Smooth surfaces with non smooth nullity

# A FLAT BUT NON-SMOOTHLY RULED SURFACE

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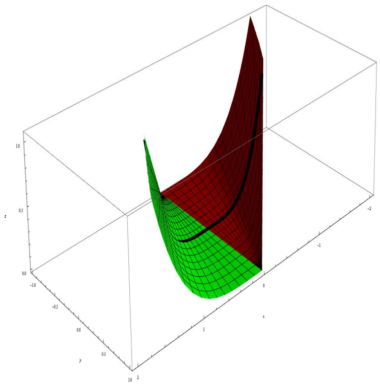
**As for everything else, so for a mathematical theory: beauty can be perceived but not explained.**

A. Cayley [a]

[a]Quoted in J. R. Newman, The World of Mathematics (New York 1956).

Here we discuss an example of flat but not $C^\infty$-ruled surface given at page 93 of *Differential Geometry Curves-Surfaces-Manifolds* by Wolfgang Kühnel, Second Edition, AMS 2006. A similar, but not explicit example, due to E. Heintze is in 3.9.4 pag.68-39 of Klingenberg's book "A course in Differential Geometry", Springer, New York, 1978.



The surface is constructed by gluing the red cone $\mathcal{C}_1$ with the green cone $\mathcal{C}_2$ along a common generatrix (the $y$-axis). The blue curve $\mathbf{c}$, the generatrix of both cones, is in the $xz$ plane. The vertex of $\mathcal{C}_1$ is the point $(0,1,0)$ while the vertex of $\mathcal{C}_2$ is the point $(0,-1,0)$. The resulting surface is $C^\infty$ near the origin $(0,0,0)$ but is not $C^\infty$-ruled around $(0,0,0)$. See below for more 3D-plots of the surface. Let $\mathbf{c} \in \mathbb{R}^3$ be a curve in the $xz$-plane give by

$$\mathbf{c}(x) = (x, 0, z(x))$$

where the function $z(x)$ is $C^\infty$, $z(0) = 0$ and $\frac{\mathrm{d}^n z}{\mathrm{d}x^n}\big|_0 = 0$ for $n = 1, 2, \cdots$. We assume also that $\frac{\mathrm{d}^2 z(x)}{\mathrm{d}x^2} > 0$ for $x \neq 0$.

Let $\mathcal{C}_1$ be the cone containing $\mathbf{c}$ with vertex at the point $(0,1,0)$. The tangent plane $\mathrm{T_O}\mathcal{C}_1$ at the origin O is the $xy$-plane. Since $\mathcal{C}_1$ is a $C^\infty$ surface near O there exist a $C^\infty$-function $f_1(x,y)$ such that near O the cone $\mathcal{C}_1$ is given by the graph $(x, y, f_1(x,y))$.

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A simple calculation shows that near $(0,0)$

$$f_1(x,y) = z(\frac{x}{1-y})(1-y)\,.$$

So for every $n, m \in \mathbb{N}$ we have

$$\left.\frac{\partial^{n+m} f_1}{\partial x^n \partial y^m}\right|_{(0,y)} = 0 \tag{1}$$

Let now $\mathcal{C}_2$ be the cone containing $\mathbf{c}$ but with vertex at the point $(0,-1,0)$. The cone $\mathcal{C}_2$ is also given by a graph of a function $f_2(x,y)$ near O. By the same argument $f_2$ also satisfy equation (1).

Then the function $f$ defined as

$$f(x,y) = \begin{cases} f_1(x,y) & \text{if } x \leq 0 \\ f_2(x,y) & \text{if } x > 0 \end{cases}$$

is a $C^\infty$-function near $(0,0)$.

**Claim:** *The graph $(x, y, f(x,y))$ is an example of $C^\infty$-surface with zero Gaussian curvature but is not a $C^\infty$-ruled surface near the origin* O .

*Proof.* We already seen that $f$ is $C^\infty$. Since the graph is the union of two cones and the Gauss curvature $\kappa(x,y)$ is smooth it follows that $\kappa(x,y) \equiv 0$ since $\kappa$ restricted to each cone is identically zero.

Assume that the graph of $f$ is ruled near O i.e. the graph is the image of a map $(s,t) \to \alpha(t) + s\beta(t)$, with $\alpha, \beta \in C^\infty$, $\alpha(0) = \mathrm{O}$ and $\alpha'(0) \wedge \beta(0) \neq 0$. So $\beta(0) = (0,b,0)$ with $b \neq 0$. Since $\alpha'(0)$ is tangent to the $xy$-plane it follows that the projection of $\alpha$ to the $xy$-plane is transversal to the $y$-axis at $(0,0)$. This imply the existence of $C^\infty$-functions $t(x), y(x)$ such that

$$\alpha(t(x)) = (x, y(x), f(x, y(x)))\,.$$

Notice that the hypothesis $\frac{\mathrm{d}^2 z(x)}{\mathrm{d}x^2} > 0$ imply that the only straight lines contained in the cones $\mathcal{C}_1, \mathcal{C}_2$ are the straight lines connecting one of the vertices with the generatrix $\mathbf{c}$. Indeed, the shape operator of the graph in a point $(x, y, f(x,y))$ is not zero if $x \neq 0$. Hence any segment contained in the graph must be tangent to the kernel of the shape operator i.e. must be part of a straight lines connecting one of the vertices with the generatrix $\mathbf{c}$ as claimed. More precisely, along $\alpha(t(x))$ we have the following continuous vector field $\mathrm{R}(x)$ pointing in the direction of the rulings:

$$\mathrm{R}(x) = \begin{cases} (x, y(x) - 1, f(x, y(x))) & \text{if } x < 0 \\ (-x, -y(x) - 1, -f(x, y(x))) & \text{if } x \geq 0 \end{cases}$$

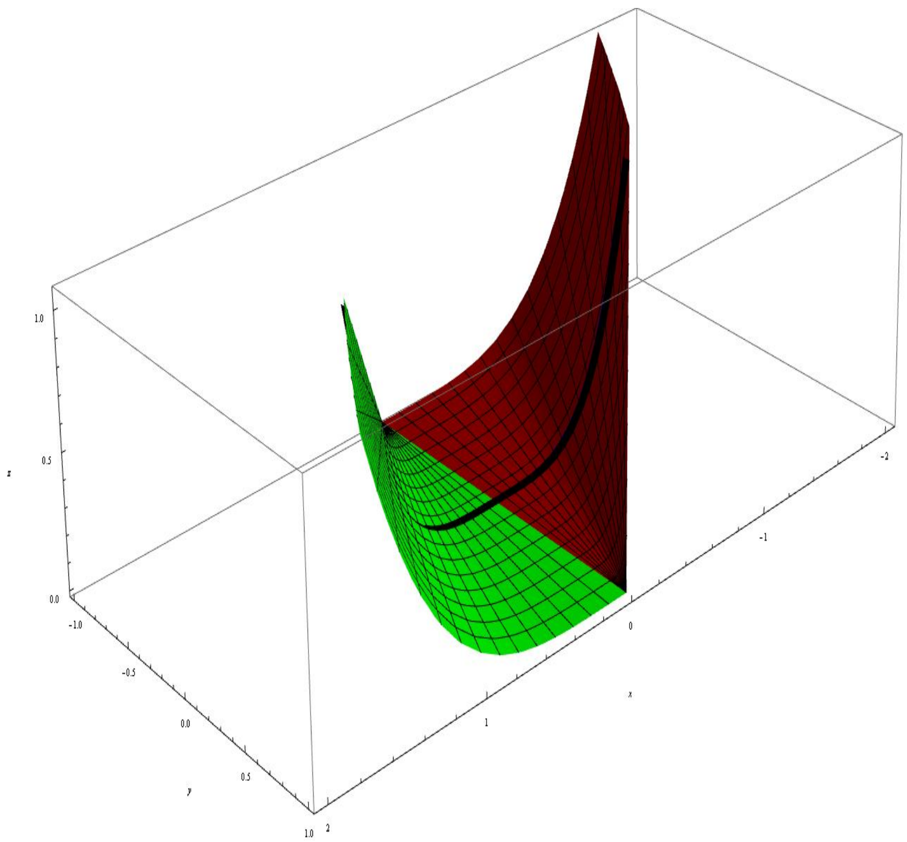
Notice that $\mathrm{R}(x)$ never vanish.

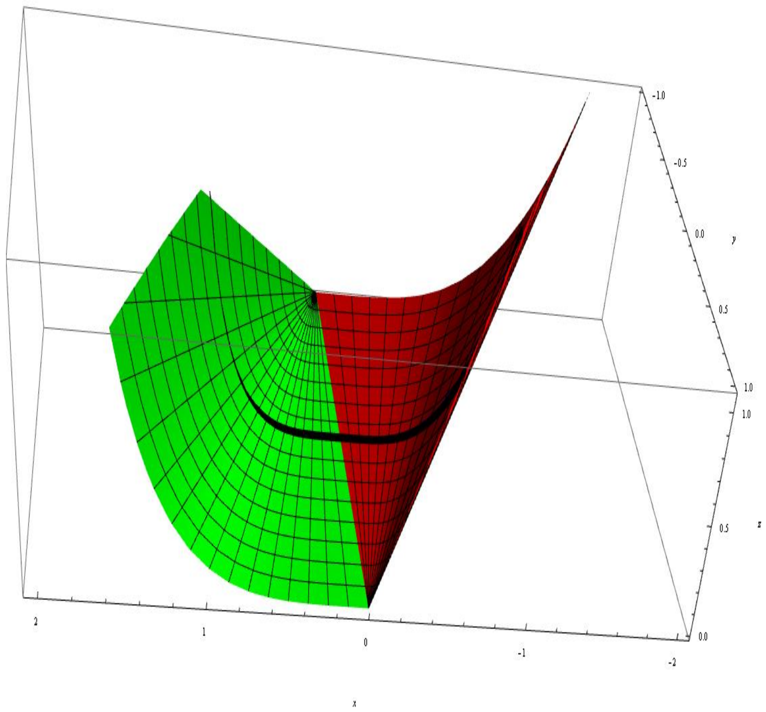
Then there exist a function $m(x)$ such that

$$\beta(t(x)) = m(x)\mathrm{R}(x)\,.$$

Since both $\beta(t(x))$ and $\mathrm{R}(x)$ are continuos and never vanish along $\alpha$ it follows that $m(x)$ is continuos and never zero. The first component of $\beta(t(x))$ is $-m(x)|x|$ which is not derivable at $x = 0$ because $m(0) \neq 0$ since $\beta(0) \neq 0$. This is a contradiction since $\beta(t(x))$ is $C^\infty$ . $\square$

**Proposition A.** *The above non-ruled $C^\infty$-surface has the property that each of its points is part of a segment included in the surface.*

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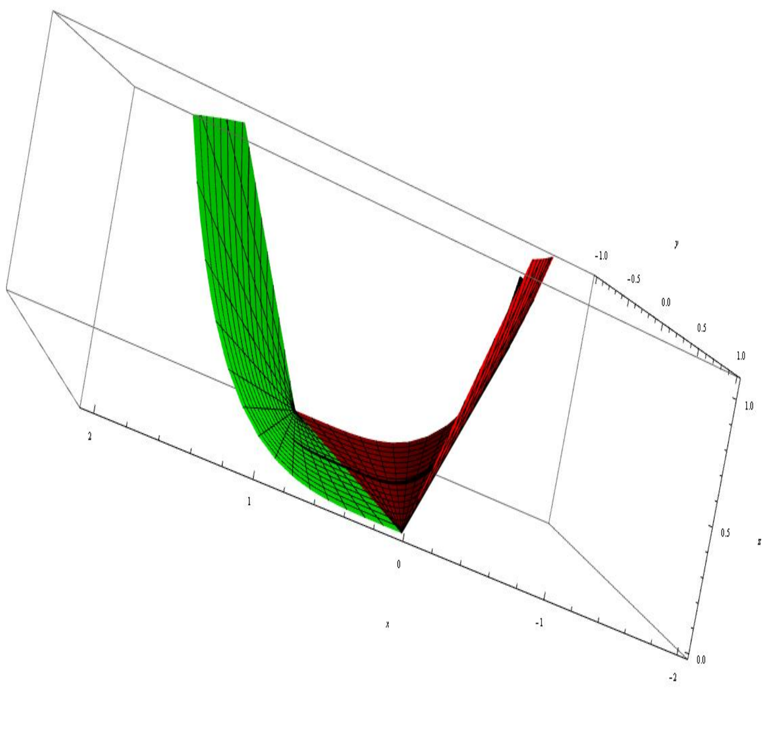
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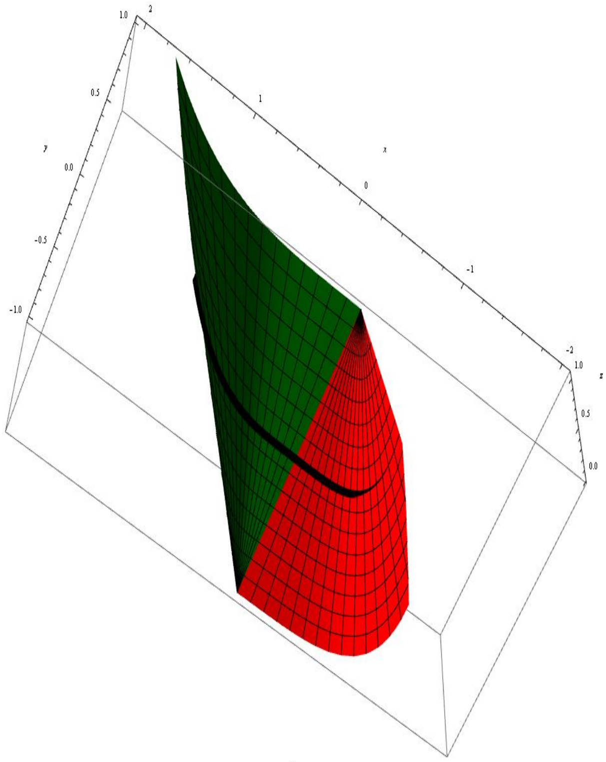

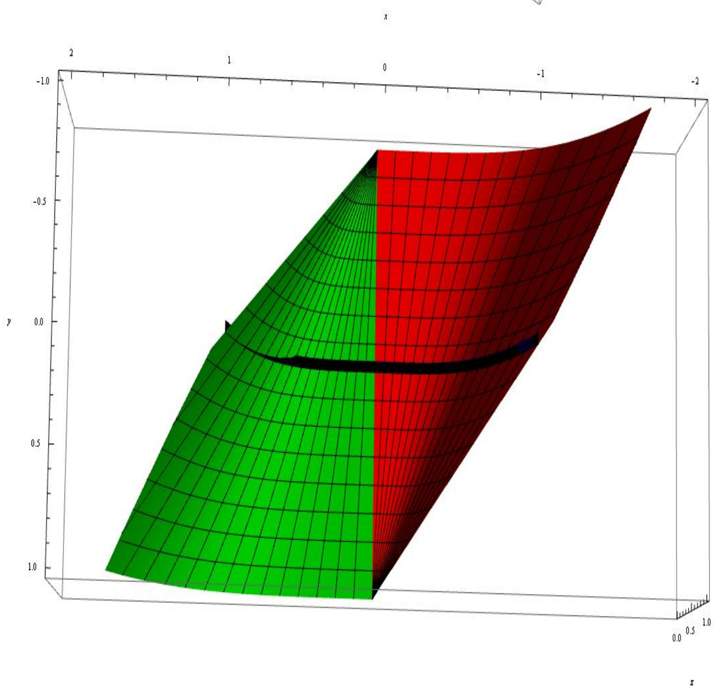

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