Quasi-Kahler manifolds with trivial Chern holonomy

QUASI-KÄHLER MANIFOLDS WITH TRIVIAL CHERN HOLONOMY

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Abstract. In this paper we study almost complex manifolds admitting a quasi-Kähler Chern-flat metric (Chern-flat means that the holonomy of the Chern connection is trivial). We prove that in the compact case such manifolds are all nilmanifolds. Some partial classification results are established and we prove that a quasi-Kähler Chern-flat structure can be tamed by a symplectic form if and only if the ambient space is isomorphic to a flat torus.

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1. Introduction

A Kähler manifold is a complex manifold (M, J) equipped with a compatible Hermitian metric g whose Levi-Civita connection preserves J. The condition $\nabla J = 0$ imposes a strong constraint on the curvature of g. In particular if (M, J, g) is a compact flat Kähler manifold, then it is isomorphic to a flat torus. Kähler manifolds can be seen as a subclass of *almost Hermitian manifolds*. An almost Hermitian manifold is a smooth manifold equipped with a not-necessarily integrable $\mathrm{U}(n)$-structure (g, J). Such a manifold differs from the Kähler one by the non-vanishing of the covariant derivative ∇J, so in the almost Hermitian case it is useful to study the geometry related to connections with torsion preserving g and J, in place of the geometry of the Levi-Civita connection. This kind of connection is usually called *Hermitian* (see e.g. [7]).

Any almost Hermitian manifold (M, g, J) always admits a unique Hermitian connection $\widetilde{\nabla}$ whose torsion has everywhere vanishing $(1, 1)$-part. This connection was introduced by Ehresmann and Libermann in [5] and for a Kähler manifold coincides with the Levi-Civita connection. In the complex

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case, $\widetilde{\nabla}$ coincides with the connection used by Chern in [2] and is called the *Chern connection* (in many papers it is also called the *canonical Hermitian connection*, the *second canonical Hermitian connection* or simply the *canonical connection*). Such a connection and its associated curvature tensor $\widetilde{R}$ plays a central role in almost Hermitian geometry. For instance, $\widetilde{R}$ appears in the Sekigawa's proof of the Goldberg conjecture for Einstein manifolds of non-negative scalar curvature (see [12]) and recently Tosatti, Weinkove and Yau proved a Donaldson-type conjecture under a positivity assumption on $\widetilde{R}$ (see [13]). The role of $\widetilde{\nabla}$ is also important in nearly Kähler geometry, since in this special case it has parallel and totally skew-symmetric torsion (see e.g. [6, 9] and the references therein).

We recall that a *quasi-Kähler* structure is an almost Hermitian structure whose Kähler form ω satisfies $\overline{\partial}\omega = 0$. The aim of this paper is to study almost complex manifolds admitting a compatible quasi-Kähler metric for which the holonomy associated to its Chern connection is trivial. Such manifolds will be called *quasi-Kähler Chern-flat*. The point is that for quasi-Kähler manifolds, the Chern connection can be described by a very simple formula involving only the Levi-Civita connection and the almost complex structure.

The canonical example of a quasi-Kähler Chern-flat manifold is the Iwasawa manifold M equipped with the almost Hermitian structure associated to the almost complex structure J_3 defined in [1, p.151]. The structure J_3 is the unique invariant almost complex structure on M such that its associated fundamental form ω_3 with respect to a canonical metric of M is quasi-Kähler and not-symplectic. The pair (M, J_3) is a compact almost complex manifold obtained as a quotient of an almost complex 2-step nilpotent Lie algebra by a lattice. We prove that all compact quasi-Kähler Chern-flat manifolds have this structure.

Theorem 1.1. *A compact almost complex manifold (M, J) is quasi-Kähler Chern-flat if and only if it is isomorphic to a 2-step nilpotent nilmanifold equipped with a left-invariant almost complex structure whose almost complex Lie algebra $(\mathfrak{g}, J)$ satisfies*

$$[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = 0\,, \quad [\mathfrak{g}^{1,0}, \mathfrak{g}^{1,0}] \subseteq \mathfrak{g}^{0,1}\,, \tag{1.1}$$

being $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ the splitting of $\mathfrak{g} \otimes \mathbb{C}$ in terms of the eigenspaces of J.

The proof of Theorem 1.1 relies upon a theorem of Palais, which can be found in [10]. In view of Theorem 1.1, the problem of classifying quasi-Kähler Chern-flat manifolds reduces to that of classifying 2-step nilpotent Lie algebras equipped with an almost complex structure satisfying (1.1). It is rather natural to call such Lie algebras *quasi-Kähler Chern flat*. In section (4.1) we show that any n-dimensional 2-step nilpotent Lie algebra has a naturally corresponding $2n$-dimensional quasi-Kähler Chern flat Lie algebra. For

instance, the Lie algebra associated to the Iwasawa manifold equipped with the almost complex structure J_3 is the quasi-Kähler Chern flat Lie algebra associated to the 3-dimensional Heisenberg group. In section (4.2) we classify the 8-dimensional quasi-Kähler Chern Lie algebras (the 6-dimensional case was recently carried out by the authors in [4]) and in section (4.3) we describe all the quasi-Kähler Chern flat Lie algebras which have center of complex dimension equal to 1. At the end of section 4, we determine the vector space of the infinitesimal deformations of quasi-Kähler Chern flat Lie algebras (see section 4.4).

In the last part of the paper we study the problem of taming an almost complex structure admitting a compatible quasi-Kähler Chern-flat metric with a symplectic form. This problem arises from a conjecture of Donaldson and from the work of Tosatti, Weinkove and Yau (see [3, 13]). We prove the following

Theorem 1.2. *Let (M, ω) be a compact symplectic manifold. Assume that there exist an almost complex structure on M tamed by ω and such that the pair (ω, J) induces a quasi-Kähler Chern-flat metric g. Then (M, J, ω) is a Kähler torus.*

This theorem was recently proved by the authors in the case of the Iwasawa manifold in [4].

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Notation. When a coframe $\{\alpha_1, \dots, \alpha_n\}$ is given we will denote the r-form $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_r}$ by $\alpha_{i_1 \dots i_r}$ and in the indicial expressions the summation sign over repeated indeces is omitted.

2. Preliminaries and first results

Let (M, J) be an almost complex manifold. Then the complexified tangent bundle $T \otimes \mathbb{C}$ of M splits in $T \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$. Consequently the vector space Ω^p of the complex p-forms on M splits as $\Omega^p = \oplus_{r+s=p} \Omega^{r,s}$ and the de Rham differential operator d decomposes accordingly to this type decomposition of complex p-form as $\mathrm{d} = A + \partial + \overline{\partial} + \overline{A}$. More precisely, in the almost complex context the de Rham operator satisfies

$$\mathrm{d}\left(\Omega^{r,s}\right) \subset \Omega^{r+2,s-1} \bigoplus \Omega^{r+1,s} \bigoplus \Omega^{r,s+1} \bigoplus \Omega^{r-1,s+2}$$

where the operator $A \colon \Omega^{r,s} \to \Omega^{r+2,s-1}$ is defined as the projection of d onto $\Omega^{r+2,s-1}$. It turns out that A is a tensor which vanishes if and only J is a genuine complex structure, i.e. J is integrable.

Let us fix now an almost Hermitian metric g on (M, J). It's well known that there exists a unique connection $\widetilde{\nabla}$ on M satisfying the following properties

$$\widetilde{\nabla} g = 0\,, \quad \widetilde{\nabla} J = 0\,, \quad \mathrm{Tor}^{1,1}(\widetilde{\nabla}) = 0\,,$$

where $\mathrm{Tor}^{1,1}(\widetilde{\nabla})$ denotes the $(1,1)$-part of the torsion of $\widetilde{\nabla}$. The connection $\widetilde{\nabla}$ is usually called the *canonical Hermitian connection* or the *Chern connection* of g. The aim of this paper is to study almost complex manifolds admitting an almost Hermitian metric having the holonomy of the Chern connection trivial. We will adopt the following conventionally definitions:

Definition 2.1. *An almost Hermitian metric having trivial Chern holonomy will be called in this paper a* Chern-flat metric*.*

We will refer to an almost complex manifold admitting an almost Hermitian metric with trivial Chern holonomy as a Chern-flat manifold*.*

As a first result on Chern-flat manifolds we have the following

Proposition 2.2. *An almost complex manifold (M, J) is Chern-flat if and only if there exists a global $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ on M such that $[Z_i, Z_{\overline{j}}] = 0$, for every $i, j = 1, \dots, n$.*

Proof. Assume that there exists a J-compatible almost Hermitian metric g having the holonomy of the Chern connection trivial. Then there exists a global unitary frame $\{Z_1, \dots, Z_n\}$ on M such that

$$\widetilde{\nabla}_i Z_j = \widetilde{\nabla}_i Z_{\overline{j}} = 0\,, \quad i, j = 1, \dots, n\,.$$

Since $\mathrm{Tor}^{1,1}(\widetilde{\nabla}) = 0$, then we get

$$0 = \widetilde{\nabla}_i Z_{\overline{j}} - \widetilde{\nabla}_{\overline{j}} Z_i - [Z_i, Z_{\overline{j}}] = [Z_i, Z_{\overline{j}}]\,, \quad i, j = 1, \dots, n\,.$$

On the other hand assume that there exists a global $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ such that $[Z_i, Z_{\overline{j}}] = 0$ and let g be the metric

$$g = \sum_{i=1}^{n} \zeta_i \odot \zeta_{\overline{i}}\,,$$

$\{\zeta_1, \dots, \zeta_n\}$ being the dual frame of $\{Z_1, \dots, Z_n\}$. Then we have

$$0 = \widetilde{\nabla}_i Z_{\overline{j}} - \widetilde{\nabla}_{\overline{j}} Z_i - [Z_i, Z_{\overline{j}}] = \widetilde{\nabla}_i Z_{\overline{j}} - \widetilde{\nabla}_{\overline{j}} Z_i\,, \quad i, j = 1, \dots, n\,,$$

which implies $\widetilde{\nabla}_i Z_{\overline{j}} = 0$ since $\widetilde{\nabla}$ preserves J. Furthermore, since the frame $\{Z_1, \dots, Z_n\}$ is unitary with respect to g, we get

$$g(\widetilde{\nabla}_i Z_j, Z_{\overline{k}}) = Z_i g(Z_j, Z_{\overline{k}}) - g(Z_j, \widetilde{\nabla}_i Z_{\overline{k}}) = 0\,,$$

which implies $\widetilde{\nabla}_i Z_j = 0$, for $i, j = 1, \dots, n$. Hence there exists a global frame of type $(1,0)$ on M which is preserved by $\widetilde{\nabla}$ and this is equivalent to requiring $\mathrm{Hol}(\widetilde{\nabla}) = 0$. $\square$

2.1. **Left-invariant Chern-flat structures on Lie groups.** Let G be a Lie group equipped with a left-invariant almost complex structure J. Then J induces an almost complex structure, which we denote with the same letter, on the Lie algebra $\mathfrak{g}$ associated to G. By an *almost complex structure on a Lie algebra* $\mathfrak{g}$ we just mean an endomorphism J of $\mathfrak{g}$ satisfying $J^2 = -\mathrm{I}$. Such an endomorphism has no relation with the bracket of $\mathfrak{g}$. Anyway J allows us to split the complexification of $\mathfrak{g}$ in $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$. The space $\mathfrak{g}^{1,0}$ can be clearly identified with the space of left-invariant vector fields of type $(1,0)$ on the Lie group (G, J). In view of Proposition 2.2, the existence of left-invariant almost Hermitian Chern-flat metrics on (G, J) can be characterized in terms of $(\mathfrak{g}, J)$.

Proposition 2.3. *Let (G, J) be a Lie group equipped with a left-invariant almost complex structure. If the almost complex Lie algebra $(\mathfrak{g}, J)$ associated to (G, J) satisfies*

$$[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = 0\,,$$

then every left-invariant almost Hermitian metric g on (G, J) is Chern-flat.

Proof. Let g be an arbitrary left-invariant almost Hermitian metric on (G, J) and let $\widetilde{\nabla}$ be the Chern connection of g. Since g is left-invariant, then we can find a left-invariant unitary frame $\{Z_1, \dots, Z_n\}$ on (G, J). Again using $\mathrm{Tor}^{1,1}(\widetilde{\nabla}) = 0$ and $\widetilde{\nabla} g = 0$ we get that $\widetilde{\nabla}_i Z_j = \widetilde{\nabla}_i Z_{\bar{j}}$ for every $i, j = 1, \dots, n$. Hence $\mathrm{Hol}(\widetilde{\nabla}) = 0$. □

Proposition 2.3 justifies the following

Definition 2.4. *An almost complex Lie algebra $(\mathfrak{g}, J)$ is called* Chern-flat *if it satisfies $[\mathfrak{g}^{1,0}, \mathfrak{g}^{0,1}] = 0$.*

The condition to be *Chern-flat* can be also characterized as follows:

Proposition 2.5. *An almost complex Lie algebra $(\mathfrak{g}, J)$ is Chern-flat if and only if J satisfies $[JX, Y] = [X, JY]$ for every $X, Y \in \mathfrak{g}$.*

Proposition 2.5 has the following immediate consequence

Corollary 2.6. *Let $(\mathfrak{g}, J)$ be a Chern-flat Lie algebra, then the center of $\mathfrak{g}$ is J-invariant.*

Proof. Let X be an arbitrary vector belonging to the center of $\mathfrak{g}$ and let $Y \in \mathfrak{g}$. Then we have $[JX, Y] = [X, JY] = 0$. Hence JX belongs to the center of $\mathfrak{g}$. □

3. Quasi-Kähler Chern-flat manifolds and proof of Theorem 1.1

In this section we introduce quasi-Kähler Chern-flat manifolds and we prove Theorem 1.1.

We recall that an almost Hermitian metric g on an almost complex manifold (M, J) is called *quasi-Kähler* if the Kähler 2-form $\omega_g(\cdot,\cdot) := g(J\cdot,\cdot)$ associated to (g, J) is $\overline{\partial}$-closed. In this case the Chern connection of g is simply the connection described by the formula

$$\widetilde{\nabla} = \nabla - \frac{1}{2} J \nabla J \,,$$

where ∇ is the Levi-Civita connection associated to g (see for instance [7]).

Definition 3.1. *In this paper we refer to an almost complex manifold admitting a compatible almost Hermitian metric which is quasi-Kähler and Chern-flat as a* quasi-Kähler Chern-flat manifold*.*

The canonical example of a quasi-Kähler Chern-flat manifold is the Iwasawa manifold equipped with the almost complex structure J_3 defined in [1, p. 151]. This almost complex manifold is defined as follows:
Let G be the 3-dimensional *complex Heisenberg group*

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} : z_i \in \mathbb{C} \,, \mathrm{i} = 1, 2, 3 \right\}$$

and let M be the compact manifold $M = G/\Gamma$, where Γ is the co-compact lattice of G formed by the matrices with integral entries. Then M is a 2-step nilpotent nilmanifold usually called the *Iwasawa manifold*. This manifold admits a global frame $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ satisfying the following structure equations

$$[X_1, X_2] = X_3 \,, \quad [X_4, X_5] = -X_3 \quad [X_2, X_4] = X_6 \,, \quad [X_5, X_1] = X_6 \,.$$

The structure J_3 can be defined as the almost complex structure

$$\begin{aligned} J_3 X_1 &= X_4 \,, & J_3 X_2 &= X_5 \,, & J_3 X_3 &= X_6 \,, \\ J_3 X_4 &= -X_1 \,, & J_3 X_5 &= -X_2 \,, & J_3 X_6 &= -X_3 \,. \end{aligned}$$

Such an almost complex structure induces the $(1, 0)$-frame

$$Z_1 = X_1 - \mathrm{i}\, X_4 \,, \quad Z_2 = X_2 - \mathrm{i}\, X_5 \,, \quad Z_3 = X_3 - \mathrm{i}\, X_6 \,.$$

Clearly

$$[Z_1, Z_2] = 2\, Z_{\overline{3}} \,, \quad [Z_{\overline{1}}, Z_{\overline{2}}] = 2\, Z_3 \,.$$

Let $\{\zeta_1, \zeta_2, \zeta_3\}$ be the coframe associated to $\{Z_1, Z_2, Z_3\}$. Then it easy to show that the canonical metric $g = \zeta_1 \odot \zeta_{\overline{1}} + \zeta_2 \odot \zeta_{\overline{2}} + \zeta_3 \odot \zeta_{\overline{3}}$ is quasi-Kähler and Chern-flat with respect to J_3.

Remark 3.2. The notation used in this paper is slightly different from the one taken into account in [1], where the Lie algebra associated to the complex Heisenberg group is described using the following structure equations

$$[e_1, e_3] = -e_5 \,, \quad [e_2, e_4] = e_5 \,, \quad [e_1, e_4] = -e_6 \,, \quad [e_2, e_3] = -e_6 \,.$$

The following relations describe an explicit isomorphism between our frame and the one taken into account in [1]

$$\begin{aligned} X_1 &\leftrightarrow e_1\,, \quad X_4 \leftrightarrow -e_2\,,\\ X_2 &\leftrightarrow e_4\,, \quad X_5 \leftrightarrow e_3\,,\\ X_3 &\leftrightarrow e_5\,, \quad X_6 \leftrightarrow e_6\,. \end{aligned}$$

With respect to the frame $\{e_i\}$, the fundamental 2-form ω_3 associated to the pair (g, J_3) is

$$\omega_3 = -e^{12} - e^{34} + e^{56}\,,$$

where e^{ij} means $e^i \wedge e^j$. In view of [1] the almost complex structure J_3 is the unique almost complex structure on the Iwasawa manifold which is quasi-Kähler with respect to the metric g_3 and it is invariant under the natural action of $\mathbb{T}^3$ on the space of the g_3-compatible almost Hermitian structures on M.

The Iwasawa manifold is an example of a compact 2-step nilpotent nilmanifold equipped with a left-invariant almost complex structure which is quasi-Kähler and Chern-flat. A 2-*step nilpotent nilmanifold* is by definition a homogeneous space which is obtained by a quotient of a 2-step nilpotent Lie group G with a discrete subgroup Γ. Any left-invariant almost complex structure J on a nilmanifold induces an almost complex structure J on the Lie algebra $\mathfrak{g}$ associated to G.

Now we prove Theorem 1.1:

Proof of Theorem 1.1. Let (M, J) be a compact almost complex manifold and assume that there exists a quasi-Kähler Chern-flat metric g compatible with J on M. Then we can find a global unitary frame on M $\{Z_1, \dots, Z_n\}$ satisfying

$$\widetilde{\nabla}_i Z_j = \widetilde{\nabla}_{\bar{i}} Z_j = 0$$

for every $i, j = 1, \dots, n$, where $\widetilde{\nabla}$ denotes the Chern connection of g. Such a condition reads in terms of brackets as

$$[Z_i, Z_{\bar{j}}] = 0\,, \quad [Z_i, Z_j] \in \Gamma(T^{0,1}M)\,.$$

We can write $[Z_i, Z_j] = \sum_{r=1}^n c_{ij}^{\bar{r}} Z_{\bar{r}}$. The Jacobi identity implies

$$\begin{aligned} 0 =& \mathfrak{S}[[Z_i, Z_j], Z_{\bar{k}}] = [[Z_i, Z_j], Z_{\bar{k}}] = [c_{ij}^{\bar{r}} Z_{\bar{r}}, Z_{\bar{k}}]\\ =& c_{ij}^{\bar{r}}[Z_{\bar{r}}, Z_{\bar{k}}] - Z_{\bar{k}}(c_{ij}^{\bar{r}}) Z_{\bar{r}} = c_{ij}^{\bar{r}} c_{\bar{r}\bar{k}}^{l} Z_l - Z_{\bar{k}}(c_{ij}^{\bar{r}}) Z_{\bar{r}}\,, \quad i, j, k = 1, \dots, n\,, \end{aligned}$$

where the symbol $\mathfrak{S}$ denotes the cyclic sum. Then

$$c_{ij}^{\bar{r}} c_{\bar{r}\bar{k}}^{l} = 0\,, \quad Z_{\bar{k}}(c_{ij}^{\bar{r}}) = 0\,.$$

Thus, the functions $c_{ij}^{\bar{r}}$'s are holomorphic maps on M. Since M is compact, the $c_{ij}^{\bar{r}}$'s are constant. In view of [10], the universal cover of M inherits a structure of the Lie group and M has a natural structure of homogeneous

space obtained as a quotient of a Lie group with a lattice. Since the finite dimensional Lie algebra generated by the fields $\{Z_1, \dots, Z_n\}$ is 2-step nilpotent, we get that M is indeed a 2-step nilpotent nilmanifold. The almost complex structure J is clearly left-invariant on M. Note that the almost complex Lie algebra associated to the universal cover of M satisfies conditions (1.1).

On the other hand assume that $M = G/\Gamma$ is a 2-step nilpotent nilmanifold equipped with a left-invariant almost complex structure J such that the almost complex Lie algebra $(\mathfrak{g}, J)$ associated to (G, J) satisfies (1.1). Fix an arbitrary $(1,0)$-coframe $\{\zeta_1, \dots, \zeta_n\}$ on $(\mathfrak{g}, J)$ and let

$$g := \sum_{i=1}^{n} \zeta_i \odot \zeta_{\bar{i}} \,.$$

Then g induces an almost Hermitian metric on (M, J). The fundamental form of g is $\omega_g = \mathrm{i} \sum_{i=i}^{n} \zeta_r \wedge \zeta_{\bar{r}}$. Since conditions (1.1) read in terms of a left-invariant coframe as

$$\partial \zeta_r = \overline{\partial} \zeta_r = 0 \,,$$

then

$$\overline{\partial} \omega_g = \mathrm{i} \sum_{r=1}^{n} \left(\overline{\partial}(\zeta_r) \wedge \zeta_{\bar{r}} + \overline{\partial}(\zeta_{\bar{r}}) \wedge \zeta_r \right) = 0$$

and g is a quasi-Kähler metric. Finally Proposition 2.3 implies that g is Chern-flat, as required. □

Remark 3.3. The condition $\mathrm{Hol}(\widetilde{\nabla}) = 0$ is quite strong. For instance we have the following results:

- in almost Kähler manifolds and in nearly Kähler manifolds condition $\mathrm{Hol}(\widetilde{\nabla}) = 0$ implies the integrability (see [4, 15]);
- in dimension 4 there are no strictly quasi-Kähler Chern-flat metrics (see [4]).

4. Quasi-Kähler Chern-flat Lie algebras

In view of Theorem 1.1, the problem of classifying quasi-Kähler Chern-flat manifolds reduces to that of classifying 2-step nilpotent Lie algebras equipped with an almost complex structure satisfying (1.1). It is quite natural to introduce the following definition:

Definition 4.1. *An almost complex Lie algebra* $(\mathfrak{g}, J)$ *satisfying equations* (1.1) *is said to be* quasi-Kähler and Chern-flat.

The following easy-prove Proposition will be useful

Proposition 4.2. *Let* $(\mathfrak{g}, J)$ *be an almost complex Lie algebra. The following facts are equivalent:*

1. $(\mathfrak{g}, J)$ *is quasi-Kähler and Chern-flat;*
2. $\partial \Lambda^{1,0} \mathfrak{g}^* = \overline{\partial} \Lambda^{1,0} \mathfrak{g}^* = 0$;

3. *the almost complex structure J satisfies*
$$J[X,Y] = -[JX,Y] = -[X,JY]$$
for every $X, Y \in \mathfrak{g}$.

The following result can be viewed as a corollary of Theorem 1.1.

Corollary 4.3. *Let (M,J) be a quasi-Kähler Chern-flat manifold. Then every left-invariant almost Hermitian metric on (M,J) is quasi-Kähler and Chern-flat.*

Proof. Let g be a left-invariant almost Hermitian metric on (M,J), then we can find a left-invariant $(1,0)$-coframe $\{\zeta_1,\dots,\zeta_n\}$ with respect to which g writes as $g = \sum_{i=1}^n \zeta_i \odot \zeta_{\bar{i}}$. Now using (1.1) it easy to see that the frame $\{Z_1,\dots,Z_n\}$ dual to $\{\zeta_1,\dots,\zeta_n\}$ satisfies
$$\widetilde{\nabla}_i Z_j = \widetilde{\nabla}_{\bar{i}} Z_j = 0\,, \quad i = 1,\dots,n\,.$$
This implies that every left-invariant almost Hermitian metric g on (M,J) is quasi-Kähler and Chern-flat. □

4.1. **Examples of quasi-Kähler Chern-flat Lie algebras.** In this subsection we show that it is possible to construct a quasi-Kähler Chern-flat Lie algebra $\mathfrak{h}_{\overline{\mathbb{C}}}$ starting from an arbitrary 2-step nilpotent real Lie algebra $\mathfrak{h}$.

Let $(\mathfrak{h}, [\,,\,]_{\mathfrak{h}})$ be a 2-step nilpotent Lie algebra and let $(\mathfrak{h}_{\overline{\mathbb{C}}} = \mathfrak{h}\otimes\mathbb{C}, J_{\mathfrak{h}})$ be the almost complex Lie algebra having the bracket defined by the relation
$$[X\otimes a, Y\otimes b] := [X,Y]_{\mathfrak{h}} \otimes \overline{ab}$$
and as almost complex structure the one defined by
$$J_{\mathfrak{h}}(X\otimes a) := X \otimes \mathrm{i}\, a\,.$$

Proposition 4.4. *$(\mathfrak{h}_{\overline{\mathbb{C}}} = \mathfrak{h}\otimes\mathbb{C}, J_{\mathfrak{h}})$ is a quasi-Kähler Chern-flat Lie algebra.*

Proof. Since $\mathfrak{h}$ is a 2-step nilpotent Lie algebra, then we have $[[X\otimes a, Y\otimes b], Z\otimes c] = [[X,Y],Z]\otimes ab\bar{c} = 0$. So $(\mathfrak{h}_{\overline{\mathbb{C}}} = \mathfrak{h}\otimes\mathbb{C}, J_{\mathfrak{h}})$ is also a 2-step nilpotent Lie algebra. To finish the proof it is enough to observe that the condition 3 of Proposition 4.2 hold by the construction of the bracket. □

Now we give an example of the above construction. Let $\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\{X,Y,Z\}$ be the 3-dimensional real Heisenberg Lie algebra, i.e. $\mathfrak{h}$ is the 2-step nilpotent Lie algebra with bracket $[X,Y] = Z$. Let $(\mathfrak{h}_{\overline{\mathbb{C}}}, J_{\mathfrak{h}})$ be its associated quasi-Kähler Chern-flat Lie algebra. Then $\mathfrak{h}_{\overline{\mathbb{C}}}$ is the Lie algebra of the Iwasawa manifold described in Section 3 equipped with the almost complex structure J_3. An explicit isomorphism between the two Lie algebras can be described as follows:
$$X_1 \equiv X\otimes 1\ ,\ X_4 \equiv X\otimes \mathrm{i},$$
$$X_2 \equiv Y\otimes 1\ ,\ X_5 \equiv Y\otimes \mathrm{i},$$
$$X_3 \equiv Z\otimes 1\ ,\ X_6 \equiv Z\otimes \mathrm{i}\ .$$

4.2. **Quasi-Kähler Chern-flat Lie algebras in low dimension.** In this subsection we take into account quasi-Kähler Chern-flat Lie algebras in low dimension. In complex dimension 2 there exists only the abelian case. In view of [4] in complex dimension 3 there are the abelian Lie algebra and the Lie algebra of the complex Heisenberg group. The 4-dimensional case is described by the following

Theorem 4.5. *Let $(\mathfrak{g}, J)$ be a non-complex quasi-Kähler Chern-flat Lie algebra of complex dimension 4. Then there exists a $(1,0)$-frame $\{Z_1, \dots, Z_4\}$ on $\mathfrak{g}$ such that*

$$[Z_1, Z_2] = Z_{\overline{3}}$$

and the other brackets involving the vectors of the frame vanish.

Proof. Let $\{Z_1, \dots, Z_4\}$ be an arbitrary complex frame of type $(1,0)$ on $\mathfrak{g}$. Since J is not integrable, there exists at least a bracket involving two vectors of the frame different from zero. We may assume that $[Z_1, Z_2] \neq 0$ and we can write $[Z_1, Z_2] = \sum_{i=1}^4 c_{12}^{\bar{i}} Z_{\bar{i}}$. The Jacobi identity implies that it is not possible that $[Z_1, Z_2] \in \mathrm{Span}_{\mathbb{C}}\{Z_1, Z_2\}$. Indeed if $[Z_1, Z_2] = c_{12}^{\overline{1}} Z_{\overline{1}} + c_{12}^{\overline{2}} Z_{\overline{2}}$, then

$$[[Z_1, Z_2], Z_{\overline{1}}] = 0\,, \quad [[Z_1, Z_2], Z_{\overline{2}}] = 0$$

imply $[Z_1, Z_2] = 0$ which is a contradiction. So we can assume that $c_{12}^{\overline{3}} \neq 0$. Replacing Z_3 with $Z_3 = \frac{1}{c_{12}^{\overline{3}}}\left(Z_3 - \sum_{i \neq 3} c_{12}^{i} Z_i\right)$, we obtain

$$[Z_1, Z_2] = Z_{\overline{3}}\,.$$

Now we observe that the Jacobi identity implies that the vector Z_3 belongs to the center of $\mathfrak{g}$ since

$$0 = [[Z_1, Z_2], Z_{\overline{k}}] = [Z_{\overline{3}}, Z_{\overline{k}}]\,, \quad k = 1, \dots, 4\,.$$

The last step consists to show that we may assume $[Z_1, Z_4] = [Z_2, Z_4] = 0$. Setting $[Z_1, Z_4] = c_{14}^{\overline{s}} Z_{\overline{s}}$, we have

$$0 = [[Z_1, Z_4], Z_{\overline{1}}] = -c_{14}^{\overline{2}} Z_{\overline{3}} - c_{14}^{\overline{4}} [Z_{\overline{1}}, Z_{\overline{4}}]\,,$$
$$0 = [[Z_2, Z_4], Z_{\overline{2}}] = c_{14}^{\overline{1}} Z_{\overline{3}} - c_{24}^{\overline{4}} [Z_{\overline{2}}, Z_{\overline{4}}]\,,$$

i.e.

$$c_{14}^{\overline{2}} Z_{\overline{3}} = -c_{14}^{\overline{4}} [Z_{\overline{1}}, Z_{\overline{4}}]\,, \qquad c_{14}^{\overline{1}} Z_{\overline{3}} = c_{24}^{\overline{4}} [Z_{\overline{2}}, Z_{\overline{4}}]$$

which imply $[Z_1, Z_4] = c_{14}^{\overline{1}} Z_{\overline{1}} + c_{14}^{\overline{3}} Z_{\overline{3}}$, $[Z_2, Z_4] = c_{24}^{\overline{2}} Z_{\overline{2}} + c_{24}^{\overline{3}} Z_{\overline{3}}$. Hence if $[Z_1, Z_4] \neq 0$, then it is necessary to be $c_{14}^{\overline{3}} \neq 0$. So in the case $[Z_1, Z_4] \neq 0$ if we replace Z_4 with $\frac{1}{c_{14}^{\overline{3}}} Z_4 - Z_3$ we force $[Z_1, Z_4]$ to be a multiple of $Z_{\overline{1}}$ and the Jacobi identity implies that $[Z_1, Z_4] = 0$. Analogously if $[Z_2, Z_4] \neq 0$, then one replaces Z_4 with $\frac{1}{c_{24}^{\overline{3}}} Z_4 - Z_3$ obtaining $[Z_1, Z_4] = [Z_2, Z_4] = 0$. $\square$

In view of this last theorem, any Chern-flat quasi-Kähler Lie algebra of complex dimension 4 is reducible. In complex dimension 5 things work differently: for instance the almost complex Lie algebra $(\mathfrak{g}, J)$ admitting a $(1,0)$-frame $\{Z_1, \dots, Z_5\}$ satisfying the following structure equations

$$[Z_1, Z_2] = Z_{\overline{3}}\,, \quad [Z_2, Z_4] = Z_{\overline{5}}$$

gives rise to an example of a irreducible quasi-Kähler Chern-flat Lie algebra.

4.3. **Chern-flat Lie algebras with small center.** In this subsection we study quasi-Kähler Chern-flat Lie algebras having the center of complex dimension equal to one. Such a condition is admissible when the Lie algebra has complex dimension odd, only and in this case the space reduces to a standard model:

Theorem 4.6. *Let $(\mathfrak{g}, J)$ be a quasi-Kähler Chern-flat Lie algebra having center $\mathfrak{z}$ of complex dimension one. Then the complex dimension of $\mathfrak{g}$ is odd. Moreover, there exists a $(1,0)$-frame $\{Z_1, \dots, Z_n\}$ on $\mathfrak{g}$ such that*

$$[Z_i, Z_j] = Z_{\overline{n}}\,, \text{ for } i, j = 1, \dots, n-1,\ i < j\,.$$

where $Z_{\overline{n}}$ is a generator of $\mathfrak{z}^{(0,1)}$.

Proof. Since $(\mathfrak{g}, J)$ is Chern-flat, then the center of $\mathfrak{g}$ is J-invariant. Let $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$ be a decomposition $\mathfrak{g}$, where $\mathfrak{v}$ is a J-invariant subspace of $\mathfrak{g}$. Then $\mathfrak{g}^{1,0}$ splits in $\mathfrak{g}^{1,0} = \mathfrak{v}^{1,0} \oplus \mathfrak{z}^{1,0}$. The Lie bracket $[\ ,\ ]$ gives rise to a non-degenerated skew-symmetric two form ω on $\mathfrak{v}^{1,0}$. Namely,

$$[X, Y] := \omega(X, Y)\overline{A}$$

where $X, Y \in \mathfrak{v}^{1,0}$ and A is a generator of $\mathfrak{z}^{1,0}$. It is clear that ω must be non-degenerate since otherwise $\dim_{\mathbb{C}} \mathfrak{z} > 1$. This imply that $\dim_{\mathbb{C}} \mathfrak{v}$ is even. Let $\dim_{\mathbb{C}} \mathfrak{v} = 2m$, then $\dim_{\mathbb{C}} \mathfrak{g} = 2m + 1$. Now we consider the following skew-symmetric 2-form on $\mathbb{C}^{2k}$:

$$\Omega_k = \sum_{1 \le i < j \le 2k} dz_i \wedge dz_j$$

Notice that

$$\Omega_k = \Omega_{k-1} + \alpha \wedge \beta + dz_{2k-1} \wedge dz_{2k}\,,$$

where $\alpha = \sum_{1 \le i \le 2k-2} dz_i$ and $\beta = dz_{2k-1} + dz_{2k}$. Then

$$\Omega_k^k = \underbrace{\Omega \wedge \cdots \wedge \Omega}_{k-times} = \Omega_{k-1}^{k-1} \wedge dz_{2k-1} \wedge dz_{2k}\,.$$

It follows that $\Omega_k^k = \bigwedge_{i=1}^{i=2k} dz_i$. So Ω_k is not degenerated for all $k \in \mathbb{N}$. Since any two non-degenerated skew-symmetric 2-forms on $\mathbb{C}^{2k}$ are equivalent, we get that there exists a complex basis $(Z_1, \cdots Z_{2k})$ of $\mathfrak{v}^{1,0}$ such that:

$$\omega = \sum_{i<j} Z_i \wedge Z_j\,.$$

Hence $\{Z_1, \dots, Z_{n-1}, Z_n = A\}$ is the desired $(1,0)$-frame. $\square$

4.4. **Deformations of quasi-Kähler Chern-flat structures on Lie algebras.** Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra and let $\mathcal{M}$ be the moduli space of Chern-flat quasi-Kähler almost complex structures on $\mathfrak{g}$. The goal of this section is to determine the virtual tangent space to $\mathcal{M}$ at an arbitrary $[J] \in \mathcal{M}$. Such a space is defined as the quotient of all infinitesimal deformations of J by the subspace of the deformations obtained by Lie derivations.

Let $\{J_t\}_{t\in(-\delta,\delta)}$ a family of left-invariant Chern-flat quasi-Kähler almost complex structures on $\mathfrak{g}$ satisfying $J_0 = J$ and let $L := \frac{\mathrm{d}}{\mathrm{d}t} J_{t|t=0}$. Since for any t we have

$$J_t^2 = -\mathrm{I}\,, \quad J_t[X,Y] = -[J_tX,Y] = -[X,J_tY]\,,$$

then L satisfies

$$LJ = -JL\,, \quad L[X,Y] = -[LX,Y] = -[X,LY] \tag{4.1}$$

for every $X, Y \in \mathfrak{g}$. Moreover, if ϕ_t is a smooth family of endomorphisms of $\mathfrak{g}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_t J \phi_t^{-1}{}_{|t=0} = \mathcal{L}_X J\,,$$

being $\mathcal{L}$ the Lie derivative and X the infinitesimal vector associated to the path ϕ_t. We have

$$(\mathcal{L}_X J)Y = [X, JY] - J[X,Y] = -2J[X,Y]\,,$$

for every $X, Y \in \mathfrak{g}$. Hence the virtual tangent space to the moduli space $\mathcal{M}$ at an arbitrary point $[J]$ is defined by

$$T_{[J]}\mathcal{M} = \frac{\{L \in \mathrm{End}(\mathfrak{g}) \mid LJ = -JL\,,\, L[X,Y] = -[LX,Y] = -[X,LY]\ \forall X,Y \in \mathfrak{g}\}}{\{J[X,\cdot\,] \mid X \in \mathfrak{g}\}}\,.$$

Note that conditions (4.1) imply that if $\mathfrak{z}$ denotes the center of $\mathfrak{g}$, then

$$L([\mathfrak{g},\mathfrak{g}]) = 0\,, \quad L(\mathfrak{g}) \subseteq \mathfrak{z}\,.$$

Indeed, if $Z \in \mathfrak{g}^{1,0}$ then $L(Z) \in \mathfrak{g}^{0,1}$ and if Z_1, Z_2 are vectors of type $(1,0)$ one has

$$L[Z_1, Z_2] = [L(Z_1), Z_2] = 0$$

and if $X \in \mathfrak{z}$, then

$$[L(X), Y] = -L[X,Y] = 0$$

for any $Y \in \mathfrak{g}$.

We have the following

Proposition 4.7. *Let $\mathfrak{g}$ be a 2-step nilpotent Lie algebra such that $\dim_{\mathbb{C}}\mathfrak{z} = 1$. Then $\mathcal{M}$ is discrete.*

Proof. Assume that $\dim_{\mathbb{C}}\mathfrak{z} = 1$. We show that for every quasi-Kähler Chern-flat almost complex structure J on $\mathfrak{g}$ one has $T_{[J]}\mathcal{M} = 0$. Let L be an endomorphism of $\mathfrak{g}$ satisfying (4.1); then it has to be

$$L(Z) = l(Z)\,\overline{A}\,, \quad \forall Z \in \mathfrak{g}^{1,0}$$

for some $l \in \mathfrak{g}^*$, where A is a fixed generator of $\mathfrak{z}^{1,0}$. Since $\mathfrak{z}$ is a J-invariant subspace of $\mathfrak{g}$, we can split $\mathfrak{g}$ in $\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$, being $\mathfrak{v}$ a J-invariant complement of $\mathfrak{z}$. The form l can be viewed as a 1-form on $\mathfrak{v}$. Since the bracket of $\mathfrak{g}$ can be identified with a non-degenerate 2-form ω on $\mathfrak{v}$ by the relation $[Z_1, Z_2] = \omega(Z_1, Z_2)\,\overline{A}$, then there exists a vector X_L of $\mathfrak{g}$ such that $l(Y)\overline{A} = [X_L, Y]$ for every $Y \in \mathfrak{g}$. That easily implies that $T_{[J]}\mathcal{M}$ is trivial, as required. □

Note that this last Proposition accords to Theorem 4.6.

5. Proof of Theorem 1.2

In this section we consider almost complex structures tamed by a symplectic form. We recal that a symplectic form ω on a manifold M *tames* an almost complex J if $\omega_x(v, J_x v) \geq 0$ for every $x \in M$, $v \in T_x M$ and $\omega_x(v, J_x v) = 0$ if and only if $v = 0$.

In [4] we proved that the almost complex structure J_3 on the Iwasawa manifold can not be tamed by a symplectic form such that the pair (ω, J) induces a quasi-Kähler Chern-flat metric. Theorem 1.2 is the analogue of this result in an arbitrary quasi-Kähler Chern-flat manifold. In order to prove Theorem 1.2 we need to work explicitly with the operator A. Here we recall that such operator is a linear map

$$A\colon \Omega^{r,s} \to \Omega^{r+2,s-1}$$

which is defined by composing the de Rham operator

$$\mathrm{d}\colon \Omega^{r,s} \to \Omega^{r+2,s-1} \bigoplus \Omega^{r+1,s} \bigoplus \Omega^{r,s+1} \bigoplus \Omega^{r-1,s+2}$$

with the projection onto $\Omega^{r+2,s-1}$. If $\{Z_1, \dots, Z_n\}$ is a local $(1,0)$-frame with associated coframe $\{\zeta^1, \dots, \zeta^n\}$, then

$$A(\zeta^{\overline{r}})(Z_l, Z_k) = -\zeta^{\overline{r}}([Z_r, Z_k])$$

and we can write

$$A\zeta^{\overline{r}} = -\sum_{l,k=1}^{n} c^{\overline{r}}_{lk}\zeta^l \wedge \zeta^k\,,$$

where

$$[Z_r, Z_k] = \sum_{i=1}^{n} c^i_{rk} Z_i + \sum_{l=1}^{n} c^{\overline{l}}_{rk} Z_{\overline{l}}\,.$$

Theorem 1.2 can be viewed as a consequence of the following

Lemma 5.1. *Let (M, J) be a compact Chern-flat quasi-Kähler manifold and let $\beta \in \Omega^{2,0}$ be a $(2,0)$-form satisfying*

$$\overline{\partial}\beta + A\overline{\beta} = 0\,,$$

then β is closed, i.e.

$$\mathrm{d}\beta = 0\,.$$

Proof. In view of Theorem 1.1, J induces a structure of nilmanifold on M. Let G be the universal cover of M and let $(\mathfrak{g}, J)$ be the almost complex Lie algebra associated to G. We can find a $(1,0)$-frame $\{Z_1, \dots, Z_l, W_1, \dots, W_m\}$ on $\mathfrak{g}$ such that $\{W_1, \dots, W_m, W_{\overline{1}}, \dots, W_{\overline{m}}\}$ is a frame of the complexified of the center of $\mathfrak{g}$. With respect to this frame we have

$$[Z_i, Z_j] = \sum_{k=1}^{m} c_{ij}^{\overline{k}} \, W_{\overline{k}} \,, \quad i, j = 1, \dots, l \,.$$

Let $\{\zeta_1, \dots, \zeta_l, \eta_1, \dots, \eta_m\}$ be the coframe dual to $\{Z_1, \dots, Z_l, W_1, \dots, W_m\}$; then

$$\mathrm{d}\zeta_j = 0 \,, \quad j = 1, \dots, l \,; \quad \mathrm{d}\eta_i = c_{\overline{rs}}^{i} \, \zeta_{\overline{rs}} \,, \quad i = 1, \dots, m \,.$$

Identifying β with its pull-back on the universal cover of M, we can write

$$\beta = a_{ij}\zeta_{ij} + a_{ij}b_{ij}\zeta_i \wedge \eta_j + d_{ij}\eta_{ij}$$

and

$$\overline{\beta} = \overline{a_{ij}} \wedge \zeta_{\overline{ij}} + \overline{b_{ij}}\zeta_{\overline{i}} \wedge \eta_{\overline{j}} + \overline{d_{ij}}\eta_{\overline{ij}} \,,$$

where a_{ij}, b_{ij}, d_{ij} are smooth maps on M. Since the coframe $\{\zeta_1, \dots, \zeta_l, \eta_1, \dots, \eta_m\}$ is $(\partial + \overline{\partial})$-closed, we have

$$\overline{\partial}\beta = \overline{\partial}(a_{ij}) \wedge \zeta_{ij} + \overline{\partial}(b_{ij}) \wedge \zeta_i \wedge \eta_j + \overline{\partial}(d_{ij}) \wedge \eta_{ij} \,.$$

Furthermore

$$A\overline{\beta} = \overline{b_{rs}} c_{ij}^{\overline{s}} \, \zeta_{\overline{r}ij} + 2d_{\overline{rs}} c_{ij}^{\overline{s}} \, \eta_{\overline{r}} \wedge \zeta_{ij} = (\overline{b_{rs}} c_{ij}^{\overline{s}} \, \zeta_{\overline{r}} + 2d_{\overline{rs}} c_{ij}^{\overline{s}} \, \eta_{\overline{r}}) \wedge \zeta_{ij} \,.$$

Hence equation $\overline{\partial}\beta + A\overline{\beta} = 0$ reads in terms of a, b, d's as

$$\begin{cases} \overline{\partial}(a_{ij}) = \overline{b_{rs}} c_{ij}^{\overline{s}} \, \zeta_{\overline{r}} + 2d_{\overline{rs}} c_{ij}^{\overline{s}} \, \eta_{\overline{r}} \\ \overline{\partial}(b_{ij}) = \overline{\partial}(d_{ij}) = 0 \end{cases}$$

Since M is compact the b_{ij}'s and the d_{ij}'s have to be constant on M. Moreover the a_{ij}'s satisfies

$$\partial\overline{\partial}(a_{ij}) = 0 \,.$$

Again by the compactness of M we get the a_{ij}'s are constant. Hence β is a left-invariant form. Consequently it satisfies

$$\partial\beta = \overline{\partial}\beta = 0 \,.$$

Finally equation $\overline{\partial}\beta + A\overline{\beta} = 0$ implies $A\beta = 0$. Hence β is a closed form on M. □

Now we can prove Theorem 1.2

Proof of Theorem 1.2. Since by hypothesis the metric g induced on M by the pair (ω, J) is quasi-Kähler and Chern-flat, then J induces a structure of nilmanifold on M. Let $(\mathfrak{g}, J)$ be the Lie algebra associated to (M, J) and let ω_g be the Kähler form of (g, J). Then it has to be

$$\omega = \omega_g + \beta + \overline{\beta} \,.$$

being β a $(2,0)$-form on M. Since ω_g is $\overline{\partial}$-closed, condition on ω to be closed in terms of ω_g and β reads as

$$\begin{cases} A\omega_g + \partial\beta = 0 \\ \overline{\partial}\beta + A\overline{\beta} = 0\,. \end{cases}$$

In view of Lemma 5.1 such equations imply $\mathrm{d}\omega_g = 0$. There follows that g is an almost-Kähler metric on (M, J). Since g is Chern-flat, then J is integrable (see [4]). Hence g is a Kähler metric and M is a torus. □

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