A FLAT BUT NON-SMOOTHLY RULED SURFACE

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As for everything else, so for a mathematical theory: beauty can be perceived but not explained.
A. Cayley


The surface is constructed by gluing the red cone $C_1$ with the green cone $C_2$ along a common generatrix (the $y$-axis). The blue curve $c$, the generatrix of both cones, is in the $xz$ plane. The vertex of $C_1$ is the point $(0,1,0)$ while the vertex of $C_2$ is the point $(0,-1,0)$. The resulting surface is $C^\infty$ near the origin $(0,0,0)$ but is not $C^\infty$-ruled around $(0,0,0)$. See below for more 3D-plots of the surface.

Let $c \in \mathbb{R}^3$ be a curve in the $xz$-plane give by

\[ c(x) = (x,0,z(x)) \]

where the function $z(x)$ is $C^\infty$, $z(0) = 0$ and $\frac{d^n z}{dx^n}$ $|_0 = 0$ for $n = 1, 2, \ldots$.

We assume also that $\frac{d^2 z(x)}{dx^2} > 0$ for $x \neq 0$.

Let $C_1$ be the cone containing $c$ with vertex at the point $(0,1,0)$. The tangent plane $T_O C_1$ at the origin $O$ is the $xy$-plane. Since $C_1$ is a $C^\infty$ surface near $O$ there exist a $C^\infty$-function $f_1(x,y)$ such that near $O$ the cone $C_1$ is given by the graph $(x,y,f_1(x,y))$.

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A simple calculation shows that near \((0, 0)\)

\[
f_1(x, y) = z\left(\frac{x}{1-y}\right)(1-y).
\]

So for every \(n, m \in \mathbb{N}\) we have

\[
\frac{\partial^{n+m} f_1}{\partial x^n \partial y^m}\bigg|_{(0, y)} = 0 \tag{1}
\]

Let now \(C_2\) be the cone containing \(c\) but with vertex at the point \((0, -1, 0)\). The cone \(C_2\) is also given by a graph of a function \(f_2(x, y)\) near \(O\). By the same argument \(f_2\) also satisfy equation (1).

Then the function \(f\) defined as

\[
f(x, y) = \begin{cases} f_1(x, y) & \text{if } x \leq 0 \\ f_2(x, y) & \text{if } x > 0 \end{cases}
\]

is a \(C^\infty\)-function near \((0, 0)\).

**Claim:** The graph \((x, y, f(x, y))\) is an example of a \(C^\infty\)-surface with zero Gaussian curvature but is not a \(C^\infty\)-ruled surface near the origin \(O\).

**Proof.** We already seen that \(f\) is \(C^\infty\). Since the graph is the union of two cones and the Gauss curvature \(\kappa(x, y)\) is smooth it follows that \(\kappa(x, y) \equiv 0\) since \(\kappa\) restricted to each cone is identically zero.

Assume that the graph of \(f\) is ruled near \(O\) i.e. the graph is the image of a map \((s, t) \rightarrow \alpha(t) + s\beta(t)\), with \(\alpha, \beta \in C^\infty\), \(\alpha(0) = 0\) and \(\alpha'(0) \wedge \beta(0) \neq 0\). So \(\beta(0) = (0, b, 0)\) with \(b \neq 0\). Since \(\alpha'(0)\) is tangent to the \(xy\)-plane it follows that the projection of \(\alpha\) to the \(xy\)-plane is transversal to the \(y\)-axis at \((0, 0)\). This imply the existence of \(C^\infty\)-functions \(t(x), y(x)\) such that

\[
\alpha(t(x)) = (x, y(x), f(x, y(x))).
\]

Notice that the hypothesis \(\frac{d^2 z(x)}{dx^2} > 0\) imply that the only straight lines contained in the cones \(C_1, C_2\) are the straight lines connecting one of the vertices with the generatrix \(c\). Indeed, the shape operator of the graph in a point \((x, y, f(x, y))\) is not zero if \(x \neq 0\). Hence any segment contained in the graph must be tangent to the kernel of the shape operator i.e. must be part of a straight lines connecting one of the vertices with the generatrix \(c\) as claimed. More precisely, along \(\alpha(t(x))\) we have the following continuous vector field \(R(x)\) pointing in the direction of the rulings:
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\[
R(x) = \begin{cases} 
(x, y(x) - 1, f(x, y(x))) & \text{if } x < 0 \\
(-x, -y(x) - 1, -f(x, y(x))) & \text{if } x \geq 0
\end{cases}
\]

Notice that \(R(x)\) never vanish.

Then there exist a function \(m(x)\) such that
\[
\beta(t(x)) = m(x)R(x).
\]

Since both \(\beta(t(x))\) and \(R(x)\) are continuos and never vanish along \(\alpha\) it follows that \(m(x)\) is continuos and never zero. The first component of \(\beta(t(x))\) is \(-m(x)|x|\) which is not derivable at \(x = 0\) because \(m(0) \neq 0\) since \(\beta(0) \neq 0\). This is a contradiction since \(\beta(t(x))\) is \(C^\infty\). □

Proposition A. The above non-ruled \(C^\infty\)-surface has the property that each of its points is part of a segment included in the surface.
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