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ON UNIVARIATE AND BIVARIATE AGING FOR DEPENDENT LIFETIMES WITH ARCHIMEDEAN SURVIVAL COPULAS

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Let $\boldsymbol{X} = (X, Y)$ be a pair of exchangeable lifetimes whose dependence structure is described by an Archimedean survival copula, and let $\boldsymbol{X}_t = [(X - t, Y - t)|X > t, Y > t]$ denotes the corresponding pair of residual lifetimes after time t, with $t \geq 0$. This note deals with stochastic comparisons between $\boldsymbol{X}$ and $\boldsymbol{X}_t$: we provide sufficient conditions for their comparison in usual stochastic and lower orthant orders. Some of the results and examples presented here are quite unexpected, since they show that there is not a direct correspondence between univariate and bivariate aging.

This work is mainly based on, and related to, recent papers by Bassan and Spizzichino ([4] and [5]), Averous and Dortet-Bernadet [2], Charpentier ([6] and [7]) and Oakes [16].

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1. INTRODUCTION

Let X be a random variable, and for each real $t \in \{t : \Pr\{X > t\} > 0\}$ let

$$X_t = [X - t \big| X > t]$$

denotes a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$. When X is a lifetime of a device then X_t can be interpreted as the residual lifetime of the device at time t, given that the device is alive at time t.

In the literature one can find several characterizations of aging notions by means of stochastic comparisons between the residual lifetimes X_t, with $t \in \{t : \Pr\{X > t\} > 0\}$. These characterizations serve a few purposes; they can be used when one wants to prove analytically that some random variable has an aging property, and they also throw a new light of understanding on the intrinsic meaning of the aging notions that are involved.

Among others, the following well-known aging notion can be defined by comparisons among X and any residual lifetime X_t: given a non-negative random lifetime

X defined on $[0, +\infty)$ we say that

$$X \in \text{NBU} \iff X_t \leq_{st} X \text{ whenever } t \geq 0.$$

An exhaustive list of applications and properties of the *New Better than Used* (NBU) notion may be found in Barlow and Proschan [3]. Here $\leq_{st}$ denotes the usual stochastic order (see Section 2 for definition, and Shaked and Shanthikumar, [18], for details about this stochastic comparison).

Let us consider now a pair $\boldsymbol{X} = (X, Y)$ of exchangeable non-negative random variables, and let

$$\overline{F}(x, y) = \Pr(X > x, Y > y) \text{ and } \overline{G}(x) = \overline{F}(x, 0) = \Pr(X > x)$$

be the corresponding joint survival function and marginal univariate survival function, respectively. Assume that $\overline{F}$ is a continuous survival function which is strictly decreasing on each argument, and that $\overline{G}(0) = 1$. Different concepts of aging for bivariate or multivariate lifetimes have been considered in the literature. For example, a multivariate extension of the NBU notion for exchangeable lifetimes, based on a Bayesian approach, has been recently defined and studied in Bassan and Spizzichino [5]; considered a new item with lifetime X and a used one with lifetime Y, they said that the exchangeable pair $\boldsymbol{X} = (X, Y)$ is multivariate NBU if, conditionally on the knowledge of the age of the used item, survival probabilities of the new item are greater than the survival probabilities of the used one, i. e., if, and only if,

$$\Pr(X > t + s | X > t) \leq \Pr(Y > s | X > t) \quad \forall\, t, s \geq 0.$$

Here we are interested in a bivariate extension of the NBU property based on a more traditional approach, assuming that the two items age together (like for example the multivariate extension provided in Marshall and Shaked [14]). For this, let us denote with $\boldsymbol{X}_t = [(X - t, Y - t)\ | X > t, Y > t]$ the pair of the residual lifetimes at time $t \geq 0$, i. e., the pair of non-negative random variables having joint survival function

$$\overline{F}_t(x, y) = \Pr(X > t + x, Y > t + y | X > t, Y > t) = \frac{\overline{F}(x + t, y + t)}{\overline{F}(t, t)}.$$

As a natural generalization of the NBU notion, one can in fact consider the stochastic inequality

$$\boldsymbol{X}_t \leq_{st} \boldsymbol{X} \text{ for all } t \geq 0. \tag{1}$$

Condition (1) is of course of interest in different fields of applied probability, like reliability and actuarial sciences. In reliability theory, in particular, it provides sufficient condition for the usual stochastic comparison of two systems having the same coherent life function τ but builded using new components or used components: in fact, for every $t \geq 0$ one has $\tau(\boldsymbol{X}_t) \leq_{st} \tau(\boldsymbol{X})$ if (1) holds, since coherent functions are non-decreasing in their arguments (see also Theorem 6.B.16(a) in Shaked and Shanthikumar [18]).

We will denote with $\mathcal{A}^+$ the class of bivariate lifetimes that satisfy (1). Similarly, one can consider the negative aging class $\mathcal{A}^-$ reversing the inequalities in (1) (as a generalization of the univariate negative aging property NWU, *New Worst than Used*), and the class $\mathcal{A}^0$ of bivariate lifetimes such that in (1) the equality holds for every $t \geq 0$ (as a generalization of the lack of memory property satisfied in the univariate case by the exponential distribution). This last class has been already extensively considered in the literature; see, for example, Ghurye and Marshall [12], where different characterizations of multivariate lifetimes in $\mathcal{A}^0$ are provided.

It is a well-know fact that the dependence structure of $\boldsymbol{X}$ can be usefully described by its survival copula K, defined as

$$K(u,v) = \overline{F}\left(\overline{G}^{-1}(u), \overline{G}^{-1}(v)\right),$$

where $(u,v) \in [0,1] \times [0,1]$ (see, e. g., Nelsen [15]). This function, together with the marginal survival function $\overline{G}$, allows for a different representation of $\overline{F}$ in terms of the pair $(\overline{G}, K)$, which is useful to analyze dependence properties between X and Y.

In this note we will restrict our attention to exchangeable bivariate lifetimes $\boldsymbol{X} = (X, Y)$ having an Archimedean survival copula, i. e., pairs such that $\overline{F}$ can be written in the form

$$\overline{F}(x,y) = W(R(x) + R(y)) \tag{2}$$

for a suitable one-dimensional, continuous, strictly positive, decreasing and convex survival function W and for a suitable continuous and strictly increasing function $R : [0,\infty) \to [0,\infty)$ such that $R(0) = 0$ and $\lim_{x\to\infty} R(x) = \infty$. Note that, in this case,

$$K(u,v) = W(W^{-1}(u) + W^{-1}(v)) \quad \text{and} \quad \overline{G}(x) = W(R(x)),$$

where the inverse W^{-1} is said to be the *generator* of the Archimedean copula K. We refer the reader to Nelsen [15] for an exhaustive monograph on copulas and Archimedean copulas.

Joint survival functions $\overline{F}$ that admit this representation have been called in different manners in the literature. We will use here the nomenclature introduced by Bassan and Spizzichino: a vector $\boldsymbol{X}$ with joint survival function of the form (2) is said to be defined by a *time-transformed exponential model*, denoted TTE(W, R). Further details on this model can be found for example in Bassan and Spizzichino [5].

We will provide here some simple sufficient conditions for a bivariate vector $\boldsymbol{X}$ defined by a time-transformed exponential model to be in the above defined aging classes. Also, some sufficient conditions for stochastic comparisons similar to (1) will be described. Only the case of bivariate lifetimes will be discussed here, being easy the generalization of the subsequent results and examples to vectors of lifetimes having more than two components.

2. PRELIMINARIES

Assume that $\boldsymbol{X}$ is described by a TTE(W, R) model as defined above. Then it is not hard to verify that the corresponding vector $\boldsymbol{X}_t$ of residual lifetimes at time t

is given by a TTE(W_t, R_t) model such that

$$\overline{F}_t(x,y) = W_t(R_t(x) + R_t(y)),$$

where

$$W_t(x) = \frac{W(2R(t)+x)}{W(2R(t))} \quad \text{and} \quad R_t(x) = R(t+x) - R(t).$$

Thus, the survival copula of $\boldsymbol{X}_t$ is defined by

$$K_t(u,v) = W_t\big(W_t^{-1}(u) + W_t^{-1}(v)\big),$$

while the univariate marginal survival function is given by

$$\overline{G}_t(x) = W_t(R_t(x)) = \frac{\overline{F}(x+t,t)}{\overline{F}(t,t)}.$$

Few preliminary properties should be recalled in order to present the subsequent sufficient conditions and related examples. The first one, which is essentially due to Genest and MacKay [11] and Averous and Dortet-Bernadet [2], deals with comparisons between the survival copulas K and K_t, and, therefore, with comparisons between the dependence structures of $\boldsymbol{X}$ and $\boldsymbol{X}_t$.

Property 1. Let $\boldsymbol{X}$ and $\boldsymbol{X}_t$ be defined as above. Then one has $K(u,v) = \;[\geq, \leq]$ $K_t(u,v)$ for all $(u,v) \in [0,1] \times [0,1]$ if, and only if,

$$\begin{aligned} W(W^{-1}(u) + W^{-1}(v)) &= \;[\geq, \leq]\; W_t(W_t^{-1}(u) + W_t^{-1}(v)) \\ &= \frac{W(2R(t) + W_t^{-1}(u) + W_t^{-1}(v))}{W(2R(t))}, \end{aligned}$$

i. e., if and only if

$$W^{-1} \circ W_t \quad \text{is an additive [superadditive, subadditive] function.} \tag{3}$$

Note that inequalities between $K(u,v)$ and $K_t(u,v)$, for all (u,v) and t, only depend on W (i. e., on K), and not on the function R. This fact has been also noticed in Foschi and Spizzichino [10], where conditions like (3) but for more general survival copulas are provided.

It should be also pointed out that, as proved in recent works by Charpentier, [6] and [7] and Oakes [16], within the class of Archimedean copulas the equality in (3) is satisfied if, and only if, K is a Clayton copula, i. e., if and only if $W(x) = (x+1)^{-\theta}$ for some positive θ, or, equivalently,

$$K(u,v) = \max\left\{(u^{-\frac{1}{\theta}} + v^{-\frac{1}{\theta}} - 1)^{-\theta}, 0\right\}.$$

These copulas have been introduced in Clayton [8], and further studied and applied, for example, in Cook and Johnson [9], Juri and Wüthrich [13] and Charpentier [7], among others.

The next statement deals with comparisons between the univariate marginal survival functions $\overline{G}$ and $\overline{G}_t$, and it can be easily proved by straightforward calculations.

Property 2. It holds $\overline{G}(s) = [\geq, \leq]\ \overline{G}_t(s)$ for every $s, t \geq 0$ if, and only if,

$$\overline{F}(t+s,t) = [\leq, \geq]\ \overline{F}(t,t) \cdot \overline{F}(s,0),$$

i. e., if and only if

$$W(R(t+s)+R(t)) = [\leq, \geq]\ W(2R(t)) \cdot W(R(s)) \quad \forall\, t, s \geq 0. \tag{4}$$

Note that in the case when $\boldsymbol{X}$ has a Clayton survival copula, then equation (4) becomes

$$R(t+s) = [\geq, \leq]\ R(t) + R(s) + 2R(t)R(s), \quad \forall\, t, s \geq 0. \tag{5}$$

We also recall the definition of some stochastic comparisons and of a positive dependence order that will be mentioned in the next sections. Given two bivariate random vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$, having joint survival functions $\overline{F}_{\boldsymbol{X}}$ and $\overline{F}_{\boldsymbol{Y}}$, respectively, we say that

(i) $\boldsymbol{X}$ *is smaller than* $\boldsymbol{Y}$ *in usual stochastic order (shortly* $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$*) if, and only if,* $\boldsymbol{E}[h(\boldsymbol{X})] \leq \boldsymbol{E}[h(\boldsymbol{Y})]$ *for every non-decreasing function* $h : \boldsymbol{R}^2 \to \boldsymbol{R}$ *such that the two expectations exist;*

(ii) $\boldsymbol{X}$ *is smaller than* $\boldsymbol{Y}$ *in the upper orthant [lower orthant] order (shortly* $\boldsymbol{X} \leq_{uo} [\leq_{lo}]\ \boldsymbol{Y}$*) if, and only if* $\overline{F}_{\boldsymbol{X}}(x,y) \leq \overline{F}_{\boldsymbol{Y}}(x,y)$ $[F_{\boldsymbol{X}}(x,y) \geq F_{\boldsymbol{Y}}(x,y)]$ *for all* $(x,y) \in \boldsymbol{R}^2$*;*

(iii) $\boldsymbol{X}$ *is smaller than* $\boldsymbol{Y}$ *in the Positive Quadrant Dependence order (shortly* $\boldsymbol{X} \leq_{\mathrm{PQD}} \boldsymbol{Y}$*) if, and only if, they have the same marginals and both stochastic inequalities* $\boldsymbol{X} \leq_{uo} \boldsymbol{Y}$ *and* $\boldsymbol{X} \geq_{lo} \boldsymbol{Y}$ *hold.*

Details on these stochastic comparisons may be found in Shaked and Shanthikumar [18]. Here we just recall that $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$ implies both $\boldsymbol{X} \leq_{uo} \boldsymbol{Y}$ and $\boldsymbol{X} \leq_{lo} \boldsymbol{Y}$, while $\boldsymbol{X} \leq_{\mathrm{PQD}} \boldsymbol{Y}$ holds if, and only if, they have the same marginal distributions and $K_{\boldsymbol{X}}(u,v) \leq K_{\boldsymbol{Y}}(u,v)$ for all $(u,v) \in [0,1] \times [0,1]$, where $K_{\boldsymbol{X}}$ and $K_{\boldsymbol{Y}}$ are the survival copulas of $\boldsymbol{X}$ and $\boldsymbol{Y}$, respectively.

3. THE CLAYTON COPULA CASE

In this section we describe some conditions for a bivariate lifetime $\boldsymbol{X}$ defined via a TTE(W,R) model to be in one of the aging classes $\mathcal{A}^0$, $\mathcal{A}^+$ or $\mathcal{A}^-$. We will not consider here the trivial case of independent lifetimes X and Y (that corresponds to $W(x) = \exp(-\theta x)$ for some positive θ).

In this and the subsequent sections $\widetilde{\boldsymbol{X}} = (\tilde{X}, \tilde{Y})$ and $\widetilde{\boldsymbol{X}}_t = (\tilde{X}_t, \tilde{Y}_t)$ denote the two bivariate vectors having uniformly [0,1] distributed univariate marginals and joint distributions K and K_t, respectively. For the prosecution, it is useful also to recall that

$$\boldsymbol{X} =_{st} (\overline{G}^{-1}(\tilde{X}), \overline{G}^{-1}(\tilde{Y})) \text{ and } \widetilde{\boldsymbol{X}} =_{st} (\overline{G}(X), \overline{G}(Y)),$$

and similarly for $\boldsymbol{X}_t$ and $\widetilde{\boldsymbol{X}}_t$, as one can easily verify.

Moreover, in almost all of the examples presented in this section the function R is assumed to be a member of the family $\{R_{\alpha,b}, \alpha, b > 0\}$ where

$$R_{\alpha,b}(x) = \frac{e^{bx} - 1}{\alpha}. \tag{6}$$

It is useful to observe that, for every fixed $b > 0$, equality in (5) is satisfied for $\alpha = 2$, while inequalities $\geq$ and $\leq$ are satisfied, respectively, for $\alpha > 2$ and $\alpha < 2$.

The following statement is almost immediate.

Theorem 1. Let $\boldsymbol{X}$ be defined via a TTE(W, R) model, i.e. such that its joint survival function is of the form

$$\overline{F}(x, y) = W(R(x) + R(y)), \quad x, y \geq 0.$$

Then it satisfies the weak multivariate lack of memory property $\mathcal{A}^0$ if, and only if,

$$W(x) = (x + 1)^{-\theta}$$

for some positive constant θ, and $R(x) = R_{2,b}(x)$ where b is any strictly positive real number.

P r o o f. It is enough to observe that in order to satisfy condition $\boldsymbol{X} =_{st} \boldsymbol{X}_t$ for all $t \geq 0$, vectors $\boldsymbol{X}$ and $\boldsymbol{X}_t$ should of course have the same survival copula. This means that (3) should be satisfied with equality, i.e., that K should be a Clayton copula (with any positive value for the parameter θ). Moreover, it should also be satisfied that $\overline{G}(s) = \overline{G}_t(s)$ for every $s, t \geq 0$, i.e., (5) should be verified with equality. It is a well-known fact that the functional equation (5) is satisfied only by the function $R(x) = (e^{bx} - 1)/2$ where $b > 0$ (see, e.g., Aczél [1]). Thus, letting K be a Clayton copula, and R defined as above, we have

$$\begin{aligned} \boldsymbol{X} \ =_{st} \ (\overline{G}^{-1}(\tilde{X}), \overline{G}^{-1}(\tilde{Y})) \ &=_{st} \ (\overline{G}^{-1}(\tilde{X}_t), \overline{G}^{-1}(\tilde{Y}_t)) \\ &=_{\text{a.s}} \ (\overline{G}_t^{-1}(\tilde{X}_t), \overline{G}_t^{-1}(\tilde{Y}_t)) \ = \ \boldsymbol{X}_t, \end{aligned}$$

i.e., $\boldsymbol{X} =_{st} \boldsymbol{X}_t$ for all $t \geq 0$. Here the first equality follows from (3), while the second one from (5). □

The joint survival function and univariate marginal survival function of the vector $\boldsymbol{X}$ that satisfies the assumption of Theorem 1 are, respectively,

$$\overline{F}(x, y) = \left(\frac{e^{bx} + e^{by}}{2} \right)^{-\theta} \quad \text{and} \quad \overline{G}(x) = \left(\frac{e^{bx} + 1}{2} \right)^{-\theta}.$$

It should be observed that, because of the comments above, any bivariate joint survival function described by a TTE(W, R) model that is in the class $\mathcal{A}^0$ (i.e., that satisfy the weak multivariate lack of memory property as defined in Ghurye and Marshall [12]) should be of this kind. It is also remarkable the fact that the marginals X and Y of a pair $\boldsymbol{X}$ in this parametric family are not exponentially distributed but are NBU (as one can verify recalling that the marginal distribution is $\overline{G}(t) = W(R(t)),\ t \geq 0$).

Reasoning in a similar manner as in the previous Theorem 1, one can prove the following sufficient conditions for positive, or negative, bivariate aging.

Theorem 2. Let $\boldsymbol{X}$ be defined via a TTE(W, R) model. Then it is in the $\mathcal{A}^+$ $[\mathcal{A}^-]$ class if

$$W(x) = (x+1)^{-\theta}$$

for some positive constant θ, and the function R satisfies equation (5) with inequality $\geq$ $[\leq]$.

Proof. By assumption it holds $\widetilde{\boldsymbol{X}} =_{st} \widetilde{\boldsymbol{X}}_t$ and $\overline{G}(u) \geq [\leq]\ \overline{G}_t(u)$ for all $t, u \geq 0$, and therefore also

$$\begin{aligned}\boldsymbol{X} =_{st} (\overline{G}^{-1}(\tilde{X}), \overline{G}^{-1}(\tilde{Y})) &=_{st} (\overline{G}^{-1}(\tilde{X}_t), \overline{G}^{-1}(\tilde{Y}_t)) \\ &\geq_{\text{a.s}} (\overline{G}_t^{-1}(\tilde{X}_t), \overline{G}_t^{-1}(\tilde{Y}_t)) = \boldsymbol{X}_t,\end{aligned}$$

i.e., $\boldsymbol{X} \geq_{st} [\leq_{st}]\ \boldsymbol{X}_t$ for all $t \geq 0$. □

Considering the above result it is not hard to provide examples of bivariate vectors $\boldsymbol{X}$ that are in the $\mathcal{A}^+$ or $\mathcal{A}^-$ classes. In fact, we can again consider the case that K is a Clayton copula, and we can take a function R such that inequality $\geq$, or $\leq$, is satisfied in (5). This happens, for example, when $R(x) = R_{1,b}(x)$ with $b > 0$. In this case we have $\overline{G}_t(u) \leq \overline{G}_{t+s}(u)$ for all $u \geq 0$, and therefore the vector $\boldsymbol{X}$ is in the $\mathcal{A}^-$ class. The joint survival function and univariate marginal survival function of this vector $\boldsymbol{X}$ are, respectively,

$$\overline{F}(x,y) = \left(e^{bx} + e^{by} - 1\right)^{-\theta} \quad \text{and} \quad \overline{G}(x) = e^{-b\theta x},$$

with $\theta, b > 0$. This is an interesting case, since the marginals X and Y are exponentially distributed (thus both NBU and NWU).

Obviously, it is also immediate to provide an example of a bivariate vector $\boldsymbol{X}$ that is in the $\mathcal{A}^+$ class. Assume again that K is a Clayton copula, and take a function R such that inequality $\geq$ is satisfied in (4). This is for example the case when $R(x) = R_{3,b}(x)$ with $b > 0$. The joint survival function and univariate marginal survival function of this vector $\boldsymbol{X}$ are, respectively,

$$\overline{F}(x,y) = \left(\frac{e^{bx} + e^{by} + 1}{3}\right)^{-\theta} \quad \text{and} \quad \overline{G}(x) = \left(\frac{e^{-bx} + 2}{3}\right)^{-\theta}.$$

It is not hard to verify that in this last case the marginals X and Y are NBU.

At this point it is interesting to observe that if $\boldsymbol{X}$ has a Clayton copula, and the function R is such that

$$R(t+s) = \ [\leq, \geq]\ R(t)R(s) + R(t) + R(s)$$

for all $t, s \geq 0$, then

$$\begin{aligned}\Pr(X > t+s) &= W(R(t+s)) = (R(t+s)+1)^{-\theta} \\ &= \ [\geq, \leq]\ (R(t)+1)^{-\theta} \cdot (R(s)+1)^{-\theta} \\ &= W(R(t)) \cdot W(R(s)) \\ &= \Pr(X > t) \cdot \Pr(X > s),\end{aligned}$$

for all $t, s \geq 0$. It follows that in this case the marginals are exponentially distributed [NWU, NBU].

Thus, when the survival copula K is a Clayton copula we have the following possible cases.

- When the function R is such that $R(t+s) \leq R(t)R(s) + R(t) + R(s)$ for all $t, s \geq 0$ (thus also $R(t+s) \leq 2R(t)R(s) + R(t) + R(s)$), then $\boldsymbol{X}$ is in the $\mathcal{A}^-$ class and the marginals are NWU. This is the case, for example, of $R(x) = bx$, with $b > 0$, or $R(x) = R_{\alpha,b}(x)$ with $\alpha \in (0,1)$ and $b > 0$.
- When the function R is such that $R(t+s) = R(t)R(s) + R(t) + R(s)$ for all $t, s \geq 0$ (thus also $R(t+s) \leq 2R(t)R(s) + R(t) + R(s)$), then $\boldsymbol{X}$ is in the $\mathcal{A}^-$ class and the marginals are exponentially distributed. This is the case, of $R(x) = R_{\alpha,b}(x)$ with $\alpha = 1$ and $b > 0$.
- When the function R is such that $R(t)R(s) + R(t) + R(s) \leq R(t+s) \leq 2R(t)R(s) + R(t) + R(s)$ for all $t, s \geq 0$ then $\boldsymbol{X}$ is in the $\mathcal{A}^-$ class and the marginals are NBU. This is the case, for example, of $R(x) = R_{\alpha,b}(x)$ with $\alpha \in (1,2)$ and $b > 0$.
- When the function R is such that $R(t+s) = 2R(t)R(s) + R(t) + R(s)$ for all $t, s \geq 0$ (thus also $R(t+s) \geq R(t)R(s) + R(t) + R(s)$), then $\boldsymbol{X}$ is in the $\mathcal{A}^0$ class and the marginals are NBU. This is the case of $R(x) = R_{\alpha,b}(x)$ with $\alpha = 2$ and $b > 0$ (which is the only possible case, as discussed above).
- When the function R is such that $R(t+s) \geq 2R(t)R(s) + R(t) + R(s)$ for all $t, s \geq 0$ (thus also $R(t+s) \geq R(t)R(s) + R(t) + R(s)$), then $\boldsymbol{X}$ is in the $\mathcal{A}^+$ class and the marginals are NBU. This is the case, for example, of $R(x) = \frac{e^{bx} - 1 - bx}{2}$, with $b > 0$, or $R(x) = R_{\alpha,b}(x)$ with $\alpha \in (2, +\infty)$ and $b > 0$, or

  $$R(x) = \begin{cases} (1-x)^{-1/\theta} - 1 & \text{if } x < 1 \\ +\infty & \text{if } x \geq 1. \end{cases}$$

  Note that in this last case the marginal distributions are uniformly distributed on $[0,1] \subseteq \boldsymbol{R}$.

It is also interesting to observe that for bivariate lifetimes $\boldsymbol{X}$ having Clayton survival copula and $R(x) = R_{\alpha,b}(x)$, with $b, \alpha > 0$, the aging behaviors only depend on the value of the parameter α.

Of course, there are also cases such that the function R satisfies, for example, $R(t+s) \leq 2R(t)R(s)+R(t)+R(s)$ for all $t, s \geq 0$, but neither $R(t+s) \leq R(t)R(s)+R(t)+R(s)$ nor $R(t+s) \geq R(t)R(s)+R(t)+R(s)$ for all $t, s \geq 0$ are satisfied. Thus, there are cases such that $\boldsymbol{X}$ is in $\mathcal{A}^-$ but its marginals are neither NBU nor NWU. This happens for example when $R(x) = \frac{e^{bx}-1+bx}{2}$, with $b > 0$. However, there are some relationships between univariate and bivariate aging that can be asserted, as summarized in the following statement.

Property 3. Let the bivariate vector of lifetimes $\boldsymbol{X}$ described by a TTE(W, R) model. Let $W(x) = (x+1)^{-\theta}$, with $\theta > 0$, i. e., let $\boldsymbol{X}$ has a Clayton survival copula. The following assertions hold:

i) if the marginals of $\boldsymbol{X}$ are NWU then $\boldsymbol{X}$ is in the $\mathcal{A}^-$ class;

ii) if $\boldsymbol{X}$ is in the $\mathcal{A}^+$ class then the marginals are NBU.

What is interesting to remark is that there exist cases of bivariate lifetimes such that $\boldsymbol{X}$ is in the $\mathcal{A}^-$ class even if the marginals are NBU, while we can not have cases where $\boldsymbol{X}$ is in the $\mathcal{A}^+$ class and the marginals are NWU.

4. THE NON–CLAYTON CASE

When one considers survival copulas different from the Clayton copula, then it is not easy to find conditions for the usual stochastic comparisons between $\boldsymbol{X}$ and $\boldsymbol{X}_t$. Thus here we just mention some sufficient conditions for weaker comparisons among them, i.e, for comparisons in lower orthant order ($\leq_{lo}$). Even if this comparison is weaker than the usual stochastic order $\leq_{st}$, it has its own reasons of interest. In fact, it should be recalled that from inequality $(X_1, X_2) \leq_{lo} (Y_1, Y_2)$ it follows that, for example, $\max\{X_1, X_2\} \leq_{st} \max\{Y_1, Y_2\}$.

As we will see, all of these sufficient conditions are based on equations (3) and (4). In fact, as we have already seen, comparisons between $\boldsymbol{X}$ and $\boldsymbol{X}_t$ are essentially based on conditions for inequalities $\geq$ or $\leq$ in (3), which means on conditions for $\widetilde{\boldsymbol{X}} \geq_{\text{PQD}} \widetilde{\boldsymbol{X}}_t$ or $\widetilde{\boldsymbol{X}} \leq_{\text{PQD}} \widetilde{\boldsymbol{X}}_t$ for all $t \geq 0$, and conditions for inequalities $\geq$ or $\leq$ in (4).

As already mentioned, conditions for $\widetilde{\boldsymbol{X}} \geq_{\text{PQD}} \widetilde{\boldsymbol{X}}_t$ or $\widetilde{\boldsymbol{X}} \leq_{\text{PQD}} \widetilde{\boldsymbol{X}}_t$ for all $t \geq 0$ have been provided in Foschi and Spizzichino [10], where more general dependence structures are considered. Other sufficient conditions are described in Charpentier [7]. Here we provide a new result on this direction, giving immediate conditions for these comparisons. For this, we recall that a random variable X is said to be IFR (*Increasing Fairure Rate*) if, and only if,

$$X \in \text{IFR} \iff X_{t+s} \leq_{st} X_t \quad \text{whenever } t, s \geq 0,$$

and that similarly, reversing the above inequality, is defined the negative aging notion DFR (*Decreasing Fairure Rate*).

Property 4. Let the bivariate vector of lifetimes $\boldsymbol{X}$ described by a TTE(W, R) model. If the function W is DFR [IFR] then $\widetilde{\boldsymbol{X}} \geq_{\text{PQD}}$ $[\leq_{\text{PQD}}]$ $\widetilde{\boldsymbol{X}}_t$ for all $t \geq 0$.

Proof. Fix $t \geq 0$ and let Z be a random lifetime having distribution $F_Z(u) = 1 - W(u)$. Also, let $\widetilde{t} = 2R(t)$, and observe that $0 = 2R(0) \leq \widetilde{t}$ because the function R is non-negative.

Since W is DFR, by Theorem 2.2 and Lemma 2.1 in Pellerey and Shaked [17] it follows $Z \leq_{\text{disp}} Z_{\widetilde{t}}$, where $\leq_{\text{disp}}$ denotes the *dispersive order* (see Shaked and Shanthikumar [18] for definition and properties of this variability order), and $Z_{\widetilde{t}} = [Z - \widetilde{t}|\ Z > \widetilde{t}]$. Moreover, since W is DFR then it also follows $Z \leq_{st} Z_{\widetilde{t}}$. Thus it is possible to apply Theorem 3.B.10(b) in Shaked and Shanthikumar [18], to obtain that $\phi(Z) \geq_{\text{disp}} \phi(Z_{\widetilde{t}})$ for every non-decreasing and concave function ϕ. In particular, it follows that $\ln(Z) \geq_{\text{disp}} \ln(Z_{\widetilde{t}})$, which is equivalent to the monotonicity property

$$\frac{W^{-1}(u)}{W_t^{-1}(u)} \text{ is non-increasing in } u. \tag{7}$$

Now the assertion follows observing that (7) implies superadditivity of $W^{-1} \circ W_t$, which implies, by Proposition 4 in Avérous and Dortet-Bernadet [2], $K(u,v) \geq K_t(u,v)$ for all $u, v \in [0,1]$, i. e., $\widetilde{\boldsymbol{X}} \geq_{\text{PQD}} \widetilde{\boldsymbol{X}}_t$. The proof of the assertion in the brackets is similar. □

It should be pointed out that the DFR [IFR] property of the survival function W implies strong positive [negative] dependence properties of $\boldsymbol{X}$, as shown in Avérous and Dortet-Bernadet [2]. Examples of Archimedean survival copulas having W that is DFR are the Clayton copula and the Gumbel copula, where $W(x) = \exp(-x^{\frac{1}{\theta}})$ with $\theta \geq 1$. On the contrary, the Archimedean survival copula numbered as 9 in Nelsen [15], page 94, is defined by an IFR survival function $W(x) = \exp(\frac{1-e^x}{\theta})$ with $0 < \theta \leq 1$.

Regarding the conditions for inequalities $\geq$ or $\leq$ in (4), the following useful property holds.

Property 5. Let the bivariate vector of lifetimes $\boldsymbol{X}$ described by a TTE(W, R) model. If W is DFR [IFR] and R is concave [convex], then $\overline{G}(u) \leq$ $[\geq]$ $\overline{G}_t(u)$ for all $t, u \geq 0$.

Proof. We give only the proof of the assertion without the brackets, the other being similar. Let $t, u \geq 0$. Also let $\widetilde{t} = R(t)$ and $\delta = R(t+u) - R(t)$, and observe that $\delta \leq R(u)$, by the concavity or R, and $\widetilde{t} \geq 0$, since R is non-decreasing. Thus

$$\begin{aligned}\overline{G}_t(u) &= \frac{W(R(t+u)+R(t))}{W(2R(t))} = \frac{W(2\widetilde{t}+\delta)}{W(2\widetilde{t})} \\ &\geq \frac{W(\delta)}{W(0)} \geq \frac{W(R(u))}{W(0)} = \overline{G}(u).\end{aligned}$$

The first inequality follows from the DFR property of W (which is equivalent to require that the ratio $\frac{W(t+u)}{W(t)}$ is non-decreasing in t for every fixed $u \geq 0$), while the second inequality follows from $\delta \leq R(u)$. □

Thus, one immediately gets the following sufficient conditions for lower orthant comparisons between $\boldsymbol{X}$ and $\boldsymbol{X}_t$.

Property 6. Let the bivariate vector of lifetimes $\boldsymbol{X}$ described by a TTE(W, R) model. If W is DFR [IFR] and R is concave [convex], then $\boldsymbol{X} \leq_{lo} [\geq_{lo}] \boldsymbol{X}_t$ for all $t \geq 0$.

P r o o f. We give only the proof of the assertion without the brackets, the other being similar. In this case it holds

$$\begin{aligned}\boldsymbol{X} =_{st} (\overline{G}^{-1}(\widetilde{X}), \overline{G}^{-1}(\widetilde{Y})) &\geq_{\text{PQD}} (\overline{G}^{-1}(\widetilde{X}_t), \overline{G}^{-1}(\widetilde{Y}_t)) \\ &\leq_{\text{a.s}} (\overline{G}_t^{-1}(\widetilde{X}_t), \overline{G}_t^{-1}(\widetilde{Y}_t)) = \boldsymbol{X}_t,\end{aligned}$$

for all $t \geq 0$. The first inequality follows from Property 4, while the second one from Property 5. Thus, there exists $\boldsymbol{Z}$ such that $\boldsymbol{X} \geq_{\text{PQD}} \boldsymbol{Z} \leq_{st} \boldsymbol{X}_t$, and therefore $\boldsymbol{X} \leq_{lo} \boldsymbol{X}_t$. □

Consider now the particular case where $R(x) = x$, i.e., where the joint survival function $\overline{F}$ is Schur–constant (see Bassan and Spizzichino, [5], on this property). From Property 6 follows that under positive dependence (like for the Gumbel survival copula case) it holds $\boldsymbol{X} \leq_{lo} \boldsymbol{X}_t$ for all $t \geq 0$, while under negative dependence (like for the survival copula numbered as 9 in Nelsen [15]) it holds $\boldsymbol{X} \geq_{lo} \boldsymbol{X}_t$ for all $t \geq 0$.

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