated as follows:

$$H_{GE} = \rho \ln \rho + \frac{3}{2}\,\rho\, u^2 + \frac{9}{8}\,\rho\,(T - T_M)^2 \omega^2 + O(u^6). \tag{20}$$

Using (20) in the entropy condition (15), we find $\omega^* \approx -1$ (see Fig. 1). Consequently, the stability region of the relaxation frequency δ controlling the bulk viscosity is estimated $0 < \delta < 2\lambda$ or, taking into account Eq. (19), equivalently $0 < \nu/\xi < 2$. In particular, for high Reynolds number flows, the ratio ν/ξ tends to the lowest limit, i.e. large bulk viscosity is required to make more stable the numerical computations.

Since the bulk viscosity controls the attenuation of acoustic waves, which are fictitious when searching for the incompressible limit, increasing this tunable parameter allows one to mitigate the effects of fictitious compressibility and hence it increases the stability region of the scheme. In order to check the accuracy of the scheme, let us consider the Taylor-Green vortex flow test. Let us consider a square domain $(x, y) \in [0, 2\pi/k] \times [0, 2\pi/k]$. The Taylor-Green vortex flow has the following analytical solution [22]:

$$u_x = -u_0 \cos\left(kx\right) \sin\left(ky\right) \exp\left(-2\nu k^2 t\right), \tag{21}$$

$$u_y = u_0 \cos\left(ky\right) \sin\left(kx\right) \exp\left(-2\nu k^2 t\right), \tag{22}$$

$$p = -\frac{u_0^2}{4}\left[\cos\left(2kx\right) + \cos\left(2ky\right)\right] \exp\left(-4\nu k^2 t\right). \tag{23}$$

where the pressure $p = (c^2 \rho)/(3\,\rho_0)$. It is immediate to prove that

$$\Phi(t) = \frac{1}{2}\int_0^{2\pi/k}\int_0^{2\pi/k} (u_x^2 + u_y^2) k^2 dx dy = \frac{u_0^2}{4} \exp\left(-4\nu k^2 t\right). \tag{24}$$

The previous formula suggests a simple way to measure the actual kinematic viscosity. Introducing the simulation time $t \in [0, t_0]$, the measured kinematic viscosity is given by

$$\nu_* = -\frac{\ln\left(4\Phi(t_0)/u_0^2\right)}{4k^2 t_0}. \tag{25}$$

In the following numerical results, we set $k = 1$, $u_0 = 1$, $\rho_0 = 1$ and $t_0 = 5$. Consequently the Reynolds number becomes Re $= 2\pi/\nu$. Let us consider a homogeneous mesh made of 160×160 nodes, which implies Knudsen number Kn $= 1/160$. Let us select the Mach number as Ma $= 1/16$. Some numerical tests are reported for different kinematic viscosity ν and bulk viscosity ξ. The numerical results are reported in Table I and compared with the standard lattice BGK (LBGK) model. First of all, this test shows that the model recovers the right kinematic viscosity. Moreover, the relaxation frequency δ, controlling the bulk viscosity, does not affect the leading part of the solution. According to the previous test, even large bulk viscosities may be adopted without affecting significantly the numerical results.

To conclude, the generalized Maxwellian (3) opens a new perspective on the LB method. Various LB equilibria introduced in the past are special cases of (3). Important application of (3), considered in this paper, is a novel

TABLE I: Taylor-Green vortex flow test. Some numerical tests are reported for different kinematic viscosity ν and bulk viscosity ξ. The mesh is made of 160×160 nodes. The Knudsen number is $\mathrm{Kn} = 1/160$, the Mach number $\mathrm{Ma} = 1/16$ and finally the Reynolds number $\mathrm{Re} = 2\pi/\nu$. The actual kinematic viscosity ν_* is measured by means of Eq. (25) and the relative error $(\nu - \nu_*)/\nu$ is reported as well.

	ν/ξ	ν	Measured ν_*	Error [%]
LBGK	1	0.001	0.00102065	2.0648
present	0.1	0.001	0.00102071	2.0713
present	0.01	0.001	0.00102106	2.1058
LBGK	1	0.010	0.00998509	-0.1491
present	0.1	0.010	0.00998555	-0.1445
present	0.01	0.010	0.00998654	-0.1346
LBGK	1	0.100	0.09977323	-0.2268
present	0.1	0.100	0.09977355	-0.2264
present	0.01	0.100	0.09977230	-0.2277

class of entropic multiple–relaxation–time (E–MRT) LB models. They enjoy both the H–theorem and the additional free–tunable parameter for controlling the bulk viscosity. Hence, they combine the two most successful strategies for enhancing stability of LB for high Reynolds number simulations. Because all the results above are derived in a closed form, numerical implementation of the E–MRT LB models is straightforward. Preliminary numerical results demonstrated that efficient stabilization of the LB simulation without loss of accuracy is indeed achieved with the suggested scheme. Moreover, the implementation is not much different from the familiar LBGK scheme, unlike the standard MRT model. These results show that the present model can be used for enhancing stability instead of the most popular LBGK method. Details of the implementation and numerical results will be reported in a separate publication. I.V.K. acknowledges support of CCEM-CH.

[1] S. Succi, *The Lattice Boltzmann Equation for Fluid Dynamics and Beyond* (Oxford University Press, New York, 2001), 2nd ed.

[2] S. Succi, I. V. Karlin, and H. Chen, Rev. Mod. Phys. **74**, 1203 (2002).

[3] H. Chen, S. Kandasamy, S. Orszag, R. Shock, S. Succi, and V. Yakhot, Science **301**, 633 (2003).

[4] I. V. Karlin, A. N. Gorban, S. Succi, and V. Boffi, Phys. Rev. Lett. **81**, 6 (1998).

[5] I. V. Karlin, A. Ferrante, and H. C. Ottinger, Europhys. Lett. **47**, 182 (1999).

[6] S. Ansumali, I. V. Karlin, and H. C. Ottinger, Europhys. Lett. **63**, 798 (2003).

[7] S. Ansumali and I. V. Karlin, Physical Review Letters **95**, 260605 (2005), ISSN 0031-9007.

[8] S. S. Chikatamarla and I. V. Karlin, Phys. Rev. Lett. **97**, 190601 (2006).

[9] F. J. Higuera, S. Succi, and R. Benzi, Europhys. Lett. **9**, 345 (1989).

[10] D. d'Humières, in *Rarefied Gas Dynamics: Theory and Simulations*, edited by B. D. Shizgal and D. P. Weave (AIAA, Washington, D.C., 1992), vol. 159 of *Prog. Astronaut. Aeronaut.*, pp. 450–458.

[11] P. Lallemand and L.-S. Luo, Phys. Rev. E **61**, 6546 (2000).

[12] Y. Qian, D. d'Humières, and P. Lallemand, Europhys. Lett. **17**, 479 (1992).

[13] X. Shan and X. He, Phys. Rev. Lett. **80**, 65 (1998).

[14] A. N. Gorban and I. V. Karlin, *Invariant Manifolds for Physical and Chemical Kinetics*, no. 660 in Lecture Notes in Physics (Springer-Verlag, Berlin, 2005), 1st ed.

[15] T. Koga, The Journal of Chemical Physics **22**, 1633 (1954).

[16] N. I. Prasianakis and I. V. Karlin, Physical Review E - Statistical, Nonlinear, and Soft Matter Physics **76**, 016702 (2007).

[17] C. Cercignani, *The Boltzmann Equation and its Application* (Springer, New York, 1988).

[18] S. Brull and J. Schneider, Continuum Mech. Therm. **20**, 63 (2008).

[19] M. Junk, A. Klar, and L.-S. Luo, J. Computat. Phys. **210**, 676 (2005).

[20] P. Asinari and T. Ohwada, Comput. Math. Appl. (2008), (in press).

[21] P. Asinari, Phys. Rev. E **78**, 016701 (2008).

[22] S.A. Orszag, in *Proceedings of the Symposium on Computer Methods in Applied Sciences and Engineering*, edited by R. Glowinski and J. L. Lions (Springer-Verlag, New York, 1974), part II.