Complex submanifolds of almost complex Euclidean spaces

COMPLEX SUBMANIFOLDS OF ALMOST COMPLEX EUCLIDEAN SPACES

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Abstract. We prove that a compact Riemann surface can be realized as a pseudo-holomorphic curve of $(\mathbb{R}^4, J)$, for some almost complex structure J if and only if it is an elliptic curve. Furthermore we show that any (almost) complex $2n$-torus can be holomorphically embedded in $(\mathbb{R}^{4n}, J)$ for a suitable almost complex structure J. This allows us to embed any compact Riemann surface in some almost complex Euclidean space and to show many explicit examples of almost complex structures in $\mathbb{R}^{2n}$ which can not be tamed by any symplectic form.

1. Introduction

An almost complex structure on a $2n$-dimensional smooth manifold M is a tensor $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{I}$. An almost complex structure is called *integrable* if it is induced by a holomorphic atlas. In dimension 2 any almost complex structure is integrable, while in higher dimension this is far from true.

A smooth map $f\colon N \to M$ between two almost complex manifolds (N, J'), (M, J) is called *pseudo-holomorphic* if $f_* J = J' f_*$. When the map f is an immersion, (N, J) is said to be an *almost complex submanifold* of (M, J). If N is a compact Riemann surface, then the triple (N, J', f) is called a *pseudo-holomorphic curve*. A local existence result for pseudo-holomorphic curves appeared in [9]. The concept of pseudo-holomorphic curves was also implicit in early work of Gray [3]. Such curves were studied by Bryant in [1] and they are related with the study of harmonic maps and minimal surfaces (see [2]). The idea of using pseudo-holomorphic curves to study almost complex and symplectic manifolds is due to Gromov. In the celebrated paper [4] Gromov used pseudo-holomorphic curves to introduce new invariants of symplectic manifolds. Subsequently such curves were taken into account by

Date: February 13, 2013.
2000 *Mathematics Subject Classification.* Primary 32Q60 ; Secondary 32Q65.
This work was supported by the Project M.I.U.R. "Riemannian Metrics and Differentiable Manifolds" and by G.N.S.A.G.A. of I.N.d.A.M..

many authors (see e.g. [7, 6] and the references therein).

In the present paper we study compact pseudo-holomorphic curves embedded in $\mathbb{R}^{2n}$. If we equip $\mathbb{R}^{2n}$ with a complex structure, then it does not admit any compact complex submanifold (by the maximum principle). Thus, it is a very natural problem to ascertain if it is possible to find compact complex manifolds embedded in $\mathbb{R}^{2n}$ equipped with a non-integrable almost complex structure. We prove the following:

Theorem 1.1. *A compact Riemann surface X can be realized as a pseudo-holomorphic curve of $\mathbb{R}^4$ equipped with an almost complex structure if and only if it is an elliptic curve.*

Recall that a complex torus $\mathbb{T}^n$ is the quotient of $\mathbb{C}^n$ by a lattice Λ. If $n \geq 2$ it may or may not be algebraic (i.e., an abelian variety). We shall consider more generally $\mathbb{R}^{2n}$ equipped with an almost complex structure invariant by a lattice. We call the quotient an *almost complex torus*.

Theorem 1.2. *Any almost complex torus $\mathbb{T}^n = \mathbb{R}^{2n}/\Lambda$ can be pseudo-holomorphically embedded into $(\mathbb{R}^{4n}, J_\Lambda)$ for a suitable almost complex structure J_Λ.*

Theorem 1.1 can be applied to prove the following

Theorem 1.3. *Any compact Riemann surface can be realized as a pseudo-holomorphic curve of some $(\mathbb{R}^{2n}, J)$, where J is a suitable almost complex structure.*

The above results can be used to construct explicit examples of almost complex structures which cannot be tamed by any symplectic form.

2. Preliminaries

In the present paper all manifolds, maps, *etc.*, are taken to be smooth, i.e., C^∞. Let M be a manifold, then giving an almost complex structure on M is equivalent to give an n-dimensional subbundle Λ of $TM^* \otimes \mathbb{C}$ satisfying $\Lambda \oplus \overline{\Lambda} = TM^* \otimes \mathbb{C}$, where TM^* is the cotangent bundle to M (see e.g. [10])

Moreover, if M is a parallelizable manifold, then the choice of an almost complex structure on M is equivalent to the choice of a global complex coframe $\{\alpha_1, \ldots, \alpha_n\}$ satisfying $\Lambda \oplus \overline{\Lambda} = TM \otimes \mathbb{C}$, where $\Lambda = \operatorname{span}_{\mathbb{C}}\{\alpha_i\}$. The parallelizable case obviously includes $\mathbb{R}^{2n}$ and complex tori.

Now we consider the case of an open subset of $\mathbb{R}^n$. Let be Ω an open subset of $\mathbb{R}^{2n}$ and let J be an almost complex structure on Ω determined by a complex coframe $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$. Fix a smooth map $f : \Omega \subset \mathbb{R}^{2n} \to \mathbb{C}$ and denote by $\mathcal{Z}_f := f^{-1}(0)$ the zero set of f. We have the following easy-prove lemma which will be useful

Lemma 2.1. *Assume that $df \wedge \overline{df}|_{\mathcal{Z}_f} \neq 0$. Then the following facts are equivalent:*

(i) $\mathcal{Z}_f$ *is an almost complex hypersurface of* (Ω, J),
(ii) $\operatorname{span}_{\mathbb{C}}\{df, \overline{df}\}_{|\mathcal{Z}_f}$ *is* J*-invariant,*
(iii) $df \wedge \overline{df}_{|\mathcal{Z}_f} \in \Lambda^{1,1}$,
(iv) $df \wedge \overline{df} \wedge \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_{n|\mathcal{Z}_f} = 0\,,$
(v) $\overline{\partial f} \wedge \overline{\partial} f_{|\mathcal{Z}_f} = 0\,,$

where $\Lambda^{1,1}$ *is the space of* $(1,1)$*-forms with respect to* J.

3. Proof of the main results

In this section we prove Theorems 1.1 and 1.2. The first part of Theorem 1.1 can be seen as a corollary of the following result due to Whitney

Theorem 3.1 ([5], page 138)**.** *Any compact* n*-dimensional submanifold of* $\mathbb{R}^{2n}$ *has a non-vanishing normal vector field.*

Indeed, if J is an almost complex structure on $\mathbb{R}^4$ and $i\colon X \hookrightarrow \mathbb{R}^4$ is a pseudo-holomorphic curve of $(\mathbb{R}^4, J)$, then in view of Whitney's Theorem the normal bundle to X has to be trivial, since it is J-invariant. Then we fix an arbitrary J-Hermitian metric g on $\mathbb{R}^4$ and denote by $\{J_1, J_2, J_3\}$ the canonical quaternionic complex structure of $\mathbb{R}^4$. Now, we know that there exists a smooth map $A\colon \mathbb{R}^4 \to \mathrm{GL}(4,\mathbb{R})$ such that $J_0 = A^{-1}JA$. This implies that $W := AJ_2A^{-1}V$ is a nowhere vanishing vector filed tangent to X and, consequently, X is a torus.

Remark 3.2. The same argument can be used to prove the following more general result: *Let* J *be an almost complex structure on* $\mathbb{R}^{4n}$ *and let* (M^{2n}, J_M) *be an almost complex submanifold of* $(\mathbb{R}^{4n}, J)$*. Then the Euler characteristic of* M *vanishes.*

Now we show the proof of Theorem 1.2, which in particular implies the second part of Theorem 1.1.

Proof of Theorem 1.2. First of all we consider the case of the standard elliptic curve $\mathbb{T} = \mathbb{C}/\mathbb{Z}^2$. Let J be the almost complex structure on $\mathbb{R}^4$ determined by the following complex frame

$$\alpha_1 := dz - \mathrm{i}\, zw\, d\overline{w}\,, \quad \alpha_2 := dw + \mathrm{i}\, zw\, d\overline{z}\,,$$

where z, w are the standard complex coordinates on $\mathbb{R}^4$ and let $f\colon \mathbb{C}^2 \to \mathbb{C}$ be the map $f(z,w) = (|z|^2 - 1) + i(|w|^2 - 1)$. Then $f^{-1}(0)$ defines a smooth embedding of the torus in $\mathbb{R}^4$. We have

$$\begin{aligned}
&df \wedge \overline{df} \wedge \alpha_1 \wedge \alpha_2 = 2\,\mathrm{i}\, d|w|^2 \wedge d|z|^2 \wedge (dz \wedge dw + (zw)^2 d\overline{w} \wedge d\overline{z}) = \\
&2\,\mathrm{i}\, d|w|^2 \wedge d|z|^2 \wedge dz \wedge dw + 2\,\mathrm{i}\, d|w|^2 \wedge d|z|^2 \wedge (zw)^2 d\overline{w} \wedge d\overline{z} = \\
&2\,\mathrm{i}\, w d\overline{w} \wedge z d\overline{z} \wedge dz \wedge dw + 2\,\mathrm{i}\, \overline{w} dw \wedge \overline{z} dz \wedge (zw)^2 d\overline{w} \wedge d\overline{z} = \\
&2\,\mathrm{i}\, wz(d\overline{w} \wedge d\overline{z} \wedge dz \wedge dw + |zw|^2 dw \wedge dz \wedge d\overline{w} \wedge d\overline{z}) = \\
&2\,\mathrm{i}\, zw(1 - |zw|^2) d\overline{w} \wedge d\overline{z} \wedge dz \wedge dw\,.
\end{aligned}$$

Hence $df \wedge \overline{df} \wedge \alpha_1 \wedge \alpha_2|_{f^{-1}(0)} = 0$ and in view of Lemma 2.1 $f^{-1}(0)$ is an almost complex submanifold of $(\mathbb{R}^4, J)$.

Now we consider the general case: let $(\mathbb{T}^{2n} := \mathbb{R}^{4n}/\mathbb{Z}^{4n}, J_0)$ be the $2n$-dimensional torus equipped with the standard complex structure. In this case, the space of the $(1,0)$-forms is generated by $\{dz_1, \dots, dz_n\}$. Any almost complex structure J' on $\mathbb{T}^n$ can be described by a coframe $\{\zeta_1, \dots, \zeta_n\}$ of the form

$$\zeta_i = \sum_{j=1}^{n} \tau_{ij}\, dz_j + \sum_{\bar{j}=1}^{n} \tau_{i\bar{j}}\, dz_{\bar{j}}\,, \text{ for } i = 1, \dots, n\,,$$

where the τ_{ij}'s and the $\tau_{i\bar{j}}$'s are complex maps on $\mathbb{T}^{2n}$. Such an almost complex structure corresponds to a lattice Λ in $\mathbb{R}^{2n}$. In view of the first step of this proof, the standard complex torus $(\mathbb{T}^{2n} := \mathbb{R}^{4n}/\mathbb{Z}^{4n}, J_0)$ can be holomorphically embedded in $(\mathbb{R}^{4n}, J)$, for a suitable J. Let $\{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n\}$ be a $(1,0)$-coframe on $(\mathbb{R}^{4n}, J)$ satisfying

$$i^*(\alpha_i) = i^*(\beta_i) = dz_i\,, \quad i = 1, \dots, n\,. \tag{3.1}$$

The maps $\{\tau_{ij}\}, \{\tau_{i\bar{j}}\}$ can be regarded as periodic maps on $\mathbb{R}^{2n}$ and they can be extended on $\mathbb{R}^{4n}$. We can assume that conditions (3.1) still hold on $\mathbb{R}^{4n}$. Now set

$$\alpha_i^\tau := \sum_{k=1}^{n} \tau_{ik}\alpha_k + \sum_{k=1}^{n} \tau_{i\bar{k}}\, \beta_{\bar{k}}\,, \;\; \beta_j^\tau := \sum_{k=1}^{n} \tau_{ik}\beta_k + \sum_{k=1}^{n} \tau_{i\bar{k}}\, \alpha_{\bar{k}}\,,$$

for $i, j = 1 \dots, n$. The frame $\{\alpha_1^\tau, \dots, \alpha_n^\tau, \beta_1^\tau, \dots, \beta_n^\tau\}$ induces an almost complex structure J_Λ on $\mathbb{R}^{4n}$. A direct computation gives

$$i^*(\alpha_i^\tau) = i^*(\beta_i^\tau) = \zeta_i\,, \text{ for } i = 1, \dots, n\,.$$

Hence $i\colon (\mathbb{T}^{2n}, J') \hookrightarrow (\mathbb{R}^{4n}, J_\Lambda)$ is a pseudo-holomorphic embedding. □

Using the well-known Abel-Jacobi map one can embed any compact Riemann surface of positive genus in its Jacobian variety. Hence in view of above theorem any compact Riemann surface of positive genus can be embedded in some $(\mathbb{R}^{2N}, J)$. So, in order to prove Theorem 1.3, remains to show that $\mathbb{CP}^1$ can be embedded in some $(\mathbb{R}^{2n}, J)$ for a suitable J. We have the following

Lemma 3.3. *The Riemann sphere $\mathbb{CP}^1$ can be pseudo-holomorphically embedded in $(\mathbb{R}^6, J)$, for a suitable J.*

Proof. The sphere S^6 has a canonic almost complex structure J induced by the algebra of the octonians $\mathbb{O}$ (see e.g., [**?**, 1]). Moreover, if we intersect S^6 with a suitable 3-dimensional subspace of $\operatorname{Im}\mathbb{O}$, then we obtain a pseudo-holomorphic embedding of $\mathbb{CP}^1$ in (S^6, J) (see [1], again). Hence the stereographic projection $\pi\colon S^6 - \{\text{a point}\} \to \mathbb{R}^6$ induces a pseudo-holomorphic embedding of $\mathbb{CP}^1$ to $\mathbb{R}^6$ equipped with the almost complex structure induced by J and π. □

Remark 3.4. We recall that an almost complex structure J on a manifold M is said to be *tamed* by a symplectic form ω if $\omega_x(v, J_x v) > 0$ for every $x \in M$ and every $v \in T_x M$, $v \neq 0$. Now, using Stokes Theorem one easily gets that if J is an almost complex on $\mathbb{R}^{2n}$ admitting a pseudo-holomorphic curves, then J cannot be tamed by any symplectic form. This implies that the J_Λ's described in section 2 gives explicit examples of almost complex structures which cannot be tamed by any symplectic form.

For other examples of almost complex structures which can not be calibrated by symplectic forms see [8, 11].

Acknowledgment

We thank Simon Salamon, Paolo de Bartolomeis and Diego Matessi for useful comments and remarks.

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