A geometric proof of the Karpelevich-Mostow’s Theorem

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Abstract

In this paper we give a geometric proof of the Karpelevich’s theorem that asserts that a semisimple Lie subgroup of isometries, of a symmetric space of non compact type, has a totally geodesic orbit. In fact, this is equivalent to a well-known result of Mostow about existence of compatible Cartan decompositions.

1. Introduction.

In this paper we address the problem of giving a geometric proof of the following theorem of Karpelevich.

Theorem 1.1. (Karpelevich [7]) Let $M$ be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected and semisimple subgroup $G \subset \text{Iso}(M)$ has a totally geodesic orbit $G.p \subset M$.

It is well-known that Karpelevich’s theorem is equivalent to the following algebraic theorem.

Theorem 1.2. (Mostow [8, Theorem 6]) Let $\mathfrak{g}'$ be a real semisimple Lie algebra of non compact type and let $\mathfrak{g} \subset \mathfrak{g}'$ be a semisimple Lie subalgebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition for $\mathfrak{g}$. Then there exists a Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ for $\mathfrak{g}'$ such that $\mathfrak{k} \subset \mathfrak{k}'$ and $\mathfrak{p} \subset \mathfrak{p}'$.

The proof of the above theorems is very algebraic in nature and uses delicate arguments related to automorphisms of semisimple Lie algebras.

For the real hyperbolic spaces, i.e. when $\mathfrak{g}' = \mathfrak{so}(n,1)$, there are two geometric proofs of Karpelevich’s theorem [4], [2]. The proof in [4] is based on the study of minimal orbits of isometries subgroups, i.e. orbits with zero mean curvature. The approach in [2] is based on hyperbolic dynamics. It is interesting to note that both proofs are strongly based on the fact that the boundary at infinity of real hyperbolic spaces has a simple structure.

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The only non-trivial algebraic tool that we will use is the existence of a Cartan decomposition of a non compact semisimple Lie algebra. But this can also be proved geometrically as was explained by S.K. Donaldson in [5].

Here is a brief explanation of our proof of Theorem 1.1. We first show that a simple subgroup \( G \subset \text{Iso}(M) \) has a minimal orbit \( G.p \subset M \). Then, by using a standard totally geodesic embedding \( M \hookrightarrow P \), where \( P = \text{SL}(n, \mathbb{R})/\text{SO}(n) \), we will show that \( G.p \) is, actually, a totally geodesic submanifold of \( M \).

2. Preliminaries.

The results in this section are well known and are included to orient the non-specialist reader.

The equivalence between Theorems 1.1 and 1.2 is a consequence of the following Elie Cartan’s famous and remarkable theorem.

**Theorem 2.1.** (Elie Cartan) Let \( M \) be a Riemannian symmetric space of non positive curvature without flat factor. Then the Lie group \( \text{Iso}(M) \) is semisimple of non compact type. Conversely, if \( \mathfrak{g} \) is a semisimple Lie algebra of non compact type then there exist a Riemannian symmetric space \( M \) of non positive curvature without flat factor such that \( \mathfrak{g} \) is the Lie algebra of \( \text{Iso}(M) \).

The difficult part of the proof of the above theorem is the second part. Namely, the construction of the Cartan decomposition \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \), where \( \mathfrak{k} \) is maximal compact subalgebra of \( \mathfrak{g} \) and the Killing form \( B \) of \( \mathfrak{g} \) is positive definite on \( \mathfrak{p} \). The standard and well-known proof of the existence of a Cartan decomposition is long and via the classification theory of complex semisimple Lie algebras, i.e. the existence of a real compact form (see e.g. [6]). There is also a direct and geometric proof of the existence of a Cartan decomposition [5].

On the other hand, when \( \mathfrak{g} = \text{Lie}(\text{Iso}(M)) \), where \( M \) is a Riemannian symmetric space of non positive curvature without flat factor, a Cartan decomposition of \( \mathfrak{g} \) can be constructed geometrically. Namely, \( \mathfrak{g} = \text{Lie}(\text{Iso}(M)) = \mathfrak{k} \oplus \mathfrak{p} \) where \( \mathfrak{k} \) is the Lie algebra of the isotropy group \( K_p \subset \text{Iso}(M) \) and \( \mathfrak{p} := \{ X \in \text{Lie}(\text{Iso}(M)) : (\nabla X)_p = 0 \} \).

It is well-known that the Riemannian symmetric spaces \( P = \text{SL}(n, \mathbb{R})/\text{SO}(n) \) are the universal Riemannian symmetric space of non positive curvature. Namely, any Riemannian symmetric space of non compact type \( M = G/K \) can be totally geodesically embedded in some \( P \) (up to rescaling the metric in the irreducible De Rham factors). A proof of this fact follows from the following well-known result (c.f. Theorem 1 in [5]).

**Proposition 2.2.** Let \( \mathfrak{g} \subset \mathfrak{sl}(n, \mathbb{R}) \) be a semisimple Lie subalgebra and let \( \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p} \) be a Cartan decomposition. Then there exists a Cartan decomposition \( \mathfrak{sl}(n, \mathbb{R}) = \mathfrak{a} \oplus \mathfrak{s} \) such that \( \mathfrak{t} \subset \mathfrak{a} \) and \( \mathfrak{p} \subset \mathfrak{s} \). Thus, if \( G \subset \text{SL}(n) \) is semisimple, \( G \) has a totally geodesic orbit in \( P = \text{SL}(n)/\text{SO}(n) \). Indeed, any Riemannian symmetric space of non positive curvature \( M \), without flat factor, can be totally geodesically embedded in some \( P = \text{SL}(n)/\text{SO}(n) \).

**Proof.** Notice that any Cartan decomposition of \( \mathfrak{sl}(n, \mathbb{R}) \) is given by the anti-symmetric \( \mathfrak{a} \) and symmetric matrices \( \mathfrak{s} \) w.r.t. a positive definite inner product on \( \mathbb{R}^n \). Since \( \mathfrak{g}^\ast := \mathfrak{t} \oplus i\mathfrak{p} \) is a compact Lie subalgebra of \( \mathfrak{sl}(n, \mathbb{C}) \), there exists a positive definite Hermitian form \( (\cdot | \cdot) \) of \( \mathbb{C}^n \).
invariant by $g^*$. By defining $(\ , \ ) := \text{Real}( \ | \ )$ it follows that $\mathfrak{k} \subset \mathfrak{A}$ and $\mathfrak{p} \subset \mathfrak{S}$. □

Let $S_\infty(M)$ be the sphere or boundary at infinity of $M$, i.e. $S_\infty(M)$ is the set of equivalence classes of asymptotic geodesics rays (see [5] or [3, Chapter II.8] for details).

Here is another corollary of the existence of the totally geodesic embedding $M \hookrightarrow \mathcal{P}$.

**Corollary 2.3.** Let $M$ be a Riemannian symmetric space of non positive curvature without flat factor. Then a connected and semisimple Lie subgroup $G \subset \text{Iso}(M)$ of non compact type has no fixed points in $S_\infty(M)$.

This corollary is false see Corrigendum in http://arxiv.org/abs/1104.0892

We include the following proposition.

**Proposition 2.4.** Let $M$ be a Riemannian symmetric space of non positive curvature without flat factor. Let $S = \mathbb{R}^N \times M$ be a symmetric space of non positive curvature with flat factor $\mathbb{R}^N$. If $G \subset \text{Iso}(S)$ is a connected non compact simple Lie group then $G \subset \text{Iso}(M)$.

Proof. Let $\mathfrak{g}$ be the Lie algebra of $G$. Then the projection $\pi : \mathfrak{g} \twoheadrightarrow \text{Lie}(\text{Iso}(\mathbb{R}^N))$ is injective or trivial i.e. $\pi \equiv 0$. If $\pi$ is injective then a further composition with the projection to $\text{so}(N)$ gives that $\mathfrak{g}$ must carry a bi-invariant metric. So, $\mathfrak{g}$ can not be simple and non compact. □

Let $G$ be a semisimple Lie group and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. A subspace $T \subset \mathfrak{p}$ is called a Lie triple system if $[T, [T, T]] \subset T$. It is well-known that there is a 1-1 correspondence between Lie triple systems $T$ of $\mathfrak{p}$ and totally geodesic submanifolds through the base point $[K] \in G/K$ (see [6]).

### 3. Minimal and totally geodesic orbits.

We will need the following proposition (see Lemma 3.1. in [4] or Proposition 5.5. in [1]).

**Proposition 3.1.** Let $M$ be a Riemannian symmetric space of non positive curvature without flat factor and let $G \subset \text{Iso}(M)$ be a connected group of isometries. Assume that $G$ has a totally geodesic orbit $G.p$. Then any other minimal orbit $G.q$ is also a totally geodesic submanifold of $M$. Moreover, if $G$ is semisimple then the union of totally geodesic $G$-orbits $T_G$ is a totally geodesic submanifold of $M$ which is a Riemannian product $T_G = (G.p) \times A$ where $A$ is a totally geodesic submanifold of $M$.

Proof. Let $G.p$ be the totally geodesic orbit and let $G.q \neq \{q\}$ be another orbit. Let $\gamma$ be a geodesic in $M$ that minimizes the distance between $q$ and $G.p$ (such geodesic do exists since totally geodesic submanifolds of $M$ are closed and embedded). Eventually by changing the base point $p$ by another in the orbit we may assume that $\gamma(0) = p$ and $\gamma(1) = q$. A simple computation using the Killing equation shows that $\dot{\gamma}(t)$ is perpendicular to $T_{\gamma(t)}(G.\gamma(t))$, for all $t$.

Let $X$ be a Killing field in the Lie algebra of $G$ such that $X.q \neq 0$ and let $\phi_s^X$ be the one-parameter group of isometries generated by $X$. Define $h : I \times \mathbb{R} \to M$ by $h_s(t) := \phi_s^X \cdot \gamma(t)$. Note that $X.h_s(t) = \frac{\partial h}{\partial s}$ and that, for a fixed $s$, $h_s(t)$ is a geodesic.
Let $A_{\gamma}(t)$ be the shape operator, in the direction of $\dot{\gamma}(t)$ of the orbit $G\cdot\gamma(t)$. Define $f(t) := -(A_{\gamma}(t)(X,\gamma(t)), X,\gamma(t)) = \langle \frac{D}{D\tau}\partial_{\tau}, X, h_{\tau}(t) \rangle |_{s=0}$. Now a computation as in Lemma 3.1. in [4] or Proposition 5.5. in [1] implies that $\frac{d}{dt}f(t) \geq 0$. Since $f(0) = 0$, due to the fact that $G.p$ is totally geodesic, we obtain that $f(1) = -(A_{\gamma}(1)(X,q), X,q) \geq 0$. Hence $A_{\gamma}(1)$ is negative semidefinite. Since $G.q$ is minimal, $\text{trace}(A_{\gamma}(1)) = 0$, we get that $f(t) \equiv 0$. Thus, $\langle R(\dot{\gamma}(t), X,\gamma(t))\dot{\gamma}(t), X,\gamma(t) \rangle \equiv 0$ and $\nabla_{\dot{\gamma}(t)}(X,\gamma(t)) \equiv 0$. Notice that the tangent spaces $T_{\gamma(t)}G \cdot \gamma(t)$ are parallel along $\gamma(t)$ in $M$. So the normal spaces $\nu_{\gamma(t)}G \cdot \gamma(t)$ are also parallel along $\gamma(t)$ in $M$. Since $M$ is a symmetric space of non positive curvature the condition $\langle \nabla_{\dot{\gamma}(t)} Y, \eta(t) \rangle = \langle \nabla_{\dot{\gamma}(t)} \nabla_{X} Y, \eta(t) \rangle = \langle \nabla_{X} \nabla_{\dot{\gamma}(t)} Y, \eta(t) \rangle + \langle R(\dot{\gamma}(t), X,\gamma(t))(Y), \eta(t) \rangle \equiv 0$. Since $\langle \nabla_{X} Y, \eta(0) \rangle = 0$ we get that the $G$-orbits $G\cdot\gamma(t)$ are totally geodesic submanifolds of $M$. This show the first part.

For the second part let $K' := \text{Iso}(M)_p$ be the isotropy subgroup at $p \in M$ and let $\mathfrak{p}' \subset \text{Lie}(\text{Iso}(M))$ be such that $X \in \mathfrak{p}'$ if $(\nabla X)_p = 0$. Thus, $\text{Lie}(\text{Iso}(M)) = \mathfrak{t}' \oplus \mathfrak{p}'$ is a Cartan decomposition of $\text{Lie}(\text{Iso}(M))$. Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$. Since $G.p$ is totally geodesic in $M$ we get that $\mathfrak{t} \subset \mathfrak{p}$ and $\mathfrak{p} \subset \mathfrak{p}'$. Let $\alpha := \{Y \in \mathfrak{p}' : Y \perp \mathfrak{p}$ and $[Y, p] = 0 \}$ which is a Lie triple system of $\mathfrak{p}'$. Moreover, $\mathfrak{n} := \mathfrak{p} \oplus \alpha$ is also a Lie triple system of $\mathfrak{p}'$. So, $N := exp_p(\mathfrak{n}) = exp_p(\mathfrak{p}) \times exp_p(\alpha)$ is a $G$-invariant totally geodesic submanifold of $M$. Notice that (by construction) $N \subset T_G$.

Let $G.p$ any other totally geodesic $G$-orbit. From the computation in the first part we get $R(\dot{\gamma}(t), X,\gamma(t))(\cdot) \equiv 0$ which implies $\gamma'(0) \in \alpha$. This shows $T_G \subset N$. Then $N = T_G = (G.p) \times A$ where $A := exp_p(\alpha)$ is a totally geodesic submanifold of $M$ associated to the Lie triple system $\alpha$. \(\square\)

4. Karpelevich’s Theorem for $G$ a simple Lie group.

Here is the first step to prove Theorem 1.1.

**Theorem 4.1.** Let $M$ be a Riemannian symmetric space of non positive curvature without flat factor. Then any connected, simple and non compact Lie subgroup $G \subset \text{Iso}(M)$ has a minimal orbit $G.p \subset M$.

**Proof.** Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of the Lie algebra $\mathfrak{g} := \text{Lie}(G)$ and let $K \subset G$ be the maximal compact subgroup associated to $\mathfrak{t}$. Let $\Sigma$ be the set of fixed points of $K$. Notice that $\Sigma \neq \emptyset$ by Cartan’s fixed point theorem. Since $G$ is simple all $G$-orbits $G.x$ through points in $x \in \Sigma$ are homothetic i.e. the Riemannian metric induced on $G.x$ and $G.y$ differ from a constant multiple for $x, y \in \Sigma$. Let $x_0 \in \Sigma$ be a point in $\Sigma$ and let $g_0$ be the Riemannian metric on $G.x_0 = G/K$ induced by the Riemannian metric $g = \langle \cdot, \cdot \rangle$ of $M$. So if $y \in \Sigma$ the Riemannian metric $g_y$ on $G.y$ is given by $g = \lambda(y) \cdot g_0$. Notice that if $X \in \mathfrak{p}$ is unitary at $x_0$ (i.e. $g_0(X(x_0), X(x_0)) = 1$) then $\lambda(y) = g(X(y), X(y)) = \|X(y)\|^2$. We claim that $\lambda(y)$ has a minimum in $\Sigma$. Indeed, if $y_n \to \infty \in S_\infty(\Sigma) \subset S_\infty(M)$ (where $y_n \in \Sigma$) and $\lambda(y_n) \leq \text{const}$ then the monoparametric Lie group $\psi^X_{t_1} \subset G$ associated to any unitary $X \in \mathfrak{p}$ at $x_0 \in \Sigma$ must fix $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$. Thus, since $X \in \mathfrak{p}$ is arbitrary and $\mathfrak{p}$ generate $\mathfrak{g}$ we get that $\infty \in S_\infty(\Sigma) \subset S_\infty(M)$ is a fixed point of $G$. This contradicts Corollary 2.3. So there exist $y_0 \in \Sigma$ such that $\lambda$ has a minimum. Notice that the volume element $Vol_y$ of an orbit $G.y$ is given by $\frac{1}{\lambda^n}Vol_{x_0}$, where $n = \dim(G/K)$. Now a simple computation shows that the mean curvature vector of $G.y_0$ vanish and we are done. \(\square\)
Now we are ready to prove Karpelevich’s Theorem 1.1 for $G$ a simple non compact Lie subgroup of $Iso(M)$.

**Theorem 4.2.** Let $M$ be a Riemannian symmetric space of non positive curvature. Then any connected, simple and non compact Lie subgroup $G \subset Iso(M)$ has a totally geodesic orbit $G.p \subset M$.

**Proof.** According to Proposition 2.4 we can assume that $M$ has no flat factor. Let $i : (M, g) \hookrightarrow (\mathcal{P}, h)$ be a totally geodesic embedding as in Proposition 2.2. Notice that the pull-back metric $i^*h$ can eventually differ (up to constant factors) from $g$ on each irreducible De Rham factor of $M$. Anyway, totally geodesic submanifolds of $(M, g)$ and $(M, i^*h)$ are the same since totally geodesic submanifolds are defined in terms of the same Levi-Civita connection $\nabla^g = \nabla^{i^*h}$.

Notice that $G$ also acts by isometries on $(M, i^*h)$. Indeed, $G$ can be also regarded as a subgroup of $Iso(\mathcal{P})$. Now Proposition 2.2 implies that $G$ has a totally geodesic orbit $G.p$ in $\mathcal{P}$. The above proposition shows that $G$ has a minimal orbit $G.y_0$ in $(M, i^*h)$. Since the embedding $M \hookrightarrow \mathcal{P}$ is totally geodesic we get that the $G$-orbit $G.y_0$ is also a minimal submanifold of $\mathcal{P}$. Then Proposition 3.1 implies that $G.y_0$ is a totally geodesic submanifold of $\mathcal{P}$. Thus, $G.y_0$ is a totally geodesic submanifold of $(M, i^*h)$ and so $G.y_0$ is also a totally geodesic submanifold of $(M, g)$. □

5. **Karpelevich’s Theorem.**

Let $G \subset Iso(M)$ be a semisimple, connected Lie group. Then the Lie algebra $\mathfrak{g} = Lie(G) = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a sum of a simple Lie algebra $\mathfrak{g}_1$ and a semisimple Lie algebra $\mathfrak{g}_2$. Due to Cartan’s fixed point theorem we can assume that each simple factor of $\mathfrak{g}$ is non compact. We are going to make induction on the number of simple factors of the semisimple Lie algebra $\mathfrak{g}$. Let $G_1$ (resp. $G_2$) be the simple Lie group associated to $\mathfrak{g}_1$ (resp. the semisimple Lie subgroup associated to $\mathfrak{g}_2$). Let $T_{G_1} \subset M$ be the union of the totally geodesic orbits of the simple subgroup $G_1$ acting on $M$. Notice that Theorem 4.2 implies that $T_{G_1} \neq \emptyset$ and Proposition 3.1 implies that $T_{G_1} = (G_1 \cdot p) \times A$ is a totally geodesic submanifold of $M$, where $G_1 \cdot p$ is a totally geodesic $G_1$-orbit. Notice that $G_2$ acts on $T_{G_1} = (G_1 \cdot p) \times A$. Then $G_2$ (or eventually a quotient $G_2/\sim$ of it) acts on $A$. Since $A$ is symmetric space of non positive curvature we get (by induction) that the semisimple subgroup $G_2$ (or eventually a quotient $G_2/\sim$ of it) has a totally geodesic orbit $S \subset A$. Then $(G_1 \cdot p) \times S$ is a totally geodesic orbit of $G$ and this finish our proof of Karpelevich’s Theorem 1.1.

**References**


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